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Distribution of arithmetical functions on some subsets of integers

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Abstract. This is a survey paper on the distribution of arithmetical functions on some subsets of integers. Continuous homomorphisms as arithmetical functions, and sets of uniqueness are also treated.

1. Notations, definitions and basic theorems

We shall use the usual notation: $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ for the set of positive integers, integers, rational, real and complex numbers, respectively. Let $\mathbb{Q}_x$, $\mathbb{R}_x$ be the multiplicative group of rational, real numbers, respectively.

Let $\mathcal{P}$ be the set of primes, $p(n)$ be the smallest, $P(n)$ be the largest prime factor of n. Let $x_1 = \log x$, $x_2 = \log x_1, \dots$. Let $\pi(x)$ be the number of primes up to x.

Let G be an Abelian group. $\mathcal{A}_G$ = class of additive functions (taking values from G) is defined as follows: $f : \mathbb{N} \to G$ belongs to $\mathcal{A}_G$ if $f(mn) = f(m) + f(n)$ holds for every $m, n \in \mathbb{N}$ with $(m, n) = 1$. We say that $f \in \mathcal{A}_G^*$ if $f(mn) = f(m) + f(n)$ holds without any constraint.

For $f \in \mathcal{A}_G^*$ we extend the domain to $\mathbb{Q}_x$:

$$f\left(\frac{m}{n}\right) = f(m) - f(n).$$

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If G is multiplicatively written we define $\mathcal{M}_G$, $\mathcal{M}_G^*$ as follows:

$g \in \mathcal{M}_G$ if $g : \mathbb{N} \to G,\ g(mn) = g(m) \cdot g(n) \quad \forall (m,n) = 1$,

$g \in \mathcal{M}_G^*$ if $g : \mathbb{N} \to G,\ g(mn) = g(m) \cdot g(n) \quad \forall m, n \in \mathbb{N}$.

$\mathcal{M}_G$ = class of multiplicative functions, $\mathcal{M}_G^*$ = class of completely multiplicative functions.

The following assertion can be proved easily.

Lemma 1. *Let G be a topological Abelian group, $f \in \mathcal{A}_G^*$, $f : \mathbb{Q}_x \to G$ is continuous at* 1. *Then for each $\alpha \in \mathbb{R}_x$ there exists the limit*

$$\lim_{\substack{r \to \alpha \\ r \in \mathbb{Q}_x}} f(r) =: \Phi(\alpha).$$

Φ is continuous everywhere in $\mathbb{R}_x$, furthermore $\Phi(\alpha\beta) = \Phi(\alpha) + \Phi(\beta)$ valid for all $\alpha, \beta \in \mathbb{R}_x$. Thus Φ is a continuous homomorphism of $\mathbb{R}_x$ into G.

On the other hand: *if $\phi : \mathbb{R}_x \to G$ homomorphism then $\phi/_{\mathbb{N}} \in \mathcal{A}_G^*$.*

Let $y_n (n \in \mathbb{N})$ be a sequence of real numbers,

$$F_N(u) := \frac{1}{N} \#\{n \le N \mid y_n < u\}.$$

Definition. *We say that $\{y_n (n \in \mathbb{N})\}$ has a limit distribution, if*

$$\lim_{N \to \infty} F_N(u) = F(u)$$

exists for every continuity point of F, where F is a distribution function.

Theorem of Erdős and Wintner: *$f \in \mathcal{A}$ has a limit distribution if and only if*

$$(a) \quad \sum_{|f(p)|<1} \frac{f(p)}{p}, \qquad (b) \quad \sum_{|f(p)|<1} \frac{f^2(p)}{p}, \qquad (c) \quad \sum_{|f(p)| \ge 1} 1/p$$

are convergent.

Theorem (Kátai, 1965.). *Assume that $f \in \mathcal{A}$, and* (a), (b), (c) *are convergent. Then f has a distribution on the set of shifted primes, i.e.*

$$\lim_{N \to \infty} \frac{1}{\pi(N)} \# \{p < N \mid f(p+a) < u\} = G_a(u)$$

exist for every continuity points of G_a, when $a \ne 0$.

Question: are the convergence of (a), (b), (c) necessary? I wanted to find all $f \in \mathcal{A}^*$ for which $f(p+1) = 0, \quad (p \in \mathcal{P})$.

Definition. *We say that $E \subseteq \mathbb{N}$ is a set of uniqueness (for the set of completely additive functions) if $f \in \mathcal{A}^*$, $f(E) = 0$ implies that $f(\mathbb{N}) = 0$.*

I proved: The set $\mathcal{P}+1$ can be enlarged by a finite set of primes $q_1,\dots,q_s$ such that $E=\{\mathcal{P}+1,\ q_1,\dots,q_s\}$ is a set of uniqueness. (Acta Arithmetica: **16** (1968), 1-4).

Elliott proved: $\mathcal{P}+1$ is a set of uniqueness (Acta Arithmetica **26** (1974), 11-20).

Wolke (Elemente der Math. **33** (1978), 14-16) proved: E is a set of uniqueness if and only if every $n\in\mathbb{N}$ can be written as

$$n=e_{i_1}^{r_1}\dots e_{i_h}^{r_h},\quad r_1,\dots,r_h\in\mathbb{Q}.$$

Hildebrand (Proc. London **53** (1989), 209-232) proved that the convergence of (a), (b), (c) are necessary to the existence of the limit distribution of $f(p+a)$.

From Hildebrand's theorem it follows that, if for each $\varepsilon>0$,

$$\frac{1}{\pi(x)}\#\{p\le x\ \big|\ |f(p+1)|\ge\varepsilon\}\to 0\quad (x\to\infty),$$

then $f(p+1)=0$.

Definition. *We say that $E\subseteq\mathbb{N}$ is a set of uniqueness for the class of functions in $\mathcal{A}_G^*$, if $f\in\mathcal{A}_G^*$, $f(E)=0$ implies that $f(\mathbb{N})=0$.*

Let $G=T=\text{torus}=\{z\in\mathbb{C},\ |z|=1\}$.

Meyer, Indlekofer, Dress and Wolkman, Hoffman proved: *in order that E would be a set of uniqueness for the class $\mathcal{A}_T^*$, it is necessary and sufficient that every positive integer n had a representation*

$$n=\prod_{j=1}^{s}a_j^{d_j},$$

with some integers d_j and $a_j\in E$.

Let K be the multiplicative group generated by the elements $\{p+1\mid p\in\mathcal{P}\}$.

In my paper implicitly is stated: *there is a suitable constant L, such that every integer n can be written as $a(n)\cdot k(n)$, where $k(n)\in K$, and $a(n)$ is such a rational number in the reduced form of which all prime factors are less than L.*

Elliott proved: $L=10^{387}$ is appropriate.

Recently Elliott proved: $O(Q_x\mid K)\in\{1,2,3\}$.

It is not known whether $(-1)^{\Omega(p+1)}=\text{constant}$ for every large prime p, or not?

Question: Under what condition is true, that for $f\in\mathcal{A}$, $\Delta f(n)=f(n+1)-f(n)$ has a limit distribution?

If $f(n)=c\log n$, then $\Delta f(n)\to 0\Rightarrow\exists$ limit distribution.

Hildebrand proved: *Let $f \in \mathcal{A}$, such that $\Delta f(n)$ has a limit distribution. Then $f(n) = c \log n + g(n)$,*

$$(*) \qquad \sum_{p \in \mathcal{P}} \frac{\min(g^2(p), 1)}{p} < \infty.$$

If $(*)$ holds, then the function $\Delta f(n)$ has a limit distribution.

2. Characterization of $\log n$ as an additive function

The first result is from Erdős (Annals of Math. 1946). Here we shall only list the most important results on this topic:

1. If $\Delta(n) = f(n+1) - f(n) \to 0$, then $f = c \log$. (P. Erdős, 1946)
2. If $\Delta(n) \geq 0 \quad (\forall n)$, then $f = c \log$. (P. Erdős, 1946)
3. If $\liminf \Delta^k f(n) \geq 0$, then $f = c \log$. (I. Kátai)
4. If $\Delta f(n) \geq -K$, then $f(n) = c \log n + u(n)$, $u(n)$ bounded. (Wirsing)
5. If

$$\frac{1}{x} \sum_{n \leq x} |\Delta f(n)| \to 0 \quad (x \to \infty),$$

then $f = c \log$. (I. Kátai)

6. If $f \in \mathcal{A}^*$, and $\dfrac{\Delta f(n)}{\log n} \to 0 \quad (n \to \infty)$, then $f = c \log$. (Wirsing)
7. Let $f, g \in \mathcal{A}$, $\eta_n := g(n+1) - f(n)$. If
   a) $\eta_n \to 0$, then $f(n) = g(n) = c \log n$,
   b) η_n is bounded, then $f(n) = c \log n + u(n)$,
   $g(n) = c \log n + v(n)$, and $u(n)$, $v(n)$ are bounded.
   c) If $f, g \in \mathcal{A}^*$, $\dfrac{\eta_n}{\log n} \to 0$, then $f(n) = g(n) = c \log n$.
8. Elliott characterized $f \in \mathcal{A}$, satisfying

$$f(an+b) - f(An+B) \to C, \quad (n \to \infty),$$

if $\Delta = aB - Ab \neq 0$.

Assume that $aA\Delta \neq 0$. Then there exists c, c_1, so that

$$\left| \frac{f(m)}{\log m} - \frac{f(n)}{\log n} \right| \leq c_1 \left(\frac{L(m)}{\log m} + \frac{L(n)}{\log n} \right)$$

holds uniformly for all integers m and n which satisfy $2 \leq m \leq n \leq e^m$, and are prime to $aA\Delta$.

Here

$$L(x) = \max_{n \leq x^c} |f(an+b) - f(An+B)|.$$

3. Characterization of n^s as a multiplicative function

In a series of papers: *Multiplicative functions with regularity properties I-VI.*, (Acta Mathem. Hung. 1983-1991) the following assertions are proved:

I. If $f, g \in \mathcal{M}$, and

$$\sum \frac{|g(n+1) - f(n)|}{n} < \infty,$$

then either

$$\sum \frac{|f(n)|}{n} < \infty, \quad \sum \frac{|g(n)|}{n} < \infty, \tag{!}$$

or

$$f(n) = g(n) = n^{\sigma + it}, \quad \sigma, \tau \in \mathbb{R},\ 0 \le \sigma < 1.$$

II. Assume that $f, g \in \mathcal{M}^*$, $k \ge 1$ be fixed, $f(n) \ne 0$, $g(n) \ne 0$, if $(n, k) = 1$, and $f(n) = g(n) = 0$ if $(n, k) > 1$, furthermore

$$\sum_n \frac{1}{n} |g(n+k) - f(n)| < \infty.$$

Then either (!) is satisfied or there exist $F, G \in \mathcal{M}^*$, $s \in \mathbb{C}$ with $\operatorname{Re} s < 1$, such that $f(n) = n^s F(n)$, $g(n) = n^s G(n)$, and

$$G(n+k) = F(n) \quad (n \in \mathbb{N})$$

holds.

Consequence: $\sum \frac{1}{n} |\lambda(n+1) - \lambda(n)| = \infty, \lambda =$Liouville function.

Conjecture (1984): If $f \in \mathcal{M}$, $\Delta f(n) \to 0$, then either $f(n) \to 0$ or $f(n) = n^{i\tau}$.

Conjecture proved by Wirsing in 1984, published by Wirsing, Tang and Shao Journal of Number Theory **56**, (1996).

Bassily and Kátai proved: If $f, g \in \mathcal{M}$, $g(2n+1) - cf(n) \to 0 \quad (n \to \infty)$, $c \ne 0$, then either $f(n) \to 0$, or $f(n) = n^s$, $0 \le \text{Re } s < 1$, $g(n) = f(n)$ if n odd.

Problem: Characterize $f, g \in \mathcal{M}$ with $g(An+B) + Cf(an+b) \to 0$.

Some important results are proved by B.M. Phong.

Conjecture 1. *If $f \in \mathcal{M}$ and*

$$\frac{1}{x} \sum_{n \le x} |\Delta f(n)| \to 0, \tag{$*$}$$

then either

$$\frac{1}{x}\sum_{n\le x}|f(n)|\to 0,$$

or $f(n)=n^s,\ 0\le Re\ s<1$.

Mauclaire and Murata proved: *If* $f\in\mathcal{M}$, $(*)$ *holds,* $|f(n)|=1$ $(\forall n\in\mathbb{N})$, *then* $f\in\mathcal{M}^*$.

Hildebrand proved: $\exists c>0$: *If* $g\in\mathcal{M}^*, |g(n)|=1$, *and* $|g(p)-1|\le c$ $(p\in\mathcal{P})$, *then either* $g(n)=1$ $(n\in\mathbb{N})$, *or*

$$\liminf\frac{1}{x}\sum_{n\le x}|\Delta g(n)|>0.$$

I proved: $\exists 0<\beta<1,\ \delta>0$*: If* $g\in\mathcal{M}^*, |g(n)|=1$, *and*

$$\limsup\sum_{x^\beta<p<x}\frac{|g(p)-1|}{p}<\delta,$$

$$\liminf_{x\to\infty}\frac{1}{x}\sum_{\frac{x}{2}\le n\le x}|\Delta g(n)|=0,$$

then $g(n)=1$ $(\forall n)$.

Let $\|x\|=\min_{n\in\mathbb{Z}}|x-n|$.

Bui Minh Phong proved the following theorems:

I. Let k be fixed. Then there exist suitable positive constants $\delta,\ \eta>0$ such that if $f_0,f_1,\ldots,f_k\in\mathcal{A},\ \|f_j(p)\|<\delta\quad(p\in\mathcal{P},\ j=0,\ldots,k)$ $l(n):=f_0(n)+\ldots+f_k(n+k)$, then

$$\liminf_{x\to\infty}\frac{1}{x}\sum_{n\le x}\|l(n)+\Gamma\|<\eta$$

implies that

$$l(n)+\Gamma\equiv 0(\mathrm{mod}\,1).$$

II. Let $a_1,b_1,a_2,b_2\in\mathbb{Z},\ a_1,a_2\in\mathbb{N},\Delta=a_1b_2-a_2b_1\ne 0$. Then there exist $\delta,\eta>0$ such that: if

$$f,g\in\mathcal{A},\ \|f(p)\|\le\delta,\ \|g(p)\|\le\delta\quad(p\in\mathcal{P})$$

and

$$\liminf_{x\to\infty}\frac{1}{x}\sum_{n\le x}\|f(a_1n+b_1)+g(a_2n+b_2)+\Gamma\|\le\eta,$$

then

$$f(a_1n+b_1)+g(a_2n+b_2)+\Gamma\equiv 0(\mathrm{mod}\,1)\quad(n\in\mathbb{Z}).$$

4. On additive functions mod 1.

Let $T = \mathbb{R}/\mathbb{Z}$.

Definition. *We say that* $F \in \mathcal{A}_T$ *is of* **"finite support"** *if* $F(p^\alpha) = 0$ *holds for every large* p *and* $\alpha \in \mathbb{N}$.

Let $F_0, \dots, F_{k-1} \in \mathcal{A}_T$,

$$L_n(F_0, \dots, F_{k-1}) = F_0(n) + \dots + F_{k-1}(n + k - 1).$$

Conjecture 2.

Let $\mathcal{L}_0^{(k)}$ *be the space of those* $(F_0, \dots, F_{k-1})$ *for which* $L_n(F_0, \dots, F_{k-1}) = 0 \quad (\forall n \in \mathbb{N})$.

Then every F_j *is of finite support. Furthermore* $\mathcal{L}_0^{(k)}$ *is a finite dimensional* $\mathbb{Z}$ *module.*

Remark. If $G_j(n) = \tau_j \log n \pmod 1$, $\tau_0 + \dots + \tau_{k-1} = 0$, then $L_n(G_0, \dots, G_{k-1}) \to 0 \quad (n \to \infty)$.

Conjecture 3. *If* $F_\nu \in \mathcal{A}_T$ $(\nu = 0, \dots, k-1)$, *and*

$$L_n(F_0, \dots, F_{k-1}) \to 0 \quad (n \to \infty),$$

then there exist suitable real numbers $\tau_0, \dots, \tau_{k-1}$ *such that* $\tau_0 + \dots + \tau_{k-1} = 0$ *and for the functions* $H_j(n) = F_j(n) - \tau_j \log n$ *we have*

$$L_n(H_0, \dots, H_{k-1}) = 0.$$

Remarks.

1. Conjecture 3, $k = 1$ follows from Wirsing's theorem.
2. Conjecture 2, $k = 3$ was proved by Kátai for $F_\nu \in \mathcal{A}_T^*$, and for $F_\nu \in \mathcal{A}_T$ by R. Styer.

Conjecture 4. *For every integer* $k \geq 1$ *there exists a constant* c_k *such that for every prime* p *greater than* c_k,

$$\min_{\substack{1 \leq j \\ P(j) < p}} \ \max_{\substack{i \in [-k,k] \\ i \neq 0}} P(jp + l) < p.$$

No proof for $k \geq 2$.

Proposition. *Let* $\mathcal{L}_0^{*l}$ *be the space of those* l*-tuples* $(F_0, \dots, F_{l-1})$ *of* $F_\nu \in \mathcal{A}_T^*$ *for which* $L_n(F_0, \dots, F_{l-1}) = 0 \quad (n \in \mathbb{N})$ *holds. Assume that Conjecture 4 is true for* $k = 1$. *Then* $\mathcal{L}_0^{*l}$ *is a finite dimensional space.*

5. Continuous homomorphisms as elements of $\mathcal{A}_G$, $G =$ Abelian compact group

Z. Daróczy and I. Kátai: Let $G =$ metrically compact Abelian group.

$f : \mathbb{N} \to G$. S_f = set of limit points of $f(n)(n \in \mathbb{N})$. S_f is a closed space, $f(\mathbb{N}) \subseteq S_f$.

Theorem. *Let $f \in \mathcal{A}_G^*$, $\Delta f(n) \to 0$ $(n \to \infty)$. Then there exists a continuous homomorphism $\phi : \mathbb{R}_x \to G$ such that $f(n) = \phi(n)$.*

More generally: *Let $H : S_f \to S_f$ be continuous. Assume that $f \in \mathcal{A}_G^*$, $f(n+1) - H(f(n)) \to 0$. Then $f(n) = \phi(n)$, $\phi : \mathbb{R}_x \to G$ is a continuous homomorphism.*

Conjecture 5. *Let $f \in \mathcal{A}_T^*$, $e_n = (f(n), \dots, f(n+k-1))$. Then either $f(n) = \lambda \log n (\mathrm{mod}\ 1)$ with some $\lambda \in \mathbb{R}$, or $\{e_n \mid n \in \mathbb{N}\}$ is dense in $T_k = T \times \cdots \times T$.*

Conjecture 6. (I. Kátai & M.V. Subbarao). *Assume that $f \in \mathcal{A}_G^*$, G = Abelian compact group. Assume that $S_f = G$. Let H = set of limit points of $\Delta f(n)$. Then H is a closed subspace of G, and*

$$f(n) = \varphi(n) + \alpha(n),$$

where φ is the restriction of a continuous homomorphism $\phi : \mathbb{R}_x \to G$ (i.e. $\varphi(n) = \phi(n) \quad (n \in \mathbb{N})$) *and $\alpha(\mathbb{N}) \subseteq H$, closure $\alpha(\mathbb{N}) = H$.*

A special case of Conjecture 6 can be formulated as

Conjecture 7. *Let $f \in \mathcal{M}^*$, $|f(n)| = 1 (n \in \mathbb{N})$, and*

$$\mathcal{B}_k = \{\alpha_1, \dots, \alpha_k\} = \text{limit points of } \{f(n+1)\overline{f}(n) \quad (n \in \mathbb{N})\}.$$

Then $\mathcal{B}_k = S_k = \{w | w^k = 1\}$, $f(n) = n^{i\tau} F(n)$, where $F(\mathbb{N}) = S_k$, and for every $w \in S_k$ there is an infinite sequence n_ν such that

$$F(n_\nu + 1)\overline{F}(n_\nu) = w.$$

I. Kátai and M.V. Subbarao proved: Conjecture 7 *is true if* $k = 1, 2, 3$, *and partially proved for* $k = 4$.

Wirsing proved: Under the conditions of Conjecture 7, there is an integer l, such that $f(n) = n^{i\tau} F(n)$, where $F(\mathbb{N}) = S_l$. (Annales Univ. Budapest, Sectio Computatorica, 2004.)

6. Diophantinely smooth sequences

The definition and some interesting theorems are proved by Barban.
Let $\mathcal{B} = \{a_1 < a_2 < \dots\}$ be a subsequence in $\mathbb{N}$, $A(x) = \#\{a_\nu < x\}$, $A(x, D, l) = \#\{a_\nu < x,\ a_\nu \equiv l \ (\mathrm{mod}\ D)\}$,

$$R(x, D, l) = \left| A(x, D, l) - \frac{A(x)}{\varphi(D)} \right|.$$

Assume that

$$\sum_{D\le x^\alpha} \lim_{y\le x} \lim_{(l,D)=1} R(y,D,l) \ll \frac{A(x)}{(\log x)^B},$$

where α is a suitable positive and B an arbitrary large constant.

We say that $\mathcal{A}$ is diophantinely smooth.

Examples for diophantinely smooth sequences:

1. $\mathcal{A} = \mathcal{P}$. (Linnik, Rényi, Barban, Bombieri - A.I. Vinogradov)
2. $\mathcal{A} = \{n \mid p|n \Rightarrow p \equiv 1 \pmod 4\}$. (Levin - Timofeev)
3. $\mathcal{A}_k = \{n \mid \omega(n) = k\}$. (Wolke - Zhan Math. Z. 1993.)

One can prove that $f \in \mathcal{A}$ has a limit distribution on the set of $\mathcal{B}_{+e} = \mathcal{B} + e,\ e \ne 0$, where $\mathcal{B}$ is a diophantinely smooth sequence, if it has a limit distribution on $\mathbb{N}$.

7. Distribution of additive functions on $\mathcal{A}_k$

Let $\pi_k(x) = \#\{n \le x \mid \omega(n) = k\},\ \mathcal{A}_k = \{n \mid \omega(n) = k\}$.

I. Kátai and M.V. Subbarao (Publicationes Math. 2003) proved:

Theorem: *Let $k = k(x)$ be such a sequence for which $\left|\frac{k}{x_2} - 1\right| < \delta_x,\ \delta_x \to 0$. Assume that $f \in \mathcal{A}$, and*

$$\sum_{|f(p)|\ge 1} \frac{1}{p}, \quad \sum_{|f(p)|\le 1} \frac{f(p)}{p}, \quad \sum_{|f(p)|\le 1} \frac{f^2(p)}{p}$$

are convergent.

Then

$$\lim_{x\to\infty} \frac{1}{\pi_k(x)} \#\{n \le x,\ n \in \mathcal{A}_k, f(n) < y\} = F(y),$$

where F is a distribution function.

Theorem. *Let $k = k(x)$ be as above. Let $g \in \mathcal{M},\ |g(n)| = 1$ such that*

$$\sum_p \frac{1 - g(p)}{p}$$

is convergent. Then

$$\frac{1}{\pi_k(x)} \sum_{\substack{n<x\\ n\in\mathcal{A}_k}} g(n) = (1 + o_x(1))M(g) \quad (x \to \infty).$$

Here

$$M(g) = \prod_p e_p,$$

$$e_p = \left(1 - \frac{1}{p}\right) \left(1 + \frac{g(p)}{p} + \frac{g(p^2)}{p^2} + \dots\right).$$

8. On the iterates of arithmetical functions

I. Let $\varphi(n) = \varphi_1(n)$ be Euler's totient function, $\varphi_k(n) = \varphi(\varphi_{k-1}(n))$ $(k = 2, 3, \dots)$. Let Φ be the Gaussian distribution function.

Theorem 1. (N.L. Bassily, I. Kátai and M. Wijsmuller, Journal of Number Theory, 1997)

Let

$$a_k = \frac{1}{(k+1)!}, \quad b_k = \frac{1}{\sqrt{2k+1}} a_k,$$

$k \geq 1$ *be fixed. Then*

$$\lim_{x\to\infty} x^{-1} \# \left\{ n \leq x \,\middle|\, \frac{\omega(\varphi_k(n)) - a_k \cdot x_2^{k+1}}{b_k \cdot x_2^{k+1/2}} < z \right\} = \Phi(z),$$

and

$$\lim_{x\to\infty} \frac{1}{\pi(x)} \# \left\{ p \leq x \,\middle|\, \frac{\omega(\varphi_k(p-1)) - a_k \cdot x_2^{k+1}}{b_k \cdot x_2^{k+1/2}} < z \right\} = \Phi(z).$$

Let $\Delta(n) = \Omega(n) - \omega(n)$.

In Publicationes Math. Debrecen (Bassily, Kátai, Wijsmuller) is proved:

Theorem 2. *We have for every fixed k:*

$$\Delta(\varphi_k(n)) = a_{k-1}(1 + o(1)) x_2^k x_4$$

for all but $o(x)$ integers $n \leq x$.

Theorem 3. *We have*

$$\lim_{x\to\infty} x^{-1} \# \left\{ n \leq x \,\middle|\, \frac{\Delta(\varphi(n)) - s(x)}{\sqrt{x_2} \cdot x_4} < z \right\} = \Phi(z)$$

where $s(x) = x_2 \cdot x_4 + c_1 x_2 + o(x_2)$, c_1 is a suitable constant.

II. Let $\sigma^*(n) = \sum\limits_{\substack{d \mid n \\ (d, \frac{n}{d}) = 1}} d$ = sum of unitary divisors of n. $\sigma_k^*(n) = \sigma^*(\sigma_{k-1}^*(n))$, $k = 2, 3, \dots$, $\sigma^* = \sigma_1^*$.

Erdős and Subbarao proved:

$$\frac{\sigma_2^*(n)}{\sigma_1^*(n)} \to 1 \quad \text{almost all } n.$$

I. Kátai and M. Wijsmuller proved:

$$\frac{\sigma_3^*(n)}{\sigma_2^*(n)} \to 1 \quad \text{almost all } n.$$

(Acta Math. Hung.)

We hope that

$$\frac{\sigma_{k+1}^*(n)}{\sigma_k^*(n)} \to 1 \quad \text{almost all } n$$

is true for every $k = 3, 4, \dots$.

III. In our paper (Mathematica Pannonica 1999, I. Kátai and M.V. Subbarao) we investigated the iterates of the sum of "exponential divisors: $\sigma^{(e)}(n)$. $\sigma^{(e)}$ is multiplicative and

$$\sigma^{(e)}(p^a) = \sum_{b|a} p^b.$$

Let

$$f_0(n) = n, \quad f_1(n) = \sigma^{(e)}(n), \quad f_{j+1}(n) = f_1(f_j(n)).$$

Theorem. $k \geq 1$ fix. *We have*

$$\lim_{x\to\infty} x^{-1}\#\left\{n \leq x \,\middle|\, \frac{f_j(n)}{f_{j-1}(n)} < \alpha_j,\ j = 1, \dots, k\right\} = F_k(\alpha_1, \dots, \alpha_k),$$

where F_k is strictly monotonic in each variables in $(1, \infty)^k$.

IV. Indlekofer – Kátai: (Lietuvos Mat. Rinkinis 2004). $k \geq 1$, fix. Q prime, fix: $\kappa_0, \kappa_1, \dots$ be completely additive functions,

$$\kappa_0(p) = \begin{cases} 1 & \text{if } p = Q \\ 0 & \text{if } p \neq Q \end{cases}; \quad \kappa_{j+1}(p) = \sum_{\substack{q|p-1 \\ q \in \mathcal{P}}} \kappa_j(q),$$

$$\rho_k(Q) = \rho_k(Q|x) = \prod_{\substack{p<x \\ \kappa_{k+1}(p)\neq 0 \\ p\in\mathcal{P}}} (1 - 1/p)\,.$$

$$N_k(Q|x) = \#\{n \leq x \mid Q \nmid \varphi_{k+1}(n)\}.$$

Theorem 1. *Let* $\varepsilon > 0,\ k \geq 2$ *be fixed,* $Q \in \left[x_2^{k+\varepsilon}, x_2^{k+1-\varepsilon}\right],\ Q$ *be prime. Then*

$$N_k(Q|x) = \rho_k(Q) \times (1 + O\left(1/x_2\right)),$$

and

$$\log \frac{1}{\rho_k(Q)} = A_{k+1}(x) + O(1/Q) + O\left(\frac{x_2^{2k+1}}{Q^2}\right),$$

$$A_{k+1}(x) = \frac{x_2^{k+1}}{(k+1)!(Q-1)} + O\left(\frac{x_2^{k+\varepsilon/2}}{Q}\right).$$

Theorem 2. *Let* $x_3 \cdot x_2 \leq Q \leq x_2^2$. *Then*

$$N_1(Q|x) = x\rho_1(Q)\left(1 + O\left(\frac{x_2 x_3}{Q}\right)\right),$$

$$\log \frac{1}{\rho_1(Q)} = \frac{x_2^2}{2Q} + O\left(\frac{x_2^3}{Q^2} + \frac{x_2 \log Q}{Q}\right).$$

Theorem 3. *Let* $\varepsilon > 0,\ k \geq 2,\ r \geq 1$ *be fixed.*

Let $Q_1, \dots, Q_r \in \left[x_2^{k+1/2+\varepsilon},\ x_2^{k+1-\varepsilon}\right], Q_1, \dots, Q_r$ *be distinct primes. Then*

$$\frac{N_k(Q_1, \dots, Q_r|x)}{x} = \rho_k(Q_1) \dots \rho_k(Q_r)$$
$$\left\{1 + O\left(x_2^{2k+1} \sum 1/Q_j^2\right)\right\}.$$

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