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Central Configurations and Mutual Differences

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Abstract. Central configurations are solutions of the equations $\lambda m_j \boldsymbol{q}_j = \frac{\partial U}{\partial \boldsymbol{q}_j}$, where U denotes the potential function and each $\boldsymbol{q}_j$ is a point in the d-dimensional Euclidean space $E \cong \mathbb{R}^d$, for $j = 1, \dots, n$. We show that the vector of the mutual differences $\boldsymbol{q}_{ij} = \boldsymbol{q}_i - \boldsymbol{q}_j$ satisfies the equation $-\frac{\lambda}{\alpha}\boldsymbol{q} = P_m(\Psi(\boldsymbol{q}))$, where P_m is the orthogonal projection over the spaces of 1-cocycles and $\Psi(\boldsymbol{q}) = \frac{\boldsymbol{q}}{|\boldsymbol{q}|^{\alpha+2}}$. It is shown that differences $\boldsymbol{q}_{ij}$ of central configurations are critical points of an analogue of U, defined on the space of 1-cochains in the Euclidean space E, and restricted to the subspace of 1-cocycles. Some generalizations of well known facts follow almost immediately from this approach.

Key words: central configurations; relative equilibria; n-body problem

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1 Introduction

Central configurations play an important role in the (Newtonian) n-body problem: to name two, they arise as configurations yielding homographic solutions, and as rest points in the flow on the McGehee collision manifold. Following the spirit of Albouy and Chenciner [2], in this article we study the problem of central configurations from the point of view of mutual distances; but instead of lengths we consider the space of *differences* of positions, which turns out to be a suitable group of cochains C^1 of degree 1 with coefficients in the Euclidean space E. Hence, we show that central configurations are critical points of a function defined on C^1 and restricted to the subspace of 1-cocycles, and show some consequences. The technique of embedding the central configurations problem into a suitable space of cocycles was actually already used by Moeckel in [12], in an implicit way, and again by Moeckel and Montgomery in [15]. In this article we study this approach introducing cocycles and cohomology, and show that many calculations can be significantly simplified in this way. For further details and recent remarkable advances we refer to [3, 8].

More precisely, assume $n \geq 2$, $d \geq 1$. Let $E = \mathbb{R}^d$ denote the d-dimensional Euclidean space. An element of E^n will be denoted by $\boldsymbol{q} = (\boldsymbol{q}_1, \boldsymbol{q}_2, \dots, \boldsymbol{q}_n)$ where $\forall\, j$, $\boldsymbol{q}_j \in E$. Let $\mathbb{F}_n(E)$ denote as in [4] the configuration space of n particles in E:

$$\mathbb{F}_n(E) = \big\{\boldsymbol{q} \in E^n \colon \boldsymbol{q}_i \neq \boldsymbol{q}_j\big\}.$$

If Δ is the collision set

$$\Delta = \bigcup_{i<j} \big\{\boldsymbol{q} \in E^n \colon \boldsymbol{q}_i = \boldsymbol{q}_j\big\},$$

then $\mathbb{F}_n(E) = E^n \setminus \Delta$.

For $j = 1, \dots, n$, let $m_j > 0$ be positive masses. Assume that the masses are normalized, i.e., that

$$\sum_{j=1}^{n} m_j = 1. \tag{1.1}$$

Let $\langle *, * \rangle_M$ denote the *mass-metric* on (the tangent vectors of) E^n, defined as

$$\langle \boldsymbol{v}, \boldsymbol{w} \rangle_M = \sum_{j=1}^{n} m_j \boldsymbol{v}_j \cdot \boldsymbol{w}_j,$$

where $\boldsymbol{v}_j \cdot \boldsymbol{w}_j$ is the Euclidean scalar product in (the tangent space of) E. Let $|\boldsymbol{v}_j|$ denote the Euclidean norm of a vector $\boldsymbol{v}_j$ in E. The norm corresponding to the mass-metric is $\|\boldsymbol{v}\|_M = \sqrt{\langle \boldsymbol{v}, \boldsymbol{v} \rangle_M}$.

Let $\alpha > 0$ be a fixed homogeneity parameter, and $U \colon \mathbb{F}_n(E) \to \mathbb{R}$ the potential function defined as

$$U(\boldsymbol{q}) = \sum_{1 \leq i < j \leq n} \frac{m_i m_j}{|\boldsymbol{q}_i - \boldsymbol{q}_j|^\alpha}.$$

A *central configuration* is a configuration $\boldsymbol{q} \in \mathbb{F}_n(E)$ such that there exists $\lambda \in \mathbb{R}$ such that $(\forall\, j = 1, \dots, n)$

$$\lambda m_j \boldsymbol{q}_j = \frac{\partial U}{\partial \boldsymbol{q}_j} = -\alpha \sum_{k \neq j} m_j m_k \frac{\boldsymbol{q}_j - \boldsymbol{q}_k}{|\boldsymbol{q}_j - \boldsymbol{q}_k|^{\alpha+2}}. \tag{1.2}$$

If $\boldsymbol{q}$ is a central configuration, then

$$\begin{aligned}
\lambda \sum_j m_j |\boldsymbol{q}_j|^2 &= -\alpha \sum_{j=1}^{n} \sum_{k \neq j, k=1}^{n} m_j m_k \frac{(\boldsymbol{q}_j - \boldsymbol{q}_k) \cdot \boldsymbol{q}_j}{|\boldsymbol{q}_j - \boldsymbol{q}_k|^{\alpha+2}} \\
&= -\alpha \sum_{j<k} m_j m_k \frac{(\boldsymbol{q}_j - \boldsymbol{q}_k) \cdot \boldsymbol{q}_j + (\boldsymbol{q}_k - \boldsymbol{q}_j) \cdot \boldsymbol{q}_k}{|\boldsymbol{q}_j - \boldsymbol{q}_k|^{\alpha+2}} = -\alpha \sum_{j<k} \frac{m_j m_k}{|\boldsymbol{q}_j - \boldsymbol{q}_k|^\alpha} \\
\implies \lambda \|\boldsymbol{q}\|_M^2 &= -\alpha U(\boldsymbol{q}),
\end{aligned}$$

and hence $\lambda = -\alpha \frac{U(\boldsymbol{q})}{\|\boldsymbol{q}\|_M^2} < 0$. By summing equation (1.2) in j

$$\lambda \sum_j m_j \boldsymbol{q}_j = -\alpha \sum_{j<k} m_j m_k \frac{(\boldsymbol{q}_j - \boldsymbol{q}_k) + (\boldsymbol{q}_k - \boldsymbol{q}_j)}{|\boldsymbol{q}_j - \boldsymbol{q}_k|^{\alpha+2}} = \boldsymbol{0}$$

and hence

$$\sum_j m_j \boldsymbol{q}_j = \boldsymbol{0}.$$

For an analysis of central configurations for general potential functions $U(\boldsymbol{q})$, see [6, 7]. Also, central configurations can be equivalently seen as:

(CC1) Solutions of (1.2) [13].

(CC2) Critical points of the restriction of the potential function U to the inertia ellipsoid $S = \{\boldsymbol{q} \in \mathbb{F}_n(E) \colon \|\boldsymbol{q}\|_M^2 = 1\}$ [14].

(CC3) Fixed points of the map $F \colon S \to S$ defined as $F(\boldsymbol{q}) = -\frac{\nabla_M U(\boldsymbol{q})}{\|\nabla_M U(\boldsymbol{q})\|_M}$, where ∇_M denotes the gradient with respect to the mass-metric on $\mathbb{F}_n(E)$ [6, 7].

(CC4) Critical points on $\mathbb{F}_n(E)$ of the map $\|\boldsymbol{q}\|_M^2 + U(\boldsymbol{q})$ [9].
(CC5) Critical points on $\mathbb{F}_n(E)$ of the map $\|\boldsymbol{q}\|_M^{2\alpha} U(\boldsymbol{q})^2$ (or $\|\boldsymbol{q}\|_M^{\alpha} U(\boldsymbol{q})$) [17].

In all these formulations, central configurations appear as $O(d)$-orbits in $\mathbb{F}_n(E)$, where the action of the orthogonal group $O(d)$ on $\mathbb{F}_n(E)$ is diagonal $g \cdot \boldsymbol{q} = (g\boldsymbol{q}_1, \dots, g\boldsymbol{q}_n)$.

Define the space X as

$$X = \left\{ \boldsymbol{q} \in \mathbb{F}_n(E) \colon \sum_{j=1}^{n} m_j \boldsymbol{q}_j = \boldsymbol{0} \right\}.$$

2 Central configurations and mutual differences

Let $\boldsymbol{n}$ be the set $\boldsymbol{n} = \{1, 2, \dots, n\}$ and C^0 the vector space of all maps from $\boldsymbol{n}$ to E: $C^0 = \{\boldsymbol{q} \colon \boldsymbol{n} \to E\}$. Let $\mathbb{F}_n(E) \subset C^0$ denote the inclusion sending $\boldsymbol{q} \in \mathbb{F}_n(E)$ to the map $\boldsymbol{q} \colon \boldsymbol{n} \to E$ defined by $\boldsymbol{q}(j) = \boldsymbol{q}_j$ for each $j \in \boldsymbol{n}$.

Now, let $\tilde{\boldsymbol{n}}$ denote the set of all $\binom{n}{2}$ subsets in $\boldsymbol{n}$ with two elements: $\tilde{\boldsymbol{n}} = \{\{1,2\}, \{1,3\}, \dots, \{n-1, n\}\}$. Let C^1 denote the vector space of all maps from $\tilde{\boldsymbol{n}}$ to E:

$$C^1 = \{\boldsymbol{q} \colon \tilde{\boldsymbol{n}} \to E\}.$$

It is isomorphic to $E^{\tilde{n}}$, where $\tilde{n} = \binom{n}{2}$. Note that if $E^{\boldsymbol{n}^2}$ denotes that vector space of all maps $\boldsymbol{q} \colon \boldsymbol{n}^2 \to E$, where $\boldsymbol{n}^2 = \boldsymbol{n} \times \boldsymbol{n}$ (and hence if $\boldsymbol{q} \in E^{\boldsymbol{n}^2}$, we can denote $\boldsymbol{q}_{ij} = \boldsymbol{q}((i,j)) \in E$), there is an embedding $C^1 \subset E^{\boldsymbol{n}^2}$, by sending an element $\boldsymbol{q} \in C^1$ to the map $\boldsymbol{q}' \colon \boldsymbol{n}^2 \to E$ defined by

$$\boldsymbol{q}'_{ij} = \begin{cases} \boldsymbol{q}(\{i,j\}) & \text{if } i < j, \\ -\boldsymbol{q}(\{i,j\}) & \text{if } i > j, \\ 0 & \text{if } i = j. \end{cases}$$

In fact, we are identifying elements in C^1 with the skew-symmetric elements in $E^{\boldsymbol{n}^2}$ (that is, maps $\boldsymbol{q}_{ij} + \boldsymbol{q}_{ji} = \boldsymbol{0} \in E$). Given $\boldsymbol{q} \in C^1$ with an abuse of notation we will write $\boldsymbol{q}_{ij}$ instead of $\boldsymbol{q}'_{ij}$, and ij instead (i,j).

If K is an abstract simplicial complex, recall that the simplicial chain complex of K with real coefficients, denoted by $C_*(K;\mathbb{R})$, is defined as follows: for each $k \in \mathbb{Z}$, the chain group $C_k(K;\mathbb{R})$ is the vector space of all the $\mathbb{R}$-linear combinations of k-dimensional simplexes of K; the boundary homomorphism $\partial_k \colon C_k(K;\mathbb{R}) \to C_{k-1}(K;\mathbb{R})$ is defined as $\partial_k(\sigma) = \sum_{j=0}^{k} (-1)^j \sigma d_j$ for each k-simplex σ of K and 0 otherwise, where d_j is the j-th face map. More precisely, all simplices in K can be ordered, and elements in $C_k(K;\mathbb{R})$ will be linear combinations of ordered k-simplices in K. An ordered k-simplex with vertices $x_0, \dots, x_k$ will be denoted either as $[x_0, \dots, x_k]$ or simply as $x_0 \dots x_k$. With this notation, the j-th face map sends $\sigma = [x_0, \dots, x_j, \dots, x_k]$ to $\sigma d_j = [x_0, \dots, \widehat{x}_j, \dots, x_k]$, where $\widehat{x}_j$ means that the j-th element is canceled.

By taking homomorphisms valued in an $\mathbb{R}$-vector space E, the chain complex $C_k(K;\mathbb{R})$ yields the simplicial cochain complex with coefficients in E: the cochain groups are defined as all the linear homomorsphisms $C^k(K;E) = \hom_{\mathbb{R}}(C_k(K;\mathbb{R}), E)$, and the coboundary homomorphisms $\delta^k \colon C^k(K;E) \to C^{k+1}(K;E)$ are defined for each k by

$$\delta^k(\eta) = \eta \partial_{k+1} \colon \; C_{k+1}(K;\mathbb{R}) \to E$$

for each cochain $\eta \colon C_k(K;\mathbb{R}) \to E$. The kernel of δ^k is the group of cocycles, and it is denoted as $Z^k(K;E) = \ker \delta^k \subset C^k(K;E)$.

Now, let Δ^{n-1} denote the standard (abstract) simplex with n vertices $\{1,2,\ldots,n\}$. Then the vector spaces C^k defined above for $k = 0,1$ are exactly the groups of k-dimensional simplicial cochains (with coefficients in the vector space E) of the simplicial complex Δ^{n-1}: $C^0 = C^0(\Delta^{n-1};E)$ and $C^1 = C^1(\Delta^{n-1};E)$. A 0-simplex of Δ^{n-1} is simply an element $j \in \boldsymbol{n}$, and hence a 0-dimensional cochain is an n-tuple $\boldsymbol{q}_j$, i.e., a map $\boldsymbol{q}\colon \boldsymbol{n} \to E$. Furthermore, a 1-dimensional cochain is a map $\boldsymbol{q}$ defined with values in E and as domain the set of 1-dimensional simplices of Δ^{n-1}, i.e., pairs ij with $1 \le i < j \le n$.

In simpler terms, for each i,j such that $1 \le i < j \le n$, let $\boldsymbol{q}_{ij} \in E$ denote the ij-the component of a vector in $E^{\tilde{n}}$, and for $i > j$, the variable $\boldsymbol{q}_{ij}$ is defined by the property that $\forall\, i,j,\ \boldsymbol{q}_{ij} + \boldsymbol{q}_{ji} = \boldsymbol{0}$.

The coboundary operator $\delta^0\colon C^0 \to C^1$ is the map defined by $\delta^0\boldsymbol{q} = \boldsymbol{q}\partial_1$ for each $\boldsymbol{q} \in C^0$, and hence

$$\delta^0(\boldsymbol{q})(i,j) = \boldsymbol{q}_j - \boldsymbol{q}_i \in E$$

for all i, j. In fact, for each $\boldsymbol{q}\colon \boldsymbol{n} \to E$, $\delta^0(\boldsymbol{q})(ij) = \boldsymbol{q}\partial_1[i,j] = \boldsymbol{q}(j-i) = \boldsymbol{q}_j - \boldsymbol{q}_i$. For $k=1$, the coboundary operator is defined as $\delta^1\colon C^1 \to C^2$ as

$$\delta^1(\boldsymbol{q})(ijk) = \boldsymbol{q}\partial_2(ijk) = \boldsymbol{q}(jk - ik + ij) = \boldsymbol{q}_{ij} + \boldsymbol{q}_{jk} + \boldsymbol{q}_{ki}.$$

Moreover, since the simplex Δ^{n-1} is contractible, its cohomology groups are trivial except for $k - 0$, and therefore for each $k \ge 0$

$$Z^{k+1}\big(\Delta^{n-1};E\big) = \ker\delta^{k+1} = \operatorname{Im}\delta^k.$$

With an abuse of notation, when not necessary the subscript of ∂_k and the supscript in δ^k will be omitted.

For each $\boldsymbol{q} \in C^1$, let $\boldsymbol{Q}$ be defined as $\boldsymbol{Q}_{jk} = \frac{\boldsymbol{q}_{jk}}{|\boldsymbol{q}_{jk}|^{\alpha+2}}$. Note that $\boldsymbol{Q}_{jk} = \Psi_\gamma(\boldsymbol{q}_{jk})$ with $\gamma = \alpha + 2$ where $\Psi_\gamma(\boldsymbol{x}) = \frac{\boldsymbol{x}}{|\boldsymbol{x}|^\gamma}$ for each $\boldsymbol{x} \in E$. It turns out that the map $\Psi_\gamma\colon E\setminus\{\boldsymbol{0}\} \to E\setminus\{\boldsymbol{0}\}$ is a diffeomorphism with inverse $\Psi_{\hat{\gamma}}$ where $\hat{\gamma} = \frac{\gamma}{\gamma-1}$.

Consider now that if one defines $\boldsymbol{q}_{ij} = \boldsymbol{q}_i - \boldsymbol{q}_j$, one can read equation (1.2) as (as a consequence of equation (1.1))

$$\lambda\boldsymbol{q}_j = -\alpha\sum_{k\ne j} m_k \frac{\boldsymbol{q}_{jk}}{|\boldsymbol{q}_{jk}|^{\alpha+2}} = -\alpha\sum_{k\ne j} m_k\boldsymbol{Q}_{jk},$$

and hence

$$\begin{aligned}
-\frac{\lambda}{\alpha}\boldsymbol{q}_{ij} &= \sum_{k\ne i} m_k\boldsymbol{Q}_{ik} - \sum_{k\ne j} m_k\boldsymbol{Q}_{jk} = \sum_{k\notin\{i,j\}} m_k(\boldsymbol{Q}_{ik} + \boldsymbol{Q}_{kj}) + m_j\boldsymbol{Q}_{ij} + m_i\boldsymbol{Q}_{ij}\\
&= \sum_{k\notin\{i,j\}} m_k(\boldsymbol{Q}_{ik} + \boldsymbol{Q}_{kj} + \boldsymbol{Q}_{ji}) - \sum_{k\notin\{i,j\}} m_k\boldsymbol{Q}_{ji} + m_j\boldsymbol{Q}_{ij} + m_i\boldsymbol{Q}_{ij}\\
&= \sum_{k\notin\{i,j\}} m_k(\boldsymbol{Q}_{ik} + \boldsymbol{Q}_{kj} + \boldsymbol{Q}_{ji}) + \left(\sum_k m_k\right)\boldsymbol{Q}_{ij}\\
\iff -\frac{\lambda}{\alpha}\boldsymbol{q}_{ij} &= \sum_{k\notin\{i,j\}} m_k(\boldsymbol{Q}_{ik} + \boldsymbol{Q}_{kj} + \boldsymbol{Q}_{ji}) + \boldsymbol{Q}_{ij}. \qquad (2.1)
\end{aligned}$$

Proposition 2.1. *The (linear) map $P_m\colon C^1 \to C^1$ defined by*

$$(P_m(\boldsymbol{Q}))_{ij} = \sum_{k\notin\{i,j\}} m_k(\boldsymbol{Q}_{ik} + \boldsymbol{Q}_{kj} + \boldsymbol{Q}_{ji}) + \boldsymbol{Q}_{ij}$$

is a projection from C^1 onto $Z^1 \subset C^1$, where $Z^1 = \ker\delta^1\colon C^1 \to C^2$ is the subspace of 1-cocycles.

Proof. Consider the homomorphism π_m defined on the vector space of simplicial 1-chains $C_1(\Delta^{n-1}, \mathbb{R})$ with real coefficient, as

$$\pi_m([i,j]) = [i,j] - \sum_{k \neq \{i,j\}} m_k \partial_2([i,j,k]),$$

where $\partial_2 \colon C_2 \to C_1$ is the boundary homomorphism in dimension 2. Then for any $\boldsymbol{Q} \in C^1$ and any i, j with $i \neq j$

$$P_m(\boldsymbol{Q})[i,j] = \boldsymbol{Q}(\pi_m[i,j]).$$

For each 2-simplex $[a,b,c]$ of Δ^{n-1} one has $\partial_2([a,b,k]+[b,c,k]+[c,a,k]) = \partial_2[a,b,c]$ for each $k \neq \{a,b,c\}$, and hence

$$\begin{aligned}
\pi_m \partial_2[a,b,c] = \pi_m([a,b]+[b,c]+[c,a]) &= [a,b] - \sum_{k \neq \{a,b\}} m_k \partial_2[a,b,k] \\
&+ [b,c] - \sum_{k \neq \{b,c\}} m_k \partial_2[b,c,k] + [c,a] - \sum_{k \neq \{c,a\}} m_k \partial_2[c,a,k] \\
&= \partial_2[a,b,c] - \sum_{k \notin \{a,b,c\}} m_k \partial_2[a,b,c] - m_c \partial_2[a,b,c] - m_a \partial_2[b,c,a] - m_b \partial_2[c,a,b] \\
&= \partial_2[a,b,c] - \left(\sum_k m_k \right) \partial_2[a,b,c] = 0
\end{aligned}$$

$$\implies \pi_m \partial_2 = 0.$$

As a consequence, π_m is a projection, since for each i, j, $i \neq j$,

$$\pi_m^2[i,j] = \pi_m \left([i,j] - \sum_{k \notin \{i,j\}} m_k \partial_2([i,j,k]) \right) = \pi_m[i,j] - \sum_{k \notin \{i,j\}} m_k \pi_m \partial_2[i,j,k] = \pi_m[i,j].$$

Therefore, also $P_m \colon C^1 \to C^1$ is a projection

$$P_m^2(\boldsymbol{Q})[i,j] = P_m(\boldsymbol{Q})(\pi_m[i,j]) = \boldsymbol{Q}(\pi_m^2[i,j]) = \boldsymbol{Q}(\pi_m[i,j]) = P_m(\boldsymbol{Q})[i,j].$$

Moreover, since

$$\partial_1 \pi_m = \partial_1 \tag{2.2}$$

it follows that the projection P_m is onto the subspace of all 1-cocycles in C^1, denoted in short by Z^1. In fact, since $\pi_m \partial_2 = 0$,

$$\delta^1 P_m \boldsymbol{Q} = P_m \boldsymbol{Q} \partial_2 = \boldsymbol{Q} \pi_m \partial_2 = 0 \implies \mathrm{Im}(P_m) \subset Z^1$$

and, by (2.2), for each cocycle $\boldsymbol{z} \in Z^1 \iff \boldsymbol{z} = \delta^0 \boldsymbol{x}$ one has

$$P_m \boldsymbol{z} = P_m \delta^0 \boldsymbol{x} = P_m \boldsymbol{x} \partial_1 = \boldsymbol{x} \partial_1 \pi_m = \boldsymbol{x} \partial_1 = \delta^0 \boldsymbol{x} = \boldsymbol{z}$$

and hence $\mathrm{Im}(P_m) \supset Z^1$. We can conclude, as claimed, that $\mathrm{Im}(P_m) = Z^1$. ■

As examples, for $d = 1$ and $n = 3, 4$ the matrices of the projection P_m are

$$\begin{bmatrix} m_1 + m_2 & m_3 & -m_3 \\ m_2 & m_1 + m_3 & m_2 \\ -m_1 & m_1 & m_2 + m_3 \end{bmatrix},$$

$$\begin{bmatrix} m_1+m_2 & m_3 & m_4 & -m_3 & -m_4 & 0 \\ m_2 & m_1+m_3 & m_4 & m_2 & 0 & -m_4 \\ m_2 & m_3 & m_1+m_4 & 0 & m_2 & m_3 \\ -m_1 & m_1 & 0 & m_2+m_3 & m_4 & -m_4 \\ -m_1 & 0 & m_1 & m_3 & m_2+m_4 & m_3 \\ 0 & -m_1 & m_1 & -m_2 & m_2 & m_3+m_4 \end{bmatrix}.$$

In fact, for $n=3$ the space of cochains C^1 has standard coordinates $\boldsymbol{Q}_{ij}$ for $ij \in \{12,13,23\}$, and by Proposition 2.1 the projection P_m in these coordinates is defined by

$$\begin{aligned} (P_m(\boldsymbol{Q}))_{12} &= \sum_{k\notin\{1,2\}} m_k(\boldsymbol{Q}_{1k}+\boldsymbol{Q}_{k2}+\boldsymbol{Q}_{21})+\boldsymbol{Q}_{12} = m_3(\boldsymbol{Q}_{13}+\boldsymbol{Q}_{32}+\boldsymbol{Q}_{21})+\boldsymbol{Q}_{12}, \\ (P_m(\boldsymbol{Q}))_{13} &= \sum_{k\notin\{1,3\}} m_k(\boldsymbol{Q}_{1k}+\boldsymbol{Q}_{k3}+\boldsymbol{Q}_{31})+\boldsymbol{Q}_{13} = m_2(\boldsymbol{Q}_{12}+\boldsymbol{Q}_{23}+\boldsymbol{Q}_{31})+\boldsymbol{Q}_{13}, \\ (P_m(\boldsymbol{Q}))_{23} &= \sum_{k\notin\{2,3\}} m_k(\boldsymbol{Q}_{2k}+\boldsymbol{Q}_{k3}+\boldsymbol{Q}_{32})+\boldsymbol{Q}_{23} = m_1(\boldsymbol{Q}_{21}+\boldsymbol{Q}_{13}+\boldsymbol{Q}_{32})+\boldsymbol{Q}_{23}, \end{aligned}$$

from which it follows that

$$\begin{aligned} (P_m(\boldsymbol{Q}))_{12} &= (1-m_3)\boldsymbol{Q}_{12}+m_3\boldsymbol{Q}_{13}-m_3\boldsymbol{Q}_{23}, \\ (P_m(\boldsymbol{Q}))_{13} &= m_2\boldsymbol{Q}_{12}+(1-m_2)\boldsymbol{Q}_{13}+m_2\boldsymbol{Q}_{23}, \\ (P_m(\boldsymbol{Q}))_{23} &= -m_1\boldsymbol{Q}_{12}+m_1\boldsymbol{Q}_{13}+(1-m_1)\boldsymbol{Q}_{23}. \end{aligned}$$

The same argument yields the matrix for $n=4$, in coordinates $\boldsymbol{Q}_{ij}$ for ij in the order 12, 13, 14, 23, 24, 34.

Consider the following scalar product on C^1, similar to the mass-metric on the configuration space: for $\boldsymbol{v},\boldsymbol{w}\in C^1$,

$$\langle \boldsymbol{v},\boldsymbol{w}\rangle_M = \sum_{i<j} m_i m_j \left(\boldsymbol{v}_{ij}\cdot\boldsymbol{w}_{ij}\right), \tag{2.3}$$

where as above the dot denotes the standard d-dimensional scalar product in E. It is the mass-metric on C^1, and as above $\|\boldsymbol{v}\|_M^2 = \langle\boldsymbol{v},\boldsymbol{v}\rangle_M$. It follows that

$$\begin{aligned} \langle \boldsymbol{v}, P_m(\boldsymbol{w})\rangle_M &= \sum_{i<j} m_i m_j \left(\boldsymbol{v}_{ij}\cdot (P_m(\boldsymbol{w}))_{ij}\right) \\ &= \sum_{i<j} m_i m_j \left(\boldsymbol{v}_{ij}\cdot\left(\sum_{k\notin\{i,j\}} m_k(\boldsymbol{w}_{ik}+\boldsymbol{w}_{kj}+\boldsymbol{w}_{ji})+\boldsymbol{w}_{ij}\right)\right) \\ &= \sum_{i<k}\sum_{k\notin\{i,j\}} m_i m_j m_k(\boldsymbol{v}_{ij}\cdot(\boldsymbol{w}_{ik}+\boldsymbol{w}_{kj}+\boldsymbol{w}_{ji})) + \sum_{i<j} m_i m_j \boldsymbol{v}_{ij}\cdot\boldsymbol{w}_{ij} \\ &= \sum_{a<b<c} m_a m_b m_c \big(\boldsymbol{v}_{ab}\cdot(\boldsymbol{w}_{ac}+\boldsymbol{w}_{cb}+\boldsymbol{w}_{ba}) \\ &\quad + \boldsymbol{v}_{ac}\cdot(\boldsymbol{w}_{ab}+\boldsymbol{w}_{bc}+\boldsymbol{w}_{ca}) + \boldsymbol{v}_{bc}\cdot(\boldsymbol{w}_{ba}+\boldsymbol{w}_{ac}+\boldsymbol{w}_{cb})\big) + \sum_{i<j} m_i m_j \boldsymbol{v}_{ij}\cdot\boldsymbol{w}_{ij} \\ &= -\sum_{a<b<c} m_a m_b m_c \left((\boldsymbol{v}_{ab}+\boldsymbol{v}_{bc}+\boldsymbol{v}_{ca})\cdot(\boldsymbol{w}_{ab}+\boldsymbol{w}_{bc}+\boldsymbol{w}_{ca})\right) \\ &\quad + \sum_{i<j} m_i m_j \boldsymbol{v}_{ij}\cdot\boldsymbol{w}_{ij} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i<k} \sum_{k \notin \{i,j\}} m_i m_j m_k ((\boldsymbol{v}_{ik} + \boldsymbol{v}_{kj} + \boldsymbol{v}_{ji}) \cdot \boldsymbol{w}_{ij}) + \sum_{i<j} m_i m_j \boldsymbol{v}_{ij} \cdot \boldsymbol{w}_{ij} \\
&= \sum_{i<j} m_i m_j \left(\left(\sum_{k \notin \{i,j\}} m_k (\boldsymbol{v}_{ik} + \boldsymbol{v}_{kj} + \boldsymbol{v}_{ji}) + \boldsymbol{v}_{ij} \right) \cdot \boldsymbol{w}_{ij} \right) \\
&= \langle P_m(\boldsymbol{v}), \boldsymbol{w} \rangle_M,
\end{aligned}
$$

hence the following proposition holds.

Proposition 2.2. *The projection $P_m \colon C^1 \to Z^1 \subset C^1$ is orthogonal (self-adjoint) with respect to the scalar product $\langle -, - \rangle_M$ in (2.3) defined on C^1.*

Now, consider the subspace $X \subset \mathbb{F}_n(E)$ of all configurations with center of mass in $\mathbf{0}$: $X = \{\boldsymbol{q} \in \mathbb{F}_n(E) \colon \sum_j m_j \boldsymbol{q}_j = \mathbf{0}\}$, i.e., of all $\boldsymbol{q} \in C^0$ such that $\boldsymbol{q} \sum_j m_j [j] = \mathbf{0}$. The coboundary morphism $\delta^0_{|X} \colon X \subset C^0 \to C^1$ induces an isomorphism $\delta^0_{|X} \colon X \to Z^1$. Moreover, since if $\boldsymbol{q} \in X$ then

$$
\begin{aligned}
2 \sum_{i<j} m_i m_j |\boldsymbol{q}_i - \boldsymbol{q}_j|^2 = \sum_{i,j} m_i m_j |\boldsymbol{q}_i - \boldsymbol{q}_j|^2 &= \sum_{i,j} m_i m_j \left(|\boldsymbol{q}_i|^2 - 2\boldsymbol{q}_i \cdot \boldsymbol{q}_j + |\boldsymbol{q}_j|^2 \right) \\
&= 2 \sum_i m_i |\boldsymbol{q}_i|^2 - 2 \left(\sum_j m_j \boldsymbol{q}_j \right)^2 = 2 \sum_i m_i |\boldsymbol{q}_i|^2
\end{aligned}
$$

the isomorphism $\delta^0_{|X} \colon X \to Z^1$ is an isometry, where X and Z^1 have the mass-metrics. Explicitly, for each $\boldsymbol{q} \in X$,

$$
\|\boldsymbol{q}\|_M = \|\delta^0(\boldsymbol{q})\|_M,
$$

where the two norms with the same symbol, with an abuse of notation, are actually different norms in C^0 and C^1 respectively.

Furthermore, the potential U is the composition of the restriction to X of the coboundary map δ^0 with the map $\tilde{U} \colon C^1 \to \mathbb{R}$ (partially) defined by

$$
\tilde{U}(\boldsymbol{q}) = \sum_{i<j} m_i m_j |\boldsymbol{q}_{ij}|^{-\alpha},
$$

as illustrated in the following diagram

$$
\begin{tikzcd}
C^0 \arrow[r, "\delta^0"] & C^1 \arrow[r, "\tilde{U}"] \arrow[d, bend right, "P_m"'] & \mathbb{R} \\
X \arrow[u] \arrow[r, "\cong", "\delta^0_{|X}"'] & Z^1. \arrow[u, bend right]
\end{tikzcd}
$$

Now, recall that (condition **(CC4)**) a configuration $\boldsymbol{q} \in \mathbb{F}_n(E) \subset C^0$ is a central configuration is and only if it is a critical points of the map $\|\boldsymbol{q}\|^2_M + U(\boldsymbol{q})$, defined on $\mathbb{F}_n(E)$. It is easy to see that this is equivalent to say that $\boldsymbol{q}$ is a critical point of the map $\|\boldsymbol{q}\|^2_M + U(\boldsymbol{q})$ restricted to X. But this means that $\delta^0_{|X}$ sends central configurations in X to all the critical points of the map $\|\tilde{\boldsymbol{q}}\|^2_M + \tilde{U}(\tilde{\boldsymbol{q}})$ (defined on C^1) restricted to the space of 1-cocycles Z^1.

Hence, the following theorem holds.

Theorem 2.3. *Central configurations are critical points of the function partially defined as* $C^1 \to \mathbb{R}$

$$\boldsymbol{q} \mapsto \sum_{i<j} m_i m_j \left(|\boldsymbol{q}_{ij}|^{-\alpha} + |\boldsymbol{q}_{ij}|^2 \right)$$

restricted to the space of 1*-cocycles* $Z^1 \subset C^1$.

A co-chain $\boldsymbol{q} \in C^1$ *is a central configuration if and only if there exists* $\lambda \in \mathbb{R}$ *such that* $\lambda \boldsymbol{q} = P_m(\Psi(\boldsymbol{Q}))$*, where* $\boldsymbol{Q}_{ij} = \frac{\boldsymbol{q}_{ij}}{|\boldsymbol{q}_{ij}|^{\alpha+2}}$ *for each* i*,* j *and* $P_m \colon C^1 \to Z^1 \subset C^1$ *is the orthogonal projection defined in Proposition 2.1, which sends* C^1 *onto the space of* 1*-cocycles.*

Remark 2.4. Since the function $r^{-\alpha} + r^2$ is convex on $(0, \infty)$, Theorem 2.3 implies that the restriction of $\tilde{U}$ to each component of Z^1 minus collisions is convex for $d = 1$, and hence one derives the existence (and uniqueness) of Moulton collinear central configurations.

3 Hessians and indices

Let $P \in \mathbb{F}_n(E)$ be a central configuration, with mass-norm $r = \|P\|_M$. As in the case $r = 1$, seen in **(CC1)**, it is a critical point of the restriction of the potential function U to the inertia ellipsoid $S = \{\boldsymbol{q} \in \mathbb{F}_n(E) \colon \|\boldsymbol{q}\|_M = r\}$. As such, its Morse index is the Hessian of the restriction $U|_S$, which is a bilinear form defined on the tangent space $T_P S$. The Hessian of $f = U|_S$ at a critical point $P \in S$ can be computed in general as $D^2 f(P)[\boldsymbol{u}(P), \boldsymbol{v}(P)] = ((D_{\boldsymbol{u}} D_{\boldsymbol{v}} f - D_{D_{\boldsymbol{u}}\boldsymbol{v}}) f)(P) = (D_{\boldsymbol{u}}(D_{\boldsymbol{v}} f))(P)$, where $\boldsymbol{u}$ and $\boldsymbol{v}$ are vector fields on S (see, e.g., [10, formula (1), p. 343]). This yields the well-known formula of the Hessian in terms of second derivatives with respect to a local chart (see also [14, Proposition 2.8.8, p. 136]). Given the mass-metric, the Hessian can be written as $D^2 f[\boldsymbol{u}, \boldsymbol{v}] = \langle D_{\boldsymbol{u}} \hat{\nabla}_M f, \boldsymbol{v} \rangle_M$ or as the self-adjoint endomorphism $T_P S \to T_P S$ defined by $\boldsymbol{u} \in T_P S \mapsto D_{\boldsymbol{u}} \hat{\nabla}_M f(P) \in T_P S$, where $\hat{\nabla}_M$ denotes the gradient induced by the mass-metric restricted to S (hence $\hat{\nabla}_M f(P)$ is the projection of $\nabla_M U(P)$ to $T_P S$, orthogonal with respect to $\langle -, - \rangle_M$).

If $\boldsymbol{N}$ denotes a vector normal to the tangent space $T_P S$ (such as $P - O$, where O denotes the origin of the Euclidean space E), the projection $\hat{\nabla}_M f(P)$ is equal to $\nabla_M U - \frac{\langle \nabla_M U, \boldsymbol{N} \rangle_M}{\|\boldsymbol{N}\|_M^2} \boldsymbol{N}$ evaluated at P. If $\boldsymbol{N} = P - O$, by Euler formula $\langle \nabla_M U, \boldsymbol{N} \rangle_M = -\alpha U$, and because P is a critical point of $f = U|_S$ and $\boldsymbol{u} \in T_P S$ the derivative $D_{\boldsymbol{u}} \frac{\alpha U(\boldsymbol{q})}{\|\boldsymbol{q}\|_M^2}$ vanishes at P, and hence

$$\begin{aligned} D_{\boldsymbol{u}} \hat{\nabla}_M f(P) &= D_{\boldsymbol{u}} \left(\nabla_M U(\boldsymbol{q}) + \frac{\alpha U(\boldsymbol{q})}{\|\boldsymbol{q}\|_M^2} \boldsymbol{N} \right)(P) = D_{\boldsymbol{u}}(\nabla_M U)(P) + \frac{\alpha U(P)}{\|P\|_M^2} D_{\boldsymbol{u}}(\boldsymbol{q}) \\ &= D_{\boldsymbol{u}} \left(\nabla_M U - \lambda \nabla_M \frac{\|\boldsymbol{q}\|_M^2}{2} \right), \end{aligned}$$

where λ is as above the constant $-\frac{\alpha U(P)}{\|P\|_M^2}$. Hence the following lemma holds.

Lemma 3.1. *If* P *is a critical point of the restriction* $U|_{S_r}$*, with the inertia ellipsoid* $S_r = \{\boldsymbol{q} \in \mathbb{F}_n(E) \colon \|\boldsymbol{q}\|_M = r\}$ *and with* λ *defined as* $\lambda = -\frac{\alpha U(P)}{\|P\|_M^2} = -\frac{\alpha U(P)}{r^2}$*, then* P *is a critical point of the function* $U(\boldsymbol{q}) - \frac{\lambda}{2}\|\boldsymbol{q}\|_M^2$*; moreover, the Hessian of* $U|_{S_r}$ *at* P *is the restriction to the tangent space* $T_P S_r$ *of the Hessian of the map* $U(\boldsymbol{q}) - \frac{\lambda}{2}\|\boldsymbol{q}\|_M^2$ *defined on* $\mathbb{F}_n(E)$*, evaluated* P.

Proposition 3.2. *If* $P \in \mathbb{F}_n(E)$ *is a central configuration, then the Morse index at* P *of the restriction* $f = U|_{S_r}$ *is equal to the Morse index at* P *of the function* $F(\boldsymbol{q}) = U(\boldsymbol{q}) - \frac{\lambda}{2}\|\boldsymbol{q}\|_M^2$*, where* λ *and* S_r *are as above. Furthermore, the direction parallel to* $P - O$ *is an eigenvector of the Hessian of* F*, with (positive) eigenvalue equal to* $-\lambda(\alpha + 2) > 0$.

Proof. Since $\nabla_M U(\boldsymbol{q})$ is homogeneous of degree $-\alpha-1$, if $\boldsymbol{N} = P-O$ one has $D_{\boldsymbol{N}}(\nabla_M U)(P) = -(\alpha+1)\nabla_M U(P) = -\lambda(\alpha+1)\boldsymbol{N}$. Therefore

$$D_{\boldsymbol{N}}\left(\nabla_M U - \lambda \nabla_M \frac{\|\boldsymbol{q}\|_M^2}{2}\right)(P) = -\lambda(\alpha+1)\boldsymbol{N} - \lambda\boldsymbol{N} = -\lambda(\alpha+2)\boldsymbol{N}. \qquad \blacksquare$$

Now, consider the function $f\colon C^1 \to \mathbb{R}$ defined on cochains in Theorem 2.3 as

$$f(\boldsymbol{q}) = \sum_{i<j} m_i m_j \left(|\boldsymbol{q}_{ij}|^{-\alpha} + |\boldsymbol{q}_{ij}|^2\right).$$

The following proposition links its Hessian with the Hessian of the function F of Proposition 3.2, for $\lambda = -2$.

Proposition 3.3. *Let $\boldsymbol{q} \in \mathbb{F}_n(E)$ be a central configuration which is a critical point of the function $F(\boldsymbol{q}) = U(\boldsymbol{q}) + \|\boldsymbol{q}\|_M^2$ in C^0 (and hence $\boldsymbol{q} \in X$). Let H be the Hessian of F at $\boldsymbol{q}$ (with respect to the mass-metric in C^0), and $\tilde{H}$ the Hessian matrix of the composition $f \circ P_m$ at $\delta^0(\boldsymbol{q}) \in Z^1 \subset C^1$ (with respect to the mass-metric in C^1). Then the non-zero eigenvalues of $\tilde{H}$ are the same as the non-zero eigenvalues of H, except for the eigenvalue 2 occurring in H with multiplicity $\dim E$ (which corresponds to the group of translations in E, or equivalently the orthogonal complement of X in C^0).*

Proof. Since U is invariant with respect to translations in C^0, H has the autospace $\boldsymbol{q}_1 = \boldsymbol{q}_2 = \cdots = \boldsymbol{q}_n \subset C^0$ (which is the tangent space of the group of translations acting on $\mathbb{F}_n(E)$, and is orthogonal to X with respect to the mass-metric), over which D^2U vanishes and $D^2\|\boldsymbol{q}\|_M^2 = 2$; hence it is an eigenspace with eigenvalue 2. The rest of eigenvalues of H correspond via the isometric embedding $\delta^0_{|X}$ to eigenvalues of the restriction of f to Z^1, and hence to the eigenvalues in Z^1 of the composition $f \circ P_m$. The orthogonal complement of Z^1, which is the kernel of P_m, yields zero eigenvalues to $\tilde{H}$. $\blacksquare$

4 Simple proofs of some corollaries

Equations (2.1) can be written as the following:

$$\frac{\lambda}{\alpha}\boldsymbol{q}_{ij} + \boldsymbol{Q}_{ij} = \sum_{k \notin \{i,j\}} m_k(\boldsymbol{Q}_{ij} + \boldsymbol{Q}_{jk} + \boldsymbol{Q}_{ki}). \tag{4.1}$$

Now, consider for each triple i, j, k the corresponding term $\boldsymbol{Q}_{ijk} = \boldsymbol{Q}_{ij} + \boldsymbol{Q}_{jk} + \boldsymbol{Q}_{ki}$. We give some very simple proofs to some well-known propositions (actually generalizing them to any homogeneity α), that follow from the following simple geometric lemma.

Lemma 4.1. *Let $\boldsymbol{q}_1$, $\boldsymbol{q}_2$ and $\boldsymbol{q}_3$ be three non-collinear points in E. Then $\boldsymbol{Q}_{123} = \boldsymbol{0}$ if and only if $\boldsymbol{q}_1$, $\boldsymbol{q}_2$ and $\boldsymbol{q}_3$ are vertices of an equilateral triangle.*

Furthermore, there exists $c \in \mathbb{R}$ such that $\boldsymbol{Q}_{123} = c\boldsymbol{q}_{12}$ if and only if $|\boldsymbol{q}_{13}| = |\boldsymbol{q}_{23}|$, that is, if and only if the triangle with vertices in $\boldsymbol{q}_1$, $\boldsymbol{q}_2$ and $\boldsymbol{q}_3$ is isosceles in $\boldsymbol{q}_3$.

Proof. If $\boldsymbol{q}_1$, $\boldsymbol{q}_2$ and $\boldsymbol{q}_3$ are not collinear (in E), then the differences $\boldsymbol{q}_{12}$, $\boldsymbol{q}_{13}$ and $\boldsymbol{q}_{23}$ generate a plane. Since $\boldsymbol{q}_{12} + \boldsymbol{q}_{23} + \boldsymbol{q}_{31} = \boldsymbol{0}$, $\boldsymbol{Q}_{123} = \boldsymbol{0}$ implies

$$\boldsymbol{Q}_{123} = \boldsymbol{Q}_{12} + \boldsymbol{Q}_{23} + \boldsymbol{Q}_{31} = \frac{\boldsymbol{q}_{12}}{|\boldsymbol{q}_{12}|^{\alpha+2}} + \frac{\boldsymbol{q}_{23}}{|\boldsymbol{q}_{23}|^{\alpha+2}} + \frac{\boldsymbol{q}_{31}}{|\boldsymbol{q}_{31}|^{\alpha+2}} = \boldsymbol{0} = \boldsymbol{q}_{12} + \boldsymbol{q}_{23} + \boldsymbol{q}_{31}.$$

By taking barycentric coordinates in the plane generated by the three points, it follows that $\boldsymbol{Q}_{123} = \boldsymbol{0}$ if and only if $|\boldsymbol{q}_{12}|^{\alpha+2} = |\boldsymbol{q}_{23}|^{\alpha+2} = |\boldsymbol{q}_{31}|^{\alpha+2}$, that is, if and only if the three points are vertices of an equilateral triangle.

If c_1, c_2 and c_3 are three non-zero real numbers such that $c_3\boldsymbol{q}_{12} + c_1\boldsymbol{q}_{23} + c_2\boldsymbol{q}_{31} = c\boldsymbol{q}_{12}$, then $(c_3 - c)\boldsymbol{q}_{12} + c_1\boldsymbol{q}_{23} + c_2\boldsymbol{q}_{31} = \boldsymbol{0}$, and as above this implies $c_3 - c = c_1 = c_2$. Hence if $\boldsymbol{Q}_{123} = c\boldsymbol{q}_{12}$, it holds that $|\boldsymbol{q}_{23}|^{\alpha+2} = |\boldsymbol{q}_{13}|^{\alpha+2}$ as claimed. ∎

Corollary 4.2. *For $n = 3$, $d \geq 2$ and $\alpha > 0$, the only non-collinear central configuration is the equilateral configuration.*

Proof. Equation (4.1) implies that, if $\boldsymbol{Q}_{ijk} \neq \boldsymbol{0}$, then the configuration is collinear (since (4.1) implies there exist three real numbers c_{12}, c_{23}, c_{31} such that $c_{12}\boldsymbol{q}_{12} = m_3\boldsymbol{Q}_{123}$, $c_{23}\boldsymbol{q}_{23} = m_1\boldsymbol{Q}_{231}$, $c_{31}\boldsymbol{q}_{31} = m_2\boldsymbol{Q}_{312}$, and it is easy to see that $\boldsymbol{Q}_{123} = \boldsymbol{Q}_{231} = \boldsymbol{Q}_{312}$). Therefore, if the configuration is not collinear, $\boldsymbol{Q}_{ijk} = \boldsymbol{0}$, and by Lemma 4.1 the configuration is an equilateral triangle. ∎

Another easy consequence of Lemma 4.1 is the following proposition (see [1, 11, 17] for its importance in estimating the number non-degenerate planar central configurations of four bodies).

Corollary 4.3. *For $n = 4$, $d \geq 2$ and $\alpha > 0$, if a central configuration has three collinear bodies, then it is a collinear configuration.*

Proof. Assume that $\boldsymbol{q}_1$, $\boldsymbol{q}_2$ and $\boldsymbol{q}_3$ are collinear, and $\boldsymbol{q}_4$ is not. Then, equation (4.1) implies that for suitable real numbers c_{12}, c_{23} and c_{31} the following equations hold:

$$\begin{aligned} c_{12}\boldsymbol{q}_{12} &= m_3\boldsymbol{Q}_{123} + m_4\boldsymbol{Q}_{124}, \qquad c_{23}\boldsymbol{q}_{23} = m_1\boldsymbol{Q}_{231} + m_4\boldsymbol{Q}_{234}, \\ c_{31}\boldsymbol{q}_{31} &= m_2\boldsymbol{Q}_{312} + m_4\boldsymbol{Q}_{314}. \end{aligned}$$

This implies that there are $\tilde{c}_{12}$, $\tilde{c}_{23}$ and $\tilde{c}_{31}$ such that

$$\boldsymbol{Q}_{124} = \tilde{c}_{12}\boldsymbol{q}_{12}, \qquad \boldsymbol{Q}_{234} = \tilde{c}_{23}\boldsymbol{q}_{23}, \qquad \boldsymbol{Q}_{314} = \tilde{c}_{31}\boldsymbol{q}_{31},$$

and by Lemma 4.1 this implies that $|\boldsymbol{q}_{14}| = |\boldsymbol{q}_{24}| = |\boldsymbol{q}_{34}|$, which is not possible since $\boldsymbol{q}_1$, $\boldsymbol{q}_2$ and $\boldsymbol{q}_3$ are collinear. ∎

Corollary 4.3 can be easily generalized to arbitrary n as follows:

Corollary 4.4. *For $n \geq 4$, $d \geq 2$ and $\alpha > 0$, if $n - 1$ of the bodies in the central configuration are collinear, then all of them are.*

Corollary 4.5. *For $n \geq 4$, $d \geq 3$ and $\alpha > 0$, if the first $n - 1$ bodies $\boldsymbol{q}_1, \ldots, \boldsymbol{q}_{n-1}$ in a central configuration belong to a plane $\pi \subset E$, and the n-th body $\boldsymbol{q}_n$ does not belong to the plane π, then the distance between $\boldsymbol{q}_n$ and any $\boldsymbol{q}_j$ does not depend on $j = 1, \ldots, n - 1$, i.e., there exists $c > 0$ such that $|\boldsymbol{q}_n - \boldsymbol{q}_j| = c$ for all $j < n$. Hence the $n - 1$ coplanar bodies are cocircular.*

Proof. For each $i, j \leq n - 1$ there exists $c_{ij} \in \mathbb{R}$ such that

$$c_{ij}\boldsymbol{q}_{ij} = \sum_{k \notin \{i,j\}} m_k\boldsymbol{Q}_{ijk} = \sum_{k \notin \{i,j,n\}} m_k\boldsymbol{Q}_{ijk} + m_n\boldsymbol{Q}_{ijn}.$$

The term $\sum\limits_{k \notin \{i,j,n\}} m_k\boldsymbol{Q}_{ijk}$ is parallel to the plane π, while the sum $c_{ij}\boldsymbol{q}_{ij} - m_n\boldsymbol{Q}_{ijn}$ is a vector parallel to the plane containing $\boldsymbol{q}_i$, $\boldsymbol{q}_j$ and $\boldsymbol{q}_n$. Being equal, they both need to be parallel to both planes, and hence they are multiples of $\boldsymbol{q}_{ij}$. Therefore, by Lemma 4.1, there exists $\tilde{c}_{ij} \in \mathbb{R}$ such that $\boldsymbol{Q}_{ijn} = \tilde{c}_{ij}\boldsymbol{q}_{ij}$, and as above this implies that $|\boldsymbol{q}_{in}| = |\boldsymbol{q}_{jn}|$. ∎

Pyramidal configurations for $d = 3$ and $\alpha = 1$ were studied in first [5]; see also [16] for applications to perverse solutions and for the value of the constant c.

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