A Common Structure in PBW Bases of the Nilpotent Subalgebra of $U_q(\mathfrak{g})$ and Quantized Algebra of Functions[⋆]

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Abstract. For a finite-dimensional simple Lie algebra $\mathfrak{g}$, let $U_q^+(\mathfrak{g})$ be the positive part of the quantized universal enveloping algebra, and $A_q(\mathfrak{g})$ be the quantized algebra of functions. We show that the transition matrix of the PBW bases of $U_q^+(\mathfrak{g})$ coincides with the intertwiner between the irreducible $A_q(\mathfrak{g})$-modules labeled by two different reduced expressions of the longest element of the Weyl group of $\mathfrak{g}$. This generalizes the earlier result by Sergeev on A_2 related to the tetrahedron equation and endows a new representation theoretical interpretation with the recent solution to the 3D reflection equation for C_2. Our proof is based on a realization of $U_q^+(\mathfrak{g})$ in a quotient ring of $A_q(\mathfrak{g})$.

Key words: quantized enveloping algebra; PBW bases; quantized algebra of functions; tetrahedron equation

2010 Mathematics Subject Classification: 17B37; 20G42; 81R50; 17B80

> *Dedicated to Professors Anatol N. Kirillov and Tetsuji Miwa who taught us the joy of doing mathematics.*

1 Introduction

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra and $U_q(\mathfrak{g})$ be the Drinfeld–Jimbo quantized enveloping algebra. $U_q(\mathfrak{g})$ has the subalgebra $U_q^+(\mathfrak{g})$ generated by the Chevalley generators $e_1, \ldots, e_n$ ($n = \operatorname{rank} \mathfrak{g}$) corresponding to the simple roots. Denote by $W = \langle s_1, \ldots, s_n \rangle$ the Weyl group of $\mathfrak{g}$ generated by the simple reflections $s_1, \ldots, s_n$. It is well known (see for example [15]) that for each reduced expression $w_0 = s_{i_1} \cdots s_{i_l}$ of the longest element $w_0 \in W$, one can associate the Poincaré–Birkhoff–Witt (PBW) basis of $U_q^+(\mathfrak{g})$ having the form

$$E_{\mathbf{i}}^A = e_{\beta_1}^{(a_1)} e_{\beta_2}^{(a_2)} \cdots e_{\beta_l}^{(a_l)}, \qquad A = (a_1, \ldots, a_l) \in (\mathbb{Z}_{\geq 0})^l,$$

where $e_{\beta_i}^{(a_i)}$'s are the divided powers of the positive root vectors determined by the choice $\mathbf{i} = (i_1, \ldots, i_l)$. See Section 2.2. Let $\left\{ E_{\mathbf{j}}^A \mid A \in (\mathbb{Z}_{\geq 0})^l \right\}$ with $\mathbf{j} = (j_1, \ldots, j_l)$ be another PBW basis

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associated with a yet different reduced expression $w_0 = s_{j_1} \cdots s_{j_l}$. Following Lusztig [14], one expands a basis in terms of another as

$$E_{\mathbf{i}}^A = \sum_{B \in (\mathbb{Z}_{\geq 0})^l} \gamma_B^A E_{\mathbf{j}}^B$$

and obtains the transition coefficient γ_B^A uniquely. We have suppressed its dependence on $\mathbf{i}, \mathbf{j}$ in the notation. Many remarkable properties are known for γ_B^A including the fact $\gamma_B^A \in \mathbb{Z}[q]$. See [14, Proposition 2.3] for example.

In this paper we show that the transition coefficients $\gamma = (\gamma_B^A)$ coincide with the matrix elements of the intertwiner between the irreducible $A_q(\mathfrak{g})$-modules labeled by two different reduced expressions of the longest element of the Weyl group of $\mathfrak{g}$. Here $A_q(\mathfrak{g})$ denotes the quantized algebra of functions associated with $\mathfrak{g}$. It is a Hopf subalgebra of the dual $U_q(\mathfrak{g})^*$ which has been studied from a variety of aspects. See [5, 11, 17, 18, 21, 22, 23] for example. Let us briefly recall the most relevant result to the present paper due to Vaksman and Soibelman [21, 22, 23]. To each reduced expression of a (not necessarily longest) element $w = s_{i_1} \cdots s_{i_r} \in W$, one can associate an irreducible representation $\pi_{\mathbf{i}}$ labeled by $\mathbf{i} = (i_1, \ldots, i_r)$ having the form

$$\pi_{\mathbf{i}} = \pi_{i_1} \otimes \cdots \otimes \pi_{i_r} : \ A_q(\mathfrak{g}) \to \mathrm{End}\big(\mathcal{F}_{q_{i_1}} \otimes \cdots \otimes \mathcal{F}_{q_{i_r}}\big),$$

where each component $\pi_i : A_q(\mathfrak{g}) \to \mathrm{End}(\mathcal{F}_{q_i})$ is the *fundamental representation* of $A_q(\mathfrak{g})$ on the q-oscillator Fock space $\mathcal{F}_{q_i} = \bigoplus_{m \geq 0} \mathbb{C}(q)|m\rangle$. See Section 4.1. The two irreducible representations $\pi_{\mathbf{i}}$ and $\pi_{\mathbf{j}}$ with $\mathbf{j} = (j_1, \ldots, j_r)$ are isomorphic if $s_{i_1} \cdots s_{i_r} = s_{j_1} \cdots s_{j_r} \in W$ are reduced expressions (Theorem 4). Thus one has the intertwiner $\Phi = \Phi_{\mathbf{i},\mathbf{j}} : \mathcal{F}_{q_{i_1}} \otimes \cdots \otimes \mathcal{F}_{q_{i_r}} \to \mathcal{F}_{q_{j_1}} \otimes \cdots \otimes \mathcal{F}_{q_{j_r}}$ characterized by

$$\pi_{\mathbf{j}}(g) \circ \Phi = \Phi \circ \pi_{\mathbf{i}}(g) \qquad \forall g \in A_q(\mathfrak{g})$$

up to an overall constant. Writing the basis of the Fock space $\mathcal{F}_{q_{i_1}} \otimes \cdots \otimes \mathcal{F}_{q_{i_r}}$ as $|A\rangle = |a_1\rangle \otimes \cdots \otimes |a_r\rangle$ with $A = (a_1, \ldots, a_r) \in (\mathbb{Z}_{\geq 0})^r$, we define the matrix elements of $\Phi = (\Phi_B^A)$ by $\Phi|B\rangle = \sum_A \Phi_B^A |A\rangle$ and the normalization $\Phi_{0,\ldots,0}^{0,\ldots,0} = 1$. Our main result (Theorem 5) is concerned with the longest element case $r = l$ and is stated for each pair $(\mathbf{i},\mathbf{j})$ as $\gamma_B^A = \Phi_B^A$, i.e.,

$$\gamma = \Phi. \tag{1}$$

For a convenience we also introduce the "checked" intertwiner $\Phi^\vee = \Phi \circ \sigma$, where $\sigma(|a_1\rangle \otimes \cdots \otimes |a_l\rangle) = |a_l\rangle \otimes \cdots \otimes |a_1\rangle$ is the reversal of the components.

Our work is inspired by recent developments in 3-dimensional (3D) integrable systems related to rank 2 cases. Recall the Zamolodchikov tetrahedron equation [27] and the Isaev–Kulish 3D reflection equation [8]:

$$R_{356}R_{246}R_{145}R_{123} = R_{123}R_{145}R_{246}R_{356}, \tag{2}$$

$$R_{456}R_{489}K_{3579}R_{269}R_{258}K_{1678}K_{1234} = K_{1234}K_{1678}R_{258}R_{269}K_{3579}R_{489}R_{456}. \tag{3}$$

They are equalities among the linear operators acting on the tensor product of 6 and 9 vector spaces, respectively. The indices specify the components in the tensor product on which the operators R and K act nontrivially. They serve as 3D analogue of the Yang–Baxter and reflection equations postulating certain factorization conditions of straight strings which undergo the scattering R and the reflection K by a boundary plane.

For $\mathfrak{g} = A_2$, Kapranov and Voevodsky [10] showed that $R = \Phi^\vee \in \mathrm{End}(\mathcal{F}_q^{\otimes 3})$ provides a solution to the tetrahedron equation (2). Moreover it was discovered by Sergeev [20] that the solution of the tetrahedron equation R in [2] (given also in [10] with misprint) is related with the

transition matrix as $\gamma = R \circ \sigma$. Thus the equality (1) for $\mathfrak{g} = A_2$ is a corollary of their results. Apart from the A_2 case, it has been shown more recently [13] that $K = \Phi^\vee$ for $\mathfrak{g} = C_2$ yields the first nontrivial solution to the 3D reflection equation (3). See also [12] for $\mathfrak{g} = B_2$. These results motivated us to investigate the general $\mathfrak{g}$ case and have led to (1). It is our hope that it provides a useful insight into higher-dimensional integrable systems from the representation theory of quantum groups.

The layout of the paper is as follows. In Section 2, we summarize the definitions of $U_q(\mathfrak{g})$ and PBW bases. In Section 3, we recall the basic facts on $A_q(\mathfrak{g})$ following Kashiwara [11]. A fundamental role is played by the Peter–Weyl type Theorem 1. The relation with the Reshetikhin–Takhtadzhyan–Faddeev realization by generators and relations [18] is explained and its concrete forms are quoted for A_n, C_n and G_2 [19] which will be of use in later sections. The construction of a certain quotient ring $A_q(\mathfrak{g})_\mathcal{S}$ of $A_q(\mathfrak{g})$ and the special elements $\sigma_i \in A_q(\mathfrak{g})$ (Definition 1) and $\xi_i \in A_q(\mathfrak{g})_\mathcal{S}$ (36) will play a key role in our proof of (1). In Section 4, we briefly review the representation theory of $A_q(\mathfrak{g})$ in [21, 22, 23] and sketch the intertwiners for the rank 2 cases. Section 5 is devoted to the proof of the main theorem $\gamma = \Phi$. It reduces to the rank 2 cases and is done without recourse to explicit formulae of γ or Φ. Our method is to identify their characterizations under the correspondence $e_i \mapsto \xi_i$. Actually, this map extends to an algebra homomorphism $U_q^+(\mathfrak{g}) \to A_q(\mathfrak{g})_\mathcal{S}$ for general $\mathfrak{g}$ as shown by Yakimov [26]. We give a direct proof of a part of his results in Section 6.

2 Quantized enveloping algebra $U_q(\mathfrak{g})$

2.1 Definition

In this paper $\mathfrak{g}$ stands for a finite-dimensional simple Lie algebra. Its weight lattice, simple roots, simple coroots, fundamental weights are denoted by P, $\{\alpha_i\}_{i\in I}$, $\{h_i\}_{i\in I}$, $\{\varpi_i\}_{i\in I}$ where I is the index set of the Dynkin diagram of $\mathfrak{g}$. The Cartan matrix $(a_{ij})_{i,j\in I}$ is given by $a_{ij} = \langle h_i, \alpha_j \rangle = 2(\alpha_i, \alpha_j)/(\alpha_i, \alpha_i)$.

The quantized enveloping algebra $U_q(\mathfrak{g})$ is an associative algebra over $\mathbb{Q}(q)$ generated by $\{e_i, f_i, k_i^{\pm 1} \mid i \in I\}$ satisfying the relations:

$$k_i k_j = k_j k_i, \qquad k_i k_i^{-1} = k_i^{-1} k_i = 1,$$

$$k_i e_j k_i^{-1} = q_i^{\langle h_i, \alpha_j \rangle} e_j, \qquad k_i f_j k_i^{-1} = q_i^{-\langle h_i, \alpha_j \rangle} f_j, \qquad [e_i, f_j] = \delta_{ij} \frac{k_i - k_i^{-1}}{q_i - q_i^{-1}},$$

$$\sum_{r=0}^{1-a_{ij}} (-1)^r e_i^{(r)} e_j e_i^{(1-a_{ij}-r)} = \sum_{r=0}^{1-a_{ij}} (-1)^r f_i^{(r)} f_j f_i^{(1-a_{ij}-r)} = 0, \qquad i \neq j. \tag{4}$$

Here we use the following notations: $q_i = q^{(\alpha_i, \alpha_i)/2}, [m]_i = (q_i^m - q_i^{-m})/(q_i - q_i^{-1}), [n]_i! = \prod_{m=1}^{n} [m]_i,$ $e_i^{(n)} = e_i^n/[n]_i!, f_i^{(n)} = f_i^n/[n]_i!.$ We normalize the simple roots so that $q_i = q$ when α_i is a short root. $U_q(\mathfrak{g})$ is a Hopf algebra. As its comultiplication we adopt the following one

$$\Delta(k_i) = k_i \otimes k_i, \qquad \Delta(e_i) = e_i \otimes 1 + k_i \otimes e_i, \qquad \Delta(f_i) = f_i \otimes k_i^{-1} + 1 \otimes f_i.$$

2.2 PBW basis

Let W be the Weyl group of $\mathfrak{g}$. It is generated by simple reflections $\{s_i \mid i \in I\}$ obeying the relations: $s_i^2 = 1$, $(s_i s_j)^{m_{ij}} = 1$ $(i \neq j)$ where $m_{ij} = 2, 3, 4, 6$ for $\langle h_i, \alpha_j \rangle \langle h_j, \alpha_i \rangle = 0, 1, 2, 3$, respectively. Let w_0 be the longest element of W and fix a reduced expression $w_0 = s_{i_1} s_{i_2} \cdots s_{i_l}$.

Then every positive root occurs exactly once in

$$\beta_1 = \alpha_{i_1}, \quad \beta_2 = s_{i_1}(\alpha_{i_2}), \quad \ldots, \quad \beta_l = s_{i_1} s_{i_2} \cdots s_{i_{l-1}}(\alpha_{i_l}).$$

Correspondingly, define elements $e_{\beta_r} \in U_q(\mathfrak{g})$ $(r = 1, \ldots, l)$ by

$$e_{\beta_r} = T_{i_1} T_{i_2} \cdots T_{i_{r-1}}(e_{i_r}). \tag{5}$$

Here T_i is the action of the braid group on $U_q(\mathfrak{g})$ introduced by Lusztig [15]. It is an algebra automorphism and is given on the generators $\{e_j\}$ by

$$T_i(e_i) = -k_i f_i, \qquad T_i(e_j) = \sum_{r=0}^{-a_{ij}} (-1)^r q_i^r e_i^{(r)} e_j e_i^{(-a_{ij}-r)}, \qquad i \neq j.$$

$U_q(\mathfrak{g})$ has a subalgebra generated by $\{e_i \mid i \in I\}$, denoted by $U_q^+(\mathfrak{g})$. It is known that $e_{\beta_r} \in U_q^+(\mathfrak{g})$ holds for any r. $U_q^+(\mathfrak{g})$ has the so-called Poincaré–Birkhoff–Witt (PBW) basis. It depends on the reduced expression $s_{i_1} s_{i_2} \cdots s_{i_l}$ of w_0. Set $\mathbf{i} = (i_1, i_2, \ldots, i_l)$ and define for $A = (a_1, a_2, \ldots, a_l) \in (\mathbb{Z}_{\geq 0})^l$

$$E_{\mathbf{i}}^A = e_{\beta_1}^{(a_1)} e_{\beta_2}^{(a_2)} \cdots e_{\beta_l}^{(a_l)}. \tag{6}$$

Then $\{E_{\mathbf{i}}^A \mid A \in (\mathbb{Z}_{\geq 0})^l\}$ forms a basis of $U_q^+(\mathfrak{g})$. We hope that the notations e_{i_r} with $i_r \in I$ and e_{β_r} with a positive root β_r can be distinguished properly from the context. In particular $e_{\beta_r}^{(a_r)} = (e_{\beta_r})^{a_r} / \prod_{m=1}^{a_r} \frac{p_r^m - p_r^{-m}}{p_r - p_r^{-1}}$ with $p_r = q^{(\beta_r, \beta_r)/2}$.

3 Quantized algebra of functions $A_q(\mathfrak{g})$

3.1 Definition

Following [11] we give the definition of the quantized algebra of functions $A_q(\mathfrak{g})$. It is valid for any symmetrizable Kac–Moody algebra $\mathfrak{g}$. Let $O_{\mathrm{int}}(\mathfrak{g})$ be the category of integrable left $U_q(\mathfrak{g})$-modules M such that, for any $u \in M$, there exists $l \geq 0$ satisfying $e_{i_1} \cdots e_{i_l} u = 0$ for any $i_1, \ldots, i_l \in I$. Then $O_{\mathrm{int}}(\mathfrak{g})$ is semisimple and any simple object is isomorphic to the irreducible module $V(\lambda)$ with dominant integral highest weight λ. Similarly, we can consider the category $O_{\mathrm{int}}(\mathfrak{g}^{\mathrm{opp}})$ of integrable right $U_q(\mathfrak{g})$-modules M^r such that, for any $v \in M^r$, there exists $l \geq 0$ satisfying $v f_{i_1} \cdots f_{i_l} = 0$ for any $i_1, \ldots, i_l \in I$. $O_{\mathrm{int}}(\mathfrak{g}^{\mathrm{opp}})$ is also semisimple and any simple object is isomorphic to the irreducible module $V^r(\lambda)$ with dominant integral highest weight λ. Let u_λ (resp. v_λ) be a highest-weight vector of $V(\lambda)$ (resp. $V^r(\lambda)$). Then there exists a unique bilinear form $(\ ,\ )$

$$V^r(\lambda) \otimes V(\lambda) \to \mathbb{Q}(q)$$

satisfying

$$(v_\lambda, u_\lambda) = 1 \qquad \text{and} \qquad (vP, u) = (v, Pu) \quad \text{for} \ \ v \in V^r(\lambda), \ \ u \in V(\lambda), \ \ P \in U_q(\mathfrak{g}).$$

Let $U_q(\mathfrak{g})^*$ be $\mathrm{Hom}_{\mathbb{Q}(q)}(U_q(\mathfrak{g}), \mathbb{Q}(q))$ and $\langle\ ,\ \rangle$ be the canonical pairing between $U_q(\mathfrak{g})^*$ and $U_q(\mathfrak{g})$. The comultiplication Δ of $U_q(\mathfrak{g})$ induces a multiplication of $U_q(\mathfrak{g})^*$ by

$$\langle \varphi \varphi', P \rangle = \langle \varphi \otimes \varphi', \Delta(P) \rangle \qquad \text{for} \ \ P \in U_q(\mathfrak{g}), \tag{7}$$

thereby giving $U_q(\mathfrak{g})^*$ the structure of $\mathbb{Q}(q)$-algebra. It also has a $U_q(\mathfrak{g})$-bimodule structure by

$$\langle x\varphi y, P \rangle = \langle \varphi, yPx \rangle \qquad \text{for } x, y, P \in U_q(\mathfrak{g}). \tag{8}$$

We define the subalgebra $A_q(\mathfrak{g})$ of $U_q(\mathfrak{g})^*$ by

$$A_q(\mathfrak{g}) = \{\varphi \in U_q(\mathfrak{g})^*; U_q(\mathfrak{g})\varphi \text{ belongs to } O_{\text{int}}(\mathfrak{g}) \text{ and } \varphi U_q(\mathfrak{g}) \text{ belongs to } O_{\text{int}}(\mathfrak{g}^{\text{opp}})\},$$

and call it the quantized algebra of functions.

The following theorem is the q-analogue of the Peter–Weyl theorem. See e.g. [11] for a proof.

Theorem 1. *As a $U_q(\mathfrak{g})$-bimodule $A_q(\mathfrak{g})$ is isomorphic to $\bigoplus_\lambda V^r(\lambda) \otimes V(\lambda)$, where λ runs over all dominant integral weights, by the homomorphisms*

$$\Psi_\lambda : V^r(\lambda) \otimes V(\lambda) \to A_q(\mathfrak{g})$$

given by

$$\langle \Psi_\lambda(v \otimes u), P \rangle = (v, Pu)$$

for $v \in V^r(\lambda)$, $u \in V(\lambda)$, and $P \in U_q(\mathfrak{g})$.

Let us now assume that $\mathfrak{g}$ is a finite-dimensional simple Lie algebra. Then $A_q(\mathfrak{g})$ turns out a Hopf algebra. See e.g. [9, Chapter 9]. Its comultiplication is also denoted by Δ.

Let $\mathcal{R}$ be the universal R matrix for $U_q(\mathfrak{g})$. For its explicit formula see e.g. [4, p. 273]. For our purpose it is enough to know that

$$\mathcal{R} \in q^{(\text{wt}\,\cdot,\text{wt}\,\cdot)} \bigoplus_{\beta \in Q^+} (U_q^+)_\beta \otimes (U_q^-)_{-\beta}, \tag{9}$$

where $q^{(\text{wt}\,\cdot,\text{wt}\,\cdot)}$ is an operator acting on the tenor product $u_\lambda \otimes u_\mu$ of weight vectors u_λ, u_μ of weight λ, μ by $q^{(\text{wt}\,\cdot,\text{wt}\,\cdot)}(u_\lambda \otimes u_\mu) = q^{(\lambda,\mu)}u_\lambda \otimes u_\mu$, $Q_+ = \bigoplus_i \mathbb{Z}_{\geq 0}\alpha_i$, and $(U_q^\pm)_{\pm\beta}$ is the subspace of $U_q^\pm(\mathfrak{g})$ spanned by root vectors corresponding to $\pm\beta$.

Fix λ, let $\{u_j^\lambda\}$ and $\{v_i^\lambda\}$ be bases of $V(\lambda)$ and $V^r(\lambda)$ such that $(v_i^\lambda, u_j^\lambda) = \delta_{ij}$, and $\varphi_{ij}^\lambda = \Psi_\lambda(v_i^\lambda \otimes u_j^\lambda)$. Let R be the so-called constant R matrix for $V(\lambda) \otimes V(\mu)$. Denoting the homomorphism $U_q(\mathfrak{g}) \to \text{End}(V(\lambda))$ by π_λ, it is given as

$$R \propto (\pi_\lambda \otimes \pi_\mu)(\sigma\mathcal{R}), \tag{10}$$

where σ stands for the exchange of the first and second components. The scalar multiple is determined appropriately depending on $\mathfrak{g}$. The reason we apply σ is that it agrees to the convention of [18]. R satisfies

$$R\Delta(x) = \Delta'(x)R \qquad \text{for any } x \in U_q(\mathfrak{g}),$$

where $\Delta' = \sigma \circ \Delta$. Define matrix elements $R_{ij,kl}$ by $R(u_k^\lambda \otimes u_l^\mu) = \sum_{i,j} R_{ij,kl} u_i^\lambda \otimes u_j^\mu$. Define the right action of R on $V^r(\lambda) \otimes V^r(\mu)$ in such a way that $((v_i^\lambda \otimes v_j^\mu)R, u_k^\lambda \otimes u_l^\mu) = (v_i^\lambda \otimes v_j^\mu, R(u_k^\lambda \otimes u_l^\mu))$ holds. Then we have $(v_i^\lambda \otimes v_j^\mu)R = \sum_{k,l} R_{ij,kl} v_k^\lambda \otimes v_l^\mu$. From

$$\sum_{m,p} R_{ij,mp}\langle \varphi_{mk}^\lambda \varphi_{pl}^\mu, x \rangle = \sum_{m,p} R_{ij,mp}\langle \varphi_{mk}^\lambda \otimes \varphi_{pl}^\mu, \Delta(x) \rangle = \sum_{m,p} R_{ij,mp}(v_m^\lambda \otimes v_p^\mu, \Delta(x)(u_k^\lambda \otimes u_l^\mu))$$

$$= ((v_i^\lambda \otimes v_j^\mu)R, \Delta(x)(u_k^\lambda \otimes u_l^\mu)) = (v_i^\lambda \otimes v_j^\mu, R\Delta(x)(u_k^\lambda \otimes u_l^\mu))$$

$$= \sum_{m,p}(v_i^\lambda \otimes v_j^\mu, \Delta'(x)(u_m^\lambda \otimes u_p^\mu))R_{mp,kl}$$

$$= \sum_{m,p} (v_j^\mu \otimes v_i^\lambda, \Delta(x)(u_p^\mu \otimes u_m^\lambda)) R_{mp,kl}$$

$$= \sum_{m,p} \langle \varphi_{jp}^\mu \otimes \varphi_{im}^\lambda, \Delta(x) \rangle R_{mp,kl} = \sum_{m,p} \langle \varphi_{jp}^\mu \varphi_{im}^\lambda, x \rangle R_{mp,kl}$$

for any $x \in U_q(\mathfrak{g})$, we have

$$\sum_{m,p} R_{ij,mp} \varphi_{mk}^\lambda \varphi_{pl}^\mu = \sum_{m,p} \varphi_{jp}^\mu \varphi_{im}^\lambda R_{mp,kl}. \tag{11}$$

We call such a relation RTT relation.

3.2 Right quotient ring $A_q(\mathfrak{g})_{\mathcal{S}}$

For later use we require a certain right quotient ring of $A_q(\mathfrak{g})$ by a suitable multiplicatively closed subset $\mathcal{S}$. We first review the general construction from [16, Chapter 2].

Let R be a noncommutative ring with 1 and $\mathcal{S}$ a multiplicatively closed subset of R. The following condition is called the right Ore condition:

(Ore) For any $r \in R$, $s \in \mathcal{S}$, $r\mathcal{S} \cap sR \neq \varnothing$.

Set

$$\mathrm{ass}\,\mathcal{S} = \{r \in R \mid rs = 0 \text{ for some } s \in \mathcal{S}\}.$$

Then under the right Ore condition $\mathrm{ass}\,\mathcal{S}$ turns out a two-sided ideal. Let $^- : R \to R/\mathrm{ass}\,\mathcal{S}$ denote the canonical projection. Suppose

(reg) $\overline{\mathcal{S}}$ consists of regular elements, namely, elements x such that both $xr = 0$ and $rx = 0$ imply $r = 0$.

Then a theorem in [16, Chapter 2] states

Theorem 2 (Theorem 2.1.12 of [16])**.** *The right quotient ring $R_{\mathcal{S}}$ exists, if and only if (Ore) and (reg) are satisfied.*

By passing to the images by $^-$, it suffices to consider the case when $\mathrm{ass}\,\mathcal{S} = 0$, and then elements of $R_{\mathcal{S}}$ are of the form r/s. For $r_i/s_i \in R/\mathcal{S}$ $(i = 1, 2)$ the addition and multiplication formulae are given by

$$r_1/s_1 + r_2/s_2 = (r_1 u + r_2 u')/(s_1 u), \qquad (r_1/s_1)(r_2/s_2) = (r_1 v')/(s_2 v), \tag{12}$$

where u, u', v, v' are so chosen that $s_1 u = s_2 u'$ $(u \in \mathcal{S}, u' \in R)$, $r_2 v = s_1 v'$ $(v \in \mathcal{S}, v' \in R)$.

Let us return to our case where $R = A_q(\mathfrak{g})$.

Definition 1. For any $i \in I$, let $u_{w_0 \varpi_i}$ (resp. v_{ϖ_i}) be a lowest (resp. highest) weight vector of $V(\varpi_i)$ (resp. $V^r(\varpi_i)$). Set

$$\sigma_i = \Psi_{\varpi_i}(v_{\varpi_i} \otimes u_{w_0 \varpi_i}).$$

The following proposition is proven in [9]. However, we dare to prove again, since conventions might be different.

Proposition 1 (Corollary 9.1.4 of [9])**.** *Let $\varphi_{\lambda\mu}$ be an element of $A_q(\mathfrak{g})$ such that $k_i \varphi_{\lambda\mu} = q_i^{\langle h_i, \mu \rangle} \varphi_{\lambda\mu}, \varphi_{\lambda\mu} k_i = q_i^{\langle h_i, \lambda \rangle} \varphi_{\lambda\mu}$ for any $i \in I$. Then the following commutation relation holds:*

$$q^{(\varpi_i, \lambda)} \sigma_i \varphi_{\lambda\mu} = q^{(w_0 \varpi_i, \mu)} \varphi_{\lambda\mu} \sigma_i.$$

In particular, $\sigma_i \sigma_j = \sigma_j \sigma_i$ for any i, j.

Proof. Without loss of generality one can assume $\varphi_{\lambda\mu} = \Psi_\nu(v_\lambda \otimes u_\mu)$ for some $\nu, v_\lambda \in V^r(\nu)$, $u_\mu \in V(\nu)$ such that $k_i u_\mu = q_i^{\langle h_i, \mu \rangle} u_\mu$, $v_\lambda k_i = q_i^{\langle h_i, \lambda \rangle} v_\lambda$. In view of (9), (10) we have

$$R(u_{w_0\varpi_i} \otimes u_\mu) = q^{(w_0\varpi_i, \mu)} u_{w_0\varpi_i} \otimes u_\mu, \qquad (v_{\varpi_i} \otimes v_\lambda)R = q^{(\varpi_i, \lambda)} v_{\varpi_i} \otimes v_\lambda.$$

Then (11) implies the commutation relation. The second relation follows from the first one, since $(\varpi_i, \varpi_i) = (w_0\varpi_i, w_0\varpi_i)$. ∎

Let n be the rank of $\mathfrak{g}$ and define

$$\mathcal{S} = \left\{ \sigma_1^{m_1} \cdots \sigma_n^{m_n} \mid m_1, \ldots, m_n \in \mathbb{Z}_{\geq 0} \right\},$$

which is obviously multiplicatively closed subset of $A_q(\mathfrak{g})$.

Lemma 1. *Let s be a nonzero element in $\mathrm{Im}\,\Psi_\lambda$ satisfying $f_i s = s f_i = 0$ for any $i \in I$. Then $s \in \mathbb{Q}(q)^\times \sigma_1^{\lambda_1} \cdots \sigma_n^{\lambda_n}$ where $\mathbb{Q}(q)^\times = \mathbb{Q}(q) \setminus \{0\}$ and $\lambda_i = \langle h_i, \lambda \rangle$.*

Proof. By (7), (8) $f_i \sigma_1^{\lambda_1} \cdots \sigma_n^{\lambda_n} = \sigma_1^{\lambda_1} \cdots \sigma_n^{\lambda_n} f_i = 0$ for any i and $\sigma_1^{\lambda_1} \cdots \sigma_n^{\lambda_n}$ belongs to $\mathrm{Im}\,\Psi_\lambda$. By Theorem 1 such an element is unique up to an element of $\mathbb{Q}(q)^\times$. ∎

In particular σ_i is characterized as the unique element (up to an overall constant) in $\mathrm{Im}\,\Psi_{\varpi_i}$ such that $f_j \sigma_i = \sigma_i f_j = 0$ for all $j \in I$. We remark that Theorem 1 implies that if a nonzero element $\varphi_{\lambda,\mu} \in A_q(\mathfrak{g})$ satisfies the assumption of Proposition 1 and $f_j \varphi_{\lambda,\mu} = \varphi_{\lambda,\mu} f_j = 0$ for all j, then $\lambda = w_0\mu$ must hold.

In [9] it is shown that $A_q(\mathfrak{g})$ is an integral domain (Lemma 9.1.9), hence (reg) is satisfied, and that (Ore) is also satisfied (Lemma 9.1.10). Therefore we have the following theorem. (A proof is attached for self-containedness.)

Theorem 3. *The right quotient ring $A_q(\mathfrak{g})_{\mathcal{S}}$ exists.*

Proof. In view of Theorem 2 it is enough to show that

(1) if $\varphi \neq 0$, then $\varphi s \neq 0$ for any $s \in \mathcal{S}$,

(1') if $\varphi \neq 0$, then $s\varphi \neq 0$ for any $s \in \mathcal{S}$, and

(2) the right Ore condition is satisfied,

since (1) implies $\mathrm{ass}\,\mathcal{S} = 0$, then (1) and (1') imply $\overline{\mathcal{S}} = \mathcal{S}$ consists of regular elements.

Let us prove (1). Let $\varphi = \sum_j \varphi_j$ be the two-sided weight decomposition. If $\varphi_j s \neq 0$ for some j, $\varphi s \neq 0$ since the weights of $\varphi_j s$ are distinct. Hence we can reduce the claim when φ is a weight vector. Suppose $\varphi = \sum_\mu \varphi_\mu$, $\varphi_\mu \in \mathrm{Im}\,\Psi_\mu$ and let λ be a maximal weight, with respect to the standard ordering on weights, such that $\varphi_\lambda \neq 0$. Choose sequences $i_1, \ldots, i_k$ and $j_1, \ldots, j_l$ such that $f_{i_k} \cdots f_{i_1} \varphi_\lambda f_{j_1} \cdots f_{j_l}$ turns out a left-lowest and right-highest weight vector. Then by Lemma 1 it coincides with cs' with some $c \in \mathbb{Q}(q)^\times$, $s' \in \mathcal{S}$. Then

$$f_{i_k} \cdots f_{i_1} (\varphi s) f_{j_1} \cdots f_{j_l} = c's's + \cdots$$

with another $c' \in \mathbb{Q}(q)^\times$. By the maximality of λ the remaining part $+\cdots$ in the right-hand side does not contain the terms with the same two-sided weight. Hence $\cdots = 0$. Therefore, the left-hand side is not 0 and we conclude $\varphi s \neq 0$.

(1') is similar. For (2) we can reduce the claim when φ is a weight vector, and in this case the claim is clear from Proposition 1. ∎

3.3 Realization by generators and relations

We consider the fundamental representation $V(\varpi_1)$ of $U_q(\mathfrak{g})$ for $\mathfrak{g} = A_{n-1}, C_n, G_2$. Set $N = \dim V(\varpi_1)$. It is known [5, 18] that $A_q(\mathfrak{g})$ for $\mathfrak{g} = A_{n-1}, C_n, G_2$ is realized as an associative algebra with appropriate generators $(t_{ij})_{1\leq i,j\leq N}$ corresponding to $V^r(\varpi_1) \otimes V(\varpi_1)$ satisfying RTT relations

$$\sum_{m,p} R_{ij,mp} t_{mk} t_{pl} = \sum_{m,p} t_{jp} t_{im} R_{mp,kl}, \tag{13}$$

and additional ones depending on $\mathfrak{g}$. See below for each $\mathfrak{g}$ under consideration. In all cases, there exists a comultiplication $\Delta : A_q \to A_q \otimes A_q$ given by

$$\Delta(t_{ij}) = \sum_k t_{ik} \otimes t_{kj}. \tag{14}$$

3.3.1 A_{n-1} case

We present formulae for $A_q(A_{n-1})$. In this case $N = n$. Let u_1 and v_1 be the highest-weight vectors of $V(\varpi_1)$ and $V^r(\varpi_1)$ such that $(v_1, u_1) = 1$ and set $u_j = f_{j-1} f_{j-2} \cdots f_1 u_1$, $v_j = v_1 e_1 e_2 \cdots e_{j-1}$ for $2 \leq j \leq n$. Then the constant R matrix is given by

$$\sum_{i,j,k,l} R_{ij,kl} E_{ik} \otimes E_{jl} = q \sum_i E_{ii} \otimes E_{ii} + \sum_{i\neq j} E_{ii} \otimes E_{jj} + \left(q - q^{-1}\right) \sum_{i>j} E_{ij} \otimes E_{ji},$$

where E_{ij} is the matrix unit. Define $t_{ij} = \Psi_{\varpi_1}(v_i \otimes u_j)$. Then the RTT relations among $(t_{ij})_{1\leq i,j\leq N}$ read explicitly as follows

$$[t_{ik}, t_{jl}] = \begin{cases} 0, & i < j, \ k > l, \\ \left(q - q^{-1}\right) t_{jk} t_{il}, & i < j, \ k < l, \end{cases}$$

$$t_{ik} t_{jk} = q t_{jk} t_{ik}, \quad i < j, \qquad t_{ki} t_{kj} = q t_{kj} t_{ki}, \quad i < j.$$

In A_{n-1} case we need another condition that the quantum determinant is 1, i.e.,

$$\sum_{\sigma \in \mathfrak{S}_n} (-q)^{\ell(\sigma)} t_{1\sigma_1} \cdots t_{n\sigma_n} = 1,$$

where $\mathfrak{S}_n = W(A_{n-1})$ is the symmetric group of degree n and $\ell(\sigma)$ is the length of σ.

According to Definition 1, we have $\sigma_1 = t_{13}$ and $\sigma_2 = t_{12}t_{23} - qt_{22}t_{13}$. As an exposition, we note that $\sigma_i e_i$ in (39) is derived from

$$\langle \sigma_1 e_1, P \rangle = \langle t_{13} e_1, P \rangle = (v_1 e_1, P u_3) = (v_2, P u_3) = \langle t_{23}, P \rangle,$$

$$\langle \sigma_2 e_2, P \rangle = \langle (t_{12} \otimes t_{23} - qt_{22} \otimes t_{13}) \Delta(e_2), \Delta(P) \rangle = \langle t_{12} k_2 \otimes t_{23} e_2 - qt_{22} e_2 \otimes t_{13}, \Delta(P) \rangle$$

$$= \langle t_{12} \otimes t_{33} - qt_{32} \otimes t_{13}, \Delta(P) \rangle = \langle t_{12} t_{33} - qt_{32} t_{13}, P \rangle$$

for any $P \in U_q(A_2)$. See e.g. [17] for an extensive treatment.

3.3.2 C_n case

We present formulae for $A_q(C_n)$. In this case $N = 2n$. Let u_1 be the highest-weight vector of $V(\varpi_1)$ and define u_j for $2 \leq j \leq 2n$ recursively by $u_{j+1} = f_j u_j$ $(j \leq n)$, $-f_{2n-j} u_j$ $(j > n)$. Let $\{v_i\}$ be the dual basis to $\{u_i\}$ in $V^r(\varpi_1)$, namely, $\{v_i\}$ are determined by $(v_i, u_j) = \delta_{ij}$. Then the constant R matrix is given by

$$\sum_{i,j,k,l} R_{ij,kl} E_{ik} \otimes E_{jl} = q \sum_i E_{ii} \otimes E_{ii} + \sum_{i\neq j,j'} E_{ii} \otimes E_{jj} + q^{-1} \sum_i E_{ii} \otimes E_{i'i'}$$

$$+ \left(q - q^{-1}\right) \sum_{i>j} E_{ij} \otimes E_{ji} - \left(q - q^{-1}\right) \sum_{i>j} \epsilon_i \epsilon_j q^{\varrho_i - \varrho_j} E_{ij} \otimes E_{i'j'},$$

$$i' = 2n + 1 - i, \qquad \epsilon_i = 1, \quad 1 \le i \le n, \qquad \epsilon_i = -1, \quad n < i \le 2n,$$

$$(\varrho_1, \ldots, \varrho_{2n}) = (n - 1, n - 2, \ldots, 1, 0, 0, -1, \ldots, -n + 1).$$

Define $t_{ij} = \Psi_{\varpi_1}(v_i \otimes u_j)$. The RTT relations are given by (13) with the above $R_{ij,kl}$. Additional relations are given by

$$\sum_{j,k,l} C_{jk} C_{lm} t_{ij} t_{lk} = \sum_{j,k,l} C_{ij} C_{kl} t_{kj} t_{lm} = -\delta_{im}, \qquad C_{ij} = \delta_{i,j'} \epsilon_i q^{\varrho_j}.$$

3.3.3 G_2 case

We have $N = 7$ in this case. We adopt the basis $\{u_i\}$ of $V(\varpi_1)$ that has the representation matrices given as in [19, equation (29)], and let $\{v_i\}$ the dual basis in $V^r(\varpi_1)$. Define $t_{ij} = \Psi_{\varpi_1}(v_i \otimes u_j)$. Then $A_q(G_2)$ is generated by $(t_{ij})_{1 \le i,j \le 7}$ satisfying (i) and (ii) given below.

  (i) RTT relations (13) with the structure constants specified by $R_{ij,kl} = R^{ij}_{kl}$ in [19, equation (33)].

  (ii) Additional relations

$$g^{ij} = \sum_{k,l} t_{jl} t_{ik} g^{kl}, \qquad \sum_k f^{ij}_k t_{km} = \sum_{k,l} t_{jl} t_{ik} f^{kl}_m, \qquad (15)$$

  where g^{ij} and f^{ij}_k are given by [19, equations (30), (31)].

The relations [19, equations (20), (22)] are equivalent to (15) if the RTT relations are imposed. See the explanation after [19, Definition 7]. Note also that we use the opposite indices of the Dynkin diagram to [19].

4 Representations of $A_q(\mathfrak{g})$

4.1 General remarks

Let us recall the results in [22, 23] on the representations of $A_q(\mathfrak{g})$ necessary in this paper. Consider the simplest example $A_q(A_1)$ generated by $t_{11}, t_{12}, t_{21}, t_{22}$ with the relations

$$t_{11} t_{21} = q t_{21} t_{11}, \qquad t_{12} t_{22} = q t_{22} t_{12}, \qquad t_{11} t_{12} = q t_{12} t_{11}, \qquad t_{21} t_{22} = q t_{22} t_{21},$$

$$[t_{12}, t_{21}] = 0, \quad [t_{11}, t_{22}] = (q - q^{-1}) t_{21} t_{12}, \qquad t_{11} t_{22} - q t_{12} t_{21} = 1.$$

Let $\mathrm{Osc}_q = \langle \mathbf{a}^+, \mathbf{a}^-, \mathbf{k} \rangle$ be the q-oscillator algebra, i.e., an associative algebra with the relations

$$\mathbf{k} \mathbf{a}^+ = q \mathbf{a}^+ \mathbf{k}, \qquad \mathbf{k} \mathbf{a}^- = q^{-1} \mathbf{a}^- \mathbf{k}, \qquad \mathbf{a}^- \mathbf{a}^+ = \mathbf{1} - q^2 \mathbf{k}^2, \qquad \mathbf{a}^+ \mathbf{a}^- = \mathbf{1} - \mathbf{k}^2. \qquad (16)$$

It has a representation on the Fock space $\mathcal{F}_q = \bigoplus_{m \ge 0} \mathbb{C}(q) |m\rangle$:

$$\mathbf{k} |m\rangle = q^m |m\rangle, \qquad \mathbf{a}^+ |m\rangle = |m + 1\rangle, \qquad \mathbf{a}^- |m\rangle = (1 - q^{2m}) |m - 1\rangle. \qquad (17)$$

In what follows, the symbols $\mathbf{k}, \mathbf{a}^+, \mathbf{a}^-$ shall also be regarded as the elements from $\mathrm{End}(\mathcal{F}_q)$. It is easy to check that the following map π defines an irreducible representation of $A_q(A_1)$ on $\mathcal{F}_q$:

$$\pi : \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} \mapsto \begin{pmatrix} \mu \mathbf{a}^- & \alpha \mathbf{k} \\ -q\alpha^{-1} \mathbf{k} & \mu^{-1} \mathbf{a}^+ \end{pmatrix}, \qquad (18)$$

where α, μ are nonzero parameters.

Theorem 4 ([22, 23]).

(1) *For each vertex i of the Dynkin diagram of $\mathfrak{g}$, $A_q(\mathfrak{g})$ has an irreducible representation π_i factoring through* (18) *via $A_q(\mathfrak{g}) \twoheadrightarrow A_{q_i}(sl_{2,i})$. ($sl_{2,i}$ denotes the sl_2-subalgebra of $\mathfrak{g}$ associated to i.)*

(2) *Irreducible representations of $A_q(\mathfrak{g})$ are in one to one correspondence with the elements of the Weyl group W of $\mathfrak{g}$.*

(3) *Let $w = s_{i_1} \cdots s_{i_l} \in W$ be an reduced expression in terms of the simple reflections. Then the irreducible representation corresponding to w is isomorphic to $\pi_{i_1} \otimes \cdots \otimes \pi_{i_l}$.*

Actually the assertions (2) and (3) hold up to the degrees of freedom of the parameters α, μ in (18). See [22] for the detail. We call π_i ($i = 1, \ldots, \operatorname{rank} \mathfrak{g}$) the *fundamental representations*. For simplicity we denote $\pi_{i_1} \otimes \cdots \otimes \pi_{i_l}$ by $\pi_{i_1,\ldots,i_l}$.

A crucial corollary of Theorem 4 is the following:

If $s_{i_1} \cdots s_{i_l} = s_{j_1} \cdots s_{j_l} \in W$ are reduced expressions, then $\pi_{i_1,\ldots,i_l} \simeq \pi_{j_1,\ldots,j_l}$.

In particular, there exists the isomorphism $\Phi : \mathcal{F}_{q_{i_1}} \otimes \cdots \otimes \mathcal{F}_{q_{i_l}} \to \mathcal{F}_{q_{j_1}} \otimes \cdots \otimes \mathcal{F}_{q_{j_l}}$ characterized (up to an overall constant) by

$$\pi_{j_1,\ldots,j_l}(g) \circ \Phi = \Phi \circ \pi_{i_1,\ldots,i_l}(g) \qquad \forall g \in A_q(\mathfrak{g}).$$

Here $\pi_{i_1,\ldots,i_l}(g = t_{ij})$ for example means the tensor product representation $\sum_{r_1,\ldots,r_{l-1}} \pi_{i_1}(t_{ir_1}) \otimes \cdots \otimes \pi_{i_l}(t_{r_{l-1},j})$ obtained by the $(l-1)$-fold application of the coproduct (14).

Elements of the Fock space $|m_1\rangle \otimes \cdots \otimes |m_l\rangle \in \mathcal{F}_{q_{j_1}} \otimes \cdots \otimes \mathcal{F}_{q_{j_l}}$ will simply be denoted by $|m_1,\ldots,m_l\rangle$. We will always normalize the intertwiner by the condition $\Phi|0,0,\ldots,0\rangle = |0,0,\ldots,0\rangle$. The exchange of the ith and the jth tensor components from the left will be denoted by P_{ij}. In the remainder of this section we concentrate on $A_q(\mathfrak{g})$ of rank 2 cases $\mathfrak{g} = A_2, C_2$ and G_2, and present the concrete forms of the fundamental representations, definition of the intertwiners with a few examples of their matrix elements.

4.2 A_2 case

Let $T = (t_{ij})_{1 \le i,j \le 3}$ be the 3×3 matrix of the generators of $A_q(A_2)$. The fundamental representations $\pi_i : A_q(A_2) \to \operatorname{End}(\mathcal{F}_q)$ ($i = 1, 2$) are given by

$$\pi_1(T) = \begin{pmatrix} \mu_1 \mathbf{a}^- & \alpha_1 \mathbf{k} & 0 \\ -q\alpha_1^{-1}\mathbf{k} & \mu_1^{-1}\mathbf{a}^+ & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \pi_2(T) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mu_2 \mathbf{a}^- & \alpha_2 \mathbf{k} \\ 0 & -q\alpha_2^{-1}\mathbf{k} & \mu_2^{-1}\mathbf{a}^+ \end{pmatrix}, \tag{19}$$

where α_i, μ_i are nonzero parameters.

The Weyl group $W = \langle s_1, s_2 \rangle$ is the Coxeter system with the relations

$$s_1^2 = s_2^2 = 1, \qquad s_1 s_2 s_1 = s_2 s_1 s_2.$$

Thus we have the isomorphism $\pi_{121} \simeq \pi_{212}$. Let Φ be the corresponding intertwiner and denote by R the checked intertwiner $\Phi^\vee$ explained after (1)

$$\pi_{121}\Phi = \Phi\pi_{212}, \qquad \pi_{121}R = R\pi'_{212}, \qquad \pi'_{212} = P_{13}\pi_{212}P_{13}, \qquad R = \Phi P_{13} \in \operatorname{End}(\mathcal{F}_q^{\otimes 3}).$$

For example $\pi'_{212}(t_{ij}) = \sum_{k,l} \pi_2(t_{l,j}) \otimes \pi_1(t_{k,l}) \otimes \pi_2(t_{ik})$. Define the matrix elements of R and its parameter-free part $\mathcal{R}$ by

$$R|i,j,k\rangle = \sum_{a,b,c} R^{abc}_{ijk}|a,b,c\rangle, \qquad R^{abc}_{ijk} = \mu_1^{a-j+k}\mu_2^{b-a-k}\mathcal{R}^{abc}_{ijk}.$$

Then the following properties are valid for $\mathcal{R} = (\mathcal{R}^{abc}_{ijk})$ [13]:

$$\mathcal{R}^{abc}_{ijk} \in \mathbb{Z}[q], \qquad \mathcal{R}^{abc}_{ijk} = 0 \quad \text{unless} \ (a+b, b+c) = (i+j, j+k), \tag{20}$$

$$\mathcal{R}^{-1} = \mathcal{R}, \qquad \mathcal{R}^{abc}_{ijk} = \mathcal{R}^{cba}_{kji}, \qquad \mathcal{R}^{abc}_{ijk} = \frac{(q^2)_i (q^2)_j (q^2)_k}{(q^2)_a (q^2)_b (q^2)_c} \mathcal{R}^{ijk}_{abc}, \tag{21}$$

$$\mathcal{R}^{abc}_{ijk}|_{q=0} = \delta_{i, b+(a-c)_+} \delta_{j, \min(a,c)} \delta_{k, b+(c-a)_+}. \tag{22}$$

Here $(q^2)_a = \prod\limits_{m=1}^{a} (1 - q^{2m})$ and $(y)_+ = \max(0, y)$. Due to (20), $\mathcal{R}$ is the infinite direct sum of finite-dimensional matrices. An explicit formula of $\mathcal{R}^{abc}_{ijk}$ was obtained in [10] (unfortunately with misprint) and in [1, equation (59)] (in a different context and gauge including square roots). The formula exactly matching the present convention is [13, equation (2.20)]. The $\mathcal{R}$ satisfies [10] the tetrahedron equation (2).

Example 1. The following is the list of all the nonzero $\mathcal{R}^{abc}_{314}$:

$$\mathcal{R}^{041}_{314} = -q^2 (1 - q^4)(1 - q^6)(1 - q^8),$$
$$\mathcal{R}^{132}_{314} = (1 - q^6)(1 - q^8)(1 - q^4 - q^6 - q^8 - q^{10}),$$
$$\mathcal{R}^{223}_{314} = q^2 (1 + q^2)(1 + q^4)(1 - q^6)(1 - q^6 - q^{10}),$$
$$\mathcal{R}^{314}_{314} = q^6 (1 + q^2 + q^4 - q^8 - q^{10} - q^{12} - q^{14}),$$
$$\mathcal{R}^{405}_{314} = q^{12}.$$

Thus $\mathcal{R}^{abc}_{314}|_{q=0} = \delta_{a,1} \delta_{b,3} \delta_{c,2}$ in agreement with (22).

4.3 C_2 case

We have $(q_1, q_2) = (q, q^2)$. Let $T = (t_{ij})_{1 \le i,j \le 4}$ be the 4×4 matrix of the generators of $A_q(C_2)$. We use $\mathrm{Osc}_{q^2} = \langle \mathbf{A}^+, \mathbf{A}^-, \mathbf{K} \rangle$ in addition to $\mathrm{Osc}_q = \langle \mathbf{a}^+, \mathbf{a}^-, \mathbf{k} \rangle$ (16). The fundamental representations $\pi_i : A_q(C_2) \to \mathrm{End}(\mathcal{F}_{q_i})$ $(i = 1, 2)$ are given by

$$\pi_1(T) = \begin{pmatrix} \mu_1 \mathbf{a}^- & \alpha_1 \mathbf{k} & 0 & 0 \\ -q\alpha_1^{-1} \mathbf{k} & \mu_1^{-1} \mathbf{a}^+ & 0 & 0 \\ 0 & 0 & \mu_1 \mathbf{a}^- & -\alpha_1 \mathbf{k} \\ 0 & 0 & q\alpha_1^{-1} \mathbf{k} & \mu_1^{-1} \mathbf{a}^+ \end{pmatrix},$$

$$\pi_2(T) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \mu_2 \mathbf{A}^- & \alpha_2 \mathbf{K} & 0 \\ 0 & -q^2 \alpha_2^{-1} \mathbf{K} & \mu_2^{-1} \mathbf{A}^+ & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{23}$$

where α_i, μ_i are nonzero parameters.

The Weyl group $W = \langle s_1, s_2 \rangle$ is the Coxeter system with the relations

$$s_1^2 = s_2^2 = 1, \qquad s_2 s_1 s_2 s_1 = s_1 s_2 s_1 s_2.$$

Thus we have the isomorphism $\pi_{2121} \simeq \pi_{1212}$. Let Φ be the corresponding intertwiner and denote by K the checked intertwiner $\Phi^\vee$

$$\pi_{2121}\Phi = \Phi\pi_{1212}, \qquad \pi_{2121}K = K\pi'_{2121}, \qquad \pi'_{2121} = P_{14}P_{23}\pi_{1212}P_{14}P_{23},$$
$$K = \Phi P_{14}P_{23} \in \mathrm{End}(\mathcal{F}_{q^2} \otimes \mathcal{F}_q \otimes \mathcal{F}_{q^2} \otimes \mathcal{F}_q).$$

Define the matrix elements of K and its parameter-free part $\mathcal{K}$ by

$$K|i,j,k,l\rangle = \sum_{a,b,c,d} K_{ijkl}^{abcd}|a,b,c,d\rangle, \qquad K_{ijkl}^{abcd} = \mu_1^{2(c-k)}\mu_2^{b-j}\mathcal{K}_{ijkl}^{abcd}.$$

Then the following properties are valid for $\mathcal{K} = (\mathcal{K}_{ijkl}^{abcd})$ [13]:

$$\mathcal{K}_{ijkl}^{abcd} \in \mathbb{Z}[q], \qquad \mathcal{K}_{ijkl}^{abcd} = 0 \quad \text{unless} \quad (a+b+c, b+2c+d) = (i+j+k, j+2k+l), \quad (24)$$

$$\mathcal{K}^{-1} = \mathcal{K}, \qquad \mathcal{K}_{ijkl}^{abcd} = \frac{(q^4)_i(q^2)_j(q^4)_k(q^2)_l}{(q^4)_a(q^2)_b(q^4)_c(q^2)_d}\mathcal{K}_{abcd}^{ijkl}, \qquad (25)$$

$$\mathcal{K}_{ijkl}^{abcd}|_{q=0} = \delta_{i,a'}\delta_{j,b'}\delta_{k,c'}\delta_{l,d'}, \qquad (26)$$

$$a' = x + a + b - d, \qquad b' = c + d - x - \min(a, c+x),$$
$$c' = \min(a, c+x), \qquad d' = b + (c-a+x)_+, \qquad x = (c-a+(d-b)_+)_+.$$

Due to (24), $\mathcal{K}$ is the infinite direct sum of finite-dimensional matrices. An explicit formula of $\mathcal{K}_{ijkl}^{abcd}$ is available in [13, equations (3.27), (3.28)]. This $\mathcal{K}$ and $\mathcal{R}$ in Section 4.2 satisfy [13] the 3D reflection equation (3).

Example 2. The following is the list of all the nonzero $\mathcal{K}_{2110}^{abcd}$:

$$\mathcal{K}_{2110}^{1300} = q^8(1-q^8),$$
$$\mathcal{K}_{2110}^{2110} = -q^4(1-q^8+q^{14}),$$
$$\mathcal{K}_{2110}^{2201} = -q^6(1+q^2)(1-q^2+q^4-q^6-q^{10}),$$
$$\mathcal{K}_{2110}^{3011} = 1-q^8+q^{14},$$
$$\mathcal{K}_{2110}^{3102} = -q^{10}(1-q+q^2)(1+q+q^2),$$
$$\mathcal{K}_{2110}^{4003} = q^4.$$

Thus $\mathcal{K}_{2110}^{abcd}|_{q=0} = \delta_{a,3}\delta_{b,0}\delta_{c,1}\delta_{d,1}$ in agreement with (26).

4.4 G_2 case

We have $(q_1, q_2) = (q, q^3)$. Let $T = (t_{ij})_{1\le i,j\le 7}$ be the 7×7 matrix of the generators of $A_q(G_2)$. We use $\mathrm{Osc}_{q^3} = \langle \mathbf{A}^+, \mathbf{A}^-, \mathbf{K}\rangle$ in addition to $\mathrm{Osc}_q = \langle \mathbf{a}^+, \mathbf{a}^-, \mathbf{k}\rangle$ (16). The fundamental representations $\pi_i : A_q(G_2) \to \mathrm{End}(\mathcal{F}_{q_i})$ $(i = 1, 2)$ are given by

$$\pi_1(T) = \begin{pmatrix} \mu_1\mathbf{a}^- & \alpha_1\mathbf{k} & 0 & 0 & 0 & 0 & 0 \\ -q\alpha_1^{-1}\mathbf{k}\,\mu_1^{-1}\mathbf{a}^+ & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & (\mu_1\mathbf{a}^-)^2 & [2]_1\alpha_1\mu_1\mathbf{k}\,\mathbf{a}^- & (\alpha_1\mathbf{k})^2 & 0 & 0 \\ 0 & 0 & -q\alpha_1^{-1}\mu_1\mathbf{a}^-\,\mathbf{k} & \mathbf{a}^-\mathbf{a}^+ - \mathbf{k}^2 & \alpha_1\mu_1^{-1}\mathbf{k}\,\mathbf{a}^+ & 0 & 0 \\ 0 & 0 & (q\alpha_1^{-1}\mathbf{k})^2 & -[2]_1(\alpha_1\mu_1)^{-1}\mathbf{k}\,\mathbf{a}^+ & (\mu_1^{-1}\mathbf{a}^+)^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu_1\mathbf{a}^- & \alpha_1\mathbf{k} \\ 0 & 0 & 0 & 0 & 0 & -q\alpha_1^{-1}\mathbf{k}\,\mu_1^{-1}\mathbf{a}^+ \end{pmatrix},$$

$$\pi_2(T) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mu_2\mathbf{A}^- & \alpha_2\mathbf{K} & 0 & 0 & 0 & 0 \\ 0 & -q^3\alpha_2^{-1}\mathbf{K} & \mu_2^{-1}\mathbf{A}^+ & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu_2\mathbf{A}^- & \alpha_2\mathbf{K} & 0 \\ 0 & 0 & 0 & 0 & -q^3\alpha_2^{-1}\mathbf{K} & \mu_2^{-1}\mathbf{A}^+ & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad (27)$$

where α_i, μ_i are nonzero parameters and $[2]_1 = q + q^{-1}$ as defined after (4).

The Weyl group $W = \langle s_1, s_2 \rangle$ is the Coxeter system with the relations

$$s_1^2 = s_2^2 = 1, \quad s_2 s_1 s_2 s_1 s_2 s_1 = s_1 s_2 s_1 s_2 s_1 s_2.$$

Thus we have the isomorphism $\pi_{212121} \simeq \pi_{121212}$. Let Φ be the corresponding intertwiner and denote by F the checked intertwiner $\Phi^\vee$

$$\pi_{212121}\Phi = \Phi\pi_{121212}, \qquad \pi_{212121}F = F\pi'_{212121}, \qquad \pi'_{212121} = P_{16}P_{25}P_{34}\pi_{121212}P_{16}P_{25}P_{34},$$
$$F = \Phi P_{16}P_{25}P_{34} \in \text{End}(\mathcal{F}_{q^3} \otimes \mathcal{F}_q \otimes \mathcal{F}_{q^3} \otimes \mathcal{F}_q \otimes \mathcal{F}_{q^3} \otimes \mathcal{F}_q). \tag{28}$$

Define the matrix elements of F and its parameter-free part $\mathcal{F}$ by

$$F|i,j,k,l,m,n\rangle = \sum_{a,b,c,d,e,f} F^{abcdef}_{ijklmn}|a,b,c,d,e,f\rangle,$$
$$F^{abcdef}_{ijklmn} = \mu_1^{3c-3k+d-l+3e-3m}\mu_2^{2k-2c+l-d+3m-3e+n-f}\mathcal{F}^{abcdef}_{ijklmn}.$$

Then the following properties are valid for $\mathcal{F} = (\mathcal{F}^{abcdef}_{ijklmn})$:

$$\mathcal{F}^{abcdef}_{ijklmn} \in \mathbb{Z}[q],$$
$$\mathcal{F}^{abcdef}_{ijklmn} = 0 \quad \text{unless} \quad \begin{pmatrix} a+b+2c+d+e \\ b+3c+2d+3e+f \end{pmatrix} = \begin{pmatrix} i+j+2k+l+m \\ j+3k+2l+3m+n \end{pmatrix}, \tag{29}$$
$$\mathcal{F}^{-1} = \mathcal{F}, \qquad \mathcal{F}^{abcdef}_{ijklmn} = \frac{(q^6)_i(q^2)_j(q^6)_k(q^2)_l(q^6)_m(q^2)_n}{(q^6)_a(q^2)_b(q^6)_c(q^2)_d(q^6)_e(q^2)_f}\mathcal{F}^{ijklmn}_{abcdef}. \tag{30}$$

Due to (29), $\mathcal{F}$ is the infinite direct sum of finite-dimensional matrices. The formula for $\mathcal{F}^{abcdef}_{ijklmn}|_{q=0}$ can be deduced by the ultradiscretization (tropical form) of [3, Theorem 3.1(c)]. Although a tedious algorithm can be formulated for calculating any given $\mathcal{F}^{abcdef}_{ijklmn}$ by using (28), an explicit formula for it is yet to be constructed.

Example 3. The following is the list of all the nonzero $\mathcal{F}^{abcdef}_{010101}$:

$$\mathcal{F}^{000200}_{010101} = q^4(1-q^2)(1-q^2-q^4-q^6),$$
$$\mathcal{F}^{001001}_{010101} = -q(1-q^2)(1-q^2-q^4+q^8+q^{10}),$$
$$\mathcal{F}^{010010}_{010101} = -q(1-q^2)(1-q^2-q^4+q^8+q^{10}),$$
$$\mathcal{F}^{010101}_{010101} = 1-2q^2+2q^6+3q^8-2q^{12}-2q^{14}-q^{16},$$
$$\mathcal{F}^{020002}_{010101} = q^4(-2+2q^6+q^8+q^{10}),$$
$$\mathcal{F}^{100011}_{010101} = -q^3(1-q^2)(1-q^6-q^8),$$
$$\mathcal{F}^{100102}_{010101} = q(1-q^2-q^4-q^6+q^{10}+q^{12}+q^{14}),$$
$$\mathcal{F}^{200004}_{010101} = q^4,$$
$$\mathcal{F}^{110003}_{010101} = q(1-q+q^2)(1+q+q^2)(1-q^2-q^8).$$

5   Main theorem

In this section we fix two reduced words $\mathbf{i} = (i_1,\ldots,i_l)$, $\mathbf{j} = (j_1,\ldots,j_l)$ of the longest element $w_0 \in W$.

5.1   Definitions of γ_B^A and Φ_B^A

In the $U_q(\mathfrak{g})$ side, we defined the PBW bases $E_{\mathbf{i}}^A$, $E_{\mathbf{j}}^B$ of $U_q^+(\mathfrak{g})$ in Section 2.2. We define their transition coefficient γ_B^A by

$$E_{\mathbf{i}}^A = \sum_B \gamma_B^A E_{\mathbf{j}}^B.$$

While, in the $A_q(\mathfrak{g})$ side, we have the intertwiner $\Phi : \mathcal{F}_{q_{i_1}} \otimes \cdots \otimes \mathcal{F}_{q_{i_l}} \to \mathcal{F}_{q_{j_1}} \otimes \cdots \otimes \mathcal{F}_{q_{j_l}}$ satisfying

$$\pi_{\mathbf{j}}(g) \circ \Phi = \Phi \circ \pi_{\mathbf{i}}(g) \qquad \forall\, g \in A_q(\mathfrak{g}). \tag{31}$$

We take the parameters μ, α in (18) to be 1. This in particular means for rank 2 cases that μ_i, α_i entering $\pi_i(T)$ in (19), (23) and (27) are all 1. The intertwiner Φ is normalized by $\Phi|0,0,\ldots,0\rangle = |0,0,\ldots,0\rangle$. Under these conditions a matrix element Φ_B^A of Φ is uniquely specified by

$$\Phi|B\rangle = \sum_A \Phi_B^A |A\rangle,$$

where $A = (a_1,\ldots,a_l) \in (\mathbb{Z}_{\geq 0})^l$ and $|A\rangle = |a_1\rangle \otimes \cdots \otimes |a_l\rangle \in \mathcal{F}_{q_{j_1}} \otimes \cdots \otimes \mathcal{F}_{q_{j_l}}$ and similarly for $|B\rangle \in \mathcal{F}_{q_{i_1}} \otimes \cdots \otimes \mathcal{F}_{q_{i_l}}$. Then our main result is

Theorem 5.

$$\gamma_B^A = \Phi_B^A.$$

For any pair $(\mathbf{i}, \mathbf{j})$, from $\mathbf{i}$ one can reach $\mathbf{j}$ by applying Coxeter relations. In view of the uniqueness of γ and Φ and the fact that the braid group action T_i is an algebra homomorphism, the proof of this theorem reduces to establishing the same equality for all $\mathfrak{g}$ of rank 2. This will be done in the rest of this section.

5.2   Proof of Theorem 5 for rank 2 cases

In the rank 2 cases, there are two reduced expressions $s_{i_1} \cdots s_{i_l}$ for the longest element of the Weyl group. Denote the associated sequences $\mathbf{i} = (i_1,\ldots,i_l)$ by $\mathbf{1}$, $\mathbf{2}$ and set $\mathbf{1}' = \mathbf{2}$, $\mathbf{2}' = \mathbf{1}$. Concretely, we take them as

$$
\begin{array}{llll}
A_2: & \mathbf{1} = (1,2,1), & \mathbf{2} = (2,1,2), & (q_1,q_2) = (q,q), \\
C_2: & \mathbf{1} = (1,2,1,2), & \mathbf{2} = (2,1,2,1), & (q_1,q_2) = (q,q^2), \\
G_2: & \mathbf{1} = (1,2,1,2,1,2), & \mathbf{2} = (2,1,2,1,2,1), & (q_1,q_2) = (q,q^3),
\end{array}
$$

where q_i defined after (4) is also recalled. In order to simplify the formulae in Section 5.3, we use the PBW bases and the Fock states in yet another normalization as follows:

$$\tilde{E}_{\mathbf{i}}^A := ([a_1]_{i_1}! \cdots [a_l]_{i_l}!) E_{\mathbf{i}}^A = e_{\beta_1}^{a_1} \cdots e_{\beta_l}^{a_l},$$

$$|A\rangle\!\rangle := d_{i_1,a_1} \cdots d_{i_l,a_l} |A\rangle, \qquad d_{i,a} = q_i^{-a(a-1)/2} \lambda_i^a, \qquad \lambda_i = \left(1 - q_i^2\right)^{-1}, \tag{32}$$

where $A = (a_1,\ldots,a_l)$. See after (4) for the symbol $[a]_i!$. e_{β_r} is defined in (5). Accordingly we introduce the matrix elements $\tilde{\gamma}_B^A$ and $\tilde{\Phi}_B^A$ by

$$\tilde{E}_{\mathbf{i}}^A = \sum_B \tilde{\gamma}_B^A \tilde{E}_{\mathbf{i}'}^B, \qquad \Phi|B\rangle\!\rangle = \sum_A \tilde{\Phi}_B^A |A\rangle\!\rangle, \qquad \mathbf{i} = \mathbf{1}, \mathbf{2}.$$

It follows that $\gamma_B^A = \tilde{\gamma}_B^A \prod_{k=1}^{l} ([b_k]_{i_k}!/[a_k]_{i_k}!)$ and $\Phi_B^A = \tilde{\Phi}_B^A \prod_{k=1}^{l} (d_{i_k,a_k}/d_{i_k,b_k})$ for $B = (b_1, \ldots, b_l)$.

On the other hand, we know $\Phi_B^A = \Phi_A^B \prod_{k=1}^{l} ((q_{i_k}^2)_{b_k}/(q_{i_k}^2)_{a_k})$ from (21), (25) and (30). Due to the identity $(q_i^2)_m d_{i,m} = [m]_i!$, the assertion $\gamma_B^A = \Phi_B^A$ of Theorem 5 is equivalent to

$$\tilde{\gamma}_B^A = \tilde{\Phi}_A^B. \tag{33}$$

Let $\rho_{\mathbf{i}}(x) = (\rho_{\mathbf{i}}(x)_{AB})$ be the matrix for the left multiplication of $x \in U_q^+(\mathfrak{g})$:

$$x \cdot \tilde{E}_{\mathbf{i}}^A = \sum_B \tilde{E}_{\mathbf{i}}^B \rho_{\mathbf{i}}(x)_{BA}. \tag{34}$$

Let further $\pi_{\mathbf{i}}(g) = (\pi_{\mathbf{i}}(g)_{AB})$ be the representation matrix of $g \in A_q(\mathfrak{g})$:

$$\pi_{\mathbf{i}}(g)|A\rangle\!\rangle = \sum_B |B\rangle\!\rangle \pi_{\mathbf{i}}(g)_{BA}. \tag{35}$$

The following element in the right quotient ring $A_q(\mathfrak{g})_{\mathcal{S}}$ will play a key role in our proof.

$$\xi_i = \lambda_i(\sigma_i e_i)/\sigma_i, \qquad i = 1, 2. \tag{36}$$

See Definition 1 for σ_i and (39), (41), (42) for the concrete forms in rank 2 cases. In Section 5.3 we will check the following statement case by case.

Proposition 2. *For $\mathfrak{g}$ of rank 2, $\pi_{\mathbf{i}}(\sigma_i)$ is invertible and the following equality is valid:*

$$\rho_{\mathbf{i}}(e_i)_{AB} = \pi_{\mathbf{i}}(\xi_i)_{AB}, \qquad i = 1, 2, \tag{37}$$

where the right-hand side means $\lambda_i \pi_{\mathbf{i}}(\sigma_i e_i) \pi_{\mathbf{i}}(\sigma_i)^{-1}$.

Proof of Theorem 5 for rank 2 case. We write the both sides of (37) as M_{AB}^i and the one for $\mathbf{i}'$ instead of $\mathbf{i}$ as $M_{AB}^{\prime i}$. From

$$\sum_{B,C} \tilde{E}_{\mathbf{i}'}^C M_{CB}^{\prime i} \tilde{\gamma}_B^A = e_i \sum_B \tilde{E}_{\mathbf{i}'}^B \tilde{\gamma}_B^A = e_i \tilde{E}_{\mathbf{i}}^A = \sum_B \tilde{E}_{\mathbf{i}}^B M_{BA}^i = \sum_{B,C} \tilde{E}_{\mathbf{i}'}^C \tilde{\gamma}_C^B M_{BA}^i$$

we have $\sum_B M_{CB}^{\prime i} \tilde{\gamma}_B^A = \sum_B \tilde{\gamma}_C^B M_{BA}^i$. On the other hand, the action of the two sides of (31) with $g = \xi_i$ and $\mathbf{j} = \mathbf{i}'$ are calculated as

$$\pi_{\mathbf{i}'}(\xi_i) \circ \Phi|A\rangle\!\rangle = \pi_{\mathbf{i}'}(\xi_i) \sum_B |B\rangle\!\rangle \tilde{\Phi}_A^B = \sum_{B,C} |C\rangle\!\rangle M_{CB}^{\prime i} \tilde{\Phi}_A^B$$

and

$$\Phi \circ \pi_{\mathbf{i}}(\xi_i)|A\rangle\!\rangle = \Phi \sum_B |B\rangle\!\rangle M_{BA}^i = \sum_{B,C} |C\rangle\!\rangle \tilde{\Phi}_B^C M_{BA}^i.$$

Hence $\sum_B M_{CB}^{\prime i} \tilde{\Phi}_A^B = \sum_B \tilde{\Phi}_B^C M_{BA}^i$. Thus $\tilde{\gamma}_B^A$ and $\tilde{\Phi}_A^B$ satisfy the same relation. Moreover the maps $\pi_{\mathbf{i}}$ and $\rho_{\mathbf{i}}$ are both homomorphism, i.e., $\pi_{\mathbf{i}}(gh) = \pi_{\mathbf{i}}(g)\pi_{\mathbf{i}}(h)$ and $\rho_{\mathbf{i}}(xy) = \rho_{\mathbf{i}}(x)\rho_{\mathbf{i}}(y)$. We know that Φ is the intertwiner of the irreducible $A_q(\mathfrak{g})$ modules and (33) obviously holds as $1 = 1$ at $A = B = (0, \ldots, 0)$. Thus it is valid for arbitrary A and B. ∎

Conjecture 1. *The equality (37) is valid for any $\mathfrak{g}$.*

5.3   Explicit formulae for rank 2 cases: Proof of Proposition 2

Here we present the explicit formulae of (34) with $x = e_i$ and (35) with $g = \sigma_i, \sigma_i e_i$ that allow one to check Proposition 2. We use the notation $\langle i \rangle = q^i - q^{-i}$. In each case, there are two **i**-sequences, **1** and **2** = **1′** corresponding to the two reduced words. Let χ be the anti-algebra involution such that $\chi(e_i) = e_i$. Then the relation $\chi(\tilde{E}_{\mathbf{i}}^A) = \tilde{E}_{\mathbf{i}'}^{\bar{A}}$ holds, where $\bar{A} = (a_l, \ldots, a_2, a_1)$ denotes the reversal of $A = (a_1, a_2, \ldots, a_l)$. Applying χ to (34) with $x = e_i$ yields the right multiplication formula $\tilde{E}_{\mathbf{i}'}^{\bar{A}} \cdot e_i = \sum_B \tilde{E}_{\mathbf{i}'}^{\bar{B}} \rho_{\mathbf{i}}(e_i)_{BA}$ for **i′**-sequence. In view of this fact, we shall present the left and right multiplication formulae for $\mathbf{i} = \mathbf{2}$ only.

As for (35) with $g = \xi_i$ in (36), explicit formulae for $\sigma_i, \sigma_i e_i \in A_q(\mathfrak{g})$ and their image by the both representations $\pi_{\mathbf{1}}$ and $\pi_{\mathbf{2}}$ will be given. We include an exposition on how to use these data to check (37) along the simplest A_2 case. The C_2 and G_2 cases are similar.

5.3.1 A_2 case
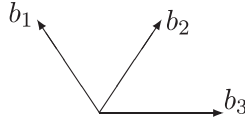
The q-Serre relations are

$$e_1^2 e_2 - [2]_1 e_1 e_2 e_1 + e_2 e_1^2 = 0, \qquad e_2^2 e_1 - [2]_1 e_2 e_1 e_2 + e_1 e_2^2 = 0,$$

where $[m]_1 = \langle m \rangle / \langle 1 \rangle$. Let b_1, b_2, b_3 be the generator for positive roots: $b_1 = e_2$, $b_2 = e_1 e_2 - q e_2 e_1$ and $b_3 = e_1$. In the notation of Section 2.2, they are the root vectors $b_i = e_{\beta_i}$ associated with the reduced expression $w_0 = s_2 s_1 s_2$ for $\mathbf{2} = (2, 1, 2)$. The corresponding positive roots are $(\beta_1, \beta_2, \beta_3) = (\alpha_2, \alpha_1 + \alpha_2, \alpha_1)$. In particular, $b_2 = T_2(e_1)$. Their commutation relations are $b_2 b_1 = q^{-1} b_1 b_2$, $b_3 b_1 = b_2 + q b_1 b_3$, $b_3 b_2 = q^{-1} b_2 b_3$.

Lemma 2. For $\tilde{E}_{\mathbf{2}}^{a,b,c} = b_1^a b_2^b b_3^c$, we have

$$\tilde{E}_{\mathbf{2}}^{a,b,c} \cdot e_1 = \tilde{E}_{\mathbf{2}}^{a,b,c+1},$$
$$\tilde{E}_{\mathbf{2}}^{a,b,c} \cdot e_2 = q^{c-b} \tilde{E}_{\mathbf{2}}^{a+1,b,c} + [c]_1 \tilde{E}_{\mathbf{2}}^{a,b+1,c-1},$$
$$e_1 \cdot \tilde{E}_{\mathbf{2}}^{a,b,c} = q^{a-b} \tilde{E}_{\mathbf{2}}^{a,b,c+1} + [a]_1 \tilde{E}_{\mathbf{2}}^{a-1,b+1,c},$$
$$e_2 \cdot \tilde{E}_{\mathbf{2}}^{a,b,c} = \tilde{E}_{\mathbf{2}}^{a+1,b,c}.$$

Proof. By induction, we have

$$b_3 b_1^n = q^n b_1^n b_3 + [n]_1 b_1^{n-1} b_2, \qquad b_3 b_2^n = q^{-n} b_2^n b_3,$$
$$b_3^n b_1 = q^n b_1 b_3^n + [n]_1 b_2 b_3^{n-1}, \qquad b_2^n b_1 = q^{-n} b_1 b_2^n.$$

The lemma is a direct consequence of these formulae.   ∎

Set $\tilde{E}_{\mathbf{1}}^{a,b,c} = \chi(\tilde{E}_{\mathbf{2}}^{c,b,a}) = \chi(b_3^a)\chi(b_2^b)\chi(b_1^c) = b_3^a b_2'^b b_1^c$, where $b_2' := \chi(b_2) = e_2 e_1 - q e_1 e_2$. By applying χ to the first two relations in Lemma 2, we get

$$e_1 \cdot \tilde{E}_{\mathbf{1}}^{a,b,c} = \tilde{E}_{\mathbf{1}}^{a+1,b,c}, \qquad e_2 \cdot \tilde{E}_{\mathbf{1}}^{a,b,c} = q^{a-b} E_{\mathbf{1}}^{a,b,c+1} + [a]_1 \tilde{E}_{\mathbf{1}}^{a-1,b+1,c}. \tag{38}$$

Thus we find $\rho_{\mathbf{i}'}(e_i) = \rho_{\mathbf{i}}(e_{3-i})$. This property is only valid for A_2 and not in C_2 and G_2.

Let us turn to the representations $\pi_{\mathbf{i}}$ of $A_q(A_2)$. The elements σ_i in Definition 1 and $\sigma_i e_i$ are given by

$$\sigma_1 = t_{13}, \qquad \sigma_2 = t_{12} t_{23} - q t_{22} t_{13}, \qquad \sigma_1 e_1 = t_{23}, \qquad \sigma_2 e_2 = t_{12} t_{33} - q t_{32} t_{13}. \tag{39}$$

See the exposition at the end of Section 3.3.1 and the remark after Lemma 1.

From (14) and (19) with $\alpha_i = \mu_i = 1$, we find

$$\pi_{\mathbf{1}}(\sigma_1) = \mathbf{k}_1\mathbf{k}_2, \qquad \pi_{\mathbf{1}}(\sigma_1 e_1) = \mathbf{a}_1^+\mathbf{k}_2, \qquad \pi_{\mathbf{1}}(\sigma_2) = \mathbf{k}_2\mathbf{k}_3, \qquad \pi_{\mathbf{1}}(\sigma_2 e_2) = \mathbf{a}_1^-\mathbf{a}_2^+\mathbf{k}_3 + \mathbf{k}_1\mathbf{a}_3^+,$$

where the notation like $\mathbf{k}_1\mathbf{a}_3^+ = \mathbf{k} \otimes 1 \otimes \mathbf{a}^+$ has been used. Since $\mathbf{k} \in \mathrm{End}(\mathcal{F}_q)$ is invertible, so is $\pi_{\mathbf{i}}(\sigma_i)$ and we may write

$$\pi_{\mathbf{1}}(\xi_1) = \lambda_1 \mathbf{a}_1^+\mathbf{k}_1^{-1}, \qquad \pi_{\mathbf{1}}(\xi_2) = \lambda_2(\mathbf{a}_1^-\mathbf{a}_2^+\mathbf{k}_2^{-1} + \mathbf{k}_1\mathbf{k}_2^{-1}\mathbf{a}_3^+\mathbf{k}_3^{-1}),$$

where $\lambda_1 = \lambda_2 = (1-q^2)^{-1}$. The action of each component on the ket vector $|m\rangle\!\rangle := d_{i,m}|m\rangle \in \mathcal{F}_{q_i}$ (cf. (32)) takes the form

$$\mathbf{a}^+|m\rangle\!\rangle = \lambda_i^{-1}q_i^m|m+1\rangle\!\rangle, \qquad \mathbf{a}^-|m\rangle\!\rangle = [m]_i|m-1\rangle\!\rangle, \qquad \mathbf{k}|m\rangle\!\rangle = q_i^m|m\rangle\!\rangle, \tag{40}$$

due to (17). (The formula (40) is valid also for C_2 and G_2 provided that $\mathbf{a}^+$, $\mathbf{a}^-$, $\mathbf{k}$ are interpreted as $\mathbf{A}^+$, $\mathbf{A}^-$, $\mathbf{K}$ for $i = 2$.) Thus one has

$$\pi_{\mathbf{1}}(\xi_1)|a,b,c\rangle\!\rangle = |a+1,b,c\rangle\!\rangle, \qquad \pi_{\mathbf{1}}(\xi_2)|a,b,c\rangle\!\rangle = [a]_1|a-1,b+1,c\rangle\!\rangle + q^{a-b}|a,b,c+1\rangle\!\rangle.$$

This agrees with (38) thereby proving (37) for $\mathbf{i} = \mathbf{1}$. The other case $\mathbf{i} = \mathbf{2}$ also holds due to the symmetry $\pi_{\mathbf{2}}(\xi_i) = \pi_{\mathbf{1}}(\xi_{3-i})$. Thus Proposition 2 is established for A_2.

In terms of the checked intertwiner $\mathcal{R}$ in Section 4.2, Theorem 5 implies

$$E_{\mathbf{i}}^{a,b,c} = \sum_{i,j,k} \mathcal{R}_{ijk}^{abc} E_{\mathbf{i}'}^{k,j,i}.$$

This is valid either for $\mathbf{i} = \mathbf{1}$ or $\mathbf{2}$ thanks to the middle property in (21). This relation connecting the PBW bases with the solution of the tetrahedron equation is due to [20].

5.3.2 C_2 case
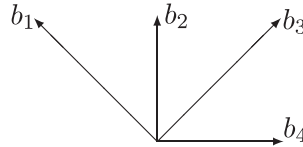
The q-Serre relations are

$$e_1^3 e_2 - [3]_1 e_1^2 e_2 e_1 + [3]_1 e_1 e_2 e_1^2 - e_2 e_1^3 = 0, \qquad e_2^2 e_1 - [2]_2 e_2 e_1 e_2 + e_1 e_2^2 = 0,$$

where $[m]_1 = \langle m\rangle/\langle 1\rangle$ and $[m]_2 = \langle 2m\rangle/\langle 2\rangle$.

Let $b_1, \ldots, b_4$ be the generator for positive roots: $b_1 = e_2$, $b_2 = e_1 e_2 - q^2 e_2 e_1$, $b_3 = \frac{1}{[2]_1}(e_1 b_2 - b_2 e_1)$ and $b_4 = e_1$. Their commutation relations are $b_2 b_1 = q^{-2} b_1 b_2$, $b_3 b_1 = -q^{-1}\langle 1\rangle[2]_1^{-1} b_2^2 + b_1 b_3$, $b_4 b_1 = b_2 + q^2 b_1 b_4$, $b_3 b_2 = q^{-2} b_2 b_3$, $b_4 b_2 = [2]_1 b_3 + b_2 b_4$, $b_4 b_3 = q^{-2} b_3 b_4$.

Lemma 3. For $\tilde{E}_{\mathbf{2}}^{a,b,c,d} = b_1^a b_2^b b_3^c b_4^d$, we have

$$\tilde{E}_{\mathbf{2}}^{a,b,c,d} \cdot e_1 = \tilde{E}_{\mathbf{2}}^{a,b,c,d+1},$$

$$\tilde{E}_{\mathbf{2}}^{a,b,c,d} \cdot e_2 = [d]_1 q^{d-2c-1} \tilde{E}_{\mathbf{2}}^{a,b+1,c,d-1} + q^{2(d-b)} \tilde{E}_{\mathbf{2}}^{a+1,b,c,d}$$
$$\qquad\qquad - \langle 1\rangle q^{2d-2c+1}[c]_2 [2]_1^{-1} \tilde{E}_{\mathbf{2}}^{a,b+2,c-1,d} + [d-1]_1 [d]_1 \tilde{E}_{\mathbf{2}}^{a,b,c+1,d-2},$$

$$e_1 \cdot \tilde{E}_{\mathbf{2}}^{a,b,c,d} = [2]_1 [b]_1 q^{2a-b+1} \tilde{E}_{\mathbf{2}}^{a,b-1,c+1,d} + q^{2a-2c} \tilde{E}_{\mathbf{2}}^{a,b,c,d+1} + [a]_2 \tilde{E}_{\mathbf{2}}^{a-1,b+1,c,d},$$

$$e_2 \cdot \tilde{E}_{\mathbf{2}}^{a,b,c,d} = \tilde{E}_{\mathbf{2}}^{a+1,b,c,d}.$$

Proof. By induction, we have

$$b_4 b_1^n = b_1^n b_4 q^{2n} + [n]_2 b_1^{n-1} b_2, \qquad b_4 b_2^n = [2]_1 [n]_1 b_2^{n-1} b_3 q^{-n+1} + b_2^n b_4,$$

$$b_4 b_3^n = q^{-2n} b_3^n b_4, \qquad b_4^n b_1 = [n]_1 b_2 b_4^{n-1} q^{n-1} + b_1 b_4^n q^{2n} + [n-1]_1 [n]_1 b_3 b_4^{n-2},$$

$$b_3^n b_1 = -q^{1-2n} \langle 1 \rangle [n]_2 [2]_1^{-1} b_2^2 b_3^{n-1} + b_1 b_3^n, \qquad b_3^n b_2 = q^{-2n} b_2 b_3^n, \qquad b_2^n b_1 = q^{-2n} b_1 b_2^n.$$

The lemma is a direct consequence of these formulae. ∎

Set $\tilde{E}_{\mathbf{1}}^{a,b,c,d} = \chi\big(\tilde{E}_{\mathbf{2}}^{d,c,b,a}\big)$. The left multiplication formula for this basis is deduced from the above lemma by applying χ. One can adjust the definition of $E_{\mathbf{i}}^A$ in (6) with that in [24] by setting $v = q^{-1}$.

Let us turn to the representations $\pi_{\mathbf{i}}$ of $A_q(C_2)$. The elements σ_i in Definition 1 and $\sigma_i e_i$ are given by

$$\sigma_1 = t_{14}, \qquad \sigma_2 = t_{13} t_{24} - q t_{23} t_{14}, \qquad \sigma_1 e_1 = t_{24}, \qquad \sigma_2 e_2 = t_{13} t_{34} - q t_{33} t_{14}. \tag{41}$$

From (14) and (23) with $\alpha_i = \mu_i = 1$, we have

$$\pi_{\mathbf{1}}(\sigma_1) = -\mathbf{k}_1 \mathbf{K}_2 \mathbf{k}_3,$$

$$\pi_{\mathbf{1}}(\sigma_1 e_1) = -\mathbf{a}_1^+ \mathbf{K}_2 \mathbf{k}_3,$$

$$\pi_{\mathbf{1}}(\sigma_2) = -\mathbf{K}_2 \mathbf{k}_3{}^2 \mathbf{K}_4,$$

$$\pi_{\mathbf{1}}(\sigma_2 e_2) = -\mathbf{a}_1^{-2} \mathbf{A}_2^+ \mathbf{k}_3{}^2 \mathbf{K}_4 - [2]_1 \mathbf{a}_1^- \mathbf{k}_1 \mathbf{a}_3^+ \mathbf{k}_3 \mathbf{K}_4 - \mathbf{k}_1{}^2 \mathbf{A}_2^- \mathbf{a}_3^{+2} \mathbf{K}_4 - \mathbf{A}_4^+ \mathbf{k}_1{}^2 \mathbf{K}_2,$$

$$\lambda_1^{-1} \pi_{\mathbf{1}}(\xi_1) = \mathbf{a}_1^+ \mathbf{k}_1^{-1},$$

$$\lambda_2^{-1} \pi_{\mathbf{1}}(\xi_2) = \mathbf{a}_1^{-2} \mathbf{A}_2^+ \mathbf{K}_2^{-1} + \mathbf{k}_1{}^2 \mathbf{A}_2^- \mathbf{K}_2^{-1} \mathbf{a}_3^{+2} \mathbf{k}_3^{-2}$$
$$+ [2]_1 \mathbf{a}_1^- \mathbf{k}_1 \mathbf{K}_2^{-1} \mathbf{a}_3^+ \mathbf{k}_3^{-1} + \mathbf{k}_1{}^2 \mathbf{k}_3^{-2} \mathbf{A}_4^+ \mathbf{K}_4^{-1},$$

$$\pi_{\mathbf{2}}(\sigma_1) = -\mathbf{k}_2 \mathbf{K}_3 \mathbf{k}_4,$$

$$\pi_{\mathbf{2}}(\sigma_1 e_1) = -\mathbf{K}_1 \mathbf{k}_2 \mathbf{a}_4^+ - \mathbf{K}_1 \mathbf{a}_2^- \mathbf{A}_3^+ \mathbf{k}_4 - \mathbf{A}_1^- \mathbf{a}_2^+ \mathbf{K}_3 \mathbf{k}_4,$$

$$\pi_{\mathbf{2}}(\sigma_2) = -\mathbf{K}_1 \mathbf{k}_2^2 \mathbf{K}_3,$$

$$\pi_{\mathbf{2}}(\sigma_2 e_2) = -\mathbf{A}_1^+ \mathbf{k}_2^2 \mathbf{K}_3,$$

$$\lambda_1^{-1} \pi_{\mathbf{2}}(\xi_1) = \mathbf{A}_1^- \mathbf{a}_2^+ \mathbf{k}_2^{-1} + \mathbf{K}_1 \mathbf{a}_2^- \mathbf{k}_2^{-1} \mathbf{A}_3^+ \mathbf{K}_3^{-1} + \mathbf{K}_1 \mathbf{K}_3^{-1} \mathbf{a}_4^+ \mathbf{k}_4^{-1},$$

$$\lambda_2^{-1} \pi_{\mathbf{2}}(\xi_2) = \mathbf{A}_1^+ \mathbf{K}_1^{-1}.$$

We find that $\pi_{\mathbf{i}}(\sigma_i)$ is invertible. Comparing these formulae with Lemma 3 by using (40), the equality (37) is directly checked. Thus Proposition 2 is established for C_2.

In terms of the checked intertwiner $\mathcal{K}$ in Section 4.3, Theorem 5 implies

$$E_{\mathbf{2}}^{a,b,c,d} = \sum_{i,j,k,l} \mathcal{K}_{ijkl}^{abcd} E_{\mathbf{1}}^{l,k,j,i}.$$

Thus the solution to the 3D reflection equation [13] is identified with the transition coefficient of the PBW bases for $U_q^+(C_2)$.

5.4 G_2 case
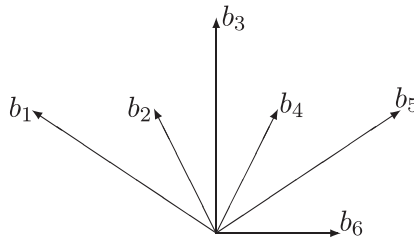
The q-Serre relations are

$$e_1^4 e_2 - [4]_1 e_1^3 e_2 e_1 + [4]_1[3]_1/[2]_1^{-1} e_1^2 e_2 e_1^2 - [4]_1 e_1 e_2 e_1^3 + e_2 e_1^4 = 0,$$
$$e_2^2 e_1 - [2]_2 e_2 e_1 e_2 + e_1 e_2^2 = 0,$$

where we remind that $[m]_1 = \langle m \rangle / \langle 1 \rangle$ and $[m]_2 = \langle 3m \rangle / \langle 3 \rangle$.

Let $b_1, \ldots, b_6$ be the generator for positive roots: $b_1 = e_2$, $b_2 = e_1 e_2 - q^3 e_2 e_1$, $b_4 = \frac{1}{[2]_1}(e_1 b_2 - q b_2 e_1)$, $b_5 = \frac{1}{[3]_1}(e_1 b_4 - q^{-1} b_4 e_1)$, $b_3 = \frac{1}{[3]_1}(b_4 b_2 - q^{-1} b_2 b_4)$ and $b_6 = e_1$. Their commutation relations are as follows: $b_2 b_1 = b_1 b_2 q^{-3}$, $b_3 b_1 = \langle 1 \rangle^2 b_2^3 q^{-3} [3]_1^{-1} + b_1 b_3 q^{-3}$, $b_4 b_1 = b_1 b_4 - b_2^2 \langle 1 \rangle q^{-1}$, $b_5 b_1 = b_1 b_5 q^3 - b_2 b_4 \langle 1 \rangle q^{-1} - (q^4 + q^2 - 1) b_3 q^{-3}$, $b_6 b_1 = b_1 b_6 q^3 + b_2$, $b_3 b_2 = b_2 b_3 q^{-3}$, $b_4 b_2 = b_2 b_4 q^{-1} + b_3 [3]_1$, $b_5 b_2 = b_2 b_5 - b_4^2 \langle 1 \rangle q^{-1}$, $b_6 b_2 = q b_2 b_6 + b_4 [2]_1$, $b_4 b_3 = b_3 b_4 q^{-3}$, $b_5 b_3 = \langle 1 \rangle^2 b_4^3 q^{-3} [3]_1^{-1} + b_3 b_5 q^{-3}$, $b_6 b_3 = b_3 b_6 - b_4^2 \langle 1 \rangle q^{-1}$, $b_5 b_4 = b_4 b_5 q^{-3}$, $b_6 b_4 = [3]_1 b_5 + b_4 b_6 q^{-1}$, $b_6 b_5 = b_5 b_6 q^{-3}$.

Lemma 4. *For $\tilde{E}_2^{a,b,c,d,e,f} = b_1^a b_2^b \cdots b_6^f$, we have*

$$\tilde{E}_2^{a,b,c,d,e,f} \cdot e_1 = \tilde{E}_2^{a,b,c,d,e,f+1},$$

$$\begin{aligned}
\tilde{E}_2^{a,b,c,d,e,f} \cdot e_2 = &-\langle 1 \rangle [e]_2 q^{-3c-d+3f-1} \tilde{E}_2^{a,b+1,c,d+1,e-1,f} \\
&+ \langle 1 \rangle^2 [e-1]_2 [e]_2 [3]_1^{-1} q^{-3e+3f+3} \tilde{E}_2^{a,b,c,d+3,e-2,f} \\
&- \langle 3 \rangle [d-1]_1 [d]_1 q^{-3c-2d+3e+3f+1} \tilde{E}_2^{a,b+1,c+1,d-2,e,f} \\
&- \langle 1 \rangle [d]_1 q^{-6c-d+3(e+f)} \tilde{E}_2^{a,b+2,c,d-1,e,f} + [f-1]_1 [f]_1 q^{-3e+f-2} \tilde{E}_2^{a,b,c,d+1,e,f-2} \\
&+ [3]_1 [d]_1 [f]_1 q^{2f-2d} \tilde{E}_2^{a,b,c+1,d-1,e,f-1} + [f]_1 q^{-3c-d+2f-2} \tilde{E}_2^{a,b+1,c,d,e,f-1} \\
&+ q^{-3(b+c-e-f)} \tilde{E}_2^{a+1,b,c,d,e,f} + \langle 1 \rangle^2 [c]_2 [3]_1^{-1} q^{3(-2c+e+f+1)} \tilde{E}_2^{a,b+3,c-1,d,e,f} \\
&- \langle 3 \rangle [d-2]_1 [d-1]_1 [d]_1 q^{3(-d+e+f+2)} \tilde{E}_2^{a,b,c+2,d-3,e,f} \\
&- \langle 1 \rangle [e]_2 [f]_1 q^{-3e+2f} \tilde{E}_2^{a,b,c,d+2,e-1,f-1} \\
&- [e]_2 q^{-3d+3f}(q^{2d+1}[3]_1 - [2]_2) \tilde{E}_2^{a,b,c+1,d,e-1,f} \\
&+ [f-2]_1 [f-1]_1 [f]_1 \tilde{E}_2^{a,b,c,d,e+1,f-3},
\end{aligned}$$

$$\begin{aligned}
e_1 \cdot \tilde{E}_2^{a,b,c,d,e,f} = &-\langle 1 \rangle [c]_2 q^{3a+b-3c+2} \tilde{E}_2^{a,b,c-1,d+2,e,f} + [3]_1 [b-1]_1 [b]_1 q^{3a-b+2} \tilde{E}_2^{a,b-2,c+1,d,e,f} \\
&+ [3]_1 [d]_1 q^{3a+b-2d+2} \tilde{E}_2^{a,b,c,d-1,e+1,f} + q^{3a+b-d-3e} \tilde{E}_2^{a,b,c,d,e,f+1} \\
&+ [2]_1 [b]_1 q^{3(a-c)} \tilde{E}_2^{a,b-1,c,d+1,e,f} + [a]_2 \tilde{E}_2^{a-1,b+1,c,d,e,f},
\end{aligned}$$

$$e_2 \cdot \tilde{E}_2^{a,b,c,d,e,f} = \tilde{E}_2^{a+1,b,c,d,e,f}.$$

Proof. By induction, we have

$$\begin{aligned}
b_6 b_1^n &= q^{3n} b_1^n b_6 + [n]_2 b_1^{n-1} b_2, \\
b_6 b_2^n &= [3]_1 q^{2-n} [n-1]_1 [n]_1 b_2^{n-2} b_3 + q^n b_2^n b_6 + [2]_1 [n]_1 b_2^{n-1} b_4, \\
b_4 b_3^n &= q^{-3n} b_3^n b_4, \qquad b_6 b_3^n = b_3^n b_6 - \langle 1 \rangle q^{2-3n} [n]_2 b_3^{n-1} b_4^2, \\
b_6 b_4^n &= [3]_1 q^{2-2n} [n]_1 b_4^{n-1} b_5 + q^{-n} b_4^n b_6, \qquad b_6 b_5^n = q^{-3n} b_5^n b_6,
\end{aligned}$$

and

$$\begin{aligned}
b_6^n b_1 = &\; q^{n-2} [n-1]_1 [n]_1 b_4 b_6^{n-2} + q^{3n} b_1 b_6^n \\
&+ q^{2(n-1)} [n]_1 b_2 b_6^{n-1} + [n-2]_1 [n-1]_1 [n]_1 b_5 b_6^{n-3}, \\
b_5^n b_1 = &\; \langle 1 \rangle^2 q^{-3(n-1)} [n-1]_2 [n]_2 [3]_1^{-1} b_4^3 b_5^{n-2} + q^{3n} b_1 b_5^n \\
&- q^{-3}(q^4 + q^2 - 1)[n]_2 b_3 b_5^{n-1} - q^{-1} \langle 1 \rangle [n]_2 b_2 b_4 b_5^{n-1},
\end{aligned}$$

$$b_5^n b_2 = b_2 b_5^n - \langle 1 \rangle q^{2-3n}[n]_2 b_4^2 b_5^{n-1}, \qquad b_5^n b_4 = q^{-3n} b_4 b_5^n,$$

$$b_4^n b_1 = -\langle 3 \rangle q^{6-3n}[n-2]_1[n-1]_1[n]_1 b_3^2 b_4^{n-3} - \langle 1 \rangle q^{-n}[n]_1 b_2^2 b_4^{n-1}$$
$$\qquad\qquad - \langle 3 \rangle q^{1-2n}[n-1]_1[n]_1 b_2 b_3 b_4^{n-2} + b_1 b_4^n,$$

$$b_4^n b_2 = [3]_1 q^{2-2n}[n]_1 b_3 b_4^{n-1} + q^{-n} b_2 b_4^n, \qquad b_4^n b_3 = q^{-3n} b_3 b_4^n,$$

$$b_3^n b_1 = q^{-3n} b_1 b_3^n + \langle 1 \rangle^2 q^{3-6n}[n]_2[3]_1^{-1} b_2^3 b_3^{n-1}, \qquad b_3^n b_2 = q^{-3n} b_2 b_3^n, \qquad b_2^n b_1 = q^{-3n} b_1 b_2^n.$$

The lemma is a direct consequence of these formulae.   ■

A part of the above results have also been obtained in [25].

Let us turn to the representations $\pi_{\mathbf{i}}$ of $A_q(G_2)$. The elements σ_i in Definition 1 and $\sigma_i e_i$ are given by

$$\sigma_1 = t_{17}, \qquad \sigma_2 = t_{26}t_{17} - qt_{27}t_{16}, \qquad \sigma_1 e_1 = t_{27}, \qquad \sigma_2 e_2 = t_{36}t_{17} - qt_{37}t_{16}. \qquad (42)$$

From (14) and (27) with $\alpha_i = \mu_i = 1$, we have

$$\pi_{\mathbf{1}}(\sigma_1) = \mathbf{k}_1 \mathbf{K}_2 \mathbf{k}_3^2 \mathbf{K}_4 \mathbf{k}_5,$$

$$\pi_{\mathbf{1}}(\sigma_2) = \mathbf{K}_2 \mathbf{k}_3^3 \mathbf{K}_4^2 \mathbf{k}_5^3 \mathbf{K}_6,$$

$$\pi_{\mathbf{1}}(\sigma_1 e_1) = \mathbf{a}_1^+ \mathbf{K}_2 \mathbf{k}_3^2 \mathbf{K}_4 \mathbf{k}_5,$$

$$\begin{aligned}
\pi_{\mathbf{1}}(\sigma_2 e_2) = {}& \mathbf{k}_1^3 \mathbf{K}_2^2 \mathbf{k}_3^3 \mathbf{K}_4 \mathbf{A}_6^+ + [2]_2 \mathbf{k}_1^3 \mathbf{A}_2^- \mathbf{K}_2 \mathbf{A}_4^+ \mathbf{K}_4 \mathbf{k}_5^3 \mathbf{K}_6 + \mathbf{a}_{1}^{-3} \mathbf{A}_2^+ \mathbf{k}_3^3 \mathbf{K}_4^2 \mathbf{k}_5^3 \mathbf{K}_6 \\
& + [3]_1 \mathbf{a}_1^{-2} \mathbf{k}_1 \mathbf{a}_3^+ \mathbf{k}_3^2 \mathbf{K}_4^2 \mathbf{k}_5^3 \mathbf{K}_6 + [3]_1 \mathbf{a}_1^- \mathbf{k}_1^2 \mathbf{K}_2 \mathbf{k}_3^2 \mathbf{K}_4 \mathbf{a}_5^+ \mathbf{k}_5^2 \mathbf{K}_6 \\
& - q[3]_1 \mathbf{k}_1^3 \mathbf{A}_2^- \mathbf{K}_2 \mathbf{k}_3^2 \mathbf{A}_4^+ \mathbf{K}_4 \mathbf{k}_5^3 \mathbf{K}_6 + [3]_1 \mathbf{k}_1^3 \mathbf{K}_2^2 \mathbf{a}_3^- \mathbf{k}_3^2 \mathbf{a}_5^{+2} \mathbf{k}_5 \mathbf{K}_6 + \mathbf{k}_1^3 \mathbf{K}_2^2 \mathbf{k}_3^3 \mathbf{A}_4^- \mathbf{a}_5^{+3} \mathbf{K}_6 \\
& + [3]_1 \mathbf{a}_1^- \mathbf{k}_1^2 \mathbf{A}_2^- \mathbf{a}_3^{+2} \mathbf{k}_3 \mathbf{K}_4^2 \mathbf{k}_5^3 \mathbf{K}_6 + [3]_1 \mathbf{a}_1^- \mathbf{k}_1^2 \mathbf{K}_2 \mathbf{a}_3^- \mathbf{k}_3 \mathbf{A}_4^+ \mathbf{K}_4 \mathbf{k}_5^3 \mathbf{K}_6 \\
& + \mathbf{k}_1^3 \mathbf{A}_2^{-2} \mathbf{a}_3^{+3} \mathbf{K}_4^2 \mathbf{k}_5^3 \mathbf{K}_6 + [3]_1 \mathbf{k}_1^3 \mathbf{A}_2^- \mathbf{K}_2 \mathbf{a}_3^+ \mathbf{k}_3 \mathbf{K}_4 \mathbf{a}_5^+ \mathbf{k}_5^2 \mathbf{K}_6 \\
& + \mathbf{k}_1^3 \mathbf{K}_2^2 \mathbf{a}_3^{-3} \mathbf{A}_4^{+2} \mathbf{k}_5^3 \mathbf{K}_6 + [3]_1 \mathbf{k}_1^3 \mathbf{K}_2^2 \mathbf{a}_3^{-2} \mathbf{k}_3 \mathbf{A}_4^+ \mathbf{a}_5^+ \mathbf{k}_5^2 \mathbf{K}_6,
\end{aligned}$$

$$\lambda_1^{-1} \pi_{\mathbf{1}}(\xi_1) = \mathbf{a}_1^+ \mathbf{k}_1^{-1},$$

$$\begin{aligned}
\lambda_2^{-1} \pi_{\mathbf{1}}(\xi_2) = {}& \mathbf{a}_1^{-3} \mathbf{A}_2^+ \mathbf{K}_2^{-1} + [2]_2 \mathbf{k}_1^3 \mathbf{A}_2^- \mathbf{k}_3^{-3} \mathbf{A}_4^+ \mathbf{K}_4^{-1} - q[3]_1 \mathbf{k}_1^3 \mathbf{A}_2^- \mathbf{k}_3^{-1} \mathbf{A}_4^+ \mathbf{K}_4^{-1} \\
& + [3]_1 \mathbf{a}_1^{-2} \mathbf{k}_1 \mathbf{K}_2^{-1} \mathbf{a}_3^+ \mathbf{k}_3^{-1} + [3]_1 \mathbf{a}_1^- \mathbf{k}_1^2 \mathbf{a}_3^- \mathbf{k}_3^{-2} \mathbf{A}_4^+ \mathbf{K}_4^{-1} + [3]_1 \mathbf{a}_1^- \mathbf{k}_1^2 \mathbf{k}_3^{-1} \mathbf{K}_4^{-1} \mathbf{a}_5^+ \mathbf{k}_5^{-1} \\
& + \mathbf{k}_1^3 \mathbf{K}_2 \mathbf{K}_4^{-1} \mathbf{k}_5^{-3} \mathbf{A}_6^+ \mathbf{K}_6^{-1} + [3]_1 \mathbf{a}_1^- \mathbf{k}_1^2 \mathbf{A}_2^- \mathbf{K}_2^{-1} \mathbf{a}_3^{+2} \mathbf{k}_3^{-2} \\
& + [3]_1 \mathbf{k}_1^3 \mathbf{A}_2^- \mathbf{a}_3^+ \mathbf{k}_3^{-2} \mathbf{K}_4^{-1} \mathbf{a}_5^+ \mathbf{k}_5^{-1} + \mathbf{k}_1^3 \mathbf{A}_2^{-2} \mathbf{K}_2^{-1} \mathbf{a}_3^{+3} \mathbf{k}_3^{-3} \\
& + [3]_1 \mathbf{k}_1^3 \mathbf{K}_2 \mathbf{a}_3^- \mathbf{k}_3^{-1} \mathbf{K}_4^{-2} \mathbf{a}_5^{+2} \mathbf{k}_5^{-2} + \mathbf{k}_1^3 \mathbf{K}_2 \mathbf{A}_4^- \mathbf{K}_4^{-2} \mathbf{a}_5^{+3} \mathbf{k}_5^{-3} \\
& + \mathbf{k}_1^3 \mathbf{K}_2 \mathbf{a}_3^{-3} \mathbf{k}_3^{-3} \mathbf{A}_4^{+2} \mathbf{K}_4^{-2} + [3]_1 \mathbf{k}_1^3 \mathbf{K}_2 \mathbf{a}_3^{-2} \mathbf{k}_3^{-2} \mathbf{A}_4^+ \mathbf{K}_4^{-2} \mathbf{a}_5^+ \mathbf{k}_5^{-1},
\end{aligned}$$

$$\pi_{\mathbf{2}}(\sigma_1) = \mathbf{k}_2 \mathbf{K}_3 \mathbf{k}_4^2 \mathbf{K}_5 \mathbf{k}_6,$$

$$\pi_{\mathbf{2}}(\sigma_2) = \mathbf{K}_1 \mathbf{k}_2^3 \mathbf{K}_3^2 \mathbf{k}_4^3 \mathbf{K}_5,$$

$$\begin{aligned}
\pi_{\mathbf{2}}(\sigma_1 e_1) = {}& \mathbf{K}_1 \mathbf{k}_2^2 \mathbf{K}_3 \mathbf{k}_4 \mathbf{a}_6^+ + \mathbf{A}_1^- \mathbf{a}_2^+ \mathbf{K}_3 \mathbf{k}_4^2 \mathbf{K}_5 \mathbf{k}_6 + \mathbf{K}_1 \mathbf{k}_2^2 \mathbf{K}_3 \mathbf{a}_4^- \mathbf{A}_5^+ \mathbf{k}_6 + \mathbf{K}_1 \mathbf{a}_2^{-2} \mathbf{A}_3^+ \mathbf{k}_4^2 \mathbf{K}_5 \mathbf{k}_6 \\
& + [2]_1 \mathbf{K}_1 \mathbf{a}_2^- \mathbf{k}_2 \mathbf{a}_4^+ \mathbf{k}_4 \mathbf{K}_5 \mathbf{k}_6 + \mathbf{K}_1 \mathbf{k}_2^2 \mathbf{A}_3^- \mathbf{a}_4^{+2} \mathbf{K}_5 \mathbf{k}_6,
\end{aligned}$$

$$\pi_{\mathbf{2}}(\sigma_2 e_2) = \mathbf{A}_1^+ \mathbf{k}_2^3 \mathbf{K}_3^2 \mathbf{k}_4^3 \mathbf{K}_5,$$

$$\begin{aligned}
\lambda_1^{-1} \pi_{\mathbf{2}}(\xi_1) = {}& \mathbf{A}_1^- \mathbf{a}_2^+ \mathbf{k}_2^{-1} + [2]_1 \mathbf{K}_1 \mathbf{a}_2^- \mathbf{K}_3^{-1} \mathbf{a}_4^+ \mathbf{k}_4^{-1} + \mathbf{K}_1 \mathbf{a}_2^{-2} \mathbf{k}_2^{-1} \mathbf{A}_3^+ \mathbf{K}_3^{-1} + \mathbf{K}_1 \mathbf{k}_2 \mathbf{a}_4^- \mathbf{k}_4^{-2} \mathbf{A}_5^+ \mathbf{K}_5^{-1} \\
& + \mathbf{K}_1 \mathbf{k}_2 \mathbf{k}_4^{-1} \mathbf{K}_5^{-1} \mathbf{a}_6^+ \mathbf{k}_6^{-1} + \mathbf{K}_1 \mathbf{k}_2 \mathbf{A}_3^- \mathbf{K}_3^{-1} \mathbf{a}_4^{+2} \mathbf{k}_4^{-2},
\end{aligned}$$

$$\lambda_2^{-1} \pi_{\mathbf{2}}(\xi_2) = \mathbf{A}_1^+ \mathbf{K}_1^{-1}.$$

We find that $\pi_{\mathbf{i}}(\sigma_i)$ is invertible. Comparing these formulae with Lemma 4 by using (40), the equality (37) is directly checked. Thus Proposition 2 is established for G_2.

In terms of the checked intertwiner $\mathcal{F}$ in Section 4.4, Theorem 5 implies

$$E_{\mathbf{2}}^{a,b,c,d,e,f} = \sum_{i,j,k,l,m,n} \mathcal{F}_{ijklmn}^{abcdef} E_{\mathbf{1}}^{n,m,l,k,j,i}.$$

6 Discussion

In view of Proposition 2 it is natural to expect that the map defined on generators of $U_q^+(\mathfrak{g})$ as $e_i \mapsto \eta_i := \sigma_i e_i/\sigma_i$ extends to an algebra homomorphism from $U_q^+(\mathfrak{g})$ to $A_q(\mathfrak{g})_{\mathcal{S}}$, namely, η_i satisfies q-Serre relations. In fact, it is true not only for rank 2 cases but also for any $\mathfrak{g}$.

Theorem 6. In $A_q(\mathfrak{g})_{\mathcal{S}}$ the following relation holds for any i, j $(i \neq j)$:

$$\sum_{r=0}^{1-a_{ij}} (-1)^r \eta_i^{(r)} \eta_j \eta_i^{(1-a_{ij}-r)} = 0.$$

Proof. By relabeling of Dynkin indices we can assume $i = 1$, $j = 2$. Set $\tau_i = \sigma_i e_i$ for $i = 1, 2$. Then from Proposition 1 we have

$$\sigma_i \tau_i = q_i \tau_i \sigma_i, \quad i = 1, 2, \qquad \sigma_i \tau_j = \tau_j \sigma_i, \quad i, j = 1, 2; \quad i \neq j. \tag{43}$$

Using (12) with these relations one verifies

$$\eta_1^r \eta_2 \eta_1^s = q_1^{(r+s)(r+s-1)/2} \left(\tau_1^r \tau_2 \tau_1^s\right)/\left(\sigma_1^r \sigma_2 \sigma_1^s\right).$$

Here we have set $s = 1 - a_{12} - r$. Recalling that σ_1 and σ_2 commute with each other, we can reduce the claim to showing

$$Z := \sum_{r=0}^{1-a_{12}} (-1)^r \tau_1^{(r)} \tau_2 \tau_1^{(s)} = 0.$$

Note that the right (resp. left) weight of Z is $(1 - a_{12})(\varpi_1 - \alpha_1) + (\varpi_2 - \alpha_2)$ (resp. $w_0((1 - a_{12})\varpi_1 + \varpi_2))$. The two weights are not related by the longest element $w_0 \in W$. Hence if we show $f_i Z = Z f_i = 0$ for any i, we can conclude $Z = 0$ by the remark after Lemma 1. The properties $f_i Z = 0$ for any i and $Z f_i = 0$ for $i \neq 1, 2$ are trivial.

First we show $Z f_2 = 0$. We have

$$\left(\tau_1^r \tau_2 \tau_1^s\right) f_2 = \tau_1^r (\tau_2 f_2)\left(\tau_1 k_2^{-1}\right)^s = \tau_1^r \sigma_2 (\beta \tau_1)^s = \beta^s \tau_1^{r+s} \sigma_2,$$

where $\beta = q_2^{-\langle h_2, \varpi_1 - \alpha_1 \rangle} = q_2^{a_{21}} = q_1^{a_{12}}$ and we have used (43). Hence,

$$Z f_2 = \left(\sum_{r+s=1-a_{12}} \frac{(-q_1^{-a_{12}})^s}{[r]_1![s]_1!} \right) (-\tau_1)^{1-a_{12}} \sigma_2 = 0.$$

In the last equality we have used the following formula:

$$\sum_{i=0}^m (-z)^i \begin{bmatrix} m \\ i \end{bmatrix} = \prod_{j=1}^m \left(1 - zq^{2j-m-1}\right),$$

where $\begin{bmatrix} m \\ i \end{bmatrix} = [m]!/([i]![m-i]!)$.

Next, we show $Zf_1 = 0$.

$$\left(\tau_1^r \tau_2 \tau_1^s\right) f_1 = \sum_{i=1}^{r} \tau_1^{r-i} \sigma_1 \left(\tau_1 k_1^{-1}\right)^{i-1} \left(\tau_2 k_1^{-1}\right) \left(\tau_1 k_1^{-1}\right)^s + \tau_1^r \tau_2 \sum_{i=1}^{s} \tau_1^{s-i} \sigma_1 \left(\tau_1 k_1^{-1}\right)^{i-1}$$

$$= \sum_{i=1}^{r} \delta \gamma^{i-1-s} \tau_1^{r-1} \tau_2 \tau_1^s \sigma_1 + \sum_{i=1}^{s} \gamma^{i-1} \tau_1^r \tau_2 \tau_1^{s-1} \sigma_1,$$

where constants γ, δ are determined by $\sigma_1\left(\tau_1 k_1^{-1}\right) = \gamma \tau_1 \sigma_1$, $\sigma_1\left(\tau_2 k_1^{-1}\right) = \delta \tau_2 \sigma_1$ and hence we have $\gamma = q_1 q_1^{-\langle h_1, \varpi_1 - \alpha_1 \rangle} = q_1^2$, $\delta = q_1^{-\langle h_1, \varpi_2 - \alpha_2 \rangle} = q_1^{a_{12}}$. Then, we obtain

$$Zf_1 = \sum_{r+s=1-a_{12}} \frac{(-1)^r}{[r]_1![s]_1!} \left(\sum_{i=1}^{r} \delta \gamma^{i-1+s} \tau_1^{r-1} \tau_2 \tau_1^s \sigma_1 + \sum_{i=1}^{s} \gamma^{i-1} \tau_1^r \tau_2 \tau_1^{s-1} \sigma_1 \right)$$

$$= \sum_{r+s=1-a_{12}} \frac{(-1)^r}{[r]_1![s]_1!} \left(\delta \gamma^s \frac{1-\gamma^r}{1-\gamma} \tau_1^{r-1} \tau_2 \tau_1^s \sigma_1 + \frac{1-\gamma^s}{1-\gamma} \tau_1^r \tau_2 \tau_1^{s-1} \sigma_1 \right)$$

$$= \sum_{r+s=1-a_{12}} \left(-(-1)^{r-1} q_1^s \tau_1^{(r-1)} \tau_2 \tau_1^{(s)} + (-1)^r q_1^{s-1} \tau_1^{(r)} \tau_2 \tau_1^{(s-1)} \right) \sigma_1 = 0$$

as desired.   ∎

Remark 1. The special case $w = w_0$ of [26, Theorem 3.7] gives Theorem 6 here. Moreover [26, Theorem 3.7] also shows that $U_q^+(\mathfrak{g})$ is isomorphic to an explicit subalgebra of $A_q(\mathfrak{g})_{\mathcal{S}}$. We would like to thank the referee for pointing this out and for giving helpful comments.

It will be interesting to investigate it further in the light of the quantum cluster algebra which has been recognized as a fundamental structure in the quantized algebra of functions [6]. The representations via multiplication on PBW bases also play a fundamental role in the study of the positive principal series representations and modular double [7].

In this paper we have not discussed the analogue of the tetrahedron and 3D reflection equations for general $\mathfrak{g}$. However, from our proof of Theorem 5, we expect that the basic constituents are R and K only, and their compatibility condition is reduced to the rank 2 cases (2) and (3).

Acknowledgments

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