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On 0-semisimplicity of linear hulls of generators for semigroups generated by idempotents

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Abstract. Let I be a finite set (without 0) and J a subset of $I \times I$ without diagonal elements. Let $S(I, J)$ denotes the semigroup generated by $e_0 = 0$ and e_i, $i \in I$, with the following relations: $e_i^2 = e_i$ for any $i \in I$, $e_i e_j = 0$ for any $(i, j) \in J$. In this paper we prove that, for any finite semigroup $S = S(I, J)$ and any its matrix representation M over a field k, each matrix of the form $\sum_{i \in I} \alpha_i M(e_i)$ with $\alpha_i \in k$ is similar to the direct sum of some invertible and zero matrices. We also formulate this fact in terms of elements of the semigroup algebra.

Introduction

We study matrix representations over a field k of semigroups generated by idempotents.

Let I be a finite set without 0 and J a subset of $I \times I$ without the diagonal elements $(i, i), i \in I$. Let $S(I, J)$ denotes the semigroup with zero generated by e_i, $i \in I \cup 0$, with the following defining relations:

1) $e_0^2 = e_0$, $e_0 e_i = e_i e_0 = e_0$ for any $i \in I \cup 0$, i. e. $e_0 = 0$ is the zero element;

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2) $e_i^2 = e_i$ for any $i \in I$;

3) $e_ie_j = 0$ for any pair $(i, j) \in J$.

Every semigroup $S(I, J) \in \mathcal{I}$ is called *a semigroup generated by idempotents with partial null multiplication* (see, e.g., [2]). The set of all semigroups $S(I, J)$ with $|I|$=n will be denoted by $\mathcal{I}_n$. Put $\mathcal{I} = \cup_{n=1}^{\infty} \mathcal{I}_n$.

With each semigroup $S = S(I, J) \in \mathcal{I}$ we associate the directed graph $\Lambda(S)$ with set of vertices $\Lambda_0(S) = \{e_i \,|\, i \in I\}$ and set of arrows $\Lambda_1(S) = \{e_i \to e_j \,|\, (i, j) \in J\}$. Denote by $\overline{\Lambda}(S)$ the directed graph which is the complement of the graph $\Lambda(S)$ to the full directed graph without loops, i.e. $\overline{\Lambda}_0(S) = \Lambda_0(S)$ and $e_i \to e_j$ belongs to $\overline{\Lambda}_1(S)$ if and only if $i \neq j$ and $e_i \to e_j$ does not belong to $\Lambda_1(S)$. Obviously, the semigroup $S \in \mathcal{I}$ is uniquely determined by each of these directed graphs.

In [1] the authors proved that a semigroup $S = S(I, J)$ is finite if and only if the graph $\overline{\Lambda}(S)$ is acyclic.

We call a quadratic matrix A over a field k *α-semisimple*, where $\alpha \in k$, if one of the following equivalent conditions holds:

a) $\operatorname{rank}(A - \alpha E)^2 = \operatorname{rank}(A - \alpha E)$ (E denotes the identity matrix);

b) $A - \alpha E$ is similar to the direct sum of some invertible and zero matrices;

c) the minimal polynomial $m_A(x)$ of A is not devided by $(x - \alpha)^2$;

d) there is a polynomial $f(x) = (x - \alpha)g(x)$ such that $g(\alpha) \neq 0$ and $f(A) = 0$.

If A is α-semisimple for all $\alpha \in k$, then it is obviously semisimple in the classical sense.

In this paper we study 0-semisimple matrices associated with matrix representations of a finite semigroup S from $\mathcal{I}$ (formulating also the received results in terms of elements of the semigroup algebra).

1. Formulation of the main results

Let S be a semigroup and k be a field. Let $M_m(k)$ denotes the algebra of all $m \times m$ matrices with entries in k.

A matrix representation of S (*of degree* m) *over* k is a homomorphism R from S to the multiplicative semigroup of $M_m(k)$. If there is a zero (resp. an identity) element $a \in S$, one can assume that the matrix $R(a)$ is zero (resp. identity)[1]. Two representation $R : S \to M_m(k)$ and $R' : S \to M_m(k)$ are called *equivalent* if there is an invertible matrix C such that $C^{-1}R(x)C = R'(x)$ for all $x \in S$.

In this paper we prove the following theorem[2].

Theorem 1. *Let $S = S(I, J)$ be a finite semigroup from $\mathcal{I}$ and R a matrix representation of S. Then, for any $\alpha_i \in k$, where i runs over I, the matrix $\sum_{i \in I} \alpha_i R(e_i)$ is 0-semisimple.*

Reformulate the theorem in terms of elements of the semigroup algebra kS^1, where $S^1 = S \cup \{1\}$. As usual, we identify the zero element of the semigroup with the zero element of the semigroup algebra; then

$$kS^1 = \{ \sum_{s \in S \setminus 0} \beta_s s + \beta_1 1 \,|\, \beta_s, \beta_1 \in k \}.$$

We call an element $g \in kS^1$ 0-*semisimple* if the minimal polynomial $m_g(x)$ of g is not devided by x^2.

Set $\mathcal{E}_I = \{e_i \,|\, i \in I\}$ and let $k\mathcal{E}_I$ denotes the k-linear hull of the generators $e_i \in \mathcal{E}_I$, i. e. $k\mathcal{E}_I = \{\sum_{i \in I} \alpha_i e_i \,|\, \alpha_i \in k\}$.

Theorem 1 is equivalent to the following one.

Theorem 2. *Let $S = S(I, J)$ be a finite semigroup from $\mathcal{I}$. Then any element $g \in k\mathcal{E}_I$ is 0-semisimple.*

Note that Theorem 1 follows from the results of [2, 3] on a normal form of matrix representations of finite semigroups $S(I, J)$, but here we prove this fact directly.

2. Proof of Theorem 1

We apply induction on $n = |I|$. The case $n = 1$ is obvious since any matrix representation of the semigroup $S(\{1\}, \varnothing)$ is given by an idempotent matrix.

Suppose that Theorem 1 is proved for all matrix representations of all finite semigroups $S(I, J) \in \mathcal{I}_n$, and prove that the theorem holds for $S(I, J) \in \mathcal{I}_{n+1}$.

[1] It is easy to show that in this case we "lose" the only indecomposable representation P of degree 1 with $P(x) = 0$ for all $x \in S \setminus a$ and $P(a) = 1$ (resp. $P(x) = 0$ for all $x \in S$).

[2] Notice that the theorem is also valid without the restrictions which has been discussed in note 1.

Let $S = S(I,J)$ be an arbitrary finite semigroup from $\mathcal{I}_{n+1}$. One may assume without loss of generality that $I = \{1, 2, \ldots, n+1\}$. We show that for a fixed matrix representation R of $S(I,J)$ and a vector $\alpha = (\alpha_1, \ldots, \alpha_{n+1}) \in k^{n+1}$, the matrix $P(\alpha) = P(\alpha_1, \ldots, \alpha_{n+1}) = \sum_{i=1}^{n+1} \alpha_i R(e_i)$ is 0-semisimple.

Put $A_1 = R(e_1), \ldots, A_{n+1} = R(e_{n+1})$. Then $A_i^2 = A_i$ for all $i \in I$, $A_i A_j = 0$ for all $(i,j) \in J$ and $P(\alpha) = \alpha_1 A_1 + \ldots \alpha_{n+1} A_{n+1}$.

Since the directed graph $\overline{\Lambda}(S)$ is acyclic (see Introduction), one can fix a vertex e_l such that there are no arrows $l \to s$, where $s \in I$. Consider the subsemigroup S' of S generated by $e'_0 = e_0$, $e'_1 = e_1, \ldots, e'_{l-1} = e_{l-1}$, $e'_l = e_{l+1}, \ldots, e'_n = e_{n+1}$. Obviously, the directed graph $\overline{\Lambda}(S')$ coincides with $\overline{\Lambda}(S) \setminus e_l$. By the induction hypothesis for the restriction T of the representation R on S', the matrix

$$\alpha_1 A_1 + \ldots + \alpha_{l-1} A_{l-1} + \alpha_{l+1} A_{l+1} + \ldots + \alpha_{n+1} A_{n+1}$$

is 0-semisimple. Denote this matrix by $P'(\alpha_1, \ldots, \alpha_{l-1}, \alpha_{l+1}, \ldots, \alpha_{n+1}) = P'(\alpha')$, where $\alpha' = (\alpha_1, \ldots, \alpha_{l-1}, \alpha_{l+1}, \ldots, \alpha_{n+1})$. Then

$$P(\alpha) = P'(\alpha') + \alpha_l A_l.$$

From the fact that there are no arrows $l \to s$ it follows that, for $j \neq l$, $e_l e_j = 0$ and consequently $A_l A_j = 0$. Then $A_l P'(\alpha') = 0$ and it remains only to apply the following statement: if A is an idempotent matrix, B is a 0-semisimple matrix and $AB = 0$ then $\gamma A + \delta B$ is 0-semisimple for any $\gamma, \delta \in k$.

Instead we prove a more general statement.

Proposition 1. *Let A and B be 0-semisimple matrices of size $m \times m$ such that $AB = 0$. Then, for any $\gamma, \delta \in k$, the matrix $\gamma A + \delta B$ is 0-semisimple.*

Because λM with $\lambda \in k$ is 0-semisimple provided that so is M, it is sufficient to consider the case $\gamma = \delta = 1$.

By condition b) of the definition of a 0-semisimple matrix there is an invertible matrix X such that

$$X^{-1} A X = \left(\begin{array}{c|c} A_0 & 0 \\ \hline 0 & 0 \end{array}\right) \tag{1}$$

where A_0 is invertible. From $AB = 0$ it follows that

$$X^{-1} B X = \left(\begin{array}{c|c} 0 & 0 \\ \hline P & Q \end{array}\right) \tag{2}$$

for some matrices P and Q (the matrices in the right parts of (1) and (2) are partitioned conformally).

From condition a) for the matrix B (see the definition of a 0-semisimple matrix) we have that

$$rank\ Q\left(\begin{array}{c|c} P & Q \end{array}\right) = rank\left(\begin{array}{c|c} P & Q \end{array}\right) \tag{3}$$

But since $rank\ Q\left(\begin{array}{c|c} P & Q \end{array}\right) \leq rank\ Q$ (by the formula $rank\ MN \leq rank\ M$) and $rank\left(\begin{array}{c|c} P & Q \end{array}\right) \geq rank\ Q$, it follows from (3) that

$$rank\left(\begin{array}{c|c} P & Q \end{array}\right) = rank\ Q \tag{4}$$

and consequently there exists an invertible matrix Y such that $P = QY$. Then

$$\left(\begin{array}{c|c} E_1 & 0 \\ \hline -Y & E_2 \end{array}\right)^{-1} \left(\begin{array}{c|c} 0 & 0 \\ \hline P & Q \end{array}\right) \left(\begin{array}{c|c} E_1 & 0 \\ \hline -Y & E_2 \end{array}\right) = \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & Q \end{array}\right) \tag{5}$$

where E_1, E_2 are the identical matrices.

From (2) and (5) it follows that the 0-semisimple matrix B is similar to the matrix

$$\left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & Q \end{array}\right)$$

and hence the matrix Q is 0-semisimple. Then (by condition b) of the definition of a 0-semisimple matrix) there is an invertible matrix Z such that

$$Z^{-1}QZ = \left(\begin{array}{c|c} Q_0 & 0 \\ \hline 0 & 0 \end{array}\right)$$

where Q_0 is invertible, and consequently

$$\left(\begin{array}{c|c} E_3 & 0 \\ \hline 0 & Z \end{array}\right)^{-1} \left(\begin{array}{c|c} 0 & 0 \\ \hline P & Q \end{array}\right) \left(\begin{array}{c|c} E_3 & 0 \\ \hline 0 & Z \end{array}\right) = \left(\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline P_0 & Q_0 & 0 \\ \hline P_1 & 0 & 0 \end{array}\right) \tag{6}$$

where E_3 is the identical matrix and $\left(\begin{array}{c} P_0 \\ \hline P_1 \end{array}\right) = Z^{-1}P$; moreover by the equality (4) we have that

$$P_1 = 0. \tag{7}$$

So, if one denotes the product of the matrices X and $\left(\begin{array}{c|c} E_3 & 0 \\ \hline 0 & Z \end{array}\right)$ by T, then (see (1), (2), (6), (7))

$$T^{-1}AT = \left(\begin{array}{c|c|c} A_0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right), \quad T^{-1}BT = \left(\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline P_0 & Q_0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right),$$

from which it follows that the matrix $A + B$ is similar to the direct sum of the invertible matrix

$$\left(\begin{array}{c|c} A_0 & 0 \\ \hline P_0 & Q_0 \end{array}\right)$$

and some zero matrix.

Proposition 1, and therefore Theorem 1, are proved.

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