Noncommutative Lagrange Mechanics*

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Abstract. It is proposed how to impose a general type of "noncommutativity" within classical mechanics from first principles. Formulation is performed in completely alternative way, i.e. without any resort to fuzzy and/or star product philosophy, which are extensively applied within noncommutative quantum theories. Newton–Lagrange noncommutative equations of motion are formulated and their properties are analyzed from the pure geometrical point of view. It is argued that the dynamical quintessence of the system consists in its kinetic energy (Riemannian metric) specifying Riemann–Levi-Civita connection and thus the *inertia* geodesics of the free motion. Throughout the paper, "noncommutativity" is considered as an internal geometric structure of the configuration space, which can not be "observed" *per se*. Manifestation of the noncommutative phenomena is mediated by the interaction of the system with noncommutative background under the consideration. The simplest model of the interaction (minimal coupling) is proposed and it is shown that guiding affine connection is modified by the quadratic analog of the Lorentz electromagnetic force (contortion term).

Key words: noncommutative mechanics; affine connection; contortion

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Dedicated to my daughter Zoe on the occasion of her birth

1 Introduction and motivation

Physical models involving "noncommutativity" have become very popular and have been extensively studied in the last decade. Strong impact comes from string theory. It is well known how the noncommutativity appears in D-branes, when the open string dynamics is analyzed in the presence of constant B-field [1]. Another plumbless source of the "noncommutativity" is pure mathematics. New mathematical ideas and structures are applicable when speculating about the microscopical structure of the space-time. There is a common belief that this structure is smashed by some fundamental uncertainty and that precise localization of its events is unreachable [2]. The concept of the point is lacking sense and physics is modeled on "noncommutative space-time." The mathematical language of noncommutative physics shifts from visual geometry to rather abstract algebra. The focus of *Noncommutative Geometry* is nowadays really wide. It includes quantum groups, K-theory and Fredholm Modules, (cyclic) cohomology and index theory, deformation quantization, fuzzy geometry and so on [3].

There is a lot of articles and dissertations elaborating the ongoing noncommutative business. Since my aim is not to trace out its history and all detailed circumstances, I refer here to survey articles [4] that will certainly fill this gap (see also patulous reference lists therein).

Noncommutative quantum mechanics is the subject of studies at [5, 6]. The common point of the first group of papers lies in simple replacement of the canonical Poisson brackets on the

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phase space (cotangent bundle) by modified NC-brackets. Switching from Classical to Quantum Mechanics is performed in the spirit of Dirac's canonical quantization dictum or via the path integral approach. However, in [6], there are proposed various field theoretical models which yield, under some circumstances, to the effective quantum dynamics corresponding to the motion on the noncommutative (Moyal) plane.

This article is about a similar subject, but noncommutativity is handled from the classical perspective only. Some effort concerning this direction was done in [7]. Authors of those papers consider noncommutative classical mechanics in the fashion of symplectic and/or Poisson geometry. They substitute the canonical two-form (Poisson brackets) on the phase space by modified quantity, which appears in the dequantization limit $\hbar \to 0$ in the Weyl–Wigner–Moyal star product. What they call as noncommutative mechanics is the standard classical mechanics in the Hamilton's picture, but with the "modified" symplectic two-form. The main objection to such "noncommutative treatment" is that according to Darboux theorem all symplectic structures are locally equivalent, i.e. when performing a suitable change of coordinates one gets from the canonical two-form endowing the phase space the two-form, which comes from the Weyl–Wigner–Moyal "dequantization". Therefore noncommutative dynamical equations derived in [7] are ordinary Hamilton equation of motion, but they are only rewritten in different coordinates. The only essence would be, if one would be able to canonize some class of coordinates as special and physically privileged. But than the covariance and treasury of the coordinate free formulation of Hamilton mechanics become broken and results start to depend on the observer.

The following paper, therefore, deals noncommutative effects in completely alternative way, i.e. without any resorts to its Hamiltonian and/or Lagrangian precursors. Our starting point is dynamics which is represented by classical equations of motion and its relation to an affine connection on a underlying configuration space M. This connection can be specified by kinetic energy which defines the system and by external forces which produce a disturbance from the free geodesic motion. The noncommutativity is imposed as an additional internal Poisson structure (non-constant bi-vector) on the configuration manifold M. Its presence together with the kinetic energy gives rise to a natural nonzero contortion term. From the point of view of "commutative RLC-connection" this term can be interpreted as a background noncommutative "Lorentz-like" force which affects the parallel transport and thus the free motion of the system. It will be shown that this new contortion force is always perpendicular to the actual velocity and therefore the total mechanical energy is conserved. Moreover, one can convince him/herself that this extra noncommutative background force field can not be derived from a potential energy function. Therefore the concept of Lagrangian/Hamiltonian is completely missing and there rises a natural question what will be a reasonable quantum analog of the proposed classical mechanics whose dynamics is governed by the Newton–Lagrange noncommutative equations of motion. It will be clear that the Moyal–Kontsevich star product which can be considered due to present (non-constant) Poisson structure on M does not provide satisfying answer.

2 Preliminaries

In the following section we recall some elementary geometry picturing Lagrange mechanics [8].

An autonomous mechanical system with m degrees of freedom occupies configuration space, a m-dimensional smooth manifold M. Its dynamics is governed by the kinetic energy (non-singular, positive) tensor $g \in \Gamma(S^2 T^*M)$ and the co-vectorial strength field $Q \in \Gamma(T^*M)$ (in general velocity dependent) describing forces that act within the system:

$$\frac{d}{dt}\left(\frac{\partial g}{\partial \dot{x}^a}\right) - \frac{\partial g}{\partial x^a} = Q_a, \quad a = 1, \ldots, m, \qquad \text{and} \qquad g(x, \dot{x}) = \tfrac{1}{2} g_{kl}(x) \dot{x}^k \dot{x}^l.$$

In the special case of exact strength $Q = -dU$ (with potential $U \in C^\infty(M)$), one can introduce the Lagrangian function $\mathrm{L} : TM \to \mathbb{R}$:

$$\big(p \in M, v \in T_pM\big) \in TM \longmapsto \mathrm{L}(p,v) := g_p(v,v) - U(p) \in \mathbb{R}$$

and write down the celebrated Euler–Lagrange equations.

Natural setting for classical mechanics is a tangent bundle of the configuration space M. Dynamics with a given initial position and velocity is then determined by a symmetric affine connection on M associated with the Riemannian kinetic tensor field g.

Let me remind the reader how it works. A general affine connection on M is defined as a m-dimensional horizontal distribution on $TM \xrightarrow{\tau} M$, $(p,v) \overset{\tau}{\mapsto} p$, which is invariant under the action of scalar multiplication in fibers: $(p,v) \overset{\alpha}{\mapsto} (p, \alpha \cdot v)$ $(\alpha \in \mathbb{R})$. Changed into the coins:

$$\forall\ (p,v) \in TM \qquad T_{(p,v)}\big(TM\big) = \mathrm{Ver}_{(p,v)} \oplus \mathrm{Hor}_{(p,v)}, \qquad \text{and} \qquad \mathrm{Ver}_{(p,v)} \cap \mathrm{Hor}_{(p,v)} = \big\{0\big\}.$$

Using a local coordinate chart $(x^a, \dot{x}^a)$ on a patch of TM, one can span the space $\mathrm{Hor}_{(p,v)}$ by a collection of m vectors:

$$\partial_{x^a}\Big|_{(p,v)}^{\mathrm{Hor}} := \frac{\partial}{\partial x^a}\Big|_{(p,v)} - A_a^b(x,\dot{x}) \frac{\partial}{\partial \dot{x}^b}\Big|_{(p,v)}.$$

Requiring the invariance w.r.t. the mentioned α-scaling: $\alpha_*\big(\mathrm{Hor}_{(p,v)}\big) = \mathrm{Hor}_{(p,\alpha\cdot v)}$, we conclude that $A_a^b(x,\dot{x}) = \dot{x}^c \mathit{\Gamma}_{ca}^b(x)$. Here $\mathit{\Gamma}$ stands for a set of m^3 Christoffel symbols.

Prescribing an affine connection on M, one is able to perform a horizontal lift of smooth curve $\gamma : [0,1] \to M$ to a tangent bundle curve $\widehat{\gamma} : [0,1] \to TM$. Concisely, suppose that at a given time t we are occupying the tangent bundle point $\widehat{\gamma}(t) = (p,v)$ and a tangent vector to $\gamma(t)$ at $p = \tau(p,v)$ is $\dot{\gamma}(p) \in T_pM$. Then, after lapsing an infinitesimally short time ε, the new tangent bundle position $\widehat{\gamma}(t+\varepsilon)$ will be specified by an ε-step in the direction of the horizontal lift $\dot{\gamma}(p)\big|_{(p,v)}^{\mathrm{Hor}}$. In other words, in coordinate patch $(x^a, \dot{x}^a)$ we have the following system of coupled ordinary differential equations for $\widehat{\gamma}$:

$$x^a(t+\varepsilon) = x^a(t) + \varepsilon\dot{\gamma}^a(t), \qquad \dot{x}^a(t+\varepsilon) = \dot{x}^a(t) - \varepsilon\dot{\gamma}^c(t)\,\dot{x}^b(t)\mathit{\Gamma}_{bc}^a\big(x(t)\big). \tag{1}$$

Here $\dot{\gamma}^a(t)$ stands for the components of the tangent vector $\dot{\gamma}(t) \in T_{\gamma(t)}M$ to γ with respect to the base coordinate basis $\{\partial_{x^a}|_{\gamma(t)}\}$.

Specifying the starting point for $\widehat{\gamma}$ in TM, i.e. the initial conditions for the above differential system:

$$\Big\{x^a(0) = x^a(\gamma(0)),\ \dot{x}^a(0) = v^a\Big\} \iff v = v^a \partial_{x^a}\Big|_{\gamma(0)}$$

we are able to perform a parallel transport of the vector $v \in T_{\gamma(0)}M$ along γ. Parallel transported vector is the solution of (1) at the final time $t = 1$, i.e. $\dot{x}^a(1)\partial_{x^a}|_{\gamma(1)}$.

For a fixed connection, there is a special class of curves that are called geodesics. Curve $\gamma : [0,1] \to M$ is a geodesic line on M, if parallel transport of $\dot{\gamma}(0)$ along γ coincides at each time t with its actual tangent vector $\dot{\gamma}(t)$. In other words, being in a local chart on M, the geodesic γ satisfies the system of second order differential equations:

$$\ddot{x}^a\big(\gamma(t)\big) = -\dot{x}^c\big(\gamma(t)\big)\dot{x}^b\big(\gamma(t)\big)\mathit{\Gamma}_{bc}^a\big(x\big(\gamma(t)\big)\big).$$

With any affine connection one can associate its torsion and curvature (for more details see [9]). Torsion of two vectors $u, v \in T_pM$ is a vector $\mathbf{T}_p(u,v)$ at point p, which closes an infinitesimal geodesic parallelogram framed on u, v up to second order. Thus

$$\mathbf{T}_p(u,v) = u^b v^c \mathbf{T}_{bc}^a(p)\partial_{x^a}\Big|_p = u^b v^c \Big\{\mathit{\Gamma}_{cb}^a - \mathit{\Gamma}_{bc}^a\Big\}(p)\partial_{x^a}\Big|_p.$$

It is clear that torsion is an element of $\Gamma\big(\bigwedge^2 T^*M \otimes TM\big)$, i.e. it is skew-symmetric in the subscript indices.

Curvature associated with tangent vectors $u, v \in T_pM$ is a linear map, $(1,1)$-tensor, $\mathbf{R}_p(u,v) : T_pM \to T_pM$. Again, analogically as before, the curvature measures how a vector w changes, up to second order, after parallel transport along an infinitesimal closed parallelogram spanned on vectors u and v. Concisely,

$$\mathbf{R}_p\,(u,v)(w) = w^b u^c v^d\,\mathbf{R}^a_{bcd}(p)\partial_{x^a}\Big|_p = w^b u^c v^d \Big\{\mathit{\Gamma}^a_{bd,c} - \mathit{\Gamma}^a_{bc,d} + \mathit{\Gamma}^i_{bd}\mathit{\Gamma}^a_{ic} - \mathit{\Gamma}^i_{bc}\mathit{\Gamma}^a_{id}\Big\}(p)\partial_{x^a}\Big|_p.$$

We shall now describe how this is related to the Lagrange mechanics. The kinetic energy tensor g, due to its symmetry, can be equivalently interpreted as a quadratic function on TM w.r.t. fiber (doted) coordinates. Using a local chart, one has:

$$g = \tfrac{1}{2} g_{ab}(x)\dot{x}^a\dot{x}^b.$$

Lets define the horizontal distribution at tangent bundle point $(p\,,v)$ by

$$\mathrm{Hor}_{(p,v)} := \Big\{ w\big|_{(p,v)} = w^a\partial_{x^a}\big|^{\mathrm{Hor}}_{(p,v)} \in T_{(p,v)}(TM);\; w\big|_{(p,v)}(g) = 0 \Big\}.$$

Requiring also that it is torsionless, one gets a system of equations for the Riemann–Levi-Civita connection:

$$\partial_{x^c}(g_{ab}) \equiv g_{ab,c} = g_{ai}\mathit{\Gamma}^i_{bc} + g_{bi}\mathit{\Gamma}^i_{ac}, \qquad 0 = \mathit{\Gamma}^a_{cb} - \mathit{\Gamma}^a_{bc}.$$

Regularity of the $(m\times m)$ matrix g ensures the unique solution of the above algebraic equations:

$$\mathit{\Gamma}^a_{bc} = \tfrac{1}{2}\,g^{ai}\big\{g_{ib,\,c} + g_{ic,\,b} - g_{bc,\,i}\big\} \equiv \left\{ {a \atop b\,c} \right\}_{\mathrm{RLC}}$$

and the existence of the canonical (musical[1]) isomorphism $\sharp_g : T^*M \to TM$. Therefore the external strength described by the co-vectorial field $Q = Q_a(x,\dot{x})dx^a$ can be turned into a force vector field $\sharp_g(Q) = (g^{ab}Q_b)\partial_a = F^a\partial_a = F$.

Suppose that a mechanical system at time t is surrounded at given point $p \in M$ and possesses the velocity $v \in T_pM$. Then the new position and velocity after the infinitesimal time interval ε is determined by the Lagrange dynamical vector field evaluated at (p,v)

$$\mathcal{L}\Big|_{(p,v)} = v\Big|^{\mathrm{Hor}}_{(p,v)} + F\Big|^{\mathrm{Ver}}_{(p,v)} = v^a\partial_{x^a}\Big|^{\mathrm{Hor}}_{(p,v)} +\; F^a(p\,,v)\partial_{\dot{x}^a}\Big|_{(p,v)} \in T_{(p,v)}(TM). \tag{2}$$

Expressing this dynamics in local coordinates $(x^a,\dot{x}^a)$, one gets the following system of differential equations:

$$x^a(t+\varepsilon) = x^a(t) + \varepsilon\dot{x}^a(t),$$
$$\dot{x}^a(t+\varepsilon) = \dot{x}^a(t) - \varepsilon\dot{x}^c(t)\dot{x}^b(t)\mathit{\Gamma}^a_{bc}\big(x(t)\big) + \varepsilon F^a\big(x(t),\dot{x}(t)\big).$$

It is straightforward to see that these equations are exactly the Euler–Lagrange equations from the beginning of this introductory paragraph. One just needs to replace the general $\mathit{\Gamma}^a_{bc}$ by the RLC Christoffel symbols $\left\{ {a \atop b\,c} \right\}_{\mathrm{RLC}}$.

[1] $\sharp_g$ symbolizes the index raising with the help of dual (inverse) matrix $g^{ai}(\cdot g_{ib} = \delta^a_b)$, for the index lowering with g_{ai} there is symbol $\flat_g$.

3 Noncommutative Lagrange mechanics

In the previous section we have seen that the mechanical system would evolve geodesically. The only disturbance of such motion is caused by the presence of external forces. Thus we can conclude that the pivotal object defining the "theory" consists in its internal kinetic energy tensor g (lets call it a metric). States of the mechanical system are labeled by points of the tangent bundle of the underlying configuration space M (no internal degrees of freedom are considered).

"Noncommutativity" is an internal property of M coming a priori from the Nature. In general it is specified by nonconstant Poisson brackets defined on $C^\infty(M)$, i.e. by a certain Poisson bi-vector field $\Pi \in \Gamma(\bigwedge^2 TM)$. Let us recall that locally:

$$\Pi = \tfrac{1}{2}\Pi^{ab}(x)\big\{\partial_{x^a}\otimes\partial_{x^b} - \partial_{x^b}\otimes\partial_{x^a}\big\} \iff \{\mathrm{f},\mathrm{h}\} := \Pi(d\mathrm{f}, d\mathrm{h}) \equiv \mathrm{f}_{,a}\Pi^{ab}\mathrm{h}_{,b}.$$

In terms of the bi-vector Π the Jacobi identity is equivalent to the vanishing of the Schouten brackets of the bi-vector Π with itself, i.e.

$$0 = [\Pi,\Pi]_{\text{Schouten}} \iff \Big\{\Pi^{ai}(\Pi^{bc})_{,i} + \text{cyclic permutation in } (a,b,c)\Big\} = 0.$$

The presence of the Poisson structure on M enables us to quantize the algebra of functions on M (see [10]). Quantization is a formal one parameter deformation of the ordinary pointwise product on $C^\infty(M)$. Resulting noncommutative associative star-product algebra $\big(C^\infty(M)[[\hbar]], \star_\hbar\big)$ satisfies (in the semiclassical regime):

$$\lim_{\hbar\to 0}\frac{1}{\hbar}\big(\mathrm{f}\star_\hbar \mathrm{h} - \mathrm{h}\star_\hbar \mathrm{f}\big) = \Pi(d\mathrm{f}, d\mathrm{h}).$$

What physicists are quite often doing when discussing the noncommutative theories (see e.g. [4]), is a replacement of the ordinary pointwise product in the governing action (Lagrangian density) by an appropriate star-product $\star_\hbar$ and going to quantum theory. The techniques and dictionary used for analyzing of the appearing NC-problems are classical (commutative) ones.

If the noncommutativity is manifestly present in Nature, then it should somehow more conceptually affect matters also on the classical level. Not only in partial changes in the guiding Lagrangian and/or Hamiltonian, but directly, modifying the classical equations of motion. These are rather more fundamental than the Lagrangian and/or Hamiltonian themselves. Therefore what I am going to do below, is to pursue this direction.

Suppose that the classical "theory" is specified by the metric g and occupies the noncommutative configuration space (M, Π). Can one use those ingredients to create something new? Let us consider the $(1,1)$-tensor lowering one index of Π with the help of g (this I called the minimal coupling of the "theory" with the "noncommutativity"), i.e.

$$\mathrm{Contr}(g\otimes\Pi) = dx^a g_{ai}(x)\Pi^{ib}(x)\partial_{x^b} = dx^a \Pi_a^b(x)\partial_{x^b} = \flat_g \Pi.$$

With any $(1,1)$-tensor A one can associate the special $(1,2)$-tensor called its Nijenhuis torsion [11]. It is defined as follows:

$$N_A(X,Y) := \big[A(X), A(Y)\big] - A\Big(\big[A(X), Y\big] + \big[X, A(Y)\big] - A\big(\big[X,Y\big]\big)\Big).$$

Here the arguments X and Y are arbitrary vector fields on M and $[\,,\,]$ stands for their commutator. Checking $C^\infty(M)$-linearity and exhibiting its properties w.r.t. argument interchange, one immediately concludes that $N_A \in \Gamma(\bigwedge^2 T^*M \otimes TM)$, i.e. it is skew-symmetric in the subscript co-vectorial indices.

What do we get when substituting $\flat_g\Pi$ for A? Nijenhuis torsion of $\flat_g\Pi$ is expressed in the local coordinate chart on M as follows:

$$N_{\flat_g\Pi} = dx^b \wedge dx^c\, \tfrac{1}{2}\Big\{\Pi^i_b(\Pi^a_c)_{,i} + \Pi^a_i(\Pi^i_b)_{,c}\Big\}\partial_{x^a}. \tag{3}$$

The $(1,2)$-tensor $N_{\flat_g\Pi}$ appears on M very naturally. It reflects the properties of both entering entities: "theory" g and "noncommutativity" Π. Therefore it is quite reasonable to postulate: *the affine connection for the NC-classical mechanics is the metric g connection, whose torsion is $N_{\flat_g\Pi}$*. This asks us to solve the modified Riemann–Levi-Civita equations:

$$g_{ab,\,c} = g_{ai}\Gamma^i_{bc} + g_{bi}\Gamma^i_{ac}, \qquad \mathbf{T}^a_{bc} = \Gamma^a_{cb} - \Gamma^a_{bc},$$

with

$$\mathbf{T}^a_{bc} = \Pi^i_b(\Pi^a_c)_{,i} - \Pi^i_c(\Pi^a_b)_{,i} + \Pi^a_i(\Pi^i_b)_{,c} - \Pi^a_i(\Pi^i_c)_{,b} = \big(N_{\flat_g\Pi}\big)^a_{bc}. \tag{4}$$

The final formulae for Γ's are given as:

$$\Gamma^a_{bc} = \tfrac{1}{2}g^{ai}\big\{g_{ib,c} + g_{ic,b} - g_{bc,i}\big\} - \tfrac{1}{2}g^{ai}\big\{g_{ij}\mathbf{T}^j_{bc} + g_{bj}\mathbf{T}^j_{ci} + g_{cj}\mathbf{T}^j_{bi}\big\} = \left\{ {a \atop b\,c} \right\}_{\rm RLC} - \mathbf{K}^a_{bc}.$$

The dynamics of a given state (p = position, v = velocity) at time t is governed by the infinitesimal flow of the Lagrange dynamical vector field (2) evaluated at $(p\,,v) \in TM$. Let us stress that its horizontal part is now specified by the modified set of Christoffel symbols. Apart from the standard (commutative) RLC-term there appears the additional tensorial part $\mathbf{K}^a_{bc}$ called the *contortion*. Thus the NC-Lagrange equations can be written as follows:

$$\ddot{x}^a + \dot{x}^c\dot{x}^b \left\{ {a \atop b\,c} \right\}_{\rm RLC} = F^a + \dot{x}^c\dot{x}^b\mathbf{K}^a_{bc}, \qquad a = 1,\ldots,m. \tag{5}$$

Using the optics of the standard Lagrange mechanics: the "noncommutativity" can be treated as an additional "internal" force. Its origin comes from the presence of the nonzero contortion. To reveal its physical consequences let us recall what geometrically the contortion is responsible for. Parallel transport (1) along γ determined by $\Gamma^a_{bc} = \left\{ {a \atop b\,c} \right\}_{\rm RLC} - \mathbf{K}^a_{bc}$ can be split into two steps:

$$v^a \longmapsto v^a_{\rm r} = v^a + \varepsilon v^b\big[\mathbf{K}^a_{bc}\dot{\gamma}^c\big] + o(\varepsilon) \longmapsto v^a_{\rm r} - \varepsilon\dot{\gamma}^c v^b_{\rm r}\left\{ {a \atop b\,c} \right\}_{\rm RLC} + o(\varepsilon).$$

The first arrow is an infinitesimal linear transformation determined by $\mathbf{K}(\cdot,\dot{\gamma})$ that takes place in the same tangent space as the initial vector v. The second one is just the standard Levi-Civita parallel transport of $v_{\rm r}$. Since the considered connection is compatible with the metric, both v and $v_{\rm r}$ have the same lengths and therefore $\mathbf{K}(\cdot,\dot{\gamma})$ is a generator of some rotation (it depends on the tangent vector $\dot{\gamma}$ to the transporting curve γ). Thus the effect of additional noncommutative "Lorentz-like" force is an infinitesimal rotation of actual velocity that precedes the next "commutative" RLC-step. Let us recall that mechanics with non zero torsion is the subject of study at [12], similar problem from the general relativity point of view is analyzed in [13].

Example. Suppose that M is ordinary two-dimensional plane, which is endowed by the flat metric $g = \frac{1}{2}\{dx \otimes dx + dy \otimes dy\}$. Noncommutativity on M is imposed by a general Poisson bracket:

$$\{x, y\} = \Theta(x, y), \qquad \text{here} \quad \Theta(x, y) \text{ is any (well-behaved) function on } \mathbb{R}^2.$$

The dynamics of a free particle is then governed by the NC-force, whose Cartesian components are given as follows:

$$F^x_{\rm NC} = \dot{y}\Theta\{\dot{x}\Theta_{,y} - \dot{y}\Theta_{,x}\}, \qquad \text{and} \qquad F^y_{\rm NC} = -\dot{x}\Theta\{\dot{x}\Theta_{,y} - \dot{y}\Theta_{,x}\}.$$

4 Concluding remarks

• The dynamical system we have considered was an autonomous one. But in a general metric, noncommutativity, as well as the external strengths, could be explicitly time dependent. In that case, one should work with the connection on the first jet-extension of $M \times \mathbb{R}$. Emerging complications are mainly technical.

• The Lorentz-like force specifying the coupling of the noncommutative background with the system under consideration comes from pure geometrical considerations. Very similar magnetic-like background effect of the noncommutativity emerges also in the standard Moyal plane approach, see [5, 6, 7]. Here it is due to the affine connection with the nonzero contortion. There it is so, because

$$[x_a, x_b] = i\Theta_{ab} \qquad (a, b = 1, 2)$$

in two-dimensions provides an effective coupling with the axial magnetic field $\mathrm{B}_3 = \Theta_{12}$. Both approaches to the noncommutative mechanics are therefore ideologically independent and physically inequivalent. This can be easily seen for example, if $\Theta_{ab} = \mathrm{const}\,\epsilon_{ab}$. Then there is a nonzero magnetic field $\mathbf{B} = (0, 0, \mathrm{B}_3)$, but since $\mathbf{K}^c_{ab}$ depends on derivatives of Θ_{ab}, the contortion term is zero.

• The question of symmetries of equations of motion has an obvious solution. The base vector field $V = V^i(x)\partial_{x^i} \in \Gamma(TM)$ is a generator of symmetry of the considered dynamical system specified by (5), if its complete (natural, also called horizontal) tangent bundle lift

$$V^{\mathrm{c}} = V^a(x)\partial_{x^a} + \dot{x}^b\big(V^a(x)\big)_{,b}\partial_{\dot{x}^a},$$

commutes with the Lagrange dynamical field $\mathcal{L}$. Let me remind the reader that $\mathcal{L}$ and V^{c} are both the vector fields defined on the tangent bundle TM.

• The additional contortion dependent force emerging in (5) is not potential-generated neither generalized potential-generated, i.e. there does not exist a tangent bundle potential function $U(x, \dot{x})$, such that

$$\mathbf{K}_{abc}\dot{x}^b\dot{x}^c \equiv g_{ai}\mathbf{K}^i_{bc}\dot{x}^b\dot{x}^c = -\frac{\partial U}{\partial x^a} + \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{x}^a}\right).$$

The proposed NC-dynamics is therefore not derivable from some "noncommutative" Lagrangian and/or Hamiltonian function in the obvious fashion. The question about the possible quantization thus remains open (at least for the author of this paper). Since we do not have Hamiltonian H, we can not write:

$$\dot{x} = \{x, H\}_{\star_\hbar}, \qquad \dot{p} = \{p, H\}_{\star_\hbar}, \qquad \text{where} \qquad \{\mathrm{f}, \mathrm{h}\}_{\star_\hbar} := \mathrm{f} \star_\hbar \mathrm{h} - \mathrm{h} \star_\hbar \mathrm{f}$$

and the Moyal–Kontsevich $\star_\hbar$-product is defined with respect to the given Poisson structure Π which specifies the noncommutative background under consideration.

• Suppose that the external forces are potential-generated, then the classical energy can be introduced as standardly:

$$\mathrm{E} = \tfrac{1}{2}g_{ab}(x)\dot{x}^a\dot{x}^b + U(x) = T + U.$$

Verification that E is conserved by the NC-dynamics is straightforward[2].

[2] Concisely, take the scalar product of (5) with the actual velocity $\dot{x}^a$ and use the mentioned "Lorentz-likeness" of the additional contortion term $\dot{x}^a\mathbf{K}_{abc}\dot{x}^b\dot{x}^c = 0 = g(v, \mathbf{K}(v, v))$, then

$$g_{ab}\dot{x}^a\left[\ddot{x}^b + \dot{x}^i\dot{x}^j\left\{ {b \atop j\,i} \right\}_{\mathrm{RLC}}\right] = g_{ab}\dot{x}^a\ddot{x}^b + \tfrac{1}{2}g_{ab,i}\dot{x}^a\dot{x}^b\dot{x}^i (= \dot{T}) \stackrel{(5)}{=} -\dot{x}^a U_{,a} + 0 = -\dot{U} \implies \dot{\mathrm{E}} = 0.$$

• The dynamics in the Hamiltonian "picture" is just a simple combination of cotangent bundle parallel transport[3] and vertical lift of the co-vectorial strength field Q. Suppose that at some time t the mechanical system occupies a cotangent bundle state ($p = \text{position}, \alpha = \text{momentum}$), then at infinitesimally close time the new T^*M-position is determined by the flow of the Hamilton dynamical field:

$$\mathcal{H}\Big|_{(p,\alpha)} = \big(\sharp_g \alpha\big)\Big|^{\text{Hor}}_{(p,\alpha)} + Q\Big|^{\text{Ver}}_{(p,\alpha)} \in T_{(p,\alpha)}(T^*M).$$

• The configuration (Riemannian) space (M, g) could be at any time replaced by a pseudo-Riemannian space-time manifold. Then one can easily impose "noncommutativity" within general relativity modifying the underlaying metric connection by non zero torsion terms. General relativity with torsion is analyzed at [13].

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