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RESEARCH ARTICLE

Multi-solid varieties and Mh-transducers

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Abstract. We consider the concepts of colored terms and multi-hypersubstitutions. If $t \in W_\tau(X)$ is a term of type τ, then any mapping $\alpha_t : Pos^{\mathcal{F}}(t) \to I\!N$ of the non-variable positions of a term into the set of natural numbers is called a coloration of t. The set $W^c_\tau(X)$ of colored terms consists of all pairs $\langle t, \alpha_t\rangle$. Hypersubstitutions are maps which assign to each operation symbol a term with the same arity. If M is a monoid of hypersubstitutions then any sequence $\rho = (\sigma_1, \sigma_2, \ldots)$ is a mapping $\rho : I\!N \to M$, called a multi-hypersubstitution over M. An identity $t \approx s$, satisfied in a variety V is an M-multi-hyperidentity if its images $\rho[t \approx s]$ are also satisfied in V for all $\rho \in M$. A variety V is M-multi-solid, if all its identities are $M-$multi-hyperidentities. We prove a series of inclusions and equations concerning M-multi-solid varieties. Finally we give an automata realization of multi-hypersubstitutions and colored terms.

Introduction

Let $\mathcal{F}$ be a set of *operation symbols*, and $\tau : \mathcal{F} \to N$ be a *type* or *signature*.

Let X be a finite set of variables, then the set $W_\tau(X)$ of *terms of type* τ with variables from X is the smallest set, such that

(i) $X \cup \mathcal{F}_0 \subseteq W_\tau(X)$;

(ii) if f is $n-$ary operation symbol and $t_1, \ldots, t_n$ are terms, then the "string" $f(t_1 \ldots t_n)$ is a term.

An algebra $\mathcal{A} = \langle A; \mathcal{F}^{\mathcal{A}}\rangle$ of type τ is a pair consisting of a set A and a set $\mathcal{F}^{\mathcal{A}}$ of operations defined on A. If $f \in \mathcal{F}$, then $f^{\mathcal{A}}$ denotes a $\tau(f)$-ary

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operation on the set A. An identity $s \approx t$ is satisfied in the algebra $\mathcal{A}$ (written $\mathcal{A} \models s \approx t$), if $s^{\mathcal{A}} = t^{\mathcal{A}}$. $\mathcal{A}lg(\tau)$ denotes the class of all algebras of type τ and $Id(\tau)$ - the set of all identities of type τ. The pair (Id, Mod) is a Galois connection between the classes of algebras from $\mathcal{A}lg(\tau)$ and subsets of $Id(\tau)$, where $Id(\mathcal{R}) := \{t \approx s \mid \forall \mathcal{A} \in \mathcal{R},\ (\mathcal{A} \models t \approx s)\}$ and $Mod(\Sigma) := \{\mathcal{A} | \forall t \approx s \in \Sigma,\ (\mathcal{A} \models t \approx s)\}$. The fixed points with respect to the closure operators $IdMod$ and $ModId$ form complete lattices

$$\mathcal{L}(\tau) := \{\mathcal{R} \mid \mathcal{R} \subseteq \mathcal{A}lg(\tau) \text{ and } ModId\mathcal{R} = \mathcal{R}\} \text{ and}$$
$$\mathcal{E}(\tau) := \{\Sigma \mid \Sigma \subseteq Id(\tau) \text{ and } IdMod\Sigma = \Sigma\}$$

of all varieties of type τ and of all equational theories (logics) of type τ. These lattices are dually isomorphic.

A *hypersubstitution of type* τ (briefly *a hypersubstitution*) is a mapping which assigns to each operation symbol $f \in \mathcal{F}$ a term $\sigma(f)$ of type τ, which has the same arity as the operation symbol f (see [5]). The set of all hypersubstitutions of type τ is denoted by $Hyp(\tau)$. If σ is a hypersubstitution, then it can be uniquely extended to a mapping $\hat{\sigma} : W_\tau(X) \to W_\tau(X)$ on the set of all terms of type τ, as follows

(*i*) if $t = x_j$ for some $j \geq 1$, then $\hat{\sigma}[t] = x_j$;

(*ii*) if $t = f(t_1, \ldots, t_n)$, then $\hat{\sigma}[t] = \sigma(f)(\hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_n])$,

where f is an n-ary operation symbol and $t_1, \ldots, t_n$ are terms.

The set $Hyp(\tau)$ is a monoid.

Let M be a submonoid of $Hyp(\tau)$. An algebra $\mathcal{A}$ is said to M-hypersatisfy an identity $t \approx s$ if for every hypersubstitution $\sigma \in M$, the identity $\hat{\sigma}[t] \approx \hat{\sigma}[s]$ holds in $\mathcal{A}$. A variety V is called *M-solid* if every identity of V is M-hypersatisfied in V.

The closure operator is defined on the set of identities of a given type τ as follows: $\chi_M[u \approx v] := \{\hat{\sigma}[u] \approx \hat{\sigma}[v] \mid \sigma \in M\}$ and $\chi_M[\Sigma] = \bigcup\limits_{u \approx v \in \Sigma} \chi_M[u \approx v]$.

Given an algebra $\mathcal{A} = \langle A; \mathcal{F}^{\mathcal{A}} \rangle$ and a hypersubstitution σ, then $\sigma[\mathcal{A}] = \langle A; (\sigma(\mathcal{F}^{\mathcal{A}})\rangle := \langle A; (\sigma(f)^{\mathcal{A}})_{f \in \mathcal{F}} \rangle$ is called the *derived algebra.* The closure operator ψ_M on the set of algebras of a given type τ, is defined as follows: $\psi_M[\mathcal{A}] = \{\sigma[\mathcal{A}] \mid \sigma \in M\}$ and $\psi_M[\mathcal{R}] = \bigcup\limits_{\mathcal{A} \in \mathcal{R}} \psi_M[\mathcal{A}]$.

It is well known [5] that if M is a monoid of hypersubstitutions of type τ, then the class of all M-solid varieties of type τ forms a complete sublattice of the lattice $\mathcal{L}(\tau)$ of all varieties of type τ.

Our aim is to transfer these results to another kind of hypersubstitution and to coloured terms.

In Section 1 we present two constructions which produce composed terms. The first one is inductive and the resulting term is obtained by simultaneous replacement of a subterm in all places where it occurs in a given term with another term of the same type. The positional composition gives a composed term as a result of the replacement of subterms in given positions with other terms of the same type. The positional composition of colored terms is an associative operation. In [4] the authors studied colored terms which are supplied with one coloration. This is a very "static" concept where each term has one fixed coloration. Here we consider composition of terms which produces an image of terms and coloration of this image. This "dynamical" point of view gives us an advantage when studying multi-hypersubstitutions, multi-solid varieties etc.

In Section 2 we use colored terms to investigate the monoid of multi-hypersubstitutions. It is proved that the lattice of multi-solid varieties is a sublattice of the lattice of solid varieties. A series of assertions are proved, which characterize multi-solid varieties and the corresponding closure operators. We study multi-solid varieties by deduction of a fully invariant congruence. The completeness theorem for multi-hyperequational theories is proved.

A tree automata realization of multi-hypersubstitutions is given in Section 3.

1. Composition of colored terms

The concept of the composition of mappings is fundamental in almost all mathematical theories. Usually we consider composition as an operation which inductively replaces some variables with other objects such as functions, terms, etc. Here, we consider a more general case when the replacement can be applied to objects which may be variables or subfunctions, subterms, etc. which are located at a given set of positions.

If t is a term, then the set $var(t)$ consisting of these elements of X which occur in t is called the set of *input variables (or variables)* for t. If $t = f(t_1, \ldots, t_n)$ is a non-variable term, then f is *root symbol (root)* of t and we will write $f = root(t)$. For a term $t \in W_\tau(X)$ the set $Sub(t)$ of its subterms is defined as follows: if $t \in X \cup \mathcal{F}_0$, then $Sub(t) = \{t\}$ and if $t = f(t_1, \ldots, t_n)$, then $Sub(t) = \{t\} \cup Sub(t_1) \cup \ldots \cup Sub(t_n)$.

The *depth* of a term t is defined inductively: if $t \in X \cup \mathcal{F}_0$ then $Depth(t) = 0$; and if $t = f(t_1, \ldots, t_n)$, then
$Depth(t) = max\{Depth(t_1), \ldots, Depth(t_n)\} + 1$.

Definition 1. *Let $r, s, t \in W_\tau(X)$ be three terms of type τ. By $t(r \leftarrow s)$ we will denote the term, obtained by simultaneous replacement of every occurrence of r as a subterm of t by s. This term is called the inductive* *composition of the terms t and s, by r. I.e.*

(i) $t(r \leftarrow s) = t$ *if* $r \notin Sub(t)$;

(ii) $t(r \leftarrow s) = s$ *if* $t = r$ *and*

(iii) $t(r \leftarrow s) = f(t_1(r \leftarrow s), \ldots, t_n(r \leftarrow s))$, *if* $t = f(t_1, \ldots, t_n)$ *and* $r \in Sub(t)$, $r \neq t$.

If $r_i \notin Sub(r_j)$ when $i \neq j$, then $t(r_1 \leftarrow s_1, \ldots, r_m \leftarrow s_m)$ means the inductive composition of $t, r_1, \ldots, r_m, s_1, \ldots, s_m$. In the particular case when $r_j = x_j$ for $j = 1, \ldots, m$ and $var(t) = \{x_1, \ldots, x_m\}$ we will briefly write $t(s_1, \ldots, s_m)$ instead of $t(x_1 \leftarrow s_1, \ldots, x_m \leftarrow s_m)$.

Let τ be a type and $\mathcal{F}$ be its set of operation symbols. Denote by $maxar = \max\{\tau(f) | f \in \mathcal{F}\}$ and $\mathbb{N}_{\mathcal{F}} := \{m \in \mathbb{N} \mid m \leq maxar\}$. Let $\mathbb{N}_{\mathcal{F}}^*$ be the set of all finite strings over $\mathbb{N}_{\mathcal{F}}$. The set $\mathbb{N}_{\mathcal{F}}^*$ is naturally ordered by $p \preceq q \iff p$ is a prefix of q. The Greek letter ε, as usual denotes the empty word (string) over $\mathbb{N}_{\mathcal{F}}$.

For any term t, the set of positions $Pos(t) \subseteq \mathbb{N}_{\mathcal{F}}^*$ of t is inductively defined as follows: $Pos(t) = \{\varepsilon\}$ if $t \in X \cup \mathcal{F}_0$ and $Pos(t) := \{\varepsilon\} \bigcup_{1 \leq i \leq n} (iPos(t_i))$, if $t = f(t_1, \ldots, t_n)$, $n \geq 0$, where $iPos(t_i) := \{iq | q \in Pos(t_i)\}$, and iq is concatenation of the strings i and q from $\mathbb{N}_{\mathcal{F}}^*$.

For a given position $p \in Pos(t)$, the length of p is denoted by $l(p)$.

Any term can be regarded as a tree with nodes labelled with the operation symbols and its leaves labelled as variables or nullary operation symbols.

Let $t \in W_\tau(X)$ be a term of type τ and let $sub_t : Pos(t) \to Sub(t)$ be the function which maps each position in a term t to the subterm of t, whose root node occurs at that position.

Definition 2. *Let $t, r \in W_\tau(X)$ be two terms of type τ and $p \in Pos(t)$ be a position in t. The positional composition of t and r on p is a term $s := t(p; r)$ obtained from t when replacing the term $sub_t(p)$ by r on the position p, only.*

More generally, the positional composition of terms is naturally defined for the compositional pairs $t(p_1, \ldots, p_m; t_1, \ldots, t_m)$, also.

Remark 1. The positional composition has the following properties:

1. If $\langle\langle p_1, p_2\rangle, \langle t_1, t_2\rangle\rangle$ is a compositional pair of t, then

$$t(p_1, p_2; t_1, t_2) = t(p_1; t_1)(p_2; t_2) = t(p_2; t_2)(p_1; t_1);$$

2. If $S = \langle p_1, \ldots, p_m\rangle$ and $T = \langle t_1, \ldots, t_m\rangle$ with

$$(\forall p_i, p_j \in S)\ (i \neq j \implies p_i \not\preceq p_j \ \&\ p_j \not\preceq p_i)$$

and π is a permutation of the set $\{1, \ldots, m\}$, then

$$t(p_1, \ldots, p_m; t_1, \ldots, t_m) = t(p_{\pi(1)}, \ldots, p_{\pi(m)}; t_{\pi(1)}, \ldots, t_{\pi(m)}).$$

3.If $t, s, r \in W_\tau(X)$, $p \in Pos(t)$ and $q \in Pos(s)$, then $t(p; s(q; r)) = t(p; s)(pq; r)$.

We will denote respectively " $\leftarrow$ " for inductive composition and ";" for positional composition.

The concept of colored terms is important when studying deductive closure of sets of identities and the lattice of varieties of a given type. Colored terms (trees) are useful tools in Computer Science, General Algebra, Theory of Formal Languages, Programming, Automata theory etc.

Let t be a term of type τ. Let us denote by $Sub^{\mathcal{F}}(t)$ the set of all subterms of t which are not variables, i.e. whose roots are labelled by an operation symbol from $\mathcal{F}$ and let $Pos^{\mathcal{F}}(t) := \{p \in Pos(t) \mid sub_t(p) \in Sub^{\mathcal{F}}(t)\}$. Any function $\alpha_t : Pos^{\mathcal{F}}(t) \to I\!N$ is called a *coloration* of the term t.

For a given term t, $Pos^X(t)$ denotes the set of all its variable positions i.e. $Pos^X(t) := Pos(t) \setminus Pos^{\mathcal{F}}(t)$.

By C_t we denote the set of all colorations of the term t i.e. $C_t := \{\alpha_t \mid \alpha_t : Pos^{\mathcal{F}}(t) \to I\!N\}$. If $p \in Pos^{\mathcal{F}}(t)$ then $\alpha_t(p) \in I\!N$ denotes the value of the function α_t which is associated with the root operation symbol of the subterm $s = sub_t(p)$, and $\alpha_t[p] \in C_s$ denotes the "restriction" of the function α_t on the set $Pos^{\mathcal{F}}(s)$ defined by $\alpha_t[p](q) = \alpha_t(pq)$ for all $q \in Pos^{\mathcal{F}}(s)$.

Definition 3. *The set $W^c_\tau(X)$ of all colored terms of type τ is defined as follows:*

(i) $X \subset W^c_\tau(X)$;

(ii) *If $f \in \mathcal{F}$, then $\langle f, q\rangle \in W^c_\tau(X)$ for each $q \in I\!N$;*

(iii) *If $t = f(t_1, \ldots, t_n) \in W_\tau(X)$, then $\langle t, \alpha_t\rangle \in W^c_\tau(X)$ for each $\alpha_t \in C_t$.*

Let $\langle t, \alpha_t\rangle \in W^c_\tau(X)$. The set $Sub_c(\langle t, \alpha_t\rangle)$ of colored subterms of $\langle t, \alpha_t\rangle$ is defined as follows:

For $x \in X$ we have $Sub_c(x) := \{x\}$
and if $\langle t, \alpha_t\rangle = \langle f, q\rangle(\langle t_1, \alpha_{t_1}\rangle, \ldots, \langle t_n, \alpha_{t_n}\rangle)$, then

$$Sub_c(\langle t, \alpha_t\rangle) := \{\langle t, \alpha_t\rangle\} \cup Sub_c(\langle t_1, \alpha_{t_1}\rangle) \cup \ldots \cup Sub_c(\langle t_n, \alpha_{t_n}\rangle).$$

Let $\langle t, \alpha_t\rangle$, $\langle r, \alpha_r\rangle$ and $\langle s, \alpha_s\rangle$ be colored terms of type τ. Their inductive composition $\langle t, \alpha_t\rangle(\langle r, \alpha_r\rangle \leftarrow \langle s, \alpha_s\rangle)$ is defined as follows:

(i) if $t = x_i \in X$, then

$$x_i(\langle r, \alpha_r\rangle \leftarrow \langle s, \alpha_s\rangle) := \begin{cases} \langle s, \alpha_s\rangle & \text{if } \ r = x_i; \\ x_i & \text{otherwise} \end{cases}$$

(ii) if $t = f(t_1, t_2, \ldots, t_n)$, $\alpha_t(\varepsilon) = q$, $\alpha_t[i] = \alpha_{t_i}$ for $i = 1, 2, \ldots, n$, then $\langle t, \alpha_t\rangle(\langle r, \alpha_r\rangle \leftarrow \langle s, \alpha_s\rangle) := \langle s, \alpha_s\rangle$ when $\langle r, \alpha_r\rangle = \langle t, \alpha_t\rangle$ and

$$\langle t, \alpha_t\rangle(\langle r, \alpha_r\rangle \leftarrow \langle s, \alpha_s\rangle) :=$$
$$\langle f, q\rangle(\langle t_1, \alpha_{t_1}\rangle(\langle r, \alpha_r\rangle \leftarrow \langle s, \alpha_s\rangle), \ldots, \langle t_n, \alpha_{t_n}\rangle(\langle r, \alpha_r\rangle \leftarrow \langle s, \alpha_s\rangle))$$

otherwise.

If $\langle r_i, \alpha_{r_i}\rangle \notin Sub_c(\langle r_j, \alpha_{r_j}\rangle)$ for $i \neq j$, then the denotations $\langle t, \alpha_t\rangle(\langle r_1, \alpha_{r_1}\rangle \leftarrow \langle s_1, \alpha_{s_1}\rangle, \ldots, \langle r_m, \alpha_{r_m}\rangle \leftarrow \langle s_m, \alpha_{s_m}\rangle)$ is clear.

Let $\langle t, \alpha_t\rangle, \langle s, \alpha_s\rangle \in W^c_\tau(X)$ be two colored terms of type τ and let $p \in Pos(t)$. The positional composition of the colored terms $\langle t, \alpha_t\rangle$ and $\langle s, \alpha_s\rangle$ at the position p is defined as follows:

$$\langle t, \alpha_t\rangle(p; \langle s, \alpha_s\rangle) := \langle t(p; s), \alpha\rangle,$$

where

$$\alpha(q) = \begin{cases} \alpha_s(k) & \text{if } \ \ q = pk, \text{ for some } k \in Pos(s); \\ \alpha_t(q) & \text{otherwise.} \end{cases}$$

The positional composition of colored terms can be defined for a sequence $(p_1, \ldots, p_m) \in Pos(t)^m$ of positions with

$$(\forall p_i, p_j \in S)\ (i \neq j \implies p_i \not\prec p_j \ \&\ p_j \not\prec p_i).$$

It is denoted by

$$\langle t, \alpha_t\rangle(p_1, \ldots, p_m; \langle s_1, \alpha_{s_1}\rangle, \ldots, \langle s_m, \alpha_{s_m}\rangle).$$

Theorem 1. *If $t, s, r \in W_\tau(X)$, $p \in Pos(t)$ and $q \in Pos(s)$, then*

$$\langle t, \alpha_t\rangle(p; \langle s, \alpha_s\rangle(q; \langle r, \alpha_r\rangle)) = \langle t, \alpha_t\rangle(p; \langle s, \alpha_s\rangle)(pq; \langle r, \alpha_r\rangle),$$

where $\alpha_t \in C_t$, $\alpha_s \in C_s$, $\alpha_r \in C_r$.

Proof. Let us consider the non-trivial case when t, r and s are not variables. Thus we obtain $\langle s, \alpha_s\rangle(q; \langle r, \alpha_r\rangle) = \langle s(q; r), \alpha\rangle$, where

$$\alpha(m) = \begin{cases} \alpha_r(k) & \text{if } \ \ m = qk, \text{ for some } k \in Pos^{\mathcal{F}}(r); \\ \alpha_s(m) & \text{otherwise} \end{cases}$$

and $\langle t, \alpha_t\rangle(p; \langle s(q; r), \alpha\rangle) = \langle t(p; s(q; r)), \beta\rangle$, where

$$\beta(l) = \begin{cases} \alpha(v) & \text{if } \ \ l = pv, \text{ for some } v \in Pos^{\mathcal{F}}(s(q; r)); \\ \alpha_t(l) & \text{otherwise} \end{cases} \quad =$$

$$= \begin{cases} \alpha_r(k) & \text{if} \quad l = pqk, \text{ for some } k \in Pos^{\mathcal{F}}(r); \\ \alpha_s(v) & \text{if} \quad l = pv, \text{ for some } v \in Pos^{\mathcal{F}}(s), \ q \not\prec v; \\ \alpha_t(l) & \text{otherwise.} \end{cases}$$

On the other side we obtain $\langle t, \alpha_t \rangle (p; \langle s, \alpha_s \rangle) = \langle t(p; s), \gamma \rangle$, where

$$\gamma(m) = \begin{cases} \alpha_s(v) & \text{if} \quad m = pv, \text{ for some } v \in Pos^{\mathcal{F}}(s); \\ \alpha_t(m) & \text{otherwise} \end{cases}$$

and $\langle t(p; s), \gamma \rangle = \langle t(p; s)(pq; r), \delta \rangle$, where

$$\delta(l) = \begin{cases} \alpha_r(k) & \text{if} \quad l = pqk, \text{ for some } k \in Pos^{\mathcal{F}}(r); \\ \gamma(l) & \text{otherwise} \end{cases} =$$

$$= \begin{cases} \alpha_r(k) & \text{if} \quad l = pqk, \text{ for some } k \in Pos^{\mathcal{F}}(r); \\ \alpha_s(v) & \text{if} \quad l = pv, \text{ for some } v \in Pos^{\mathcal{F}}(s), \ q \not\prec v; \\ \alpha_t(l) & \text{otherwise.} \end{cases}$$

Clearly, $\delta = \beta$ and $t(p; s(q; r)) = t(p; s)(pq; r)$. $\square$

The following example illustrates the positions, subterms and positional composition of colored terms.

Example 1. Let $\tau = (2)$, $\mathcal{F} = \{f\}$. The colorations of terms in the example are presented as bold superscripts of the operation symbols.

Let $\langle t, \alpha_t \rangle = f^{\mathbf{1}}(f^{\mathbf{1}}(x_1, x_2), f^{\mathbf{2}}(x_1, x_2))$, $\langle s, \alpha_s \rangle = f^{\mathbf{3}}(f^{\mathbf{2}}(x_1, x_2), x_2)$ and $\langle r, \alpha_r \rangle = f^{\mathbf{3}}(x_1, x_2)$ be three colored terms of type τ.

Then we have $Pos(t) = \{\varepsilon, 1, 2, 11, 12, 21, 22\}$, $\langle sub_t(2), \alpha_t[2] \rangle = f^{\mathbf{2}}(x_1, x_2)$, $\langle sub_t(12), \alpha_t[12] \rangle = x_2$ and
$Sub_c(\langle s, \alpha_s \rangle) = \{\langle s, \alpha_s \rangle, \langle f(x_1, x_2), \alpha_s[1] \rangle, x_1, x_2\}$.

For the positional composition we have
$\langle t, \alpha_t \rangle (2; \langle s, \alpha_s \rangle (12; \langle r, \alpha_r \rangle)) =$
$f^{\mathbf{1}}(f^{\mathbf{1}}(x_1, x_2), f^{\mathbf{3}}(f^{\mathbf{2}}(x_1, x_2), x_2))(212; \langle r, \alpha_r \rangle) =$
$f^{\mathbf{1}}(f^{\mathbf{1}}(x_1, x_2), f^{\mathbf{3}}(f^{\mathbf{2}}(x_1, f^{\mathbf{3}}(x_1, x_2)), x_2)) =$
$\langle t, \alpha_t \rangle (2; \langle s, \alpha_s \rangle)(212; \langle r, \alpha_r \rangle)$.

2. Multi-hypersubstitutions and deduction of identities

Definition 4. *[4] Let M be a submonoid of $Hyp(\tau)$ and let ρ be a mapping of $\mathbb{N}$ into M i.e. $\rho : \mathbb{N} \to M$. Any such mapping is called a multi-hypersubstitution of type τ over M.*

Denote by σ_q the image of $q \in \mathbb{N}$ under ρ i.e. $\rho(q) = \sigma_q \in Hyp(\tau)$.

Let $\sigma \in M$ and $\rho \in Mhyp(\tau, M)$. If there is a natural number $q \in \mathbb{N}$ with $\rho(q) = \sigma$, then we will write $\sigma \in \rho$.

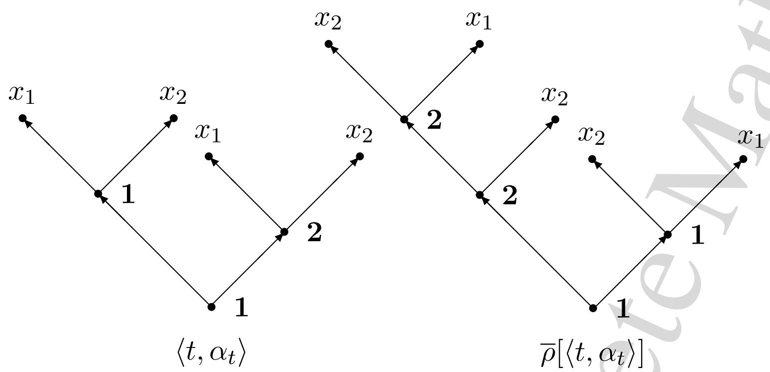

Figure 1: Multi-hypersubstitution of colored terms

We will define the extension $\overline{\rho}_{\alpha_t}$ of a multi-hypersubstitution ρ to the set of colored subterms of a term.

Let $\langle t, \alpha_t \rangle$ be a colored term of type τ, $\alpha_t \in C_t$, with $p \in Pos(t)$, $s = sub_t(p)$ and $\alpha_t(p) = m$. Then we set:

(i) if $s = x_j \in X$, then $\overline{\rho}_{\alpha_t}[s] := x_j$;

(ii) if $s = f(s_1, \ldots, s_n)$, then $\overline{\rho}_{\alpha_t}[s] := \sigma_m(f)(\overline{\rho}_{\alpha_t}[s_1], \ldots, \overline{\rho}_{\alpha_t}[s_n])$.

The extension of ρ on α_s assigns inductively a coloration $\overline{\rho}_t[\alpha_s]$ to the term $\overline{\rho}_{\alpha_t}[s]$, as follows:

(i) if $s = f(x_1, \ldots, x_n)$, then $\overline{\rho}_t[\alpha_s](q) := m$ for all $q \in Pos^{\mathcal{F}}(\overline{\rho}_{\alpha_t}[s])$;

(ii) if $s = f(s_1, \ldots, s_n)$, and $q \in Pos^{\mathcal{F}}(\overline{\rho}_{\alpha_t}[s])$, then

$$\overline{\rho}_t[\alpha_s](q) = \begin{cases} m & \text{if } \quad q \in Pos^{\mathcal{F}}(\sigma_m(f)); \\ \overline{\rho}_t[\alpha_{s_j}](k) & \text{if } \quad q = lk, \text{ for some } j,\ j \le n, \\ & k \in Pos^{\mathcal{F}}(\overline{\rho}_{\alpha_t}[s_j]),\ l \in Pos^X(\sigma_m(f)). \end{cases}$$

Definition 5. *The mapping $\overline{\rho}$ on the set $W^c_\tau(X)$ is defined as follows:*

(i) $\overline{\rho}[x] := x$ *for all* $x \in X$;

(ii) *if* $t = f(t_1, \ldots, t_n)$ *and* $\alpha_t \in C_t$, *then* $\overline{\rho}[\langle t, \alpha_t \rangle] := \langle \overline{\rho}_{\alpha_t}[t], \overline{\rho}_t[\alpha_t] \rangle$.

Example 2. Let $\langle t, \alpha_t \rangle$ and $\langle s, \alpha_s \rangle$ be the colored terms of type τ from Example 1.

Let $\sigma_1(f) = f(x_2, x_1)$, $\sigma_2(f) = f(f(x_2, x_1), x_2)$ and $\sigma_3(f) = f(x_1, x_2)$ be hypersubstitutions of type τ and $M = \{\sigma_1, \sigma_2, \sigma_3, \ldots\}$ be a submonoid of $Hyp(\tau)$. Let $\rho \in Mhyp(\tau, M)$ with $\rho(m) := \sigma_m$. Then we obtain

$\overline{\rho}[\langle t, \alpha_t \rangle]$

$= \overline{\rho}[f^{\mathbf{1}}(f^{\mathbf{1}}(x_1, x_2), f^{\mathbf{2}}(x_1, x_2))]$

$= f^{\mathbf{1}}(f^{\mathbf{2}}(f^{\mathbf{2}}(x_2, x_1), x_2), f^{\mathbf{1}}(x_2, x_1))$

and $\overline{\rho}[\langle s, \alpha_s \rangle] = \overline{\rho}[f^{\mathbf{3}}(f^{\mathbf{2}}(x_1, x_2), x_2)] = f^{\mathbf{3}}(f^{\mathbf{2}}(f^{\mathbf{2}}(x_2, x_1), x_2), x_2)$. The image of $\langle t, \alpha_t \rangle$ under ρ is shown in Figure 1.

Proposition 1. *Let $\rho \in Mhyp(\tau, M)$ be a multi-hypersubstitution and $t, s \in W_\tau(X_n)$ with $\alpha_t \in C_t$, $\alpha_s \in C_s$ and $p \in Pos(t)$. Then*

$$\overline{\rho}[\langle t, \alpha_t\rangle(p; \langle s, \alpha_s\rangle)] = \overline{\rho}[\langle t, \alpha_t\rangle(p; x_j)](x_j \leftarrow \overline{\rho}[\langle s, \alpha_s\rangle]),$$

for each $j,\ j > n$.

Proof. We will use induction on the length $l(p)$ of the position p.

First, let us observe that the case $l(p) = 0$ is trivial. So, our basis of induction is $l(p) = 1$. Then $t = f(t_1, \ldots, t_{p-1}, t_p, t_{p+1}, \ldots, t_m)$ for some $f \in \mathcal{F}_m$. Hence, for each $j,\ j > n$, we have

$\overline{\rho}[\langle t, \alpha_t\rangle(p; \langle s, \alpha_s\rangle)] = \overline{\rho}[\langle f(t_1, \ldots, t_{p-1}, s, t_{p+1}, \ldots, t_m), \alpha_{t'}\rangle] =$
$= \overline{\rho}[\langle f(t_1, \ldots, t_{p-1}, x_j, t_{p+1}, \ldots, t_m)(x_j \leftarrow s), \alpha_{t'}\rangle] =$
$= \overline{\rho}[\langle t, \alpha_t\rangle(p; x_j)](x_j \leftarrow \overline{\rho}[\langle s, \alpha_s\rangle]),$

where $\alpha_{t'}$ is a coloration of the term $t' = f(t_1, \ldots, t_{p-1}, s, t_{p+1}, \ldots, t_m)$ for which $\alpha_{t'}(q) = \alpha_t(q)$ when $q \in Pos^{\mathcal{F}}(t) \setminus \{p\}$ and $\alpha_{t'}(pq) = \alpha_s(q)$ when $q \in Pos^{\mathcal{F}}(s)$.

Our inductive supposition is that when $l(p) < k$, the proposition is true, for some $k \in \mathbb{N}$.

Let $l(p) = k$ and $p = qi$ where $q \in \mathbb{N}^*_{\mathcal{F}}$ and $i \in \mathbb{N}$. Hence $q \in Pos^{\mathcal{F}}(t)$ and $l(q) < k$. Let $u, v \in Sub(t)$ be subterms of t for which $u = sub_t(p)$ and $v = sub_t(q)$. Then we have $v = g(v_1, \ldots, v_{i-1}, v_i, v_{i+1}, \ldots, v_j)$ with $v_i = u$ for some $g \in \mathcal{F}_j$. By the inductive supposition, for every $j,\ j > n$ and $l,\ l > n$, we obtain

$\overline{\rho}[\langle t, \alpha_t\rangle(p; \langle s, \alpha_s\rangle)] =$
$= \overline{\rho}[\langle t, \alpha_t\rangle(q; x_j)(x_j \leftarrow \overline{\rho}[\langle v, \alpha_t[q]\rangle(i; \langle s, \alpha_s\rangle)] =$
$= \overline{\rho}[\langle t, \alpha_t\rangle(qi; x_l)](x_l \leftarrow \overline{\rho}[\langle s, \alpha_s\rangle])$
$= \overline{\rho}[\langle t, \alpha_t\rangle(p; x_j)](x_j \leftarrow \overline{\rho}[\langle s, \alpha_s\rangle]).$

□

A binary operation is defined in the set $Mhyp(\tau, M)$ as follows:

Definition 6. *Let $\rho^{(1)}$ and $\rho^{(2)}$ be two multi-hypersubstitutions over the submonoid M. Then the composition $\rho^{(1)} \circ_{ch} \rho^{(2)} : \mathbb{N} \to M$ maps each color $q \in \mathbb{N}$ as follows:*

$$(\rho^{(1)} \circ_{ch} \rho^{(2)})(q) := \rho^{(1)}(q) \circ_h \rho^{(2)}(q) := \sigma_q^{(1)} \circ_h \sigma_q^{(2)}.$$

Lemma 1. *For every two multi-hypersubstitutions $\rho^{(1)}$ and $\rho^{(2)}$ over M and for each colored term $\langle t, \alpha_t\rangle$ of type τ, it holds*

$$\overline{(\rho^{(1)} \circ_{ch} \rho^{(2)})}[\langle t, \alpha_t\rangle] = \overline{\rho}^{(1)}[\overline{\rho}^{(2)}[\langle t, \alpha_t\rangle]].$$

So, $Mhyp(\tau, M)$ is a monoid, where $\rho_{id} = (\sigma_{id}, \sigma_{id}, \ldots)$ is the identity multi-hypersubstitution.

Let $\mathcal{A} = \langle A, \mathcal{F}\rangle$ be an algebra of type τ. Each colored term $\langle t, \alpha_t\rangle$ defines a term-operation on the set A, as follows $\langle t, \alpha_t\rangle^{\mathcal{A}} = t^{\mathcal{A}}$.

Let $\rho = (\sigma_1, \sigma_2, \ldots)$ be any multi-hypersubstitution over a monoid M of hypersubstitutions of type τ, and let

$$\rho(\mathcal{F}) = \{t \in W_\tau(X) \mid \exists\ \sigma \in \rho \text{ and } f \in \mathcal{F} \text{ such that } t = \sigma(f)\}.$$

The algebra $\rho[\mathcal{A}] := \langle A, \rho(\mathcal{F})^{\rho[\mathcal{A}]}\rangle$ is called a *derived algebra* under the multi-hypersubstitution ρ.

Let $\mathcal{R}$ be a class of algebras of type τ. The operator ψ^c_M is defined as follows:

$\psi^c_M[\mathcal{A}] := \{\rho[\mathcal{A}] \mid \rho \in Mhyp(\tau, M)\}$ and $\psi^c_M[\mathcal{R}] := \{\psi^c_M[\mathcal{A}] \mid \mathcal{A} \in \mathcal{R}\}$.

Lemma 2. *For each $\langle t, \alpha_t\rangle \in W^c_\tau(X)$ it holds*

$$\langle t, \alpha_t\rangle^{\rho[\mathcal{A}]} = \overline{\rho}[\langle t, \alpha_t\rangle]^{\mathcal{A}}.$$

Proof. If $t = x_j \in X$, then $\overline{\rho}[\langle t, \alpha_t\rangle]^{\mathcal{A}} = x_j^{\mathcal{A}}$ and $\langle t, \alpha_t\rangle^{\rho[\mathcal{A}]} = x_j^{\rho[\mathcal{A}]} = x_j^{\mathcal{A}}$. Let us assume that $t = f(t_1, \ldots, t_n)$, $\alpha_t(\varepsilon) = q \in I\!N$, $\langle f, q\rangle^{\rho[\mathcal{A}]} = \rho(\langle f, q\rangle)^{\mathcal{A}}$ and $\langle t_i, \alpha_{t_i}\rangle^{\rho[\mathcal{A}]} = \overline{\rho}[\langle t_i, \alpha_{t_i}\rangle]^{\mathcal{A}}$ for all $i = 1, \ldots, n$, then we have

$$\langle t, \alpha_t\rangle^{\rho[\mathcal{A}]} = \langle f, q\rangle^{\rho[\mathcal{A}]}(\langle t_1, \alpha_t[1]\rangle^{\rho[\mathcal{A}]}, \ldots, \langle t_n, \alpha_t[n]\rangle^{\rho[\mathcal{A}]}) =$$

$$= \rho(\langle f, q\rangle)^{\mathcal{A}}(\overline{\rho}[\langle t_1, \alpha_t[1]\rangle]^{\mathcal{A}}, \ldots, \overline{\rho}[\langle t_n, \alpha_t[n]\rangle]^{\mathcal{A}}) = \overline{\rho}[\langle t, \alpha_t\rangle]^{\mathcal{A}}.$$

$\square$

Let ρ be a multi-hypersubstitution. By $\rho[t \approx s]$ we will denote $\rho[t \approx s] := \{\overline{\rho}_{\alpha_t}[t] \approx \overline{\rho}_{\alpha_s}[s] \mid \alpha_t \in C_t,\ \alpha_s \in C_s\}$. Let $\Sigma \subseteq Id(\tau)$ be a set of identities of type τ. The operator χ^c_M, is defined as follows:

$\chi^c_M[t \approx s] := \{\rho[t \approx s] \mid \rho \in Mhyp(\tau, M)\}$

and $\chi^c_M[\Sigma] := \{\chi^c_M[t \approx s] \mid t \approx s \in \Sigma\}$.

Definition 7. *An identity $t \approx s \in Id\mathcal{A}$ in the algebra $\mathcal{A}$ is called an M-multi-hyperidentity in $\mathcal{A}$, if for each multi-hypersubstitution $\rho \in Mhyp(\tau, M)$ and for every two colorations $\alpha_t \in C_t$, $\alpha_s \in C_s$, the identity $\overline{\rho}_{\alpha_t}[t] \approx \overline{\rho}_{\alpha_s}[s]$ is satisfied in $\mathcal{A}$. When $t \approx s$ is an M-multi-hyperidentity in $\mathcal{A}$ we will write $\mathcal{A} \models_{Mh} t \approx s$, and the set of all M-multi-hyperidentities in $\mathcal{A}$ is denoted by $HC_M Id\mathcal{A}$.*

Algebras in which all identities are M-multi-hyperidentities are called *M-multi-solid* i.e. an algebra $\mathcal{A}$ of type τ is M-multi-solid, if $\chi^c_M[Id\mathcal{A}] \subseteq Id\mathcal{A}$. So, if $V \subseteq \mathcal{A}lg(\tau)$ is a variety of type τ, it is called *M-multi-solid*, when $\chi^c_M[IdV] \subseteq IdV$.

Theorem 2. *Let* $\Sigma \subseteq Id(\tau)$ *and* $\mathcal{R} \subseteq \mathcal{A}lg(\tau)$. *Then*

(*i*) $\chi_M[\Sigma] \subseteq \chi^c_M[\Sigma]$ *for all* $\Sigma \subseteq Id(\tau)$;

(*ii*) $\psi_M[\mathcal{R}] = \psi^c_M[\mathcal{R}]$ *for all* $\mathcal{R} \subseteq Alg(\tau)$.

Proof. (*i*) Let $\sigma \in M$. Then we consider the multi-hypersubstitution $\rho \in Mhyp(\tau, M)$ with $\rho(q) = \sigma$ for all $q \in I\!N$. If $t \in W_\tau(X)$ then we will show that $\widehat{\sigma}[t] = \overline{\rho}_{\alpha_t}[t]$ for $\alpha_t \in C_t$. We will prove more, that $\widehat{\sigma}[r] := \overline{\rho}_{\alpha_r}[r]$ for each $r \in Sub(t)$. That will be proved by induction on the depth of the term r.

If $r \in X$ then $\widehat{\sigma}[r] = \overline{\rho}_{\alpha_t}[r] = r$.

Let us assume that $r = f(r_1, \ldots, r_n)$ with $r_1, \ldots, r_n \in Sub(t)$. Our inductive supposition is that $\widehat{\sigma}[r_k] = \overline{\rho}_{\alpha_t}[r_k]$ for $1 \leq k \leq n$. Then we have

$$\overline{\rho}_{\alpha_t}[r] = \rho(\alpha_t(\varepsilon))(f)(\overline{\rho}_{\alpha_t}[r_1], \ldots, \overline{\rho}_{\alpha_t}[r_n]) = \sigma(f)(\widehat{\sigma}[r_1], \ldots, \widehat{\sigma}[r_n]) = = \widehat{\sigma}[f(r_1, \ldots, r_n)] = \hat{\sigma}[r].$$

Let $t \approx s \in \Sigma$. Thus we have $\hat{\sigma}[t] = \overline{\rho}_{\alpha_t}[t]$, $\hat{\sigma}[s] = \overline{\rho}_{\alpha_s}[s]$ and

$$\hat{\sigma}[t] \approx \hat{\sigma}[s] \in \chi_M[\Sigma] \iff \overline{\rho}_{\alpha_t}[t] \approx \overline{\rho}_{\alpha_s}[s] \in \chi^c_M[\Sigma].$$

Hence $\hat{\sigma}[t] \approx \hat{\sigma}[s] \in \chi^c_M[\Sigma]$.

(*ii*) Let $\mathcal{A} \in \mathcal{R}$ and $q \in I\!N$ be the color of f in the fundamental colored term $\langle f(x_1, \ldots, x_n), q\rangle$. Let $\rho \in Mhyp(\tau, M)$. Then we consider the hypersubstitution $\sigma \in M$ with $\sigma(f) = \rho(q)(f)$ and from Lemma 2 we obtain

$$\langle f, q\rangle^{\rho[\mathcal{A}]} = \overline{\rho}[\langle f(x_1, \ldots, x_n), q\rangle]^{\mathcal{A}} = (\rho(q)(f)(x_1, \ldots, x_n))^{\mathcal{A}}$$

$$= \sigma(f)(x_1, \ldots, x_n)^{\mathcal{A}}.$$

This shows that $\psi^c_M[\mathcal{A}] \subseteq \psi_M[\mathcal{A}]$.

Let $\sigma \in Hyp(\tau)$, $\mathcal{A} \in \mathcal{R}$ and $\rho \in Mhyp(\tau, M)$ with $\rho(q) = \sigma$ for all $q \in I\!N$. From Lemma 2 we obtain

$$\sigma(f)^{\mathcal{A}} = \hat{\sigma}[f(x_1, \ldots, x_n)]^{\mathcal{A}} = \overline{\rho}[\langle f(x_1, \ldots, x_n), q\rangle]^{\mathcal{A}} =$$

$$= \langle f(x_1, \ldots, x_n), q\rangle]^{\rho[\mathcal{A}]} = \langle f, q\rangle^{\rho[\mathcal{A}]}.$$

This shows that $\psi_M[\mathcal{A}] \subseteq \psi^c_M[\mathcal{A}]$. Altogether we have $\psi^c_M[\mathcal{A}] = \psi_M[\mathcal{A}]$ and thus $\psi^c_M[\mathcal{R}] = \psi_M[\mathcal{R}]$. □

Theorem 3. *Each M-multi-solid variety is M-solid, but the converse assumption is in general not true i.e. there are M−solid varieties which are not M−multi-solid.*

Proof. Let V be an M-multi-solid variety. If $t \approx s \in \chi_M[IdV]$, then $t \approx s \in \chi_M^c[IdV]$ because of (i) of Theorem 2. Hence every M-multi-solid variety is M-solid one.

Let RB be the variety of rectangular bands, which is of type $\tau = (2)$. The identities satisfied in RB are:

$$Id_{RB} = \{f(x_1, f(x_2, x_3)) \approx f(f(x_1, x_2), x_3) \approx f(x_1, x_3),\ f(x_1, x_1) \approx x_1\}.$$

In [5] it was proved that RB is a solid variety, which means that for each identity $t \approx s$ with $Id_{RB} \models t \approx s$ and for each hypersubstitution σ we have $Id_{RB} \models \hat{\sigma}[t] \approx \hat{\sigma}[s]$.

Let M, ρ , $\langle t, \alpha_t \rangle$ and $\langle s, \alpha_s \rangle$ be as in Example 2 i.e.

$$\langle t, \alpha_t \rangle = f^{\mathbf{1}}(f^{\mathbf{1}}(x_1, x_2), f^{\mathbf{2}}(x_1, x_2)) \quad \text{and} \quad \langle s, \alpha_s \rangle = f^{\mathbf{3}}(f^{\mathbf{2}}(x_1, x_2), x_2).$$

Since

$$t = f(f(x_1, x_2), f(x_1, x_2)) \approx f(x_1, x_2) \approx f(x_1, f(x_2, x_2)) \approx f(f(x_1, x_2), x_2),$$

it follows that $Id_{RB} \models t \approx s$. Following the results in Example 2 we have

$\overline{\rho}_{\alpha_t}[t] = f(f(f(x_2, x_1), x_2), f(x_2, x_1))$ and

$\overline{\rho}_{\alpha_s}[s] = f(f(f(x_2, x_1), x_2), x_2)$.

Thus we obtain $Id_{RB} \models \overline{\rho}_{\alpha_t}[t] \approx f(x_2, x_1)$ and $Id_{RB} \models \overline{\rho}_{\alpha_s}[s] \approx x_2$. Hence $Id_{RB} \not\models \overline{\rho}_{\alpha_t}[t] \approx \overline{\rho}_{\alpha_s}[s]$, and RB is not M-multi-solid. $\square$

The operators χ_M^c and ψ_M^c are connected by the condition that

$$\psi_M^c[\mathcal{A}] \quad \textit{satisfies} \quad u \approx v \iff \mathcal{A} \quad \textit{satisfies} \quad \chi_M^c[u \approx v].$$

ψ_M^c and χ_M^c are additive and closure operators.

Let $\mathcal{R} \subset \mathcal{A}lg(\tau)$ and $\Sigma \subset Id(\tau)$. Then we set:

$$HC_M Mod\Sigma := \{\mathcal{A} \in \mathcal{A}lg(\tau) \mid t \approx s \in \Sigma \implies \mathcal{A} \models_{Mh} t \approx s\};$$

$$HC_M Id\mathcal{R} := \{t \approx s \in Id(\tau) \mid \chi_M^c[t \approx s] \subseteq Id\mathcal{R}\};$$

and

$$HC_M Var\mathcal{R} := HC_M Mod HC_M Id\mathcal{R}.$$

A set Σ of identities of type τ is called an $M-$*multi-hyperequational theory* if $\Sigma = HC_M Id HC_M Mod\Sigma$ and a class $\mathcal{R}$ of algebras of type τ is called an $M-$*multi-hyperequational class* if $\mathcal{R} = HC_M Mod HC_M Id\mathcal{R}$.

Theorem 4. *Let $\Sigma \subseteq Id(\tau)$ and $\mathcal{R} \subseteq \mathcal{A}lg(\tau)$. If $\Sigma = HC_M Id\mathcal{R}$ and $\mathcal{R} = HC_M Mod\Sigma$, then $\mathcal{R} = Mod\Sigma$ and $\Sigma = Id\mathcal{R}$.*

Proof. Let $\Sigma = HC_M Id\mathcal{R}$ and $\mathcal{R} = HC_M Mod\Sigma$. First, we will prove that $\Sigma = \chi_M^c[\Sigma]$ and $\mathcal{R} = \psi_M^c[\mathcal{R}]$.

χ_M^c and ψ_M^c are closure operators and hence we have $\Sigma \subseteq \chi_M^c[\Sigma]$ and $\mathcal{R} \subseteq \psi_M^c[\mathcal{R}]$.

We will prove the converse inclusions. Let $r \approx v \in \chi_M^c[\Sigma]$. Then there is an identity $t \approx s \in \Sigma$ with $r \approx v \in \chi_M^c[t \approx s]$, i.e. $\chi_M^c[r \approx v] \subseteq \chi_M^c[\chi_M^c[t \approx s]] \subseteq \chi_M^c[t \approx s]$. From $t \approx s \in \Sigma$ it follows $t \approx s \in HC_M Id\mathcal{R}$,

$\chi^c_M[t \approx s] \subseteq Id\mathcal{R}$ and $\chi^c_M[r \approx v] \subseteq \chi^c_M[t \approx s] \subseteq Id\mathcal{R}$. Hence $r \approx v \in HC_M Id\mathcal{R}$ and $r \approx v \in \Sigma$, i.e. $\Sigma = \chi^c_M[\Sigma]$.

Let $\mathcal{A} \in \psi^c_M[\mathcal{R}]$. Then there is an algebra $\mathcal{B} \in \mathcal{R}$ with $\mathcal{A} \in \psi^c_M[\mathcal{B}]$, i.e. $\mathcal{A} = \rho[\mathcal{B}]$ for some $\rho \in Mhyp(\tau, M)$. Hence $\psi^c_M[\mathcal{A}] = \psi^c_M[\rho[\mathcal{B}]] \subseteq \psi^c_M[\mathcal{B}]$. Since $\mathcal{B} \in \mathcal{R} = HC_M Mod\Sigma$ we have $\chi^c_M[\Sigma] \subseteq Id\mathcal{B}$ i.e. $\psi^c_M[\mathcal{B}] \subseteq Mod\Sigma$ and $\psi^c_M[\mathcal{A}] \subseteq \psi^c_M[\mathcal{B}] \subseteq Mod\Sigma$. Hence $\mathcal{A} \in HC_M Mod\Sigma$ and $\mathcal{A} \in \mathcal{R}$, i.e. $\mathcal{R} = \psi^c_M[\mathcal{R}]$.

Now, we obtain

$Mod\Sigma = \{\mathcal{A} \mid \Sigma \subseteq Id\mathcal{A}\} = \{\mathcal{A} \mid \chi^c_M[\Sigma] \subseteq Id\mathcal{A}\} = HC_M Mod\Sigma = \mathcal{R}$ and

$Id\mathcal{R} = \{r \approx v \mid \mathcal{R} \models r \approx v\} = \{r \approx v \mid \psi^c_M[\mathcal{R}] \models r \approx v\} = HC_M Id\mathcal{R} = \Sigma$.

Hence $\mathcal{R} = Mod\Sigma$ and $\Sigma = Id\mathcal{R}$. □

Proposition 2. *Let $\mathcal{A} \in \mathcal{A}lg(\tau)$, $\mathcal{R} \subseteq \mathcal{A}lg(\tau)$, $t \approx s \in Id(\tau)$, $\Sigma \subseteq Id(\tau)$, $\alpha_t \in C_t$ and $\alpha_s \in C_s$ and let ρ be a multi-hypersubstitution over M. Then we have:*

(i) $\mathcal{A} \models \overline{\rho}_{\alpha_t}[t] \approx \overline{\rho}_{\alpha_s}[s] \iff \rho[\mathcal{A}] \models t \approx s$;

(ii) $\mathcal{A} \models_{Mh} t \approx s \iff \chi^c_M[t \approx s] \subseteq Id\mathcal{A} \iff \psi^c_M[\mathcal{A}] \models t \approx s$;

(iii) $\Sigma \subseteq HC_M Id\mathcal{R} \iff \chi^c_M[\Sigma] \subseteq Id\mathcal{R} \iff \Sigma \subseteq Id\psi^c_M[\mathcal{R}]$

(iv) $\mathcal{R} \subseteq HC_M Mod\Sigma \iff \psi^c_M[\mathcal{R}] \subseteq Mod\Sigma \iff \mathcal{R} \subseteq Mod\chi^c_M[\Sigma]$

(v) $HC_M Id\mathcal{R} \subseteq Id\mathcal{R}$;

(vi) $Var\mathcal{R} \subseteq HC_M Var\mathcal{R}$;

(vii) The pair $\langle HC_M Id, HC_M Mod\rangle$ forms a Galois connection.

Proof. (i) follows from Theorem 4 and Lemma 2.

(ii) Let ρ be an arbitrary multi-hypersubstitution over the monoid M. Let us assume that $\mathcal{A} \models_{Mh} t \approx s$. Hence $\mathcal{A} \models \overline{\rho}_{\alpha_t}[t] \approx \overline{\rho}_{\alpha_s}[s]$ i.e. $\mathcal{A} \models \chi^c_M[t \approx s]$ and $\psi^c_M[\mathcal{A}] \models t \approx s$.

(iii) and (v) follow from (ii).

(iv) follows from (iii).

(vi) From (ii), it follows that $HC_M Var\mathcal{R}$ is a variety. Then $\mathcal{R} \subseteq HC_M Var\mathcal{R}$, and $Var\mathcal{R} \subseteq HC_M Var\mathcal{R}$.

(vii) follows from Theorem 4. □

Proposition 3. *For every $\Sigma \subseteq Id(\tau)$ and $\mathcal{R} \subseteq \mathcal{A}lg(\tau)$ the following hold:*

(i) $\chi^c_M[HC_M Id\mathcal{R}] = HC_M Id\mathcal{R} = Id\psi^c_M[\mathcal{R}]$;

(ii) $\psi^c_M[HC_M Mod\Sigma] = HC_M Mod\Sigma = Mod\chi^c_M[\Sigma]$;

(iii) $Var\psi^c_M[\mathcal{R}] = HC_M Var\mathcal{R} = ModHC_M Id\mathcal{R}$;

(iv) $Mod\chi^c_M[\Sigma] = HC_M Mod\Sigma = IdHC_M Mod\Sigma$;

(iv) $Id\psi^c_M[\mathcal{R}] = HC_M Id\mathcal{R} = ModHC_M Id\mathcal{R}$.

Proof. (i) From Proposition 2 we obtain

$HC_M Id\mathcal{R} \subseteq HC_M Id\mathcal{R}$

$\Longrightarrow \chi^c_M[\chi^c_M[HC_M Id\mathcal{R}]] \subseteq \chi^c_M[HC_M Id\mathcal{R}] \subseteq Id\mathcal{R}$

$\Longrightarrow \chi^c_M[HC_M Id\mathcal{R}] \subseteq HC_M Id\mathcal{R}$.

The converse inclusion is obvious.

(ii) can be proved in an analogous way as (i).

(iii) We have consequently,
$HC_M Var\mathcal{R} = HC_M ModHC_M Id\mathcal{R}$
$= Mod\chi^c_M[HC_M Id\mathcal{R}] = ModHC_M Id\mathcal{R}$
$= ModId\psi^c_M[\mathcal{R}] = Var\psi^c_M[\mathcal{R}]$.

(iv) and (v) can be proved in an analogous way as (iii). □

For given set Σ of identities the set $IdMod\Sigma$ of all identities satisfied in the variety $Mod\Sigma$ is the deductive closure of Σ, which is the smallest fully invariant congruence containing Σ (see [1, 2, 9]).

A remarkable fact is, that there exists a variety $V \subset \mathcal{A}lg(\tau)$ with $IdV = \Sigma$ if and only if Σ is a fully invariant congruence [2].

A congruence $\Sigma \subseteq Id(\tau)$ is called a fully invariant congruence if it additionally satisfies the following axioms (some authors call them "deductive rules", "derivation rules", "productions" etc.):

(i) (variable inductive substitution)
$(t \approx s \in \Sigma)$ & $(r \in W_\tau(X))$ & $(x \in var(t)) \implies t(x \leftarrow r) \approx s(x \leftarrow r) \in \Sigma$;

(ii) (term positional replacement)
$(t \approx s \in \Sigma)$ & $(r \in W_\tau(X))$ & $(sub_r(p) = t) \implies r(p; s) \approx r \in \Sigma$.

For any set of identities Σ the smallest fully invariant congruence containing Σ is called the $D-$closure of Σ and it is denoted by $D(\Sigma)$.

In [5] totally invariant congruences are studied as fully invariant congruences which preserve the hypersubstitution images i.e. if $t \approx s \in \Sigma$ then $\hat{\sigma}[t] \approx \hat{\sigma}[s] \in \Sigma$ for all $\sigma \in Hyp(\tau)$.

We extend that results, to the case of multi-hypersubstitutions over a given submonoid $M \subseteq Hyp(\tau)$.

Definition 8. *A fully invariant congruence Σ is Mh-deductively closed if it additionally satisfies*

Mh_1 *(Multi-Hypersubstitution)*
$(t \approx s \in \Sigma)$ & $(\rho \in Mhyp(\tau, M)) \implies \rho[t \approx s] \subseteq \Sigma$.

For any set of identities Σ the smallest $Mh-$deductively closed set containing Σ is called the $Mh-$closure of Σ and it is denoted by $Mh(\Sigma)$. It is clear that for each fully invariant congruence Σ we have $Mh(\Sigma) = \chi^c_M[\Sigma]$.

Let Σ be a set of identities of type τ. For $t \approx s \in Id(\tau)$ we say $\Sigma \vdash_{Mh} t \approx s$ ("Σ Mh-proves $t \approx s$") if there is a sequence of identities $t_1 \approx s_1, \ldots, t_n \approx s_n$, such that each identity belongs to Σ or is a result of applying any of the derivation rules of fully invariant congruence or Mh_1-rule to previous identities in the sequence and the last identity $t_n \approx s_n$ is $t \approx s$.

Let $t \approx s$ be an identity and $\mathcal{A}$ be an algebra of type τ. Then $\mathcal{A} \models_{Mh} t \approx s$ means that $\mathcal{A} \models Mh(t \approx s)$ (see Definition 7).

Let $\Sigma \subseteq Id(\tau)$ be a set of identities and $\mathcal{A}$ be an algebra of type τ. Then $\mathcal{A} \models_{Mh} \Sigma$ means that $\mathcal{A} \models Mh(\Sigma)$. For $t, s \in W_\tau(X)$ we say $\Sigma \models_{Mh} t \approx s$ (read: "Σ $Mh-$yields $t \approx s$") if, given any algebra $\mathcal{B} \in \mathcal{A}lg(\tau)$,

$$\mathcal{B} \models_{Mh} \Sigma \quad \Rightarrow \quad \mathcal{B} \models_{Mh} t \approx s.$$

Remark 2. In a more general case we have $\chi_M^c[\Sigma] \subseteq Mh(\Sigma)$ and there are examples when $\chi_M^c[\Sigma] \neq Mh(\Sigma)$.

It is easy to see that each $Mh-$deductively closed set is a totally invariant congruence [5]. On the other side from Theorem 3 it follows that the totally invariant congruence Id_{RB} is not $Mh-$deductively closed.

Lemma 3. *For any set $\Sigma \subseteq Id(\tau)$ of identities and $t \approx s \in Id(\tau)$ the following equivalences hold:*

$$\Sigma \vdash_{Mh} t \approx s \iff \chi_M^c[\Sigma] \vdash t \approx s \iff Mh(\Sigma) \vdash t \approx s.$$

Theorem 5. (Completeness Theorem for Multi-hyperequational Logic.) *For $\Sigma \subseteq Id(\tau)$ and $t \approx s \in Id(\tau)$ we have:*

$$\Sigma \models_{Mh} t \approx s \iff \Sigma \vdash_{Mh} t \approx s.$$

Proof. From $HC_M IdHC_M Mod\Sigma = IdHC_M Mod\Sigma = IdMod\chi_M^c[\Sigma]$ we obtain that $\Sigma \models_{Mh} t \approx s$ is equivalent to $t \approx s \in HC_M IdHC_M Mod\Sigma$. From Theorem 14.19 [2] we have $HC_M IdHC_M Mod\Sigma \models t \approx s$ and $HC_M IdHC_M Mod\Sigma \vdash t \approx s$. Hence $\chi_M^c[\Sigma] \vdash t \approx s$ and $\Sigma \vdash_{Mh} t \approx s$.

The converse implication follows from the fact that $Mh(\Sigma)$ is a fully invariant congruence which is closed under the rule Mh_1. □

Corollary 1. *Let M be a submonoid of $Hyp(\tau)$. Then the class of all $M-$multi-solid varieties of type τ is a complete sublattice of the lattice $\mathcal{L}(\tau)$ of all varieties of type τ and dually, the class of all $M-$multi-hyperequational theories of type τ is a complete sublattice of the lattice $\mathcal{E}(\tau)$ of all equational theories (fully invariant congruences) of type τ.*

Lemma 4. *For any set $\Sigma \subseteq Id(\tau)$ of identities and $t \approx s \in Id(\tau)$ the following equivalence holds:*

$$\Sigma \models_{Mh} t \approx s \iff Mh(\Sigma) \models t \approx s.$$

Proof. " $\Rightarrow$ " Let $\Sigma \models_{Mh} t \approx s$. We have to prove $Mh(\Sigma) \models t \approx s$. Let $\mathcal{A} \in \mathcal{A}lg(\tau)$ be an algebra for which $\mathcal{A} \models Mh(\Sigma)$. This implies $\mathcal{A} \models_{Mh} \Sigma$ and $\mathcal{A} \models_{Mh} t \approx s$, because of $\Sigma \models_{Mh} t \approx s$. Hence $\mathcal{A} \models Mh(t \approx s)$. On the other side, we have $t \approx s \in Mh(t \approx s)$ and $\mathcal{A} \models t \approx s$.

" $\Leftarrow$ " Let $Mh(\Sigma) \models t \approx s$. Since $\chi^c_M[\Sigma] \subseteq Mh(\Sigma)$ and from Proposition 2 (*ii*) we have $\Sigma \models_{Mh} t \approx s$. □

Corollary 2. *For any set $\Sigma \subseteq Id(\tau)$ of identities and $t \approx s \in Id(\tau)$ it holds:*

$$\Sigma \models_{Mh} t \approx s \iff \chi^c_M[\Sigma] \models t \approx s.$$

Example 3. Let τ be an arbitrary type.

(*i*) Let $t \in W_\tau(X)$ be a term of type τ. Let us consider the following two functions $Left : W_\tau(X) \to X$ and $Right : W_\tau(X) \to X$, which assign to each term t the leftmost and the rightmost variable of t. For instance, if $t = g(f(x_1, x_2), x_3, x_4)$, then $Left(t) = x_1$ and $Right(t) = x_4$.

Let us denote by $K_1 \subset Hyp(\tau)$ the set of all hypersubstitutions which preserve the functions $Left$ and $Right$ i.e. $\sigma \in K_1$, iff for all $t \in W_\tau(X)$ we have $Left(t) = Left(\hat{\sigma}[t])$ and $Right(t) = Right(\hat{\sigma}[t])$. Then K_1 is a submonoid of $Hyp(\tau)$. Let us consider the monoid $Mhyp(\tau, K_1)$ of multi-hypersubstitutions, generated by K_1. It is a submonoid of $Mhyp(\tau, Hyp(\tau))$. The variety RB of rectangular bands is K_1-multi-solid.

(*ii*) Let $Vstr : W_\tau(X) \to X^*$ be a mapping, which assigns to each term t the string of the variables in t. For instance, if

$$t = f(x_1, g(f(x_1, x_2), x_3, x_2), x_4), \quad \text{then} \quad Vstr(t) = x_1x_1x_2x_3x_2x_4.$$

This mapping is defined inductively as follows: if $t = x_j \in X$, then $Vstr(t) := x_j$; and if $t = f(t_1, \ldots, t_n)$, then
$Vstr(t) := Vstr(t_1)Vstr(t_2) \ldots Vstr(t_n)$.

Let $K_2 \subset Hyp(\tau)$ be the set of all hypersubstitutions which preserve $Vstr$ i.e. $\sigma \in K_2$, if and only if for each $t \in W_\tau(X)$ it holds $Vstr(t) = Vstr(\hat{\sigma}[t])$. Then K_2 is a submonoid of $Hyp(\tau)$.

Let us consider the monoid $Mhyp(\tau, K_2)$ of multi-hypersubstitutions, generated by K_2. It is a submonoid of $Mhyp(\tau, Hyp(\tau))$ and $K_2 \subseteq K_1$.

It is not difficult to prove that the variety RB is K_2-multi-solid.

Finally, let us note that if Σ is the set of identities satisfied in RB, and if we add to K_1 and K_2 the hypersubstitution $\sigma \in Hyp(\tau)$ with $\sigma(f) = f(x_2, x_1)$, then $M_1 := K_1 \cup \{\sigma\}$ and $M_2 := K_2 \cup \{\sigma\}$ are monoids, again such that $\chi^c_{M_1}[\Sigma] = Id(\tau)$, but $\chi^c_{M_2}[\Sigma] \neq Id(\tau)$.

3. Tree automata realization

We consider an automata realization of the multi-hypersubstitutions. This concept will allow to use computer programmes in the case of finite monoids of hypersubstitutions to obtain the images of terms under multi-hypersubstitutions.

In Computer Science terms are used as data structures and they are called trees. The operation symbols are labels of the internal nodes of trees and variables are their leaves.

The concept of tree automata was introduced in the 1960s in various papers such as [10]. Gécseg & Steinby's book [6] is a good survey of the theory of tree automata and [3] is a development of this theory. Tree automata are classified as tree recognizers and tree transducers. Our aim is to define tree automata which interpret the application of multi-hypersubstitutions of the colored terms of a given type.

Definition 9. *A colored tree transducer of type* τ *is a tuple* $\underline{A} = \langle X, \mathcal{F}, P\rangle$ *where as usual* X *is a set of variables,* $\mathcal{F}$ *is a set of operation symbols and* P *is a finite set of productions (rules of derivation) of the forms*
(i) $x \to x,\ x \in X$;
(ii) $\langle f, q\rangle(\xi_1, \cdots, \xi_n) \to \langle r, \alpha_r\rangle(\xi_1, \cdots, \xi_n)$,
with $\langle r, \alpha_r\rangle(\xi_1, \cdots, \xi_n) \in W^c_\tau(X \cup \chi_m)$, $\langle f, q\rangle \in \mathcal{F}^c$, $\xi_1, \cdots, \xi_n \in \chi_n$,
where $\chi_n = \{\xi_1, \cdots, \xi_n\}$ *is an auxiliary alphabet.*
(All auxiliary variables ξ_j *belong to a set* $\chi_m = \{\xi_1, \cdots, \xi_m\}$ *where* $m = maxar$ *is the maximum of the arities of all operation symbols in* $\mathcal{F}$*.)*

Definition 10. *Let* $M \subseteq Hyp(\tau)$ *be a submonoid of* $Hyp(\tau)$ *and* $\rho \in Mhyp(\tau, M)$ *be a multi-hypersubstitution over* M*. A colored tree transducer* $\underline{A}^\rho = \langle X, \mathcal{F}, P\rangle$ *is called* $Mh-$*transducer over* M*, if for the rules (ii) in* P *we have* $r = \sigma(q)(f)$ *and* $\alpha_r(p) = q$ *for all* $p \in Pos^{\mathcal{F}}(r)$*.*

The $Mh-$transducer $\underline{A}^\rho$ runs over a colored term $\langle t, \alpha_t\rangle$ starting at the leaves of t and moves downwards, associating along the run a resulting colored term (image) with each subterm inductively: if $t = x \in X$ then the $Mh-$transducer $\underline{A}^\rho$ associates with t the term $x \in W^c_\tau(X)$, if $x \to x \in P$; if $\langle t, \alpha_t\rangle = \langle f, q\rangle(\langle t_1, \alpha_{t_1}\rangle \ldots, \langle t_n, \alpha_{t_n}\rangle)$ then with $\langle t, \alpha_t\rangle$ the $Mh-$transducer $\underline{A}^\rho$ associates the colored term $\langle s, \alpha_s\rangle = \langle u, \alpha_u\rangle(\langle t_1, \alpha_{t_1}\rangle \ldots, \langle t_n, \alpha_{t_n}\rangle)$, if $\langle f, q\rangle(\xi_1, \ldots, \xi_n) \to \langle u, \alpha_u\rangle(\xi_1, \cdots, \xi_n) \in P$, where $u = \sigma(q)(f)$ and $\alpha_u(p) = q$ for all $p \in Pos^{\mathcal{F}}(u)$.

For trees $\langle t, \alpha_t\rangle$, and $\langle s, \alpha_s\rangle$ we say $\langle t, \alpha_t\rangle$ *directly derives* $\langle s, \alpha_s\rangle$ *by* $\underline{A}^\rho$, if $\langle s, \alpha_s\rangle$ can be obtained from $\langle t, \alpha_t\rangle$ by replacing of an occurrence of a subtree $\langle f, q\rangle(\langle r_1, \alpha_{r_1}\rangle, \cdots, \langle r_n, \alpha_{r_n}\rangle)$, in $\langle t, \alpha_t\rangle$ by
$\langle u, \alpha_u\rangle(\langle r_1, \alpha_{r_1}\rangle, \cdots, \langle r_n, \alpha_{r_n}\rangle) \in W^c_\tau(X \cup \chi_m)$.

If $\langle t, \alpha_t \rangle$ directly derives $\langle s, \alpha_s \rangle$ in $\underline{A}^\rho$, we write $\langle t, \alpha_t \rangle \to_{\underline{A}^\rho} \langle s, \alpha_s \rangle$. Furthermore, we say $\langle t, \alpha_t \rangle$ *derives* $\langle s, \alpha_s \rangle$ *in* $\underline{A}^\rho$, if there is a sequence $\langle t, \alpha_t \rangle \to_{\underline{A}^\rho} \langle s_1, \alpha_{s_1} \rangle \to_{\underline{A}^\rho} \langle s_2, \alpha_{s_2} \rangle \to_{\underline{A}^\rho} \cdots \to_{\underline{A}^\rho} \langle s_n, \alpha_{s_n} \rangle = \langle s, \alpha_s \rangle$ of direct derivations or if $\langle t, \alpha_t \rangle = \langle s, \alpha_s \rangle$. In this case we write $\langle t, \alpha_t \rangle \Rightarrow^*_{\underline{A}^\rho} \langle s, \alpha_s \rangle$. Clearly, $\Rightarrow^*_{\underline{A}^\rho}$ is the reflexive and transitive closure of $\to_{\underline{A}^\rho}$.

Let us denote by $Thyp(\tau, M)$ the set of all $Mh-$transducers of type τ over M.

A term $\langle t, \alpha_t \rangle$ is translated to the term $\langle s, \alpha_s \rangle$ by the $Mh-$transducer $\underline{A}^\rho$ if there exists a run of $\underline{A}^\rho$ such that it associates with $\langle t, \alpha_t \rangle$ the colored term $\langle s, \alpha_s \rangle$. In this case we will write $\underline{A}^\rho(\langle t, \alpha_t \rangle) = \langle s, \alpha_s \rangle$.

Lemma 5. $\underline{A}^\rho(\langle t, \alpha_t \rangle) = \langle s, \alpha_s \rangle \iff \overline{\rho}[\langle t, \alpha_t \rangle] = \langle s, \alpha_s \rangle$.

Proof. For a variable-term x we have $\underline{A}^\rho(x) = x = \overline{\rho}[x]$. Suppose that for $i = 1, \ldots, n$ we have $\underline{A}^\rho(\langle t_i, \alpha t_i \rangle) = \overline{\rho}[\langle t_i, \alpha t_i \rangle]$. Then we obtain

$$\underline{A}^\rho(\langle f, q \rangle(\langle t_1, \alpha t_1 \rangle, \ldots, \langle t_n, \alpha t_n \rangle)) =$$
$$\underline{A}^\rho(\langle f, q \rangle)(\underline{A}^\rho(\langle t_1, \alpha t_1 \rangle), \ldots, \underline{A}^\rho(\langle t_n, \alpha t_n \rangle)) = \overline{\rho}[\langle t, \alpha_t \rangle] = \overline{\rho}[\langle t, \alpha_t \rangle],$$

where $\alpha_t(\varepsilon) = q$ and $\alpha_t[i] = \alpha_{t_i}$ for $i = 1, \ldots, n$. □

The product (superposition) of two $Mh-$transducers $\underline{A}^{\rho_1}$ and $\underline{A}^{\rho_2}$ is defined by the following equation

$$\underline{A}^{\rho_1} \circ \underline{A}^{\rho_2}(\langle t, \alpha_t \rangle) := \underline{A}^{\rho_1}(\underline{A}^{\rho_2}(\langle t, \alpha_t \rangle)).$$

Lemma 6. *Let $M \subseteq Hyp(\tau)$ be a monoid of hypersubstitutions. Then the superposition of $Mh-$transducers of a given type is associative i.e.*

$$\underline{A}^{\rho_1} \circ (\underline{A}^{\rho_2} \circ \underline{A}^{\rho_3})(\langle t, \alpha_t \rangle) = (\underline{A}^{\rho_1} \circ \underline{A}^{\rho_2}) \circ \underline{A}^{\rho_3}(\langle t, \alpha_t \rangle)$$

for all $\rho_1, \rho_2, \rho_3 \in MHyp(\tau, M)$ and for all $\langle t, \alpha_t \rangle \in W^c_\tau(X)$.

Theorem 6. *Let $M \subseteq Hyp(\tau)$ be a monoid of hypersubstitutions. Then the set $Thyp(\tau, M)$ is a monoid which is isomorphic to the monoid of all multi-hypersubstitutions over M i.e.*

$$Thyp(\tau, M) \cong Mhyp(\tau, M).$$

Proof. We define a mapping $\varphi : Mhyp(\tau, M) \to Thyp(\tau, M)$ by $\varphi(\rho) := \underline{A}^\rho$.

To show that φ is a homomorphism we will prove that $\underline{A}^{\rho_1} \circ \underline{A}^{\rho_2} = \underline{A}^{\rho_1 \circ_{ch} \rho_2}$, so that $\varphi(\rho_1) \circ \varphi(\rho_2) = \varphi(\rho_1 \circ_{ch} \rho_2)$.

We have $\underline{A}^{\rho_1} \circ \underline{A}^{\rho_2}(\langle t, \alpha_t \rangle) = \underline{A}^{\rho_1}(\underline{A}^{\rho_2}(\langle t, \alpha_t \rangle)) = \underline{A}^{\rho_1}(\overline{\rho}_2[\langle t, \alpha_t \rangle]) = \overline{\rho}_1[\overline{\rho}_2[\langle t, \alpha_t \rangle]] = \overline{\rho_1 \circ_{ch} \rho_2}[\langle t, \alpha_t \rangle] = A^{\rho_1 \circ_{ch} \rho_2}(\langle t, \alpha_t \rangle)$.

To see that φ is one-to-one, let $\underline{A}^{\rho_1} = \underline{A}^{\rho_2}$. Then from Lemma 5 for all $\langle t, \alpha_t\rangle \in W^c_\tau(X)$ we have $\overline{\rho}_1[\langle t, \alpha_t\rangle] = \overline{\rho}_2[\langle t, \alpha_t\rangle]$. Hence for all $f \in \mathcal{F}$ and $q \in I\!N$ we have $\overline{\rho}_1[\langle f, q\rangle] = \overline{\rho}_2[\langle f, q\rangle]$ and therefore $\rho_1 = \rho_2$. $\square$

References

[1] G. Birkhoff, *Lattice theory,* (3rd ed.) Amer. Math. Soc., Providence, 1967

[2] S. Burris and H. Sankappanavar, *A Course in Universal Algebra*, The millennium edition, 2000

[3] H. Comon, M. Dauchet, R. Gilleron, F. Jacquemard, D. Lugiez, S. Tison, M. Tommasi,*Tree Automata, Techniques and Applications,* 1999, http://www.grappa.univ-lille3.fr/tata/

[4] K. Denecke, J. Koppitz and Sl. Shtrakov, Multi-Hypersubstitutions and Coloured Solid Varieties, *J. Algebra and Computation,* J. Algebra and Computation, Volume 16, Number 4, August, 2006, pp.797-815.

[5] K. Denecke, D. Lau, R. Pöschel and D. Schweigert, Hyperidentities, Hyperequational Classes and Clone Congruences,*General Algebra 7, Verlag Hölder-Pichler-Tempsky,* Wien 1991, Verlag B.G. Teubner Stuttgart, pp.97-118

[6] F. Gécseg, M. Steinby, *Tree Automata*, Akadémiai Kiadó, Budapest 1984

[7] E. Graczýnska, *On connection between identities and hyperidentities*, Bull.Sect.Logic 17(1988),34-41.

[8] G. Gratzer, *Universal Algebra*, D. van Nostrand Co., Princetown, 1968.

[9] R. McKenzie, G. Mc Nulty and W. Taylor, *Algebras, Lattices, Varieties,* Vol. I, Belmont, California 1987.

[10] J. W. Thatcher and J.B. Wright, Generalized finite automata, *Notices Amer. Math. Soc.*, **12.** (1965), abstract No. 65T-649,820.

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