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An application of the concept of a generalized central element

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Dedicated to Professor V. V. Kirichenko on the occasion of his 65th birthday

Abstract. With the help of the concept of a generalized central element we study finite groups with the given system of S-quasinormally embedded subgroups.

1. Introduction

In 2001 L. A. Shemetkov proposed a new concept of a generalized central element (see [1] for details). Following the terminology of L.A. Shemetkov [1], we say that an element x of a finite group G is Q-central in G if there exists a central chief factor H/K of G such that $x \in H \setminus K$. Using this concept, the following result was obtained in [2].

Theorem 1.1 [2]. *A finite group G is p-nilpotent if and only if every element in $G_p \setminus \Phi(G_p)$ is Q-central in G.*

Here G_p is a Sylow p-subgroup of G; $\Phi(G_p)$ is the Frattini subgroup of G_p.

In fact, L. A. Shemetkov gives a definition of a generalized central element within the framework of a general approach [1], considering arbitrary function

$$f : \{\text{groups}\} \longrightarrow \{\text{group classes}\}.$$

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In this case, L. A. Shemetkov defines a Qf-central element in the following way: an element x of a finite group G is Qf-central in G if there exists a chief factor H/K of G such that $G/C_G(H/K) \in f(H/K)$ and $x \in H \setminus K$. With the help of the concept of a Qf-central element a characterization of saturated formations was obtained in [3]. We also mention articles [4–5] in which Shemetkov's concept was applied.

A subgroup H of a finite group G is said to be S-quasinormal in G if it permutes with every Sylow subgroup of G. This concept was introduced by O. Kegel in [6] and has been studied extensively by several authors. More recently, A. Ballester-Bolinches and M. C. Pedraza-Aguilera [7] introduced the following interesting definition. A subgroup H of a finite group G is said to be S-quasinormally embedded in G if for each prime p dividing the order of H, a Sylow p-subgroup of H is also a Sylow p-subgroup of some S-quasinormal subgroup of G.

In this paper we show that the concept of a generalized central element is also useful for the study of S-quasinormally embedded subgroups. First, we use Theorem 1.1 for proving the following result.

Theorem 1.2. *Let H be a normal subgroup of a finite group G, and p the smallest prime dividing $|H|$. Assume that every maximal subgroup of H_p is S-quasinormally embedded in G. Then H is p-nilpotent, and its non-Frattini G-chief p-factors are cyclic.*

Second, we apply Theorem 1.2 for proving the following.

Theorem 1.3. *Let H be a normal subgroup of a finite group G. Assume that all maximal subgroups of all Sylow subgroups in H are S-quasinormally embedded in G. Then H has an ordered Sylow tower, and every non-Frattini G-chief factor of H is cyclic.*

Third, we will show that some results in [7–8] are corollaries of Theorem 1.3.

2. Preliminaries

We use standard notations (see [9]). If L/K is a chief factor of a finite group G and $L \subseteq H \trianglelefteq G$, then L/K we call a G-chief factor of H; the chief factor L/K is called non-Frattini if L/K is not contained in the Frattini subgroup of G/K. If p is a prime, then $O^p(G)$ is the subgroup generated by all p'-elements of G, and $O^{p'}(G)$ is the subgroup generated by all p-elements of G. We say that a finite group H of order $p_1^{\alpha_1} p_2^{\alpha_2} \dots p_n^{\alpha_n}$, $p_1 > p_2 > \dots > p_n$, has an ordered Sylow tower if H has a normal subgroup of order $p_1^{\alpha_1} p_2^{\alpha_2} \dots p_i^{\alpha_i}$ for every $i = 1, 2, \dots, n$.

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We need some notations from the formation theory. A formation is a group class closed under taking homomorphic images and finite subdirect products. A formation $\mathfrak{F}$ of finite groups is called saturated if for every finite group G, $G/\Phi(G) \in \mathfrak{F}$ always implies $G \in \mathfrak{F}$. Let a function f associate with every prime p a formation $f(p)$. A chief factor H/K of a finite group G is called f-central in G if $G/C_G(H/K) \in f(p)$ for every prime divisor p of $|H/K|$. The class $LF(f)$ of all finite groups whose all chief factors are f-central, is a formation. Following L. A. Shemetkov [12], f is called *a local satellite* of the formation $\mathfrak{F} = LF(f)$. A local satellite f of $\mathfrak{F} = LF(f)$ is called: 1) integrated if $f(p) \subseteq \mathfrak{F}$ for every prime p; 2) canonical if it is integrated and $f(p) = \mathfrak{N}_p f(p)$, for every prime p (here $\mathfrak{N}_p$ is the class of all finite p-groups). It is known that every non-empty saturated formation of finite groups has a unique canonical satellite (see [9], Theorem IV,3.7).

Lemma 2.1 ([7]). *Let U be a S-quasinormally embedded subgroup of a finite group G. If $U \leq H \leq G$ and $K \trianglelefteq G$, then:*

(a) *U is S-quasinormally embedded in H;*

(b) *UK is S-quasinormally embedded in G and UK/K is S-quasinormally embedded in G/K.*

Lemma 2.2 ([10]). *Let G be a finite group.*

(a) *An S-quasinormal subgroup of G is subnormal in G.*

(b) *If $H \leq K \leq G$ and H is S-quasinormal in G, then H is S-quasinormal in K.*

Lemma 2.3 ([10]). *A p-subgroup H of a finite group G is S-quasinormal in G iff $N_G(H) \supseteq O^p(G)$.*

Lemma 2.4. *Let G be a finite group. If G_p is normal in a subnormal subgroup H of G, then G_p is normal in G.*

Proof. By the condition, there exists a subnormal series

$$G_p \trianglelefteq H_0 = H \trianglelefteq H_1 \trianglelefteq \cdots \trianglelefteq H_n = G.$$

Since G_p is characteristic in H, we have that $G_p \trianglelefteq H_1$. So, G_p is characteristic in H_i for every $i = 1, 2, \ldots, n$. □

Lemma 2.5. *Let P be a normal p-subgroup of a finite group G. If a subgroup P_1 of P is S-quasinormally embedded in G, then P_1 is S-quasinormal in G.*

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Proof. By the condition, P_1 is a Sylow subgroup of some S-quasinormal subgroup H of G. Since $P_1 = P \cap H$ is normal in H, it follows from Lemma 2.2(a) and Lemma 2.4 that P_1 is normal in HS for every Sylow q-subgroup S of G, $q \neq p$. Since $G_p^x \supseteq P \supseteq P_1$ for any $x \in G$, we have that $G_p^x P_1 = P_1 G_p^x = G_p^x$. Hence, P_1 is S-quasinormal. □

Lemma 2.6 ([11], Lemma 11.6). *Let G be a finite group. If $G = AB$, then for every prime p there exists Sylow p-subgroups P, P_1 and P_2 in G, A and B such that $P = P_1P_2$.*

Lemma 2.7. *Let p be the smallest prime dividing the order of a finite group G. If G_p is cyclic, then G is p-nilpotent.*

Proof. Assume that G is non-p-nilpotent. By Frobenius theorem, G possesses a p-closed non-nilpotent subgroup S of order $p^\alpha q^\beta$ with a cyclic Sylow q-subgroup. Since S_p and S_q are cyclic and $p < q$, S is p-nilpotent, a contradiction. □

Lemma 2.8. *Let G be a finite group with a normal subgroup H such that all maximal subgroups of all Sylow p-subgroups of H are S-quasinormally embedded in G. Then for any nontrivial normal subgroup N of G, all maximal subgroups of all Sylow p-subgroups of HN/N are S-quasinormally embedded in G/N.*

Proof. Let Q/N be a Sylow p-subgroup of HN/N. Then there exists a Sylow p-subgroup P of H such that $Q = PN$. Consider a map

$$\alpha : xN \to x(P \cap N),\ x \in P.$$

Clearly, α is a isomorphism of PN/N onto $P/P \cap N$.

Let M/N be a maximal subgroup of Q/N. Then $M = (P \cap M)N$, and we have that $(M/N)^\alpha = P \cap M/P \cap N$. Since α is a isomorphism, it follows that $P \cap M$ is a maximal subgroup of P.

By the condition, $P \cap M$ is S-quasinormally embedded in G. Hence, M/N is S-quasinormally embedded in G/N by lemma 2.1(b). □

Theorem 2.9 ([11], Theorem 4.2). *Let H be a normal subgroup of a finite group G.*

(a) If $H/H \cap \Phi(G)$ belongs to a saturated formation $\mathfrak{F}$, then $H = A \times B$, where $(|A|, |B|) = 1$, $A \in \mathfrak{F}$ and $B \subseteq \Phi(G)$.

(b) If $H/H \cap \Phi(G)$ is p-supersoluble, then H is p-supersoluble.

Lemma 2.10. *Let p be the smallest prime dividing the order of a finite group G. If G is p-supersoluble, then G is p-nilpotent.*

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Proof. Assume that G is non-p-nilpotent. By Frobenius theorem, G possesses a p-closed non-nilpotent subgroup S of order $p^\alpha q^\beta$ with a cyclic Sylow q-subgroup. Clearly, S is supersoluble. Therefore, S is p-nilpotent, a contradiction. □

Theorem 2.11 ([9], Theorem A,9.13). *Let H be a normal subgroup of a finite group G. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be G-chief series of H. Then there exists a one-to-one correspondence between the chief factors of $\mathcal{H}_1$ and those of $\mathcal{H}_2$ such that corresponding factors are G-isomorphic and such that the Frattini (in G) chief factors of $\mathcal{H}_1$ correspond to the Frattini (in G) chief factors of $\mathcal{H}_2$.*

Remark. Theorem 2.11 is a generalized version of Theorem A,9.13 in [9] where the case $G = H$ was considered. The proof is the same and use Lemma A,9.12 in [9].

Lemma 2.12 ([11], Lemma 7.9). *Let H be a nilpotent normal subgroup of a finite group G. If $H \cap \Phi(G) = 1$, then H is a direct product of minimal normal subgroups of G.*

A finite group G is called quasinilpotent, if $C_G(L/K)L = G$ for every chief factor L/K of G. Every finite group G possesses the quasinilpotent radical $F^*(G)$, the largest quasinilpotent normal subgroup in G. If a finite group G is soluble, then $F^*(G)$ is the Fitting subgroup $F(G)$ of G.

Lemma 2.13 ([9]). *(a) If G is a finite group, then $C_G(F^*(G)) \subseteq F^*(G)$. In particular, if G is soluble, then $C_G(F(G)) \subseteq F(G)$.*

(b) Let H be a soluble normal subgroup of a finite group G. If $G/\Phi(H)$ is quasinilpotent, then G is quasinilpotent.

Proposition 2.14 ([9], Proposition IV,3.11). *Let f_1 and f_2 be canonical local satellites of formations $\mathfrak{F}_1$ and $\mathfrak{F}_2$ respectively. If $\mathfrak{F}_1 \subseteq \mathfrak{F}_2$, then $f_1(q) \subseteq f_2(q)$, for every prime q.*

Theorem 2.15 ([9], Example IV,3.4.f). *For any prime p, let $F(p) = \mathfrak{N}_p f(p)$, where $f(p)$ is the class of abelian finite groups A satisfying $x^{p-1} = 1$ for any $x \in A$. Then F is a canonical satellite of the class $\mathfrak{U}$ of supersoluble finite groups.*

Theorem 2.16 (P. Schmid [13], L. A. Shemetkov [14]). *Let $\mathfrak{F} = LF(f)$ with f integrated, and let H be a normal subgroup of a finite group G such that every G-chief factor of H is f-central in G. Then $G/C_G(H) \in \mathfrak{F}$.*

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Theorem 2.17 ([14], [11, Theorem 9.16]). *Let A be a group of automorphisms of a finite group G. Assume that G has a series of A-admissible subgroups*

$$G = G_0 > G_1 > \cdots > G_n = 1$$

with prime indices $|G_{i-1} : G_i|$, $i = 1, \ldots, n$. Then A is supersoluble.

3. Proofs

Proof of Theorem 1.2. Use induction on $|G|+|H|$. We consider two cases.

Case 1. Assume that $R = O_{p'}(H) \neq 1$. By Lemma 2.8, the hypothesis of the theorem is inherited by G/R and H/R. Therefore, by induction, H/R is p-nilpotent and its non-Frattini G/R-chief p-factors are cyclic. Clearly, H is p-nilpotent. Let L/K be a non-Frattini G-chief p-factor of H such that $K \supseteq R$. Then there exists a maximal subgroup M of G such that $ML = G$ and $M \cap L = K \supseteq R$. Evidently, $(M/R)(L/R) = G/R$ and $(M/R) \cap (L/R) = K/R$. So, $L/R/K/R$ is a non-Frattini cyclic chief p-factor. It follows that all non-Frattini G-chief p-factors between H and R are cyclic. Applying Theorem 2.11, we see that the theorem in Case 1 is true.

Case 2. Now we assume that $O_{p'}(H) = 1$. Using Lemma 2.7 we can assume that H_p is not cyclic. Let $m(H_p) = \{P_1, \ldots, P_n\}$ be the set of all maximal subgroups of H_p, $n \geq 2$. By the assumption, P_i is S-quasinormally embedded in G. Hence, there exists an S-quasinormal subgroup H_i of G such that P_i is a Sylow p-subgroup of H_i. By Lemma 2.2, H_i is subnormal in G. Therefore, by the well-known Wielandt's theorem, $H_i \cap H$ is subnormal in H. Clearly, P_i is a Sylow p-subgroup in $H_i \cap H$. We see that there exists a chief factor H/B_i of H such that $B_i \supseteq H_i \cap H$. We consider two subcases of Case 2.

Case 2.1. Assume that $B_j \supseteq H_p$ for some j, $1 \leq j \leq n$. From this it follows that $O^{p'}(H) \neq H$. Since the theorem is true for G and $O^{p'}(H)$ by induction, we have that $O^{p'}(H)$ is p-nilpotent. Since $O_{p'}(H) = 1$, we have that H_p is normal in G and different from H. By induction, the theorem is true for G and H_p. Applying Lemma 2.12, we see that $H_p/H_p \cap \Phi(G)$ is a direct product of complemented cyclic minimal normal subgroups. Therefore, $H/H_p \cap \Phi(G)$ is p-supersoluble. Applying Lemma 2.10, Theorem 2.9 and Theorem 2.11, we see that the theorem in this case is valid.

Case 2.2. Now we assume that $B_i \not\supseteq H_p$ for any $i = 1, 2, \ldots, n$. It is clear that the order of a Sylow p-subgroup of H/B_i is equal to p. By Lemma 2.7, H/B_i is p-nilpotent. Since H/B_i is simple, we have that $|H/B_i| = p$. If $x \in H_p$ and $x \notin P_i$, then $\langle x \rangle P_i = H_p$. Clearly, $\langle x \rangle$ is not contained in B_i. Thus, x is Q-central in H. We have that every element in $H_p \setminus \Phi(H_p)$ is Q-central in H. By Theorem 1.1, H is p-nilpotent. Since $O_{p'}(H) = 1$, we have that $H = H_p$ is a p-group. Assume that $H \not\subseteq \Phi(G)$. By Lemma 2.12, $H/H \cap \Phi(G)$ is a direct product of minimal normal subgroups in $G/H \cap \Phi(G)$; we prove that all of them are cyclic. Let $P/H \cap \Phi(G)$ be a minimal normal subgroup in $G/H \cap \Phi(G)$ contained in $H/H \cap \Phi(G)$. Then $P \not\subseteq \Phi(G)$ and there exists a maximal subgroup M in G such that $MP = G$. By Lemma 2.6, $G_p = M_pP$, where $M_p \neq G_p$. Let R_1 be a maximal subgroup in G_p such that $R_1 \supseteq M_p$. Evidently, $G_p = R_1P = R_1H$ and $|G_p : R_1| = p$. It is clear that $R_1 \cap H$ is maximal in H. By the condition, $R_1 \cap H$ is S-quasinormally embedded in G. Therefore, by Lemma 2.5, $R_1 \cap H$ is S-quasinormal in G. Using $R_1 \cap H \lhd G_p$ and Lemma 2.3, we have that $R_1 \cap H$ is normal in G. It follows that $R_1 \cap P = (R_1 \cap H) \cap P$ is a normal subgroup of G. From $G_p = R_1P$ and $|G_p : R_1| = p$ it follows that $|P : R_1 \cap P| = p$. Since $M_p \supseteq H \cap \Phi(G)$ and $M_p \subseteq R_1$, we have that $R_1 \cap P \supseteq H \cap \Phi(G)$ and $|P : R_1 \cap P| = p$. This contradicts the minimality P. So, all G-chief factors between H and $H \cap \Phi(G)$ are cyclic. Now we apply Theorem 2.11. □

Proof of Theorem 1.3. Use induction on $|G|+|H|$. Applying Theorem 1.2, we see that H has an ordered Sylow tower. Let p be the smallest prime dividing $|H|$. Then H is p-nilpotent. So, H has a normal p-complement R. If $R = 1$, the result is true by Theorem 1.2. Therefore, we can assume that $R \neq 1$. By Lemma 2.8, the hypothesis of the theorem is inherited by G/R and H/R. Therefore, by Theorem 1.2, all non-Frattini G/R-chief factors of H/R are cyclic. Let L/K be a non-Frattini G-chief p-factor of H such that $K \supseteq R$. Then there exists a maximal subgroup M of G such that $ML = G$ and $M \cap L = K \supseteq R$. Evidently, $(M/R)(L/R) = G/R$ and $(M/R) \cap (L/R) = K/R$. So, $L/R/K/R$ is a non-Frattini cyclic chief p-factor. It follows that all non-Frattini G-chief p-factors between H and R are cyclic. Since $|R| < |H|$, the theorem is true for G and R by induction. So, all non-Frattini G-chief factors of R are cyclic. Applying Theorem 2.11, we see that every non-Frattini G-chief factor of H is cyclic. □

4. Corollaries

Corollary 1.3.1. *Let H be a nilpotent normal subgroup of a finite group G. Assume that all maximal subgroups of all Sylow subgroups in H are S-quasinormally embedded in G. Then every G-chief factor of $H/H \cap \Phi(G)$ is cyclic.*

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Proof. By Lemma 2.12,

$$H/H \cap \Phi(G) = H_1/H \cap \Phi(G) \times \cdots \times H_n/H \cap \Phi(G),$$

where $H_i/H \cap \Phi(G)$ is a complemented minimal normal subgroup in $G/H \cap \Phi(G)$, $i = 1, \ldots, n$. It follows that $H_i/H \cap \Phi(G)$ is a non-Frattini G-chief factor. By Theorem 1.3, $H_i/H \cap \Phi(G)$ is cyclic. □

Corollary 1.3.2. *Let H be a normal subgroup of a finite group G. Assume that all maximal subgroups of all Sylow subgroups in $F^*(H)$ are S-quasinormally embedded in G. Then H is supersoluble, and every non-Frattini G-chief factor of H is cyclic.*

Proof. By Theorem 1.3, $F = F^*(H)$ is soluble. Therefore, F is the Fitting subgroup in H. Assume that $\Phi(F) \neq 1$. By Lemma 2.13(b), $F^*(H/\Phi(F)) = F/\Phi(F)$. Applying Lemma 2.8, we see that by induction the result is true for $G/\Phi(F)$ and $H/\Phi(F)$. Since $\Phi(F) \subseteq \Phi(G)$, from Theorem 2.11 it follows that in this case the result is true for G and H. Therefore, we can assume that $\Phi(F) = 1$. Then F is elementary abelian. Now we consider two cases.

Case 1: H is soluble. Assume that $D = F \cap \Phi(G) \neq 1$. By Theorem 2.9(a), F/D is the Fitting subgroup in H/D. Applying Lemma 2.8, we see that the result is true for G/D and its normal subgroup H/D. Since $(F/D) \cap (\Phi(G/D))$ is trivial, from Theorem 2.11 it follows that the result is true for G and H. Therefore, we can assume that $D = F \cap \Phi(G) = 1$. By Corollary 1.3.1, $F = L_1 \times \cdots \times L_t$, where L_i is normal in G and has prime order, $i = 1, 2, \ldots, t$. By Theorem 2.17, $G/C_G(F)$ is supersoluble. Therefore, all G-chief factors of $HC_G(F)/C_G(F)$ are cyclic. Since $HC_G(F)/C_G(F)$ is G-isomorphic to $H/H \cap C_G(F) = H/C_H(F)$, it follows that all G-chief factors of $H/C_H(F)$ are cyclic. By Lemma 2.13, $C_H(F) \subseteq F$. We have that all G-chief factors of G/F are cyclic. Since $F = L_1 \times \cdots \times L_t$, where L_i is normal in G and has prime order for any i, it follows that all G-chief factors of H are cyclic.

Case 2: H is non-soluble. If $H \neq G$, then applying Lemma 2.8, we see that by induction the result is true for H; in particular, H is soluble, a conradiction. Therefore, we can assumme that $H = G$. Let M be a maximal subgroup in F. We are going to prove that M is normal in G. Evidently, $|F : M| = p$, where p is a prime. We have that $M = M_p \times T$, where M_p is maximal in F_p and T is a Hall p'-subgroup in F. Clearly, T is normal in G. By the condition, M_p is S-quasinormally imbedded in G. By Lemma 2.5, M_p is S-quasinormal in G. By Lemma 2.3, $N_G(M_p)$ contains $O^p(G)$. Hence, $FM_pO^p(G)$ is a subgroup in which M_p is normal. Assume that M_p is non-normal in G. Then $FM_pO^p(G) \neq G$. Since $G/O^p(G)$ is

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a p-group, there exists a normal subgroup R in G such that R contains $FM_pO^p(G)$ and $|G : R| = p$. It is clear that $R \supseteq F$ and $F = F^*(R)$. Applying Lemma 2.8, we see that by induction R is supersoluble. Since $|G : R| = p$, it follows that G is soluble, a contradiction. □

Corollary 1.3.3. *Let $\mathfrak{F}$ be a saturated formation of finite groups containing the class $\mathfrak{U}$ of supersoluble finite groups. Assume that a finite group G contains a normal subgroup H such that $G/H \in \mathfrak{F}$. If all maximal subgroups of all Sylow subgroups in H are S-quasinormally embedded in G, then $G \in \mathfrak{F}$.*

Proof. By Theorem 1.3, H is supersoluble, and all non-Frattini G-chief factors of H are cyclic. Put $D = F(H) \cap \Phi(G)$. We consider $\bar{G} = G/D$ and its normal subgroup $\bar{H} = H/D$. By Theorem 2.9(a), $F(\bar{H}) = F(H)/D$.

Let $\mathfrak{F} = LF(f)$, where f is a canonical satellite. By Lemma 2.12, $F(\bar{H}) = \bar{L}_1 \times \cdots \times \bar{L}_t$ is a direct product of minimal normal subgroups of $\bar{G}$. By Theorem 1.3, $\bar{L}_i$ is cyclic for any i. By Proposition 2.14 and Theorem 2.15, $\bar{L}_i = L_i/D$ is f-central in $\bar{G}$ for any i. By Theorem 2.16, $\bar{G}/\bar{C} \in \mathfrak{F}$, where $\bar{C} = C_{\bar{G}}(F(\bar{H}))$. Therefore, every $\bar{G}$-chief factor of $\bar{H}\bar{C}/\bar{C}$ is f-central. Since $\bar{H}\bar{C}/\bar{C}$ and $\bar{H}/\bar{H} \cap \bar{C}$ are $\bar{G}$-isomorphic, it follows that every $\bar{G}$-chief factor of $\bar{H}/\bar{H} \cap \bar{C}$ is f-central. Since $\bar{H} \cap \bar{C} \subseteq F(\bar{H})$ by Lemma 2.13, we see that every $\bar{G}$-chief factor of $\bar{H}$ is f-central. This proves that $\bar{G} \in \mathfrak{F}$. Since $\mathfrak{F}$ is saturated, we have that $G \in \mathfrak{F}$. □

Corollary 1.3.4 ([7]). *Suppose that G is a soluble finite group with a normal subgroup H such that G/H is supersoluble. If all maximal subgroups of all Sylow subgroups of $F(H)$ are S-quasinormal in G, then G is supersoluble.*

Corollary 1.3.5 [8]. *Let $\mathfrak{F}$ be a saturated formation of finite groups containing the class $\mathfrak{U}$ of supersoluble finite groups. Assume that a finite group G contains a normal subgroup H such that $G/H \in \mathfrak{F}$. If all maximal subgroups of all Sylow subgroups in $F^*(H)$ are S-quasinormally embedded in G, then $G \in \mathfrak{F}$.*

Proof. Similarly proof of Corollary 1.3.3, use Corollary 1.3.2, Proposition 2.14 and Theorem 2.15. □

References

[1] L. A. Shemetkov, *A new concept of a generalized central element*, International Algebraic Conference "Classes of groups and algebras" dedicated to the 100th Birthday of S. A. Chunikhin (abstract of lectures), Gomel: F. Scorina Gomel State Univ., 2005, p. 22–23.

[2] Kh. A. Al-Sharo, *A characteristic property of finite p-nilpotent groups*, Comm. Algebra, **32**:7 (2004), 2655–2657.

[3] Kh. A. Al-Sharo and L. A. Shemetkov, *On subgroups of prime order in a finite group*, Ukr. Math. J., **54**:6 (2002), 915–923.

[4] Kh. A. Al-Sharo, E. A. Molokova and L. A. Shemetkov, *Factorizable groups and formations*, Acta Applicandae Mathematicae, **85** (2005), 3–10.

[5] Olga Shemetkova, *Finite groups with a system of generalized central elements*, Algebra discrete math., N.**4** (2004), 59–71.

[6] O. H. Kegel, *Sylow-Gruppen und Subnormalteiler endlicher Gruppen*, Math.Z., **78** (1962), 205–221.

[7] A. Ballester-Bolinches and M. C. Pedraza-Aguilera, *Sufficient conditions for supersolvability of finite groups*, Journal of Pure and Applied Algebra, **127** (1998), 113–118.

[8] Yangming Li and Yanming Wang, *On π-quasinormally embedded subgroups of finite group*, J. Algebra, **281** (2004), 109–123.

[9] K. Doerk K. and T. Hawkes, *Finite soluble groups*, Walter de Gruyter, 1992.

[10] M. Asaad and A. A. Heliel, *On S-quasinormally embedded subgroups of finite groups*, Journal of Pure and Applied Algebra, **165** (2001), 129–135.

[11] L. A. Shemetkov, *Formations of finite groups*, Nauka: Moscow, 1978.

[12] L. A. Shemetkov, *On partially saturated formations and residuals of finite groups*, Comm. Algebra, **29**(9) (2001), 4125–4137.

[13] P. Schmid, *Lokale Formationen endlicher Gruppen*, Math.Z., **137**(1974), 31–48.

[14] L. A. Shemetkov, *Graduated formations of groups*, Math.USSR Sbornik, **23**:4 (1974), 593–611.

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