The Clifford Deformation of the Hermite Semigroup

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Abstract. This paper is a continuation of the paper [De Bie H., Ørsted B., Somberg P., Souček V., *Trans. Amer. Math. Soc.* **364** (2012), 3875–3902], investigating a natural radial deformation of the Fourier transform in the setting of Clifford analysis. At the same time, it gives extensions of many results obtained in [Ben Saïd S., Kobayashi T., Ørsted B., *Compos. Math.* **148** (2012), 1265–1336]. We establish the analogues of Bochner's formula and the Heisenberg uncertainty relation in the framework of the (holomorphic) Hermite semigroup, and also give a detailed analytic treatment of the series expansion of the associated integral transform.

Key words: Dunkl operators; Clifford analysis; generalized Fourier transform; Laguerre polynomials; Kelvin transform; holomorphic semigroup

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1 Introduction

It is well-known that the classical Dirac operator and its Fourier symbol generate via Clifford multiplication a natural Lie superalgebra $\mathfrak{osp}(1|2)$ contained in the Clifford–Weyl algebra. More surprisingly, this carries over to a natural family of deformations of the Dirac operator, see [7]. Moreover, it is possible to define a Fourier transform naturally associated to the deformed family.

The novelty of the present article is that we let group theory be the guiding principle in defining operators and transformations, in the next step followed by a study of explicit (analytic) properties for naturally arising eigenfunctions and kernel functions. Thus the main aim is to find the kernel function for the Fourier transform connected with our deformation, and also to study its associated holomorphic semigroup regarded as a particular descendant of the Gelfand–Gindikin program analyzing representations of reductive Lie groups, see, e.g., [22] and the discussion in [2].

Let us now recall the basic setup and results from [7] and also discuss further aspects of our construction. The deformation family of Dunkl–Dirac operators

$$\mathbf{D} = r^{1-\frac{a}{2}}\mathcal{D}_\kappa + br^{-\frac{a}{2}-1}\underline{x} + cr^{-\frac{a}{2}-1}\underline{x}\mathbb{E}, \qquad a,b,c \in \mathbb{R},$$

together with the radial deformation of the coordinate function

$$\underline{x}_a = r^{\frac{a}{2}-1}\underline{x}, \qquad r = \sqrt{\sum_{i=1}^m x_i^2},$$

forms a realization of $\mathfrak{osp}(1|2)$ in the Clifford–Weyl algebra. Here $\mathcal{D}_\kappa = \sum_{i=1}^{m} e_i T_i$ with T_i the Dunkl operators, $\underline{x} = \sum_{i=1}^{m} e_i x_i$ and $\mathbb{E} = \sum_{i=1}^{m} x_i \partial_{x_i}$. The e_i are generators of the Clifford algebra $\mathcal{C}l_m$. See also the next section for more details.

We will show in Proposition 3.2 that this realization builds a Howe dual pair with $\widetilde{\mathcal{G}}$. Here the group $\widetilde{\mathcal{G}}$ is the double cover (contained in the Pin group) of the finite reflection group $\mathcal{G}$ used in the construction of the Dunkl operators.

The Fourier transform is then defined by

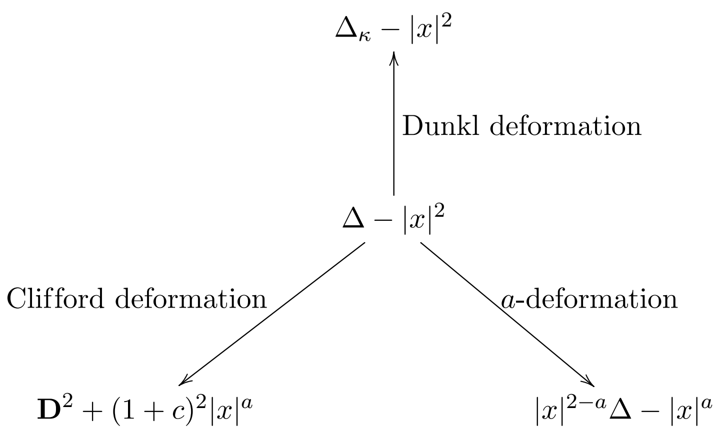
$$\mathcal{F}_{\mathbf{D}} = e^{i\frac{\pi}{2}\left(\frac{1}{2}+\frac{\mu-1}{a(1+c)}\right)} e^{\frac{-i\pi}{2a(1+c)^2}\left(\mathbf{D}^2-(1+c)^2\underline{x}_a^2\right)},$$

where $L = \mathbf{D}^2 - (1+c)^2\underline{x}_a^2$ is the generalized Hamiltonian and μ the Dunkl dimension. The main aim of the present paper is to find an integral expression for this Fourier transform,

$$\mathcal{F}_{\mathbf{D}}(f)(y) = \int_{\mathbb{R}^m} K(x,y) f(x) h(r_x) dx$$

with $h(r_x)dx$ the measure associated to $\mathbf{D}$ and $K(x,y)$ the integral kernel to be determined. Note that this ties in with recent work on generalized Fourier transforms in different contexts, e.g., analysis on minimal representations of reductive groups (see [19, 20, 21]) or integral transforms in Clifford analysis (see [6, 8]).

The deformation of the classical Hamiltonian for the harmonic oscillator is visualized in the following figure:



The Dunkl deformation is by now quite standard and described for example in [11]. The a-deformation is the subject of the paper [2] and is a scalar radial deformation of the harmonic oscillator. Our Clifford deformation is also a radial deformation but richer in the sense that Clifford algebra- (or spinor)-valued functions are involved.

In this paper we will thus find a series representation of the kernel function for our new Fourier transform $\mathcal{F}_{\mathbf{D}}$, and also study the holomorphic semigroup with generator L. The main results are Theorem 6.1 on the operator properties of the semigroup, Theorem 7.2 on the Fourier transform intertwining the Dirac operator and the Clifford multiplication, Proposition 7.2 on the Bochner identities, and Proposition 7.3 on the Heisenberg uncertainty relation. Finally in Theorem 7.3 we give the analogue of what is sometimes called the "Master formula" in the context of Dunkl operators (see, e.g., [26, Lemma 4.5(1)] or [4]).

The paper is organized as follows. In Section 2 we repeat basic notions on Clifford algebras and Dunkl operators needed in the rest of the paper. In Section 3 we construct intertwining operators to reduce our radially deformed Dirac operator to its simplest form. Subsequently, in Section 4 we discuss the representation theoretic content of our deformation and solve the spectral problem of the associated Hamiltonian. In Section 5, we obtain the reproducing kernels for spaces of spherical monogenics, which allows us to construct the kernel of the holomorphic semigroup in Section 6. Section 7 contains the results on the (deformed) Fourier transform. Further properties are collected in Section 8. Finally, we summarize some results on special functions used in the paper in Appendix A and give a list of notations in Appendix B.

2 Preliminaries

In this section we collect some basic results on Clifford algebras and Dunkl operators.

2.1 Clifford algebras

Let $\mathbb{V}$ be a vector space of dimension m with a given negative definite quadratic form and let $\mathcal{C}l_m$ be the corresponding Clifford algebra. If $\{e_i\}$ is an orthonormal basis of $\mathbb{V}$, then $\mathcal{C}l_m$ is generated by e_i, $i = 1, \ldots, m$, with the relations

$$e_ie_j + e_ie_j = 0, \quad i \neq j, \qquad e_i^2 = -1.$$

The algebra $\mathcal{C}l_m$ has dimension 2^m as a vector space over $\mathbb{R}$. It can be decomposed as $\mathcal{C}l_m = \oplus_{k=0}^{m} \mathcal{C}l_m^k$ with $\mathcal{C}l_m^k$ the space of k-vectors defined by

$$\mathcal{C}l_m^k := \operatorname{span}\{e_{i_1} \cdots e_{i_k},\ i_1 < \cdots < i_k\}.$$

The projection on the space of k-vectors is denoted by $[\cdot]_k$.

The operator $\bar{\cdot}$ is the main anti-involution on the Clifford algebra $\mathcal{C}l_m$ defined by

$$\overline{ab} = \bar{b}\bar{a}, \qquad \overline{e_i} = -e_i, \quad i = 1, \ldots, m.$$

Similarly we have the automorphism ϵ given by

$$\epsilon(ab) = \epsilon(a)\epsilon(b), \qquad \epsilon(e_i) = -e_i, \quad i = 1, \ldots, m.$$

In the sequel, we will always consider functions f taking values in $\mathcal{C}l_m$, unless explicitly mentioned. Such functions can be decomposed as

$$f(x) = f_0(x) + \sum_{i=1}^{m} e_i f_i(x) + \sum_{i<j} e_ie_j f_{ij}(x) + \cdots + e_1 \cdots e_m f_{1\ldots m}(x)$$

with $f_0, f_i, f_{ij}, \ldots, f_{1\ldots m}$ all real-valued functions.

Several important groups can be embedded in the Clifford algebra. Note that the space of 1-vectors in $\mathcal{C}l_m$ is canonically isomorphic to $\mathbb{V} \cong \mathbb{R}^m$. Hence we can define

$$\operatorname{Pin}(m) = \left\{s_1 s_2 \cdots s_n \,|\, n \in \mathbb{N}, s_i \in \mathcal{C}l_m^1 \text{ such that } s_i^2 = -1\right\},$$

i.e., the Pin group is the group of products of unit vectors in $\mathcal{C}l_m$. This group is a double cover of the orthogonal group $\mathrm{O}(m)$ with covering map $p : \operatorname{Pin}(m) \to \mathrm{O}(m)$, which we will describe explicitly in the next section.

Similarly we define

$$\operatorname{Spin}(m) = \left\{s_1 s_2 \cdots s_{2n} \,|\, n \in \mathbb{N}, s_i \in \mathcal{C}l_m^1 \text{ such that } s_i^2 = -1\right\},$$

i.e., the Spin group is the group of even products of unit vectors in $\mathcal{C}l_m$. This group is a double cover of $\mathrm{SO}(m)$. For more information about Clifford algebras and analysis, we refer the reader to [9, 16].

2.2 Dunkl operators

Denote by $\langle\cdot,\cdot\rangle$ the standard Euclidean scalar product in $\mathbb{R}^m$ and by $|x| = \langle x, x\rangle^{1/2}$ the associated norm. For $\alpha \in \mathbb{R}^m \setminus \{0\}$, the reflection r_α in the hyperplane orthogonal to α is given by

$$r_\alpha(x) = x - 2\frac{\langle \alpha, x\rangle}{|\alpha|^2}\alpha, \qquad x \in \mathbb{R}^m.$$

A root system is a finite subset $R \subset \mathbb{R}^m$ of non-zero vectors such that, for every $\alpha \in R$, the associated reflection r_α preserves R. We will assume that R is reduced, i.e. $R \cap \mathbb{R}\alpha = \{\pm\alpha\}$ for all $\alpha \in R$. Each root system can be written as a disjoint union $R = R_+ \cup (-R_+)$, where R_+ and $-R_+$ are separated by a hyperplane through the origin. R_+ is called a positive subsystem of the root system R. The subgroup $\mathcal{G} \subset \mathrm{O}(m)$ generated by the reflections $\{r_\alpha | \alpha \in R\}$ is called the finite reflection group associated with R. We will also assume that R is normalized such that $\langle\alpha,\alpha\rangle = 2$ for all $\alpha \in R$. For more information on finite reflection groups we refer the reader to [18].

If we identify α with a 1-vector in $\mathcal{C}l_m$ (and hence $\alpha/\sqrt{2}$ with an element in $\mathrm{Pin}(m)$), we can rewrite the reflection r_α as

$$r_\alpha(x) = \frac{1}{2}\alpha \underline{x} \alpha$$

with $\underline{x} = \sum_{i=1}^m e_i x_i$. Generalizing this map gives us the covering map p from $\mathrm{Pin}(m)$ to $\mathrm{O}(m)$ as

$$p(s)(x) = \epsilon(s)\underline{x}s^{-1}, \qquad s \in \mathrm{Pin}(m).$$

In particular, we obtain a double cover of the reflection group $\mathcal{G}$ as $\widetilde{\mathcal{G}} = p^{-1}(\mathcal{G})$ (see also the discussion in [1]).

A multiplicity function κ on the root system R is a $\mathcal{G}$-invariant function $\kappa : R \to \mathbb{C}$, i.e. $\kappa(\alpha) = \kappa(h\alpha)$ for all $h \in \mathcal{G}$. We will denote $\kappa(\alpha)$ by κ_α. We will always assume that the multiplicity function is real and satisfies $\kappa \geq 0$. This assumption is, e.g., necessary to obtain the subsequent formula (2.1), which is crucial for the sequel.

Fixing a positive subsystem R_+ of the root system R and a multiplicity function κ, we introduce the Dunkl operators T_i associated to R_+ and κ by (see [10, 13])

$$T_i f(x) = \partial_{x_i} f(x) + \sum_{\alpha \in R_+} \kappa_\alpha \alpha_i \frac{f(x) - f(r_\alpha(x))}{\langle \alpha, x\rangle}, \qquad f \in C^1(\mathbb{R}^m).$$

An important property of the Dunkl operators is that they commute, i.e. $T_i T_j = T_j T_i$.

The Dunkl Laplacian is given by $\Delta_\kappa = \sum_{i=1}^m T_i^2$, or more explicitly by

$$\Delta_\kappa f(x) = \Delta f(x) + 2\sum_{\alpha\in R_+} \kappa_\alpha \left(\frac{\langle \nabla f(x), \alpha\rangle}{\langle \alpha, x\rangle} - \frac{f(x) - f(r_\alpha(x))}{\langle \alpha, x\rangle^2} \right)$$

with Δ the classical Laplacian and ∇ the gradient operator. We also define the constant

$$\mu = \frac{1}{2}\Delta_\kappa |x|^2 = m + 2\sum_{\alpha\in R_+} \kappa_\alpha,$$

called the Dunkl-dimension.

It is possible to construct an intertwining operator V_κ connecting the classical derivatives ∂_{x_j} with the Dunkl operators T_j such that $T_j V_\kappa = V_\kappa \partial_{x_j}$ (see, e.g., [12]). Note that explicit formulae for V_κ are only known in a few special cases.

The weight function related to the root system R and the multiplicity function κ is given by $w_\kappa(x) = \prod_{\alpha \in R_+} |\langle \alpha, x\rangle|^{2\kappa_\alpha}$. For suitably chosen functions f and g one then has the following property of integration by parts (see [11])

$$\int_{\mathbb{R}^m} (T_i f) g w_\kappa(x) dx = -\int_{\mathbb{R}^m} f\,(T_i g)\, w_\kappa(x) dx. \tag{2.1}$$

For more information about Dunkl operators we refer the reader to [13, 25].

The starting point in the subsequent analysis is the Dunkl–Dirac operator, given by

$$\mathcal{D}_\kappa = \sum_{i=1}^{m} e_i T_i.$$

Together with the vector variable $\underline{x} = \sum_{i=1}^{m} e_i x_i$ this Dunkl–Dirac operator generates a copy of $\mathfrak{osp}(1|2)$, see [23] or the subsequent Theorem 3.1. In particular, we have

$$\mathcal{D}_\kappa^2 = -\Delta_\kappa \qquad \text{and} \qquad \underline{x}^2 = -|\underline{x}|^2 = -r^2 = -\sum_{i=1}^{m} x_i^2.$$

3 Intertwining operators

Let, for $a, b \in \mathbb{R}$, P and Q be two operators defined by

$$Pf(\underline{x}) = r^b f\left(\left(\frac{a}{2}\right)^{\frac{1}{a}} \underline{x} r^{\frac{2}{a}-1}\right), \qquad Qf(\underline{x}) = r^{-\frac{ab}{2}} f\left(\left(\frac{2}{a}\right)^{\frac{1}{2}} \underline{x} r^{\frac{a}{2}-1}\right).$$

These two operators act as generalized Kelvin transformations. Indeed, one can easily compute their composition

$$QP = PQ = \left(\frac{2}{a}\right)^{\frac{b}{2}}.$$

We will show that these operators allow to reduce the Dirac operator $\mathbf{D}$ to a simpler form.

We have the following proposition, where $\mathbb{E} = \sum_{i=1}^{m} x_i \partial_{x_i}$ denotes the Euler operator. Recall also from the introduction that $\underline{x}_a = r^{\frac{a}{2}-1}\underline{x}$.

Proposition 3.1. *One has the following intertwining relations*

$$\left(\frac{a}{2}\right)^{\frac{b-1}{2}} Q\left(\mathcal{D}_\kappa + br^{-2}\underline{x} + cr^{-2}\underline{x}\mathbb{E}\right) P = r^{1-\frac{a}{2}}\mathcal{D}_k + \beta r^{-\frac{a}{2}-1}\underline{x} + \gamma r^{-\frac{a}{2}-1}\underline{x}\mathbb{E},$$
$$\left(\frac{a}{2}\right)^{\frac{b+1}{2}} Q\underline{x}P = \underline{x}_a$$

with $\beta = 2b + bc$, $\gamma = \frac{2}{a}(1+c) - 1$.

Proof. In [7, Proposition 3], we already proved that

$$\left(\frac{a}{2}\right)^{\frac{b-1}{2}} Q\left(\mathcal{D}_\kappa\right) P = r^{1-\frac{a}{2}}\mathcal{D}_k + br^{-\frac{a}{2}-1}\underline{x} + \left(\frac{2}{a}-1\right) r^{-\frac{a}{2}-1}\underline{x}\mathbb{E} = \underline{x}_a.$$

Similarly we obtain

$$\left(\frac{a}{2}\right)^{\frac{b-1}{2}} Q\left(r^{-2}\underline{x}\right) P = r^{-\frac{a}{2}-1}\underline{x}$$

and

$$\left(\frac{a}{2}\right)^{\frac{b-1}{2}} Q\left(r^{-2}\underline{x}\mathbb{E}\right) P = br^{-\frac{a}{2}-1}\underline{x} + \left(\frac{2}{a}\right) r^{-\frac{a}{2}-1}\underline{x}\mathbb{E}.$$

This completes the proof of the proposition. ■

So we are reduced to the study of the operator

$$\mathbf{D} = \mathcal{D}_\kappa + br^{-2}\underline{x} + cr^{-2}\underline{x}\mathbb{E},$$

where $b, c \in \mathbb{R}$, $c \neq -1$. Here, the term $br^{-2}\underline{x}$ can also be removed. Indeed, we have

$$r^{-\alpha}\left(\mathcal{D}_\kappa + br^{-2}\underline{x} + cr^{-2}\underline{x}\mathbb{E}\right) r^{\alpha} = \mathcal{D}_\kappa + cr^{-2}\underline{x}\mathbb{E},$$

when $\alpha = -b/(1+c)$.

As a result of the previous discussion, we see that it is sufficient to study the function theory for the operator

$$\mathbf{D} = \mathcal{D}_\kappa + cr^{-2}\underline{x}\mathbb{E},$$

where we have put $a = 2$, $b = 0$. Furthermore, we will restrict ourselves to the case $c > -1$ for reasons that will become clear in Proposition 3.3. Similarly, we no longer need to consider $\underline{x}_a$ but can restrict ourselves to $\underline{x}$. Now we repeat the basic facts concerning this operator we need in the sequel. All the results are taken from [7], putting $a = 2$, $b = 0$.

Theorem 3.1. *The operators $\boldsymbol{D}$ and $\underline{x}$ generate a Lie superalgebra, isomorphic to $\mathfrak{osp}(1|2)$, with the following relations*

$$\begin{aligned}
&\{\underline{x}, \boldsymbol{D}\} = -2(1+c)\left(\mathbb{E}+\frac{\delta}{2}\right), && \left[\mathbb{E}+\frac{\delta}{2}, \boldsymbol{D}\right] = -\boldsymbol{D},\\
&\left[\underline{x}^2, \boldsymbol{D}\right] = 2(1+c)\underline{x}, && \left[\mathbb{E}+\frac{\delta}{2}, \underline{x}\right] = \underline{x},\\
&\left[\boldsymbol{D}^2, \underline{x}\right] = -2(1+c)\boldsymbol{D}, && \left[\mathbb{E}+\frac{\delta}{2}, \boldsymbol{D}^2\right] = -2\boldsymbol{D}^2,\\
&\left[\boldsymbol{D}^2, \underline{x}^2\right] = 4(1+c)^2\left(\mathbb{E}+\frac{\delta}{2}\right), && \left[\mathbb{E}+\frac{\delta}{2}, \underline{x}^2\right] = 2\underline{x}^2,
\end{aligned}$$

where $\delta = 1 + \frac{\mu-1}{1+c}$.

Note that the square of $\mathbf{D}$ is a complicated operator, given by

$$\mathbf{D}^2 = -\Delta_\kappa - (c\mu)\, r^{-1}\partial_r - \left(c^2+2c\right)\partial_r^2 + cr^{-2}\sum_i x_i T_i - cr^{-2}\sum_{i<j} e_i e_j (x_i T_j - x_j T_i).$$

If $\kappa = 0$, the formula for $\mathbf{D}^2$ simplifies a bit as now $\sum_i x_i T_i = r\partial_r = \mathbb{E}$.

Remark 3.1. The operator $\mathbf{D} = \mathcal{D}_\kappa + cr^{-2}\underline{x}\mathbb{E}$ is also considered from a very different perspective in [3] (in the case $\kappa = 0$), where the eigenfunctions of this operator are studied.

Let us now discuss the symmetry of the generators of $\mathfrak{osp}(1|2)$. First we define the action of the Pin group on $C^\infty(\mathbb{R}^m) \otimes \mathcal{C}l_m$ for $s \in \mathrm{Pin}(m)$ as

$$\rho(s): \; C^\infty(\mathbb{R}^m) \otimes \mathcal{C}l_m \to C^\infty(\mathbb{R}^m) \otimes \mathcal{C}l_m, \qquad f \otimes b \to f\big(p\big(s^{-1}\big)x\big) \otimes sb.$$

We then have

Proposition 3.2. *Let $s \in \widetilde{\mathcal{G}}$ and define $\mathrm{sgn}(s) := \mathrm{sgn}(p(s))$. Then one has*

$$\rho(s)\underline{x} = \mathrm{sgn}(s)\underline{x}\rho(s), \qquad \rho(s)\boldsymbol{D} = \mathrm{sgn}(s)\boldsymbol{D}\rho(s).$$

Proof. This follows immediately from the definition of ρ and the $\mathcal{G}$-equivariance of the Dunkl operators. ■

So up to sign, the Dirac operator $\mathbf{D}$ is $\widetilde{\mathcal{G}}$-equivariant. At this point it is interesting to remark that an algebraic analog of the Dunkl–Dirac operator $\mathbf{D}$ for graded affine Hecke algebras is introduced in [1] with the motivation to prove a version of Vogan's Conjecture for Dirac cohomology. The formulation is based on a uniform geometric parametrization of spin representations of Weyl groups. This Dirac operator is an algebraic variant of our family deformation of the differential Dirac operator for special values of the deformation parameters. Moreover, it satisfies the same symmetry as in Proposition 3.2, see [1, Lemma 3.4].

There is a measure naturally associated with $\mathbf{D}$. Indeed, one has

Proposition 3.3. *If $c > -1$, then for suitable differentiable functions f and g one has*

$$\int_{\mathbb{R}^m} \overline{(\boldsymbol{D}f)} g h(r) w_\kappa(x) dx = \int_{\mathbb{R}^m} \overline{f}(\boldsymbol{D}g) h(r) w_\kappa(x) dx$$

with $h(r) = r^{1-\frac{1+\mu c}{1+c}}$, provided the integrals exist.

In this proposition, $\bar{\cdot}$ is the main anti-involution on the Clifford algebra $\mathcal{C}l_m$.

4 Representation space for the deformation family of the Dunkl–Dirac operator

The function space we will work with is $\mathcal{L}^2_{\kappa,c}(\mathbb{R}^m) = L^2(\mathbb{R}^m, h(r)w_\kappa(x)dx) \otimes \mathcal{C}l_m$. This space has the following decomposition

$$\mathcal{L}^2_{\kappa,c}(\mathbb{R}^m) = L^2\big(\mathbb{R}^+, r^{\frac{\mu-1}{1+c}} dr\big) \otimes L^2(\mathbb{S}^{m-1}, w_\kappa(\xi)d\sigma(\xi)) \otimes \mathcal{C}l_m,$$

where on the right-hand side the topological completion of the tensor product is understood and with $d\sigma(\xi)$ the Lebesgue measure on the sphere $\mathbb{S}^{m-1}$. The space $L^2(\mathbb{S}^{m-1}, w_\kappa(\xi)d\sigma(\xi)) \otimes \mathcal{C}l_m$ can be further decomposed into Dunkl harmonics and subsequently into Dunkl monogenics. This leads to

$$L^2\big(\mathbb{S}^{m-1}, w_\kappa(\xi)d\sigma(\xi)\big) \otimes \mathcal{C}l_m = \bigoplus_{\ell=0}^{\infty} (\mathcal{M}_\ell \oplus \underline{x}\mathcal{M}_\ell)\,\big|_{\mathbb{S}^{m-1}},$$

where $\mathcal{M}_\ell = \ker \mathcal{D}_\kappa \cap (\mathcal{P}_\ell \otimes \mathcal{C}l_m)$ is the space of Dunkl monogenics of degree ℓ, with $\mathcal{P}_\ell$ the space of homogeneous polynomials of degree ℓ (see also [5] for more details on Dunkl monogenics).

Using this decomposition, we have obtained in [7] a basis for $\mathcal{L}^2_{\kappa,c}(\mathbb{R}^m)$. This basis is given by the set $\{\phi_{t,\ell,m}\}$ ($t, \ell \in \mathbb{N}$ and $m = 1, \ldots, \dim \mathcal{M}_\ell$), defined as

$$\begin{aligned}\phi_{2t,\ell,m} &= 2^{2t}(1+c)^{2t} t! L_t^{\frac{\gamma_\ell}{2}-1}(r^2) r^{\beta_\ell} M_\ell^{(m)} e^{-r^2/2},\\ \phi_{2t+1,\ell,m} &= -2^{2t+1}(1+c)^{2t+1} t! L_t^{\frac{\gamma_\ell}{2}}(r^2) \underline{x} r^{\beta_\ell} M_\ell^{(m)} e^{-r^2/2}\end{aligned}$$

with L_α^β the Laguerre polynomials and

$$\beta_\ell = -\frac{c}{1+c}\ell, \qquad \gamma_\ell = \frac{2}{1+c}\left(\ell + \frac{\mu-2}{2}\right) + \frac{c+2}{1+c},$$

and where $M_\ell^{(m)}$ ($m = 1, \ldots, \dim \mathcal{M}_\ell$) forms an orthonormal basis of $\mathcal{M}_\ell$, i.e.

$$\left[\int_{\mathbb{S}^{m-1}} \overline{M_\ell^{(m_1)}}(\xi) M_\ell^{(m_2)}(\xi) w_\kappa(\xi) d\sigma(\xi)\right]_0 = \delta_{m_1 m_2}$$

with $[\cdot]_0$ the projection on the scalar part of the Clifford algebra. The dimension of $\mathcal{M}_\ell$ is given by

$$\dim_{\mathbb{R}} \mathcal{M}_\ell = \dim_{\mathbb{R}} \mathcal{C}l_m \dim_{\mathbb{R}} \mathcal{P}_\ell\big(\mathbb{R}^{m-1}\big) = 2^m \frac{(\ell+m-2)!}{\ell!(m-2)!}$$

with $\mathcal{P}_\ell\big(\mathbb{R}^{m-1}\big)$ the space of homogeneous polynomials of degree ℓ in $m-1$ variables (see [9]).

Using formula (4.10) in [7] and the proof of Theorem 3 in [7], one obtains the following formulae for the action of $\mathbf{D}$ and $\underline{x}$ on the generalized Laguerre functions

$$2\mathbf{D}\phi_{t,\ell,m} = \phi_{t+1,\ell,m} + C(t,\ell)\phi_{t-1,\ell,m}, \qquad -2(1+c)\underline{x}\phi_{t,\ell,m} = \phi_{t+1,\ell,m} - C(t,\ell)\phi_{t-1,\ell,m} \tag{4.1}$$

with

$$C(2t,\ell) = 4(1+c)^2 t, \qquad C(2t+1,\ell) = 2(1+c)^2(\gamma_\ell + 2t).$$

These formulae determine the action of $\mathfrak{osp}(1|2)$ on $\mathcal{L}^2_{\kappa,c}(\mathbb{R}^m)$. Recall also that the action of $\widetilde{\mathcal{G}}$ on $\mathcal{L}^2_{\kappa,c}(\mathbb{R}^m)$ is given by ρ (see Section 3).

Subsequently, we can define a creation and annihilation operator in this setting by

$$A^+ = \mathbf{D} - (1+c)\underline{x}, \qquad A^- = \mathbf{D} + (1+c)\underline{x} \tag{4.2}$$

satisfying

$$A^+\phi_{t,\ell,m} = \phi_{t+1,\ell,m}, \qquad A^-\phi_{t,\ell,m} = C(t,\ell)\phi_{t-1,\ell,m}.$$

Now we introduce the following inner product

$$\langle f, g\rangle = \left[\int_{\mathbb{R}^m} \overline{f^c} g h(r) w_\kappa(x) dx\right]_0,$$

where $h(r)$ is the measure associated to $\mathbf{D}$ (see Proposition 3.3) and f^c is the complex conjugate of f. It is easy to check that this inner product satisfies

$$\langle \mathbf{D}f, g\rangle = \langle f, \mathbf{D}g\rangle, \qquad \langle \underline{x}f, g\rangle = -\langle f, \underline{x}g\rangle. \tag{4.3}$$

The related norm is defined by $||f||^2 = \langle f, f\rangle$.

Theorem 4.1. *We have*

$$\langle \phi_{t_1,\ell_1,m_1}, \phi_{t_2,\ell_2,m_2} \rangle = c(t_1,\ell_1)\delta_{t_1 t_2}\delta_{\ell_1 \ell_2}\delta_{m_1 m_2},$$

where $c(t,\ell)$ is a constant depending on t and ℓ.

The functions $\phi_{t,\ell,m}$ are eigenfunctions of the Hamiltonian of a generalized harmonic oscillator.

Theorem 4.2. *The functions $\phi_{t,\ell,m}$ satisfy the following second-order PDE*

$$\left(\boldsymbol{D}^2 - (1+c)^2 \underline{x}^2\right)\phi_{t,\ell,m} = (1+c)^2(\gamma_\ell + 2t)\phi_{t,\ell,m}.$$

Proof. This follows immediately from the formula (4.1). ■

Theorem 4.2 combined with the definition of A^+, A^- in (4.2) allows us to decompose the space $\mathcal{L}^2_{\kappa,c}(\mathbb{R}^m)$ under the action of $\mathfrak{osp}(1|2)$. Clearly the odd elements A^+ and A^- generate $\mathfrak{osp}(1|2)$ as they are linear combinations of $\mathbf{D}$ and $\underline{x}$. Moreover, they act between two basis vectors $\{\phi_{t,\ell,m}\}$ of $\mathcal{L}^2_{\kappa,c}(\mathbb{R}^m)$, so it is sufficient to consider vectors in an irreducible representation of $\mathfrak{osp}(1|2)$ inside the functional space. This is achieved as follows – for fixed ℓ and m each vector $\phi_{0,\ell,m}$ generates the irreducible representation

$$\underset{\circlearrowright L}{\phi_{0,\ell,m}} \underset{A^-}{\overset{A^+}{\rightleftarrows}} \underset{\circlearrowright L}{\phi_{1,\ell,m}} \underset{A^-}{\overset{A^+}{\rightleftarrows}} \underset{\circlearrowright L}{\phi_{2,\ell,m}} \underset{A^-}{\overset{A^+}{\rightleftarrows}} \underset{\circlearrowright L}{\phi_{3,\ell,m}} \underset{A^-}{\overset{A^+}{\rightleftarrows}} \underset{\circlearrowright L}{\phi_{4,\ell,m}} \underset{A^-}{\overset{A^+}{\rightleftarrows}} \cdots$$

where

$$L = \frac{1}{2}\{A^+, A^-\} = \mathbf{D}^2 - (1+c)^2\underline{x}^2$$

with the action given in Theorem 4.2. In fact this highest weight representation is labeled by ℓ only and we will denote it $\pi(\ell)$. In conclusion, we obtain the decomposition of our functional space $\mathcal{L}^2_{\kappa,c}(\mathbb{R}^m)$ into a discrete direct sum of highest weight (infinite-dimensional) Harish-Chandra modules for $\mathfrak{osp}(1|2)$:

$$\mathcal{L}^2_{\kappa,c}(\mathbb{R}^m) = \bigoplus_{\ell=0}^{\infty} \pi(\ell) \otimes \mathcal{M}_\ell.$$

These results should be compared with Theorem 3.19 and Section 3.6 in [2] (where one uses $\mathfrak{sl}_2$ instead of $\mathfrak{osp}(1|2)$). Also notice that the claim should be understood as an assertion on the deformation of the Howe dual pair for $\mathfrak{osp}(1|2)$ inside the Clifford–Weyl algebra on $\mathbb{R}^m$ acting on a fixed vector space $\mathcal{L}^2_{\kappa,c}(\mathbb{R}^m)$.

In particular, we have the following result. Recall that an operator T is essentially selfadjoint on a Hilbert space H if T is a symmetric operator with a dense domain $D(T) \subset H$ such that for a complete orthogonal set $\{f_n\}_n$ in H with $f_n \in D(H)$, there exist $\{\mu_n\}_n$ solving $T f_n = \mu_n f$ for all $n \in \mathbb{N}$.

Proposition 4.1. *Let $c > -1$ and $\kappa > 0$. The operator L acting on $\mathcal{L}^2_{\kappa,c}(\mathbb{R}^m)$ is essentially selfadjoint (i.e. symmetric and its closure is a selfadjoint operator). Moreover, L has no continuous spectrum and its discrete spectrum is given by*

$$\mathrm{Spec}(L) = \{2(1+c)\ell + 2(1+c)^2 t + (1+c)(\mu + c) \mid \ell, t \in \mathbb{N}\}.$$

Using Theorem 4.2 we can now define the holomorphic semigroup for the deformed Dirac operator by

$$\mathcal{F}_{\mathbf{D}}^{\omega} = e^{\omega\left(\frac{1}{2}+\frac{\mu-1}{2(1+c)}\right)} e^{\frac{-\omega}{2(1+c)^2}\left(\mathbf{D}^2-(1+c)^2\underline{x}^2\right)}.$$

Here, ω takes values in the right half-plane of $\mathbb{C}$. The special boundary value $\omega = i\pi/2$ corresponds to the Fourier transform. In that case, we will use the notation $\mathcal{F}_{\mathbf{D}}$. The functions $\phi_{t,\ell,m}$ are eigenfunctions of $\mathcal{F}_{\mathbf{D}}^{\omega}$ satisfying

$$\mathcal{F}_{\mathbf{D}}^{\omega}(\phi_{t,\ell,m}) = e^{-\omega t} e^{-\frac{\omega\ell}{(1+c)}} \phi_{t,\ell,m}. \tag{4.4}$$

Note that in the special case $\kappa = 0$, $c = 0$ the operator $\mathcal{F}_{\mathbf{D}}^{\omega}$ reduces to the classical Hermite semigroup (see, e.g., [17]).

Remark 4.1. One can also consider more general deformations of the Dirac operator, by adding suitable odd powers of $\Gamma = -\underline{x}\mathcal{D}_\kappa - \mathbb{E}$ to $\mathbf{D}$ as follows

$$\mathbf{D} = \mathcal{D}_\kappa + cr^{-2}\underline{x}\mathbb{E} + \sum_{j=0}^{\ell} c_j r^{-1}\left(\Gamma - \frac{\mu-1}{2}\right)^{2j+1}, \qquad c_j \in \mathbb{R}.$$

This does not alter the $\mathfrak{osp}(1|2)$ relations, as $\Gamma - \frac{\mu-1}{2}$ anti-commutes with $\underline{x}$ and has the correct homogeneity. In particular, $\Gamma - \frac{\mu-1}{2}$ can be seen as the square root of the Casimir of $\mathfrak{osp}(1|2)$, see [15, Example 2 in Section 2.5].

In the sequel of the paper, we will always assume $\kappa = 0$ or in other words, we do not consider the Dunkl deformation. This is to simplify the notation of the results. Most statements can be generalized to the Dunkl case by a suitable composition with the Dunkl intertwining operator V_κ, except the results obtained in Section 8.

Recall that for $\kappa = 0$, the Dunkl–Dirac operator $\mathcal{D}_\kappa$ reduces to the orthogonal Dirac operator $\partial_{\underline{x}} = \sum_{i=1}^{m} e_i \partial_{x_i}$ and the Dunkl dimension μ to the ordinary dimension m.

5 Reproducing kernels

In this section we determine the reproducing kernels for $\mathcal{M}_k$ and $\underline{x}\mathcal{M}_k$. We start with an auxiliary Lemma, which can be thought of as a Clifford analogue of the Funk–Hecke transform. We define the wedge product of two vectors as

$$\underline{x} \wedge \underline{y} := \sum_{j<k} e_j e_k (x_j y_k - x_k y_j).$$

Lemma 5.1. *Put $\underline{x} = r\underline{x}'$ and $\underline{y} = s\underline{y}'$ with $\underline{x}', \underline{y}' \in \mathbb{S}^{m-1}$. Furthermore, put $\lambda = (m-2)/2$ and $\sigma_m = 2\pi^{m/2}/\Gamma(m/2)$. Then one has, with $M_l \in \mathcal{M}_\ell$*

$$\int_{\mathbb{S}^{m-1}} C_k^\lambda(\langle \underline{x}', \underline{y}'\rangle) M_\ell(\underline{x}') d\sigma(\underline{x}') = \sigma_m \frac{\lambda}{\lambda+k}\delta_{k,\ell} M_\ell(\underline{y}'),$$

$$\int_{\mathbb{S}^{m-1}} C_k^\lambda(\langle \underline{x}', \underline{y}'\rangle) \underline{x}' M_\ell(\underline{x}') d\sigma(\underline{x}') = \sigma_m \frac{\lambda}{\lambda+k}\delta_{k,\ell+1} \underline{y}' M_\ell(\underline{y}'),$$

$$\int_{\mathbb{S}^{m-1}} (\underline{x}' \wedge \underline{y}') C_{k-1}^{\lambda+1}(\langle \underline{x}', \underline{y}'\rangle) M_\ell(\underline{x}') d\sigma(\underline{x}') = -\sigma_m \frac{k}{2(\lambda+k)}\delta_{k,\ell} M_\ell(\underline{y}'),$$

$$\int_{\mathbb{S}^{m-1}} (\underline{x}' \wedge \underline{y}') C_{k-1}^{\lambda+1}(\langle \underline{x}', \underline{y}'\rangle) \underline{x}' M_\ell(\underline{x}') d\sigma(\underline{x}') = \sigma_m \frac{k+2\lambda}{2(\lambda+k)}\delta_{k,\ell+1} \underline{y}' M_\ell(\underline{y}'),$$

where $C_k^\lambda(\langle \underline{x}', \underline{y}'\rangle)$ is the k-th Gegenbauer polynomial in the variable $\langle \underline{x}', \underline{y}'\rangle$.

Proof. The first integral is trivial: M_ℓ is a spherical harmonic of degree ℓ and $C_k^\lambda(\langle \underline{x}', \underline{y}'\rangle)$ is the reproducing kernel for spherical harmonics of degree k (see, e.g., [13]). The second integral immediately follows, because $\underline{x}' M_\ell(\underline{x}') \in \mathcal{H}_{\ell+1}$.

The other two integrals are a bit more complicated. We show how to obtain the last one. First rewrite $(\underline{x}' \wedge \underline{y}')\underline{x}' = \underline{y}' - \langle \underline{x}', \underline{y}'\rangle \underline{x}'$. The first term then follows from the first integral. For the second term, we use the recursive property of Gegenbauer polynomials:

$$wC_{n-1}^{\lambda+1}(w) = \frac{n}{2(n+\lambda)} C_n^{\lambda+1}(w) + \frac{n+2\lambda}{2(n+\lambda)} C_{n-2}^{\lambda+1}(w).$$

The result then follows by collecting everything. ■

We can use this lemma to determine the reproducing kernels. This is the subject of the following proposition.

Proposition 5.1. *For $k \in \mathbb{N}^*$ put*

$$P_k(\underline{x}', \underline{y}') = \frac{k+2\lambda}{2\lambda} C_k^\lambda(\langle \underline{x}', \underline{y}'\rangle) - (\underline{x}' \wedge \underline{y}') C_{k-1}^{\lambda+1}(\langle \underline{x}', \underline{y}'\rangle),$$

$$Q_{k-1}(\underline{x}', \underline{y}') = \frac{k}{2\lambda} C_k^\lambda(\langle \underline{x}', \underline{y}'\rangle) + (\underline{x}' \wedge \underline{y}') C_{k-1}^{\lambda+1}(\langle \underline{x}', \underline{y}'\rangle)$$

with $P_0(\underline{x}', \underline{y}') = C_0^\lambda(0) = 1$. Then

$$\int_{\mathbb{S}^{m-1}} P_k(\underline{x}', \underline{y}') M_\ell(\underline{x}') d\sigma(\underline{x}') = \sigma_m \delta_{k,\ell} M_\ell(\underline{y}'),$$

$$\int_{\mathbb{S}^{m-1}} P_k(\underline{x}', \underline{y}') \underline{x}' M_\ell(\underline{x}') d\sigma(\underline{x}') = 0$$

and

$$\int_{\mathbb{S}^{m-1}} Q_{k-1}(\underline{x}', \underline{y}') M_\ell(\underline{x}') d\sigma(\underline{x}') = 0,$$

$$\int_{\mathbb{S}^{m-1}} Q_{k-1}(\underline{x}', \underline{y}') \underline{x}' M_\ell(\underline{x}') d\sigma(\underline{x}') = \sigma_m \delta_{k,\ell+1} \underline{y}' M_\ell(\underline{y}').$$

Proof. This follows immediately from Lemma 5.1. ■

Remark 5.1. Note that, as expected, $P_k(\underline{x}', \underline{y}') + Q_{k-1}(\underline{x}', \underline{y}') = \frac{\lambda+k}{\lambda} C_k^\lambda(\langle \underline{x}', \underline{y}'\rangle)$, which is the reproducing kernel for the space of spherical harmonics of degree k.

Remark 5.2. When the dimension $m = 2$ and hence $\lambda = 0$, the reproducing kernel is still well-defined by using the well-known relation [27, (4.7.8)]

$$\lim_{\lambda \to 0} \lambda^{-1} C_k^\lambda(w) = (2/k) \cos k\theta, \qquad w = \cos\theta, \qquad k \geq 1.$$

We will also need the following lemma.

Lemma 5.2. *The reproducing kernels satisfy the following properties, for all $k, l \in \mathbb{N}$:*

$$\int_{\mathbb{S}^{m-1}} P_k(\underline{y}', \underline{x}') P_\ell(\underline{z}', \underline{y}') d\sigma(\underline{y}') = \sigma_m \delta_{k\ell} P_\ell(\underline{z}', \underline{x}'),$$

$$\int_{\mathbb{S}^{m-1}} P_k(\underline{y}', \underline{x}') Q_\ell(\underline{z}', \underline{y}') d\sigma(\underline{y}') = 0,$$

$$\int_{\mathbb{S}^{m-1}} Q_k(\underline{y}', \underline{x}') Q_\ell(\underline{z}', \underline{y}') d\sigma(\underline{y}') = \sigma_m \delta_{k\ell} Q_\ell(\underline{z}', \underline{x}').$$

Proof. This follows immediately using Lemma 7.6 and 7.10 from [8]. ■

Remark 5.3. Mind the order of the variables in the previous lemma. The kernels $P_k(\underline{x}', \underline{y}')$ and $Q_k(\underline{x}', \underline{y}')$ are not symmetric.

6 The series representation of the holomorphic semigroup

The aim of the present section is to investigate basic properties of the holomorphic semigroup defined by

$$\mathcal{F}^{\omega}_{\mathbf{D}} = e^{\omega\left(\frac{1}{2}+\frac{\mu-1}{2(1+c)}\right)} e^{\frac{-\omega}{2(1+c)^2}\left(\mathbf{D}^2-(1+c)^2\underline{x}^2\right)}, \qquad \operatorname{Re}\omega \geq 0,$$

acting on the space $\mathcal{L}^2_{0,c}(\mathbb{R}^m)$. We start with the following general statement.

Theorem 6.1. *Suppose $c > -1$. Then*

1. *For any $t, \ell \in \mathbb{N}$ and $m \in \{1, \ldots, \dim \mathcal{M}_\ell\}$, the function $\phi_{t,\ell,m}$ is an eigenfunction of the operator $\mathcal{F}^{\omega}_{\boldsymbol{D}}$:*

$$\mathcal{F}^{\omega}_{\boldsymbol{D}}(\phi_{t,\ell,m}) = e^{-\omega t} e^{-\frac{\omega \ell}{(1+c)}} \phi_{t,\ell,m}.$$

2. *$\mathcal{F}^{\omega}_{\boldsymbol{D}}$ is a continuous operator on $\mathcal{L}^2_{0,c}(\mathbb{R}^m)$ for all ω with $\operatorname{Re}\omega \geq 0$, in particular*

$$||\mathcal{F}^{\omega}_{\boldsymbol{D}}(f)|| \leq ||f||$$

for all $f \in \mathcal{L}^2_{0,c}(\mathbb{R}^m)$.

3. *If $\operatorname{Re}\omega > 0$, then $\mathcal{F}^{\omega}_{\boldsymbol{D}}$ is a Hilbert–Schmidt operator on $\mathcal{L}^2_{0,c}(\mathbb{R}^m)$.*
4. *If $\operatorname{Re}\omega = 0$, then $\mathcal{F}^{\omega}_{\boldsymbol{D}}$ is a unitary operator on $\mathcal{L}^2_{0,c}(\mathbb{R}^m)$.*

Proof. (1) is an immediate consequence of Theorem 4.2. For (2), let f be an element in $\mathcal{L}^2_{0,c}(\mathbb{R}^m)$ and expand it with respect to the (normalized) basis $\{\phi_{t,\ell,m}\}$ as

$$f = \sum_{t,\ell,m} a_{t,\ell,m}\phi_{t,\ell,m}.$$

Then one has, using orthogonality,

$$||\mathcal{F}^{\omega}_{\mathbf{D}}(f)||^2 = \sum_{t,\ell,m} |a_{t,\ell,m}|^2 e^{-2(\operatorname{Re}\omega)t} e^{-\frac{2(\operatorname{Re}\omega)\ell}{(1+c)}} \leq \sum_{t,\ell,m} |a_{t,\ell,m}|^2 = ||f||^2$$

because $\operatorname{Re}\omega \geq 0$.

As for (3), we have to show that the Hilbert–Schmidt norm is finite. We compute

$$\begin{aligned}||\mathcal{F}^{\omega}_{\mathbf{D}}||^2_{\mathrm{HS}} &= \sum_{t,k,\ell} ||\mathcal{F}^{\omega}_{\mathbf{D}}(\phi_{t,k,\ell})||^2 = \sum_{t,k,\ell} e^{-2(\operatorname{Re}\omega)t} e^{-\frac{2(\operatorname{Re}\omega)k}{(1+c)}} = \sum_{t,k} e^{-2(\operatorname{Re}\omega)t} e^{-\frac{2(\operatorname{Re}\omega)k}{(1+c)}} \dim_{\mathbb{R}} \mathcal{M}_k \\ &= \sum_{t,k} e^{-2(\operatorname{Re}\omega)t} e^{-\frac{2(\operatorname{Re}\omega)k}{(1+c)}} \frac{(k+m-2)!}{k!(m-2)!} 2^m = \sum_t e^{-2(\operatorname{Re}\omega)t} \sum_k e^{-\frac{2(\operatorname{Re}\omega)k}{(1+c)}} \frac{(k+m-2)!}{k!(m-2)!} 2^m.\end{aligned}$$

Using the ratio test, we see that these series are convergent for $\operatorname{Re}\omega > 0$.

(4) follows immediately, because when $\operatorname{Re}\omega = 0$ the eigenvalues all have unit norm. ■

We have already observed that $\mathcal{F}^{\omega}_{\mathbf{D}}$ is a Hilbert–Schmidt operator for $\operatorname{Re}\omega > 0$ and a unitary operator for $\operatorname{Re}\omega = 0$. The Schwartz kernel theorem implies that $\mathcal{F}^{\omega}_{\mathbf{D}}$ can be expressed by a distribution kernel $K(x, y; \omega)$, so

$$\left(\mathcal{F}^{\omega}_{\mathbf{D}} f\right)(y) = \int_{\mathbb{R}^m} K(x, y; \omega) f(x) h(r_x) dx,$$

and $K(x, y; \omega)h(r_x)$ is a tempered distribution on $\mathbb{R}^m \times \mathbb{R}^m$.

6.1 The case $\mathrm{Re}\,\omega > 0$

Using the reproducing kernels of Section 5, we can now make a reasonable ansatz for the kernel of the full holomorphic semigroup. We want to write this semigroup as

$$\mathcal{F}^{\omega}_{0,c}(f)(y) = \sigma_m^{-1} \int_{\mathbb{R}^m} K(x,y;\omega) f(x) h(r_x) dx$$

with $K(x,y;\omega) = K_0(x,y;\omega) + K_1(x,y;\omega)$ and

$$\begin{aligned} K_0 &= e^{-\frac{\coth\omega}{2}(r^2+s^2)} \sum_{k=0}^{+\infty} \alpha_k z^{\frac{k}{1+c}} \widetilde{J}_{\frac{\gamma_k}{2}-1}\left(\frac{iz}{\sinh\omega}\right) P_k(\underline{x}', \underline{y}'), \\ K_1 &= e^{-\frac{\coth\omega}{2}(r^2+s^2)} \sum_{k=0}^{+\infty} \beta_k z^{1+\frac{k}{1+c}} \widetilde{J}_{\frac{\gamma_k}{2}}\left(\frac{iz}{\sinh\omega}\right) Q_k(\underline{x}', \underline{y}'). \end{aligned} \tag{6.1}$$

Here $r = |\underline{x}|$, $s = |\underline{y}|$ and $z = |\underline{x}||\underline{y}|$. We also used the notation $\widetilde{J}_\nu(t) = (t/2)^{-\nu} J_\nu(t)$. Now we determine the complex constants $\{\alpha_k\}$ and $\{\beta_k\}$ such that this integral transform coincides with

$$\mathcal{F}^{\omega}_{\mathbf{D}} = e^{\omega\left(\frac{1}{2}+\frac{\mu-1}{2(1+c)}\right)} e^{\frac{-\omega}{2(1+c)^2}\left(\mathbf{D}^2-(1+c)^2\underline{x}^2\right)}$$

on the basis $\{\phi_{t,\ell,m}\}$.

We calculate

$$\begin{aligned} &\sigma_m^{-1} \int_{\mathbb{R}^m} K_0(x,y;\omega)\phi_{2t,\ell,m}(x) dx = \alpha_\ell M_\ell^{(m)}(\underline{y}') e^{-\frac{\coth\omega}{2}s^2} s^{\frac{\ell}{1+c}} \left(\frac{is}{2\sinh\omega}\right)^{-\gamma_\ell/2+1} \\ &\times \int_0^{+\infty} r^{\gamma_\ell/2} e^{-\frac{(\coth\omega+1)}{2}r^2} J_{\gamma_\ell/2-1}\left(\frac{irs}{\sinh\omega}\right) L_t^{\gamma_\ell/2-1}(r^2) dr = \alpha_\ell \frac{e^{-2\omega t} 2^{\gamma_\ell/2-1}}{(\coth\omega+1)^{\gamma_\ell/2}} \phi_{2t,\ell,m}(y), \end{aligned}$$

where we used the identity (see [2, Corollary 4.6])

$$2\int_0^{+\infty} r^{\alpha+1} J_\alpha(r\beta) L_j^\alpha(r^2) e^{-\delta r^2} dr = \frac{(\delta-1)^j \beta^\alpha}{2^\alpha \delta^{\alpha+j+1}} L_j^\alpha\left(\frac{\beta^2}{4\delta(1-\delta)}\right) e^{-\frac{\beta^2}{4\delta}}.$$

Similarly, we find

$$\begin{aligned} &\sigma_m^{-1} \int_{\mathbb{R}^m} K_0(x,y;\omega)\phi_{2t+1,\ell,m}(x) dx = 0, \qquad \sigma_m^{-1} \int_{\mathbb{R}^m} K_1(x,y;\omega)\phi_{2t,\ell,m}(x) dx = 0, \\ &\sigma_m^{-1} \int_{\mathbb{R}^m} K_1(x,y;\omega)\phi_{2t+1,\ell,m}(x) dx = \beta_\ell \frac{e^{-2\omega t} 2^{\gamma_\ell/2}}{(\coth\omega+1)^{\gamma_\ell/2+1}} \phi_{2t+1,\ell,m}(y). \end{aligned}$$

Hence we obtain by comparison with (4.4)

$$\alpha_\ell = e^{-\frac{\omega\ell}{(1+c)}} \frac{(\coth\omega+1)^{\gamma_\ell/2}}{2^{\gamma_\ell/2-1}} = 2e^{\frac{\omega\delta}{2}}(2\sinh\omega)^{-\gamma_\ell/2}, \qquad \beta_\ell = \frac{\alpha_\ell}{2\sinh\omega}.$$

We summarize our results in the following theorem.

Theorem 6.2. *Let* $\mathrm{Re}\,\omega > 0$ *and* $c > -1$. *Put*

$$K(x,y;\omega) = e^{-\frac{\coth\omega}{2}(r^2+s^2)} \left(A(z,w) + \underline{x}\wedge\underline{y} B(z,w)\right)$$

with

$$A(z,w) = \sum_{k=0}^{+\infty} \left(\alpha_k \frac{k+2\lambda}{2\lambda} z^{\frac{k}{1+c}} \widetilde{J}_{\frac{\gamma_k}{2}-1}\left(\frac{iz}{\sinh\omega}\right) + \frac{\alpha_{k-1}}{4\sinh\omega}\frac{k}{\lambda} z^{\frac{k+c}{1+c}} \widetilde{J}_{\frac{\gamma_{k-1}}{2}}\left(\frac{iz}{\sinh\omega}\right)\right) C_k^\lambda(w),$$

$$B(z,w)=\sum_{k=1}^{+\infty}\left(-\alpha_k z^{\frac{k}{1+c}-1}\widetilde{J}_{\frac{\gamma_k}{2}-1}\left(\frac{iz}{\sinh\omega}\right)+\frac{\alpha_{k-1}}{2\sinh\omega}z^{\frac{k+c}{1+c}-1}\widetilde{J}_{\frac{\gamma_{k-1}}{2}}\left(\frac{iz}{\sinh\omega}\right)\right)C_{k-1}^{\lambda+1}(w)$$

for $z=|\underline{x}||\underline{y}|$, $w=\langle\underline{x},\underline{y}\rangle/z$, $\alpha_{-1}=0$ *and* $\alpha_k=2e^{\frac{\omega\delta}{2}}(2\sinh\omega)^{-\gamma_k/2}$.

Then these series are convergent and the integral transform defined on $\mathcal{L}^2_{0,c}(\mathbb{R}^m)$ *by*

$$\mathcal{F}^{\omega}_{0,c}(f)(y)=\sigma_m^{-1}\int_{\mathbb{R}^m}K(x,y;\omega)f(x)h(r_x)dx$$

coincides with the operator $\mathcal{F}^{\omega}_{\boldsymbol{D}}=e^{\omega\left(\frac{1}{2}+\frac{\mu-1}{2(1+c)}\right)}e^{\frac{-\omega}{2(1+c)^2}\left(\boldsymbol{D}^2-(1+c)^2\underline{x}^2\right)}$ *on the basis* $\{\phi_{t,\ell,m}\}$.

Proof. We have already shown that the integral transform coincides with the operator $\mathcal{F}^{\omega}_{\mathbf{D}}=e^{\omega\left(\frac{1}{2}+\frac{\mu-1}{2(1+c)}\right)}e^{\frac{-\omega}{2(1+c)^2}\left(\mathbf{D}^2-(1+c)^2\underline{x}^2\right)}$ on the basis $\{\phi_{t,\ell,m}\}$. So we only have to show that the series are convergent. We do this for the term

$$\sum_{k=0}^{+\infty}(2\sinh\omega)^{-\gamma_k/2}\frac{k+2\lambda}{2\lambda}z^{\frac{k}{1+c}}\widetilde{J}_{\frac{\gamma_k}{2}-1}\left(\frac{iz}{\sinh\omega}\right)C_k^{\lambda}(w)$$
$$=(2\sinh\omega)^{-\delta/2}\sum_{k=0}^{+\infty}\frac{k+2\lambda}{2\lambda}\left(\frac{z}{2\sinh\omega}\right)^{\frac{k}{1+c}}\widetilde{J}_{\frac{\gamma_k}{2}-1}\left(\frac{iz}{\sinh\omega}\right)C_k^{\lambda}(w),$$

the other ones are treated in a similar fashion. We obtain

$$\left|\sum_{k=0}^{+\infty}\frac{k+2\lambda}{2\lambda}\left(\frac{z}{2\sinh\omega}\right)^{\frac{k}{1+c}}\widetilde{J}_{\frac{\gamma_k}{2}-1}\left(\frac{iz}{\sinh\omega}\right)C_k^{\lambda}(w)\right|$$
$$\le\frac{B(\lambda)}{2}e^{\left|\mathrm{Im}\,\frac{iz}{\sinh\omega}\right|}\sum_{k=0}^{+\infty}(k+2\lambda)\left|\frac{z}{2\sinh\omega}\right|^{\frac{k}{1+c}}\frac{1}{\Gamma(\gamma_k/2)}k^{2\lambda-1}$$

using formula (A.1) and (A.2). As the term $\Gamma(\gamma_k/2)$ is dominant, the series clearly converges. ■

6.2 The case $\mathrm{Re}\,\boldsymbol{\omega}=0$

In this case, we have the following theorem.

Theorem 6.3. *Let* $c>-1$. *Then for* $\omega=i\eta$ *with* $\eta\notin\pi\mathbb{Z}$, *we put*

$$K(x,y;i\eta)=e^{i\frac{\cot\eta}{2}(r^2+s^2)}\left(A(z,w)+\underline{x}\wedge\underline{y}B(z,w)\right)$$

with

$$A(z,w)=\sum_{k=0}^{+\infty}\left(\alpha_k\frac{k+2\lambda}{2\lambda}z^{\frac{k}{1+c}}\widetilde{J}_{\frac{\gamma_k}{2}-1}\left(\frac{z}{\sin\eta}\right)+\frac{\alpha_{k-1}}{4i\sin\eta}\frac{k}{\lambda}z^{\frac{k+c}{1+c}}\widetilde{J}_{\frac{\gamma_{k-1}}{2}}\left(\frac{z}{\sin\eta}\right)\right)C_k^{\lambda}(w),$$
$$B(z,w)=\sum_{k=1}^{+\infty}\left(-\alpha_k z^{\frac{k}{1+c}-1}\widetilde{J}_{\frac{\gamma_k}{2}-1}\left(\frac{z}{\sin\eta}\right)+\frac{\alpha_{k-1}}{2i\sin\eta}z^{\frac{k+c}{1+c}-1}\widetilde{J}_{\frac{\gamma_{k-1}}{2}}\left(\frac{z}{\sin\eta}\right)\right)C_{k-1}^{\lambda+1}(w)$$

for $z=|\underline{x}||\underline{y}|$, $w=\langle\underline{x},\underline{y}\rangle/z$, $\alpha_{-1}=0$ *and* $\alpha_k=2e^{\frac{i\eta\delta}{2}}(2i\sin\eta)^{-\gamma_k/2}$.

These series are convergent and the unitary integral transform defined in distributional sense on $\mathcal{L}^2_{0,c}(\mathbb{R}^m)$ *by*

$$\mathcal{F}^{i\eta}_{0,c}(f)(y)=\sigma_m^{-1}\int_{\mathbb{R}^m}K(x,y;i\eta)f(x)h(r_x)dx$$

coincides with the operator $\mathcal{F}^{i\eta}_{\boldsymbol{D}}=e^{i\eta\left(\frac{1}{2}+\frac{\mu-1}{2(1+c)}\right)}e^{\frac{-i\eta}{2(1+c)^2}\left(\boldsymbol{D}^2-(1+c)^2\underline{x}^2\right)}$ *on the basis* $\{\phi_{t,\ell,m}\}$.

Proof. This follows by taking the limit $\omega\to i\eta$ in Theorem 6.2. ■

7 The series representation of the Fourier transform

The Fourier transform is the very special case of the holomorphic semigroup, evaluated at $\omega = i\pi/2$. In this case, the kernel $K(x, y) = K(x, y; i\pi/2)$ is given by the following theorem.

Theorem 7.1. *Put $K(x, y) = A(z, w) + \underline{x} \wedge \underline{y} B(z, w)$ with*

$$A(z,w) = \sum_{k=0}^{+\infty} z^{-\frac{\delta-2}{2}} \left(\alpha_k \frac{k+2\lambda}{2\lambda} J_{\frac{\gamma_k}{2}-1}(z) - i\alpha_{k-1} \frac{k}{2\lambda} J_{\frac{\gamma_{k-1}}{2}}(z) \right) C_k^{\lambda}(w),$$

$$B(z,w) = \sum_{k=1}^{+\infty} z^{-\frac{\delta}{2}} \left(-\alpha_k J_{\frac{\gamma_k}{2}-1}(z) - i\alpha_{k-1} J_{\frac{\gamma_{k-1}}{2}}(z) \right) C_{k-1}^{\lambda+1}(w)$$

and $z = |\underline{x}||\underline{y}|$, $w = \langle \underline{x}, \underline{y} \rangle / z$, $\alpha_{-1} = 0$ and $\alpha_k = e^{-\frac{i\pi k}{2(1+c)}}$. These series are convergent and the integral transform defined in distributional sense on $\mathcal{L}^2_{0,c}(\mathbb{R}^m)$ by

$$\mathcal{F}_{0,c}(f)(y) = \sigma_m^{-1} \int_{\mathbb{R}^m} K(x,y) f(x) h(r_x) dx$$

coincides with the operator $\mathcal{F}_{\boldsymbol{D}} = e^{i\frac{\pi}{2}\left(\frac{1}{2}+\frac{\mu-1}{2(1+c)}\right)} e^{\frac{-i\pi}{4(1+c)^2}\left(\boldsymbol{D}^2-(1+c)^2\underline{x}^2\right)}$ on the basis $\{\phi_{t,\ell,m}\}$.

Proof. Using the well-known identity (see [27, Exercise 21, p. 371])

$$\int_0^{+\infty} r^{\alpha+1} J_\alpha(rs) L_j^\alpha(r^2) e^{-r^2/2} dr = (-1)^j s^\alpha L_j^\alpha(s^2) e^{-s^2/2}$$

we can prove in the same way as leading to Theorem 6.2 that the integral transform $\mathcal{F}_{0,c}$ coincides with

$$\mathcal{F}_{\mathbf{D}} = e^{i\frac{\pi}{2}\left(\frac{1}{2}+\frac{\mu-1}{2(1+c)}\right)} e^{\frac{-i\pi}{4(1+c)^2}\left(\mathbf{D}^2-(1+c)^2\underline{x}^2\right)}$$

on the basis $\phi_{t,\ell,m}$. The theorem also follows as a special case of Theorem 6.2, taking the limit $\omega \to i\pi/2$. ■

Remark 7.1. One can also define an analogue of the Schwartz space of rapidly decreasing functions in this context. Let $L = \mathbf{D}^2 - (1+c)^2\underline{x}^2$ and denote by $D(L)$ the domain of L in $\mathcal{L}^2_{0,c}(\mathbb{R}^m)$. Then the Schwartz space is defined by

$$\mathcal{S}_{0,c}(\mathbb{R}^m) = \bigcap_{k=0}^{\infty} D(L^k)$$

and one can check that the Fourier transform $\mathcal{F}_{0,c}$ is an isomorphism of this space.

Remark 7.2. In the limit case $c = 0$, we can check that the kernel reduces to

$$K(x,y) = \sum_{k=0}^{+\infty} \frac{k+\lambda}{\lambda} (-i)^k z^{-\lambda} J_{k+\lambda}(z) C_k^{\lambda}(w).$$

This is a well-known expansion of the classical Fourier kernel (see [29, Section 11.5]):

$$K(x,y) = \frac{e^{-i\langle \underline{x}, \underline{y} \rangle}}{\Gamma(m/2) 2^{\frac{m-2}{2}}}.$$

We can now summarize the main properties of the deformed Fourier transform in the following theorem.

Theorem 7.2. *The operator $\mathcal{F}_{0,c}$ defines a unitary operator on $\mathcal{L}^2_{0,c}(\mathbb{R}^m)$ and satisfies the following intertwining relations on a dense subset:*

$$\mathcal{F}_{0,c} \circ \boldsymbol{D} = i(1+c)\underline{x} \circ \mathcal{F}_{0,c}, \qquad \mathcal{F}_{0,c} \circ \underline{x} = \frac{i}{1+c}\boldsymbol{D} \circ \mathcal{F}_{0,c}, \qquad \mathcal{F}_{0,c} \circ \mathbb{E} = -\left(\mathbb{E} + \delta\right) \circ \mathcal{F}_{0,c}.$$

Moreover, $\mathcal{F}_{0,c}$ is of finite order if and only if c is rational.

Proof. Every f in $\mathcal{L}^2_{0,c}(\mathbb{R}^m)$ can be expanded in terms of the orthogonal basis $\phi_{t,\ell,m}$, satisfying

$$\langle \phi_{t_1,\ell_1,m_1}, \phi_{t_2,\ell_2,m_2} \rangle = \delta_{t_1 t_2}\delta_{\ell_1 \ell_2}\delta_{m_1 m_2}\langle \phi_{t_1,\ell_1,m_1}, \phi_{t_1,\ell_1,m_1} \rangle,$$

see Section 3. Note that the normalization can be computed explicitly (see [7, Theorem 6]). As the eigenvalues of $\mathcal{F}_{0,c}$ are given by (see (4.4))

$$(-i)^t e^{-i\frac{\pi \ell}{2(1+c)}}$$

which clearly live on the unit circle, we conclude that

$$\langle f, g \rangle = \langle \mathcal{F}_{0,c}(f), \mathcal{F}_{0,c}(g) \rangle$$

and that $\mathcal{F}_{0,c}$ is a unitary operator.

The intertwining relations are an immediate consequence of formula (4.1) combined with the fact that $\phi_{t,\ell,m}$ is an eigenbasis of $\mathcal{F}_{0,c}$. The formula for $\mathbb{E}$ follows from the anti-commutator (see Theorem 3.1)

$$\{\mathbf{D}, \underline{x}\} = -2(1+c)\left(\mathbb{E} + \frac{\delta}{2}\right).$$

The statement on the finite order of the Fourier transform is an immediate consequence of the explicit expression for the eigenvalues of the transform. ■

Now we collect some properties of the kernel $K(x,y)$.

Proposition 7.1. *One has, with $x, y \in \mathbb{R}^m$*

$$K(\lambda x, y) = K(x, \lambda y), \quad \lambda > 0, \qquad K(y,x) = \overline{K(x,y)},$$
$$K(0,y) = \frac{1}{2^{\gamma_0/2-1}\Gamma(\gamma_0/2)}, \qquad K(\bar{s}xs, \bar{s}ys,) = \bar{s}K(x,y)s, \quad s \in \mathrm{Spin}(m),$$

where $\bar{\cdot}$ is the anti-involution on the Clifford algebra $\mathcal{C}l_m$.

Proof. The first property is trivial. The second follows because

$$\overline{\underline{x} \wedge \underline{y}} = -\underline{x} \wedge \underline{y} = \underline{y} \wedge \underline{x}.$$

The third property follows from Theorem 7.1. Finally, the 4th equation follows because z and w are spin-invariant and

$$(\bar{s}xs) \wedge (\bar{s}ys) = \bar{s}\left(\underline{x} \wedge \underline{y}\right)s. \qquad ■$$

We can also obtain Bochner identities for the deformed Fourier transform. They are given in the following proposition.

Proposition 7.2. *Let $M_\ell \in \mathcal{M}_\ell$ be a spherical monogenic of degree ℓ. Let $f(x) = f(r)$ be a radial function. Then the Fourier transform of $f(r)M_\ell$ and $f(r)\underline{x}M_\ell$ can be computed as follows*

$$\mathcal{F}_{0,c}(f(r)M_\ell) = e^{-\frac{i\pi\ell}{2(1+c)}} M_\ell(\underline{y}') \int_0^{+\infty} r^\ell f(r) z^{-\frac{\delta-2}{2}} J_{\frac{\gamma_k}{2}-1}(z) h(r) r^{m-1} dr,$$

$$\mathcal{F}_{0,c}(f(r)\underline{x}M_\ell) = -ie^{-\frac{i\pi\ell}{2(1+c)}} \underline{y}' M_\ell(\underline{y}') \int_0^{+\infty} r^{\ell+1} f(r) z^{-\frac{\delta-2}{2}} J_{\frac{\gamma_k}{2}}(z) h(r) r^{m-1} dr$$

with $\underline{y} = s\underline{y}'$, $\underline{y}' \in \mathbb{S}^{m-1}$ and $z = rs$.

Proof. This follows immediately from Theorem 7.1 combined with Proposition 5.1. ∎

Remark 7.3. As a special case of this proposition, we reobtain the eigenfunctions of the Fourier transform by putting $f(r) = L_t^{\frac{\gamma_\ell}{2}-1}(r^2) r^{\beta_\ell} e^{-r^2/2}$, resp. $f(r) = L_t^{\frac{\gamma_\ell}{2}}(r^2) r^{\beta_\ell} e^{-r^2/2}$ (see equation (4.4)).

Now we prove the following lemma.

Lemma 7.1. *For all $f \in \mathcal{L}^2_{0,c}(\mathbb{R}^m)$ one has*

$$||\underline{x} f(x)||^2 + ||\underline{x}\left(\mathcal{F}_{0,c} f\right)(x)||^2 \geq \delta ||f(x)||^2.$$

The equality holds if and only if f is a multiple of $e^{-r^2/2}$.

Proof. Using formula (4.3) and the unitarity of $\mathcal{F}_{0,c}$, one can compute that

$$||\underline{x} f(x)||^2 + ||\underline{x}\left(\mathcal{F}_{0,c} f\right)(x)||^2 = \frac{1}{(1+c)^2} \langle \left(\mathbf{D}^2 - (1+c)^2 \underline{x}^2\right) f, f \rangle.$$

Now use the fact that the smallest eigenvalue of

$$\frac{1}{(1+c)^2}\left(\mathbf{D}^2 - (1+c)^2 \underline{x}^2\right)$$

is given by δ, see Theorem 4.2. This proves the inequality.

The equality holds when f is a multiple of an eigenfunction corresponding to the smallest eigenvalue, i.e. when f is a multiple of $e^{-r^2/2}$. ∎

This lemma allows us to obtain the Heisenberg inequality for the deformed Fourier transform

Proposition 7.3. *For all $f \in \mathcal{L}^2_{0,c}(\mathbb{R}^m)$, the deformed Fourier transform satisfies*

$$||\underline{x} f(x)|| \cdot ||\underline{x}\left(\mathcal{F}_{0,c} f\right)(x)|| \geq \frac{\delta}{2} ||f(x)||^2.$$

The equality holds if and only if f is of the form $f(x) = \lambda e^{-r^2/\alpha}$.

Proof. Using Lemma 7.1, we can continue in the same way as in the proof of Theorem 5.28 in [2]. ∎

Now we can obtain the Master formula for the kernel of the Fourier transform. We use the formula (see [14, p. 50])

$$\int_0^{+\infty} J_\nu(at) J_\nu(bt) e^{-\gamma^2 t^2} t dt = \frac{1}{2}\gamma^{-2} e^{-\frac{a^2+b^2}{4\gamma^2}} I_\nu\left(\frac{ab}{2\gamma^2}\right), \qquad \operatorname{Re}\nu > -1, \quad \operatorname{Re}\gamma^2 > 0, \quad (7.1)$$

where $I_\nu(z) = e^{-i\frac{\pi\nu}{2}} J_\nu(iz)$.

We then obtain

Theorem 7.3 (Master formula)**.** *Let $s > 0$. Then one has*

$$\int_{\mathbb{R}^m} K\left(y, x; i\frac{\pi}{2}\right) K\left(z, y; -i\frac{\pi}{2}\right) e^{-sr_y^2} h(r_y) dy = \sigma_m e^{-\frac{\omega\delta}{2}} K(z, x; \omega) e^{-\frac{|\underline{x}|^2+|\underline{z}|^2}{2}\frac{1-\cosh\omega}{\sinh\omega}}$$

with $2s = \sinh\omega$.

Proof. First observe that $K\left(y, x; i\frac{\pi}{2}\right) = K(y, x)$ and that $K(z, y; -i\frac{\pi}{2})$ is the complex conjugate of $K\left(z, y; i\frac{\pi}{2}\right)$.

We rewrite the kernel K obtained in Theorem 7.1 in terms of the reproducing kernels P_k and Q_k, i.e. as $K(x, y) = K_0(x, y) + K_1(x, y)$ with

$$K_0(x, y) = \sum_{k=0}^{+\infty} \alpha_k \left(|\underline{x}||\underline{y}|\right)^{-\frac{\delta-2}{2}} J_{\frac{\gamma_k}{2}-1}(|\underline{x}||\underline{y}|) P_k(\underline{x}', \underline{y}'),$$

$$K_1(x, y) = \sum_{k=0}^{+\infty} \beta_k \left(|\underline{x}||\underline{y}|\right)^{-\frac{\delta-2}{2}} J_{\frac{\gamma_k}{2}}(|\underline{x}||\underline{y}|) Q_k(\underline{x}', \underline{y}'),$$

where $\alpha_k = e^{-\frac{i\pi k}{2(1+c)}}$ and $\beta_k = -i\alpha_k$.

When passing to spherical co-ordinates, the integral simplifies, using Lemma 5.2, to

$$\sigma_m \sum_{k=0}^{+\infty} (|\underline{x}||\underline{z}|)^{-\frac{\delta-2}{2}} P_k(\underline{z}', \underline{x}') \int_0^{+\infty} re^{-sr^2} J_{\frac{\gamma_k}{2}-1}(r|\underline{x}|) J_{\frac{\gamma_k}{2}-1}(r|\underline{z}|) dr$$

$$+ \sigma_m \sum_{k=0}^{+\infty} (|\underline{x}||\underline{z}|)^{-\frac{\delta-2}{2}} Q_k(\underline{z}', \underline{x}') \int_0^{+\infty} re^{-sr^2} J_{\frac{\gamma_k}{2}}(r|\underline{x}|) J_{\frac{\gamma_k}{2}}(r|\underline{z}|) dr.$$

The radial integral can be computed explicitly using (7.1). Comparing with formula (6.1) and Theorem 6.2 leads to the statement of the theorem. ∎

Remark 7.4. For the Dunkl transform (see, e.g., [24, 28]) and for the Clifford–Fourier transform (see [8]) one can compute even a more general integral of the form

$$\int_{\mathbb{R}^m} K\left(y, x; i\frac{\pi}{2}\right) K\left(z, y; -i\frac{\pi}{2}\right) f(r_y) h(r_y) dy$$

with $f(r_y)$ an arbitrary radial function of suitable decay. This is done by using the addition formula for the Bessel function

$$u^{-\lambda} J_\lambda(u) = 2^\lambda \Gamma(\lambda) \sum_{k=0}^{\infty} (k+\lambda)(r^2|\underline{x}||\underline{z}|)^{-\lambda} J_{k+\lambda}(r|\underline{x}|) J_{k+\lambda}(r|\underline{z}|) C_k^\lambda(\langle \underline{x}', \underline{z}'\rangle)$$

with $u = r\sqrt{|\underline{x}|^2 + |\underline{z}|^2 - 2\langle \underline{x}, \underline{z}\rangle}$ instead of formula (7.1). Here, we cannot do that, as the orders of the Bessel functions do not match the order of the Gegenbauer polynomials.

Remark 7.5. Theorem 7.3 is the starting point for the study of a generalized heat equation, see, e.g., [26, Lemma 4.5(1)] in the context of Dunkl operators.

8 Further results for the kernel

In this section we will always be working in the non-Dunkl case, i.e. we put the multiplicity function $\kappa = 0$. Theorem 7.1 implies that the kernel of our deformed Fourier transform is a function of the type

$$K(x, y) = f(z, w) + \underline{x} \wedge \underline{y} g(z, w)$$

with f, g scalar functions of the variables $z = |\underline{x}||\underline{y}|$ and $w = \langle \underline{x}, \underline{y}\rangle/z$. On the other hand, this kernel needs to satisfy the system of PDEs

$$\mathbf{D}_y K(x,y) = -i(1+c)K(x,y)\underline{x}, \qquad (K(x,y)\mathbf{D}_x) = -i(1+c)\underline{y}K(x,y),$$

as can be deduced from Theorem 7.2. In order to rewrite this system in terms of the variables z, w, we first observe that

$$\begin{aligned}
&\partial_{\underline{x}} f(z,w) = r^{-2}\underline{x}z\partial_z f(z,w) + \left(z^{-1}\underline{y} - r^{-2}\underline{x}w\right)\partial_w f(z,w),\\
&\mathbb{E}f(z,w) = z\partial_z f(z,w), \qquad \partial_{\underline{x}}\left(\underline{x}\wedge\underline{y}\right) = (1-m)\underline{y}.
\end{aligned}$$

Using these identities, one obtains that the kernel is determined by the following 2 PDEs:

$$\begin{aligned}
&(m-1+c)g + (1+c)z\partial_z g + \frac{1}{z}\partial_w f + i(1+c)f - i(1+c)zwg = 0,\\
&(1+c)z\partial_z f - w\partial_w f - czwg - (1+c)z^2 w\partial_z g + z(w^2-1)\partial_w g + i(1+c)z^2 g = 0. \qquad (8.1)
\end{aligned}$$

Remark 8.1. Note that, contrary to the case of the classical Fourier transform and the Dunkl transform, where the kernel is uniquely determined by the system of PDEs

$$T_{j,x}K(x,y) = iy_j K(x,y), \qquad j = 1,\dots,m$$

this is not the case for the kernel of the radially deformed Fourier transform. In fact, one can observe that there exist several different types of solutions of (8.1). This is discussed in detail in [6] for a similar system of PDEs in the context of the so-called Clifford–Fourier transform (see [8]).

Now we show that it is sufficient to solve this system in dimension $m = 2$ and $m = 3$. Recall that the kernel $K(x,y)$ is given in Theorem 7.1. To know this kernel, it is hence sufficient to know the series

$$\begin{aligned}
A_\lambda &= \sum_{k=0}^{+\infty}\alpha_k(k+\lambda)J_{\frac{\gamma_k}{2}-1}(z)C_k^\lambda(w), & D_\lambda &= \sum_{k=0}^{+\infty}\alpha_{k-1}J_{\frac{\gamma_{k-1}}{2}}(z)C_k^\lambda(w),\\
B_\lambda &= \sum_{k=0}^{+\infty}\alpha_k J_{\frac{\gamma_k}{2}-1}(z)C_k^\lambda(w), & E_\lambda &= \sum_{k=1}^{+\infty}\alpha_k J_{\frac{\gamma_k}{2}-1}(z)C_{k-1}^{\lambda+1}(w),\\
C_\lambda &= \sum_{k=0}^{+\infty}\alpha_{k-1}(k+\lambda)J_{\frac{\gamma_{k-1}}{2}}(z)C_k^\lambda(w), & F_\lambda &= \sum_{k=1}^{+\infty}\alpha_{k-1}J_{\frac{\gamma_{k-1}}{2}}(z)C_{k-1}^{\lambda+1}(w),
\end{aligned}$$

because then one has

$$K = \frac{1}{2\lambda}z^{-\frac{\delta-2}{2}}\left(A_\lambda - iC_\lambda\right) + \frac{1}{2}z^{-\frac{\delta-2}{2}}\left(B_\lambda + iD_\lambda\right) - z^{-\frac{\delta}{2}}\underline{x}\wedge\underline{y}\left(E_\lambda + iF_\lambda\right).$$

Using the well-known property of the Gegenbauer polynomials $2\lambda C_{k-1}^{\lambda+1}(w) = \partial_w C_k^\lambda(w)$, we observe the following recursion relations

$$\begin{aligned}
A_{\lambda+1} &= e^{i\frac{\pi}{2(1+c)}}\frac{1}{2\lambda}\partial_w A_\lambda, & B_{\lambda+1} &= e^{i\frac{\pi}{2(1+c)}}\frac{1}{2\lambda}\partial_w B_\lambda, & C_{\lambda+1} &= e^{-i\frac{\pi}{2(1+c)}}\frac{1}{2\lambda}\partial_w C_\lambda,\\
D_{\lambda+1} &= e^{-i\frac{\pi}{2(1+c)}}\frac{1}{2\lambda}\partial_w D_\lambda, & E_\lambda &= \frac{1}{2\lambda}\partial_w B_\lambda, & F_\lambda &= \frac{1}{2\lambda}\partial_w D_\lambda.
\end{aligned}$$

We conclude that it suffices to know A_λ, B_λ, C_λ and D_λ for $\lambda = 0, 1/2$ or $m = 2, 3$. At this point, the problem of finding explicit expressions for these functions for special values of the deformation parameter c is still open.

A Properties of Laguerre and Gegenbauer polynomials

The generalized Laguerre polynomials $L_k^{(\alpha)}$ for $k \in \mathbb{N}$ are defined as

$$L_k^{(\alpha)}(t) = \sum_{j=0}^{k} \frac{\Gamma(k+\alpha+1)}{j!(k-j)!\Gamma(j+\alpha+1)}(-t)^j$$

and satisfy the orthogonality relation (when $\alpha > -1$)

$$\int_0^\infty t^\alpha L_k^{(\alpha)}(t) L_l^{(\alpha)}(t) \exp(-t) dt = \delta_{kl} \frac{\Gamma(k+\alpha+1)}{k!}.$$

The Gegenbauer polynomials $C_k^{(\alpha)}(t)$ are a special case of the Jacobi polynomials. For $k \in \mathbb{N}$ and $\alpha > -1/2$ they are defined as

$$C_k^{(\alpha)}(t) = \sum_{j=0}^{\lfloor k/2 \rfloor} (-1)^j \frac{\Gamma(k-j+\alpha)}{\Gamma(\alpha) j!(k-2j)!}(2t)^{k-2j}$$

and satisfy the orthogonality relation

$$\int_{-1}^{1} C_k^{(\alpha)}(t) C_l^{(\alpha)}(t) \left(1-t^2\right)^{\alpha-\frac{1}{2}} dt = \delta_{kl} \frac{\pi 2^{1-2\alpha}\Gamma(k+2\alpha)}{k!(k+\alpha)(\Gamma(\alpha))^2}.$$

One can prove that there exists a constant $B(\alpha)$ such that

$$\sup_{-1 \le t \le 1} \left| \frac{1}{\alpha} C_k^{(\alpha)}(t) \right| \le B(\alpha) k^{2\alpha-1}, \qquad \forall\, k \in \mathbb{N}, \tag{A.1}$$

see [2, Lemma 4.9].

The Bessel function $J_\nu(z)$ is defined using the following Taylor series

$$J_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k+\nu+1)} \left(\frac{z}{2}\right)^{2k+\nu}.$$

For $z \in \mathbb{C}$ and $\nu \ge -1/2$ one has the inequality (see, e.g., [27])

$$\left| \left(\frac{z}{2}\right)^{-\nu} J_\nu(z) \right| \le \frac{1}{\Gamma(\nu+1)} e^{|\operatorname{Im} z|}. \tag{A.2}$$

B List of notations

List of notations used in this paper:

m	dimension of $\mathbb{R}^m$,
κ	multiplicity function on root system,
μ	Dunkl-dimension,
c	deformation parameter of $\mathbf{D}$,
ω	semigroup parameter with $\operatorname{Re}\omega \ge 0$,
$\partial_{\underline{x}}$	ordinary Dirac operator,
$\mathcal{D}_k$	Dunkl Dirac operator,

$\mathbf{D}$	radially deformed Dirac operator,
$\mathcal{F}^{\omega}_{\mathbf{D}}$	exponential form of the holomorphic semigroup,
$\mathcal{F}^{\omega}_{0,c}$	integral form of the holomorphic semigroup,
$\mathcal{F}_{\mathbf{D}}$	exponential form of the Fourier transform,
$\mathcal{F}_{0,c}$	integral form of the Fourier transform.

We also have the following definitions:

$$\mu = m + 2\sum_{\alpha\in R_+} \kappa_\alpha, \qquad \lambda = \frac{m-2}{2}, \qquad \sigma_m = 2\pi^{m/2}/\Gamma(m/2), \qquad \delta = 1 + \frac{\mu-1}{1+c},$$

$$\beta_\ell = -\frac{c}{1+c}\ell, \quad \ell\in\mathbb{N}, \qquad \gamma_\ell = \frac{2}{1+c}\left(\ell + \frac{\mu-2}{2}\right) + \frac{c+2}{1+c}, \quad \ell\in\mathbb{N}.$$

Notations for variables. Let $\underline{x}$ and $\underline{y}$ be vector variables in $\mathbb{R}^m$. Then we denote

$$z = |\underline{x}||\underline{y}|, \qquad w = \langle \underline{x}, \underline{y}\rangle / z.$$

When using spherical co-ordinates, we use $\underline{x} = r\underline{x}'$ with $\underline{x}' \in \mathbb{S}^{m-1}$, hereby implicitly identifying a vector in the Clifford algebra with a vector in $\mathbb{R}^m$.

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References

[1] Barbasch D., Ciubotaru D., Trapa P.E., Dirac cohomology for graded affine Hecke algebras, *Acta Math.* **209** (2012), 197–227, arXiv:1006.3822.

[2] Ben Saïd S., Kobayashi T., Ørsted B., Laguerre semigroup and Dunkl operators, *Compos. Math.* **148** (2012), 1265–1336, arXiv:0907.3749.

[3] Cação I., Constales D., Krausshar R.S., On the role of arbitrary order Bessel functions in higher dimensional Dirac type equations, *Arch. Math. (Basel)* **87** (2006), 468–477.

[4] Cherednik I., Markov Y., Hankel transform via double Hecke algebra, in Iwahori–Hecke algebras and their representation theory (Martina–Franca, 1999), *Lecture Notes in Math.*, Vol. 1804, Springer, Berlin, 2002, 1–25, math.QA/0004116.

[5] De Bie H., De Schepper N., Clifford–Gegenbauer polynomials related to the Dunkl Dirac operator, *Bull. Belg. Math. Soc. Simon Stevin* **18** (2011), 193–214.

[6] De Bie H., De Schepper N., Sommen F., The class of Clifford–Fourier transforms, *J. Fourier Anal. Appl.* **17** (2011), 1198–1231, arXiv:1101.1793.

[7] De Bie H., Ørsted B., Somberg P., Souček V., Dunkl operators and a family of realizations of $\mathfrak{osp}(1|2)$, *Trans. Amer. Math. Soc.* **364** (2012), 3875–3902, arXiv:0911.4725.

[8] De Bie H., Xu Y., On the Clifford–Fourier transform, *Int. Math. Res. Not.* **2011** (2011), no. 22, 5123–5163, arXiv:1003.0689.

[9] Delanghe R., Sommen F., Souček V., Clifford algebra and spinor-valued functions. A function theory for the Dirac operator, *Mathematics and its Applications*, Vol. 53, Kluwer Academic Publishers Group, Dordrecht, 1992.

[10] Dunkl C.F., Differential-difference operators associated to reflection groups, *Trans. Amer. Math. Soc.* **311** (1989), 167–183.

[11] Dunkl C.F., Hankel transforms associated to finite reflection groups, in Hypergeometric Functions on Domains of Positivity, Jack Polynomials, and Applications (Tampa, FL, 1991), *Contemp. Math.*, Vol. 138, Amer. Math. Soc., Providence, RI, 1992, 123–138.

[12] Dunkl C.F., de Jeu M.F.E., Opdam E.M., Singular polynomials for finite reflection groups, *Trans. Amer. Math. Soc.* **346** (1994), 237–256.

[13] Dunkl C.F., Xu Y., Orthogonal polynomials of several variables, *Encyclopedia of Mathematics and its Applications*, Vol. 81, Cambridge University Press, Cambridge, 2001.

[14] Erdélyi A., Magnus W., Oberhettinger F., Tricomi F.G., Higher transcendental functions, Vol. II, Mc-Graw Hill, New York, 1953.

[15] Frappat L., Sciarrino A., Sorba P., Dictionary on Lie algebras and superalgebras, Academic Press Inc., San Diego, CA, 2000.

[16] Gilbert J.E., Murray M.A.M., Clifford algebras and Dirac operators in harmonic analysis, *Cambridge Studies in Advanced Mathematics*, Vol. 26, Cambridge University Press, Cambridge, 1991.

[17] Howe R., The oscillator semigroup, in The Mathematical Heritage of Hermann Weyl (Durham, NC, 1987), *Proc. Sympos. Pure Math.*, Vol. 48, Amer. Math. Soc., Providence, RI, 1988, 61–132.

[18] Humphreys J.E., Reflection groups and Coxeter groups, *Cambridge Studies in Advanced Mathematics*, Vol. 29, Cambridge University Press, Cambridge, 1990.

[19] Kobayashi T., Mano G., Integral formulae for the minimal representation of $\mathrm{O}(p,2)$, *Acta Appl. Math.* **86** (2005), 103–113.

[20] Kobayashi T., Mano G., The inversion formula and holomorphic extension of the minimal representation of the conformal group, in Harmonic Analysis, Group Representations, Automorphic Forms and Invariant Theory, *Lect. Notes Ser. Inst. Math. Sci. Natl. Univ. Singap.*, Vol. 12, World Sci. Publ., Hackensack, NJ, 2007, 151–208, math.RT/0607007.

[21] Kobayashi T., Mano G., The Schrödinger model for the minimal representation of the indefinite orthogonal group $\mathrm{O}(p,q)$, *Mem. Amer. Math. Soc.* **213** (2011), no. 1000, vi+132 pages, arXiv:0712.1769.

[22] Ol'shanskiĭ G.I., Complex Lie semigroups, Hardy spaces and the Gel'fand–Gindikin program, *Differential Geom. Appl.* **1** (1991), 235–246.

[23] Ørsted B., Somberg P., Souček V., The Howe duality for the Dunkl version of the Dirac operator, *Adv. Appl. Clifford Algebr.* **19** (2009), 403–415.

[24] Rösler M., A positive radial product formula for the Dunkl kernel, *Trans. Amer. Math. Soc.* **355** (2003), 2413–2438, math.CA/0210137.

[25] Rösler M., Dunkl operators: theory and applications, in Orthogonal Polynomials and Special Functions (Leuven, 2002), *Lecture Notes in Math.*, Vol. 1817, Springer, Berlin, 2003, 93–135, math.CA/0210366.

[26] Rösler M., Generalized Hermite polynomials and the heat equation for Dunkl operators, *Comm. Math. Phys.* **192** (1998), 519–542, q-alg/9703006.

[27] Szegö G., Orthogonal polynomials, *American Mathematical Society, Colloquium Publications*, Vol. 23, 4th ed., American Mathematical Society, Providence, R.I., 1975.

[28] Thangavelu S., Xu Y., Convolution operator and maximal function for the Dunkl transform, *J. Anal. Math.* **97** (2005), 25–55, math.CA/0403049.

[29] Watson G.N., A treatise on the theory of Bessel functions, Cambridge University Press, Cambridge, 1944.