Algebra and Discrete Mathematics
Volume 16 **(2013)**. Number 1. pp. 33 – 41

RESEARCH ARTICLE

Power graph of finite abelian groups[1]

T. Tamizh Chelvam and M. Sattanathan

Communicated by L. A. Shemetkov

Abstract. Let G be a group. The power graph $\Gamma_P(G)$ of G is a graph with vertex set $V(\Gamma_P(G)) = G$ and two distinct vertices x and y are adjacent in $\Gamma_P(G)$ if and only if either $x^i = y$ or $y^j = x$, where $2 \leq i, j \leq n$. In this paper, we obtain some fundamental characterizations of the power graph. Also, we characterize certain classes of power graphs of finite abelian groups.

Introduction

The study of algebraic structures using the properties of graphs becomes an exciting research topic in the last twenty four years, leading to many fascinating results and questions. There are many papers on assigning a graph to a group or a ring, for instance, see [1, 2, 5, 8]. Also investigation of algebraic properties of groups or rings using the associated graph becomes an exciting topic. In 2002, The directed power graph of a semi group S was defined by Kelarev and Quinn [7] as the digraph $\vec{\mathcal{G}}(S)$ with vertex set S, in which there is an arc from x to y if and only if $x \neq y$ and $y = x^m$ for some positive integer m. Motivated by this, Chakrabarty et al.[5] defined the undirected power graph $\Gamma_P(G)$ of a group G. Actually

[1]The work reported here is supported by the Special Assistance Programme (No. F510/ 9/ DRS/ 2007 (SAP-I)) of University Grants Commission, India awarded to the Department of Mathematics, Manonmaniam Sundaranar University, India for the period 2007-2012.

2010 MSC: 05C25.

Key words and phrases: power graph, planar graph, Eulerian graph, finite group.

the power graph $\Gamma_P(G)$ of G is the graph with vertex set $V(\Gamma_P(G)) = G$ and two distinct vertices $x, y \in G$ are adjacent in $\Gamma_P(G)$ if and only if either $x^i = y$ or $y^j = x$, where i and j are integers and $2 \leq i, j \leq n$. In this paper, we use the following notations and definitions.

By a graph $\Gamma = (V, E)$, we mean an undirected graph Γ with vertex set V, edge set E and has no loops or multiple edges. The degree $deg_\Gamma(v)$ of a vertex v in Γ is the number of edges incident to v and if the graph is understood, then we denote $deg_\Gamma(v)$ simply by $deg(v)$. The order of Γ is defined as $|V(\Gamma)|$ and its maximum and its minimum degrees will be denoted, respectively, by $\Delta(\Gamma)$ and $\delta(\Gamma)$. A graph Γ is regular if the degrees of all vertices of Γ are the same. A subset X of the vertices of Γ is called an *independent set* if the induced subgraph $< X >$ on X has no edges. The maximum size of an independent set in a graph Γ is called the *independence number* of Γ and denoted by $\beta_0(\Gamma)$. The length of a smallest cycle in a graph Γ is referred to as its *girth*, which is denoted by $gr(\Gamma)$. If a graph Γ contains no cycles, then $gr(\Gamma)$ is taken as ∞. A *planar* graph is a graph that can be embedded in the plane so that no two edges intersect geometrically except at a vertex which both are incident. An *Eulerian* graph has an *Eulerian trail*, a closed trail containing all vertices and edges. *Unicyclic* graphs are graphs which are connected and have just one cycle. The *union* $\Gamma_1 \cup \Gamma_2$ of two graphs $\Gamma_1 = (V_1, E_1)$ and $\Gamma_2 = (V_2, E_2)$ is the graph with vertex set $V = V_1 \cup V_2$ and edge set $E = E_1 \cup E_2$. The *join* $\Gamma_1 + \Gamma_2$ of $\Gamma_1 = (V_1, E_1)$ and $\Gamma_2 = (V_2, E_2)$ is the graph with $V = V_1 \cup V_2$ and $E = E_1 \cup E_2$ together with edges joining all vertices in V_1 with vertices in V_2.

Let G be a group with identity e. The number of elements of a group is called its *order* and it is denoted by $o(G)$. The order of an element g in a group is the smallest positive integer n such that $g^n = e$. If no such integer exists, we say g has infinite order. The order of an element g is denoted $o(g)$.
We state the following theorems for use in the subsequent discussions.

Theorem 1 ([3]). *K_5 and $K_{3,3}$ are non-planar.*

Theorem 2 ([5]). *Let G be a finite group. Then $\Gamma_P(G)$ is complete if and only if G is a cyclic group of order 1 or p^m, for some prime number p and for some $m \in \mathbb{N}$.*

1. Properties of power graph

In this section, we derive some properties of power graphs, which will be used later for further study.

Proposition 1. *Let G be a group with at least one non-self inverse element. Then $gr(\Gamma_P(G)) = 3$.*

Proof. Let G be a group with identity e. Let x be a non-self inverse element of G. Note that $<x>=<x^{-1}>$, $e \in <x>$ and thus the graph induced by the set $\{e, x, x^{-1}\}$ is K_3 in $\Gamma_P(G)$. Hence $gr(\Gamma_P(G)) = 3$. □

Remark 1. Let $x \in G$. Clearly x is adjacent to $x^2, x^3, \ldots, x^{o(x)}$ in $\Gamma_P(G)$ and so the number of edges in $\Gamma_P(G)$ is given by

$$|E(\Gamma_P(G))| \geq \frac{\sum\limits_{x \in G, x \neq e} o(x)}{2}.$$

Now we characterize groups G for which power graph $\Gamma_P(G)$ contains exactly $\frac{\sum\limits_{x \in G, x \neq e} o(x)}{2}$ edges.

Theorem 3. *Let G be a finite group with n elements. Then $\Gamma_P(G)$ is a graph with $\frac{\sum\limits_{x \neq e} o(x)}{2}$ edges if and only if every element other than identity of G is of prime order.*

Proof. Assume that $\Gamma_P(G)$ is a graph with $\frac{\sum\limits_{x \neq e} o(x)}{2}$ edges. It means that $deg(x) = o(x) - 1$ for all $e \neq x \in G$ in $\Gamma_P(G)$. Let $x \neq e$ be any element of G. Suppose order of x is not a prime. Without loss of generality, we assume that $o(x) = pq$, where p, q are distinct primes. Consider the subgroup $H =<x>$. Since $p|o(H)$, H has an element say y such that $o(y) = p$. From this $deg(y) = o(y) - 1 = p - 1$. Since $x \notin <y>$ and $y \in <x>$, y is adjacent to at least $x, y^2, \ldots, y^p = e$. This implies that $deg(y) > p - 1 = o(y) - 1$, which is a contradiction. Hence every element other than identity in the group G is of prime order.
Conversely, assume that every element other than identity of G is of prime order. To conclude that $\Gamma_P(G)$ contains $\frac{\sum\limits_{x \neq e} o(x)}{2}$ edges, it is enough to prove that $deg(x) = o(x) - 1$ for all $e \neq x \in G$ in $\Gamma_P(G)$. If $deg(x) > o(x) - 1$ for some $x \in G - e$, then there exists $y \notin <x>$ and y is adjacent to x. This implies that $x \in <y>$ and so $<x> \subseteq <y>$. Since $o(x)$ and $o(y)$

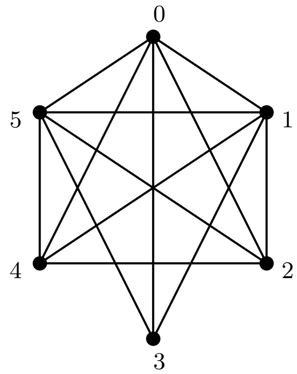

FIGURE 1. $\Gamma_P(\mathbb{Z}_6)$

are prime, we get that $< x >=< y >$, a contradiction to $y \notin < x >$. Hence $deg(x) = o(x) - 1$ for all $e \neq x \in G$ in $\Gamma_P(G)$. □

Proposition 2. *Let G be a finite group with n elements and $Z(G)$ be its center. If $deg(x) = n - 1$ in $\Gamma_P(G)$, then $x \in Z(G)$.*

Proof. Let $x \in G$ be a vertex with $deg(x) = n-1$ in $\Gamma_P(G)$ and $H =< x >$. Since $deg(x) = n - 1$, $x \in < y >$ for all $y \in G - H$. Hence x commutes with all elements in G and so $x \in Z(G)$. □

Remark 2. The converse of Proposition 2 is not true. For example, consider the group $(\mathbb{Z}_6, +_6)$ and the graph $\Gamma_P(\mathbb{Z}_6)$ given below. Here 3 $\in Z(\mathbb{Z}_6)$, whereas $deg(3) = 3 \neq 5 = 6 - 1$.

Theorem 4. *Let G be a finite group with n elements. Then the following are equivalent:*

(i) $\Gamma_P(G) \cong K_{1,n-1}$
(ii) $\Gamma_P(G)$ *is a tree*
(iii) *Every element of G is its own inverse.*

Proof. $(i) \Rightarrow (ii)$ It is trivially true.
$(ii) \Rightarrow (iii)$ Assume that $\Gamma_P(G)$ is a tree. Suppose that there exists an element $a \in G$ such that $a \neq a^{-1}$. Then the graph induced by $\{e, a, a^{-1}\}$ is a triangle in $\Gamma_P(G)$, which is a contradiction.
$(iii) \Rightarrow (i)$ Since every element of G is its own inverse, $< x >= \{e, x\}$ for all $x \in G - e$. From this $\Gamma_P(G) \cong K_{1,n-1}$. □

Theorem 5. *Let G be a finite group of order pq, where $p < q$, p and q are two distinct primes, and ϕ is the Euler function. Then*

(i) *G is cyclic if and only if* $\Gamma_P(G) \cong (K_{p-1} \cup K_{q-1}) + K_{\phi(pq)+1}$

(ii) *G is non-cyclic if and only if* $\Gamma_P(G) \cong K_1 + (qK_{p-1} \cup K_{q-1})$.

Proof. **(i)**: Let G be a cyclic group of order pq. Then G has a unique p-Sylow subgroup namely H_1 and a unique q-Sylow subgroup namely H_2. By Theorem 2, $\Gamma_P(H_1) \cong K_p$ and $\Gamma_P(H_2) \cong K_q$. Note that all elements in $G - (H_1 \cup H_2)$ are generators of G and so $|G - (H_p \cup H_q)| = \phi(pq)$. Since the generators and the identity element e of G have full degree in $\Gamma_P(G)$ and every non identity element in H_1 is not adjacent to every non identity element in H_2, $\Gamma_P(G) \cong (K_{p-1} \cup K_{q-1}) + K_{\phi(pq)+1}$.
Conversely, assume that $\Gamma_P(G) \cong (K_{p-1} \cup K_{q-1}) + K_{\phi(pq)+1}$. If G is non-cyclic, then every non identity element of G has order either p or q. From this, identity is the only vertex of full degree in $\Gamma_P(G)$, which is a contradiction. Hence G is cyclic.
(ii): Let G be a non-cyclic group. Then the number of p-Sylow subgroups of G is q and G has a unique q-Sylow subgroup. Also the identity element of G has full degree in $\Gamma_P(G)$. Hence $\Gamma_P(G) \cong K_1 + (qK_{p-1} \cup K_{q-1})$.
Conversely, assume that $\Gamma_P(G) \cong K_1 + (qK_{p-1} \cup K_{q-1})$. If G is cyclic, then G has $\phi(pq)$ generators and so $\Gamma_P(G)$ has $\phi(pq) + 1$ full degree vertices, which is a contradiction. Hence G is non cyclic. □

Theorem 6. *Let G be a finite group. Then $\Gamma_P(G)$ is Eulerian if and only if $o(G)$ is odd.*

Proof. Assume that $o(G)$ is odd. Clearly $deg(e)$ is even in $\Gamma_P(G)$. For any $e \neq x \in G$, clearly $o(x)$ is odd and so $o(x) - 1$ is even. If $deg(x) = o(x) - 1$ in $\Gamma_P(G)$, then $deg(x)$ is even . If $deg(x) > o(x) - 1$, then there exists an element $y \in G$ such that $y \notin < x >$ and $x \in < y >$. Since $< y > = < y^{-1} >$, $x \in < y^{-1} >$. From this x is adjacent to $\{e, x^2, \ldots, x^{o(x)-1}, y_1, y_1^{-1}, \ldots, y_k, y_k^{-1}\}$ for some $k \geq 1$. Since $o(G)$ is odd, G has no self inverse element and so $deg(x)$ is even in $\Gamma_P(G)$. Hence $\Gamma_P(G)$ is Eulerian.

Conversely, assume that $\Gamma_P(G)$ is Eulerian. Since $deg(e) = o(G) - 1$ is even, $o(G)$ is odd. □

Theorem 7. *Let G be a group and $o(G) = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_n^{\alpha_n}$, where $p_1, p_2, \ldots p_n$ are distinct primes. Then the independence number $\beta_0(\Gamma_P(G) \geq n$.*

Proof. By Cauchy's Theorem, G has an element a_i of order p_i for all $i = 1, 2, \ldots n$. Note that $< a_i > \cap < a_j >= \{e\}$ for all $i \neq j$. From this $\{a_1, a_2, \ldots, a_n\}$ is an independent set in $\Gamma_P(G)$ and hence the independence number $\beta_0(\Gamma_P(G) \geq n$. □

2. Power graph of finite abelian groups

In this section, we study about power graph of finite abelian groups.

Theorem 8. *Let G be an elementary abelian group of order p^n for some prime number p and positive integer n. Then $\Gamma_P(G) \cong K_1 + \cup_{i=1}^{\ell} K_{p-1}$, where $\ell = \frac{p^n-1}{p-1}$ and $\beta_0(\Gamma_P(G)) = \ell$.*

Proof. Note that there are $p^n - 1$ elements in G each with order p. Since a group of order p has exactly $p - 1$ elements of order p, G has exactly $\frac{p^n-1}{p-1}$ distinct subgroups of order p. Clearly if any two elements a and b are adjacent in $\Gamma_P(G)$, then they are in the same subgroup of G. Since G has ℓ distinct subgroups of order p, by Theorem 2, $\cup_{i=1}^{\ell} K_{p-1}$, where $\ell = \frac{p^n-1}{p-1}$, is an induced subgraph of $\Gamma_P(G)$. Since the identity element of G is adjacent to all other elements of G, $\Gamma_P(G) \cong K_1 + \cup_{i=1}^{\ell} K_{p-1}$, where $\ell = \frac{p^n-1}{p-1}$. From this $\beta_0(\Gamma_P(G)) = \ell$. □

Theorem 9. *Let G be a finite abelian group. Then $\Gamma_P(G)$ is planar if and only if either G is isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \ldots \times \mathbb{Z}_3$ or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2$ or $\mathbb{Z}_4 \times \mathbb{Z}_4 \times \ldots \times \mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_4 \times \ldots \times \mathbb{Z}_4$.*

Proof. If part:
Case (i): Let $G \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \times \ldots \times \mathbb{Z}_3$. By Theorem 8, $\Gamma_P(G) \cong \cup_\ell K_2 + K_1$, where $\ell = 1 + 3 + 3^2 + \ldots + 3^{n-1}$. Hence G is planar.
Case (ii): Suppose $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2$. Then $o(G) = 2^n$ for some $n \in \mathbb{Z}^+$. By Theorem 8, $\Gamma_P(G) \cong K_{1,2^n-1}$ and so $\Gamma_P(G)$ is planar.
Case(iii): Suppose $G \cong \mathbb{Z}_4 \times \mathbb{Z}_4 \times \ldots \times \mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_4 \times \ldots \times \mathbb{Z}_4$. Then G can be partitioned into three sets, namely A,B and C, where $A = \{e\}$, $B = \{x \in G/o(x) = 4\}$ and $C = \{x \in G/o(x) = 2\}$. Note that no two elements of order 2 are adjacent to each other so that the set C is an independent set in $\Gamma_P(G)$. Let $a, b \in B$ such that $a \neq b$. Then a and b are adjacent if and only if $a \in< b >$ or $b \in< a >$. Since $a \neq b$ and $o(a) = o(b) = 4$, $a = b^{-1}$ or $b = a^{-1}$ i.e., a and b are adjacent if and only if they are inverse of each other. Thus the subgraph induced by the set B in $\Gamma_P(G)$ is union of K_2.
Claim: No two elements of C are adjacent to the same element of B.

Suppose not, let $a \in B$, $b, c \in C$ and $b \neq c$ such that b, c are adjacent to a. Since $o(b) = o(c) = 2$ and $o(a) = 4$, $a \notin < b >$ and $a \notin < c >$. Therefore $b, c \in < a >$. Since $< a >$ has a unique element of order 2, $b = c$, which is a contradiction. Now, we arrange the elements of B such that first collection of elements which are adjacent to the first element of C, second collection of elements which are adjacent to the second element of C and so on. That the graph induced by the set $B \cup C$ is planar in $\Gamma_P(G)$. Since $\Gamma_P(G) \cong < A > + < B \cup C >$, $\Gamma_P(G)$ is planar.

Conversely assume that $\Gamma_P(G)$ is planar. Suppose $p | o(G)$, for some prime $p \geq 5$. Then G has an element x of order p. By Theorem 2, the subgraph induced by the subgroup $< x >$ is K_p. Since $p \geq 5$, by Theorem 1, $\Gamma_P(G)$ is non-planar, a contradiction. Therefore $o(G) = 2^{n_1} 3^{n_2}$, where $n_1 \geq 0$, $n_2 \geq 0$ are two integers.

Suppose $o(G) = 2^{n_1} 3^{n_2}$, for two integers $n_1 \geq 1$ and $n_2 \geq 1$. Since $2 | o(G)$ and $3 | o(G)$, G has two elements a and b such that $o(a) = 2$ and $o(b) = 3$. Since G is abelian, we have $o(ab) = \frac{o(a)o(b)}{gcd(o(a),o(b))}$, for all $a, b \in G$. Therefore $o(ab) = 6$. Now the subgraph induced by $< ab >$ must have $K_{3,3}$ as a subgraph. By Theorem 1, $\Gamma_P(G)$ is non-planar, a contradiction. Therefore $o(G)$ is either 2^n or 3^m, for some $n, m \in \mathbb{Z}^+$.

Suppose $o(G) = 3^m$ and $G \ncong \mathbb{Z}_3 \times \mathbb{Z}_3 \times \ldots \times \mathbb{Z}_3$. Since $9 | o(G)$ and G is abelian, $\mathbb{Z}_9$ must be a subgroup of G. Hence by Theorem 2, $\Gamma_P(G)$ must have K_9 as a subgraph, again $\Gamma_P(G)$ is non-planar, a contradiction. Hence if $o(G) = 3^m$, for some $m \in \mathbb{Z}^+$ then $G \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \times \ldots \times \mathbb{Z}_3$.

Suppose $o(G) = 2^n$, for some $n \in \mathbb{Z}^+$ and G is not isomorphic to one of the groups $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2$, $\mathbb{Z}_4 \times \mathbb{Z}_4 \times \ldots \times \mathbb{Z}_4$ and $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_4 \times \ldots \times \mathbb{Z}_4$. In such a case $8 | o(G)$, G must have $\mathbb{Z}_8$ as a subgroup and so by Theorem 2, K_8 must be a subgraph of $\Gamma_P(G)$, a contradiction. Hence if $o(G) = 2^n$ for some $n \in \mathbb{Z}^+$, then G is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2$ or $\mathbb{Z}_4 \times \mathbb{Z}_4 \times \ldots \times \mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_4 \times \ldots \times \mathbb{Z}_4$. □

Theorem 10. *Let G be a finite abelian group. Then $\beta_0(\Gamma_P(G)) = 2$ if and only if G is a cyclic group of order $p^n q$, where p and q are distinct primes and n is a positive integer.*

Proof. Assume that G is a cyclic group of order $p^n q$, where p and q are distinct primes and n is a positive integer. By Cauchy's Theorem, there exist two non-identity elements a and b in G such that $o(a) = p$ and $o(b) = q$. Since $< a > \cap < b > = \{e\}$, a and b are non-adjacent in $\Gamma_P(G)$. Hence $\beta_0(\Gamma_P(G)) \geq 2$. Since G is cyclic, G has a unique subgroup H of order p^n. By Theorem 2, $\Gamma_P(H) \cong K_{p^n}$. Let $x, y \in G - H$. Since the subgroup H is unique, $o(x) = p^i q$ and $o(y) = p^j q$ for some $0 \leq i, j \leq n$. From this, we have either $o(x)|o(y)$ or $o(y)|o(x)$ and so either $x \in < y >$ or $y \in < x >$. Therefore $\Gamma_P(G - H) \cong K_{p^n(q-1)}$. Since identity element is adjacent to all other elements of G, $K_1 + (K_{p^n - 1} \cup K_{p^n(q-1)})$ is a subgraph of $\Gamma_P(G)$ and so $\beta_0(\Gamma_P(G)) \leq 2$. Hence $\beta_0(\Gamma_P(G)) = 2$.

Conversely, assume that $\beta_0(\Gamma_P(G)) = 2$. First we claim that G is cyclic. Suppose G is not cyclic. Since G is a non-cyclic abelian group, there exists a prime number p such that $\mathbb{Z}_p \times \mathbb{Z}_p$ is a subgroup of G and so G has at least $\frac{p^2-1}{p-1}$ distinct subgroups of order p and $\frac{p^2-1}{p-1} > 2$. Hence G has at least three distinct elements a, b and c from three distinct subgroups of order p. This implies that $\{a, b, c\}$ is an independent set of $\Gamma_P(G)$, which is a contradiction. Hence G is cyclic. Since G is cyclic and $\beta_0(\Gamma_P(G)) = 2$, there are exactly two distinct prime divisors p and q of $o(G)$ and $o(G) = p^{n_1} q^{n_2}$ for some positive integers n_1, n_2 and $n_1 \geq n_2$. Suppose $n_2 > 1$. Since G is cyclic, G has three elements a, b and c such that $o(a) = p^{n_1}$, $o(b) = pq^{n_2 - 1}$ and $o(c) = q^{n_2}$. From this order of either of a or b or c divide orders of others and so $\{a, b, c\}$ is an independent set of $\Gamma_P(G)$, which is a contradiction. Hence G is a cyclic group of order $p^n q$, where p and q are distinct primes and n is a positive integer. □

Theorem 11. *Let G be a finite abelian group. Then $\Gamma_P(G)$ is a unicyclic graph if and only if $G \cong \mathbb{Z}_3$.*

Proof. Assume that $G \cong \mathbb{Z}_3$. Clearly $\Gamma_P(G) \cong K_3$ and hence $\Gamma_P(G)$ is a unicyclic graph.

Conversely, assume that $\Gamma_P(G)$ is a unicyclic graph. Suppose there exists a prime $p > 3$ such that $p|o(G)$. Then K_p is a subgraph of $\Gamma_P(G)$, a contradiction to $\Gamma_P(G)$ is unicyclic. Hence $o(G) = 2^n 3^m$ for some integers $n, m \geq 0$.

Suppose $o(G) = 2^n 3^m$ for some integers $n, m \geq 1$. Since $2|o(G)$ and $3|o(G)$, G has two elements a and b such that $o(a) = 2$ and $o(b) = 3$ and hence $o(ab) = 6$. Now the subgraph induced by $< ab >$ must have K_4 as a subgraph, which is a contradiction. Therefore $o(G)$ is either a power of 2 or a power of 3.

If G is an elementary abelian group of order 2^n, then $\Gamma_P(G) \cong K_{1,2^n - 1}$, which is a contradiction. On the other hand, there exists a $m > 1$ such that $\mathbb{Z}_{2^m}$ is a subgroup of G. By Theorem 2, K_{2^m} is a subgraph of $\Gamma_P(G)$, which is a contradiction. If $o(G)$ is a power of 3 and G is an elementary abelian group of order 3^n for some $n > 1$, then by Theorem 8, $\Gamma_P(G) \cong K_1 + \bigcup_{\ell} K_2$ where $\ell = \frac{3^n - 1}{2}$. Thus $\Gamma_P(G)$ is not unicyclic, which is a contradiction. On the other hand, for some $n > 1$, $\mathbb{Z}_{3^n}$ is a subgroup of G. By Theorem 8, K_{3^n} is a subgraph of $\Gamma_P(G)$, which is a contradiction. Hence $o(G) = 3$ and so $G \cong \mathbb{Z}_3$. □

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Contact information

T. Tamizh Chelvam, Department of Mathematics,
M. Sattanathan Manonmaniam Sundaranar University,
Tirunelveli 627 012, Tamil Nadu, India
E-Mail: tamche59@gmail.com,
satta_nathan@gmail.com

Received by the editors: 27.10.2011
and in final form 15.07.2012.