The Profinite Dimensional Manifold Structure of Formal Solution Spaces of Formally Integrable PDEs

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Abstract. In this paper, we study the formal solution space of a nonlinear PDE in a fiber bundle. To this end, we start with foundational material and introduce the notion of a pfd structure to build up a new concept of profinite dimensional manifolds. We show that the infinite jet space of the fiber bundle is a profinite dimensional manifold in a natural way. The formal solution space of the nonlinear PDE then is a subspace of this jet space, and inherits from it the structure of a profinite dimensional manifold, if the PDE is formally integrable. We apply our concept to scalar PDEs and prove a new criterion for formal integrability of such PDEs. In particular, this result entails that the Euler–Lagrange equation of a relativistic scalar field with a polynomial self-interaction is formally integrable.

Key words: profinite dimensional manifolds; jet bundles; geometric PDEs; formal integrability; scalar fields

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1   Introduction

Even though it appears to be unsolvable in general, the problem to describe the moduli space of solutions of a particular nonlinear PDE has led to powerful new results in geometric analysis and mathematical physics. Notably this can be seen, for example, by the fundamental work on the structure of the moduli space of Yang–Mills equations [5, 15, 46]. Among the many challenging problems which arise when studying moduli spaces of solutions of nonlinear PDEs is that the space under consideration does in general not have a manifold structure, usually not even one modelled on an infinite dimensional Hilbert or Banach space. Moreover, the solution space can possess singularities. A way out of this dilemma is to study compactifications of the moduli space like the completion of the moduli space with respect to a certain Sobolev metric, cf. [20]. Another way, and that is the one we are advocating in this article, is to consider a "coarse" moduli space consisting of so-called formal solutions of a PDE, i.e., the space of those smooth functions whose power series expansion at each point solves the PDE. In case the PDE is formally integrable in a sense defined in this article, the formal solution space turns out to be a profinite dimensional manifold. These possibly infinite dimensional spaces are ringed spaces which can be regarded as projective limits of projective systems of finite dimensional manifolds.

Profinite dimensional manifolds appear naturally in several areas of mathematics, in particular in deformation quantization, see for example [34], the structure theory of Lie-projective groups [8, 26], in connection with functional integration on spaces of connections [4], and in the

secondary calculus invented by Vinogradov [9, 27, 48] which inspired the approach in this paper, cf. in particular [27, Chapter 7]. It is to be expected that the theory of profinite dimensional manifolds as set up in this paper will have further applications, for example for the (classical) perturbation theory of PDEs in mathematical physics, where one should understand a perturbation as a deformation, i.e., a smooth family of profinite dimensional solution manifolds of a (formally integrable) PDE depending on a parameter. Work on this is in progress.

The paper consists of two main parts. The first, Section 3, lays out the foundations of the theory of profinite dimensional manifolds. Besides the papers [7] and [1], where the latter is taylored towards explaining the differential calculus by Ashtekar and Lewandowski [4], literature on profinite dimensional manifolds is scarce. Moreover, our approach to profinite dimensional manifolds is novel in the sense that we define them as ringed spaces together with a so-called pfd structure, which consists not only of one but a whole equivalence class of representations by projective systems of finite dimensional manifolds. The major point hereby is that all the projective systems appearing in the pfd structure induce the same structure sheaf, which allows to define differential geometric concepts depending only on the pfd structure and not a particular representative. One way to construct differential geometric objects is by dualizing projective limits of manifolds to injective limits of, for example, differential forms, and then sheafify the thus obtained presheaves of "local" objects. Again, it is crucial to observe that these sheaves are independent of the particular choice of a representative within the pfd structure, whereas the "local" objects obtain a filtration which depends on the choice of a particular representative. Using variants of this approach or directly the structure sheaf of smooth functions, we introduce in Section 3 tangent bundles of profinite dimensional manifolds and their higher tensor powers, vector fields, and differential forms.

The second main part is Section 4, where we introduce the formal solution space of a nonlinear PDE. We first explain the necessary concepts from jet bundle theory and on prolongations of PDEs in fiber bundles, following essentially Goldschmidt [24], cf. also [9, 36, 48, 49]. In Section 4.2.2 we introduce in the jet bundle setting a notion of an operator symbol of a nonlinear PDE such that, in the linear case, it coincides with the well-known (principal) symbol of a partial differential operator up to canonical isomorphisms. The corresponding result, Proposition 4.18, appears to be folklore; see [44] and [27, Section IV.2] for related work. Afterwards, we show that the bundle of infinite jets is a profinite dimensional manifold. This result immediately entails that the formal solution space of a formally integrable PDE is a profinite dimensional submanifold of the infinite jet bundle. Finally, in Section 4.4, we consider scalar PDEs. We prove there a widely applicable criterion for the formal integrability of scalar PDEs, which to our knowledge has not appeared in the mathematical literature yet. Moreover, we conclude from our criterion that the Euler–Lagrange equation of a relativistic scalar field with a polynomial self-interaction on an arbitrary Lorentzian manifold is formally integrable, so its formal solution space is a profinite dimensional manifold. We expect that this observation will be of avail when clarifying the Poisson structure [33, 49, 50] and quantization theory – possibly through deformation – of such scalar field theories, cf. [16, 38].

2   Some notation

Let us introduce some notation and conventions which will be used throughout the paper.

If nothing else is said, all manifolds and corresponding concepts, such as submersions, bundles etc., are understood to be smooth and finite dimensional. The symbol $\mathrm{T}^{k,l}$ stands for the functor of k-times contravariant and l-times covariant tensors, where as usual $\mathrm{T} := \mathrm{T}^{1,0}$ and $\mathrm{T}^* := \mathrm{T}^{0,1}$. If X is a manifold, then the corresponding tensor bundles will be denoted by $\pi_{\mathrm{T}^{k,l}X} \colon \mathrm{T}^{k,l}X \to X$. Moreover, we write $\mathscr{X}^\infty$ and Ω^k for the sheaves of smooth vector fields and of smooth k-forms, respectively.

Given a fibered manifold, i.e., a surjective submersion $\pi\colon E \to X$, we write $\Gamma^\infty(\pi)$ for the sheaf of smooth sections of π. Its space of sections over an open $U \subset X$ will be denoted by $\Gamma^\infty(U; \pi)$. The set of *local smooth sections of π around a point $p \in M$* is the set of smooth sections defined on some open neighborhood of p and will be denoted by $\Gamma^\infty(p; \pi)$. The stalk at p then is a quotient space of $\Gamma^\infty(p; \pi)$ and is written as $\Gamma^\infty_p(\pi)$.

The *vertical vector bundle* corresponding to the fibered manifold π is defined as the subvector bundle

$$\pi^{\mathsf{V}}\colon \ \mathsf{V}(\pi) := \ker(\mathrm{T}\pi) \longrightarrow E$$

of $\pi_{\mathrm{T}E}\colon \mathrm{T}E \to E$. If $\pi'\colon E' \to X$ is a second fibered manifold, the *vertical morphism* corresponding to a morphism $h\colon E \to E'$ of fibered manifolds over X is given by

$$h^{\mathsf{V}}\colon \ \mathsf{V}(\pi) \longrightarrow \mathsf{V}(\pi'), \qquad v \longmapsto \mathrm{T}h(v).$$

If $\pi\colon E \to X$ is a vector bundle, then the fibers of π are $\mathbb{R}$-vector spaces, hence one can apply tensor functors fiberwise to obtain the corresponding tensor bundles. In particular, $\pi^{\odot^k}\colon \mathsf{S}^k(\pi) \to X$ will stand for the *k-fold symmetric tensor product bundle of π*.

Finally, unless otherwise stated, the notions "projective system" and "projective limit" will always be understood in the category of topological spaces, where they of course exist; see [18, Chapter VIII, Section 3]. In fact, given such a projective system $(M_i, \mu_{ij})_{i,j\in\mathbb{N}, i\leq j}$, a distinguished projective limit is given as follows. Define

$$M := \left\{ (p_i)_{i\in\mathbb{N}} \in \prod_{i\in\mathbb{N}} M_i \,\middle|\, \mu_{ij}(p_j) = p_i \text{ for all } i, j \in \mathbb{N} \text{ with } i \leq j \right\}$$

to be the subspace of all threads in the product, and the continuous maps $\mu_j\colon M \to M_j$ as the restrictions of the canonical projections $\prod_{i\in\mathbb{N}} M_i \to M_j$ to M. Then one obviously has $\mu_{ij} \circ \mu_j = \mu_i$ for all $i, j \in \mathbb{N}$ with $i \leq j$. Note that a basis of the topology of M is given by the set of all open sets of the form $\mu_i^{-1}(U)$, where $i \in \mathbb{N}$ and $U \subset M_i$ is open. In the following, we will refer to the thus defined M together with the maps $(\mu_i)_{i\in\mathbb{N}}$ as *the canonical projective limit* of $(M_i, \mu_{ij})_{i,j\in\mathbb{N}, i\leq j}$, and denote it by $M = \varprojlim_{i\in\mathbb{N}} M_i$.

3   Profinite dimensional manifolds

In this section, we introduce the concept of profinite dimensional manifolds and establish the differential geometric foundations of this new category. For comparison and further reading on this topic we refer to [1, 14] and [34, Section 1.4].

3.1   The category of profinite dimensional manifolds

The following definition lies in the center of the paper:

Definition 3.1.

a) By a *smooth projective system* we understand a family $(M_i, \mu_{ij})_{i,j\in\mathbb{N}, i\leq j}$ of smooth manifolds M_i and surjective submersions $\mu_{ij}\colon M_j \to M_i$ for $i \leq j$ such that the following conditions hold true:

(SPS1) $\mu_{ii} = \mathrm{id}_{M_i}$ for all $i \in \mathbb{N}$.
(SPS2) $\mu_{ij} \circ \mu_{jk} = \mu_{ik}$ for all $i, j, k \in \mathbb{N}$ such that $i \leq j \leq k$.

b) If $(M'_a, \mu'_{ab})_{a,b\in\mathbb{N},\, a\leq b}$ denotes a second smooth projective system, a *morphism of smooth projective systems* between $(M_i, \mu_{ij})_{i,j\in\mathbb{N},\, i\leq j}$ and $(M'_a, \mu'_{ab})_{a,b\in\mathbb{N},\, a\leq b}$ is a pair $(\varphi, (F_a)_{a\in\mathbb{N}})$ consisting of a strictly increasing map $\varphi\colon \mathbb{N}\to\mathbb{N}$ and a family of smooth maps $F_a\colon M_{\varphi(a)}\to M'_a,\ a\in\mathbb{N}$ such that for each pair $a,b\in\mathbb{N}$ with $a\leq b$ the diagram

$$
\begin{array}{ccc}
M_{\varphi(a)} & \xleftarrow{\ \mu_{\varphi(a)\varphi(b)}\ } & M_{\varphi(b)} \\
{\scriptstyle F_a}\big\downarrow & & \big\downarrow{\scriptstyle F_b} \\
M'_a & \xleftarrow{\ \ \mu'_{ab}\ \ } & M'_b
\end{array}
$$

commutes. We usually denote a smooth projective system shortly by $\big(M_i, \mu_{ij}\big)$ and write

$$(\varphi, F_a)\colon\ (M_i, \mu_{ij}) \longrightarrow (M'_a, \mu'_{ab})$$

to indicate that $(\varphi, (F_a)_{a\in\mathbb{N}})$ is a morphism of smooth projective systems. If each of the maps F_a is a submersion (resp. immersion), we call the morphism (φ, F_a) a *submersion* (resp. *immersion*).

c) Two smooth projective systems (M_i, μ_{ij}) and (M'_a, μ'_{ab}) are called *equivalent*, if there are surjective submersions

$$(\varphi, F_a)\colon\ (M_i, \mu_{ij}) \longrightarrow (M'_a, \mu'_{ab}), \qquad (\psi, G_i)\colon\ (M'_a, \mu'_{ab}) \longrightarrow (M_i, \mu_{ij})$$

such that the diagrams

$$
\begin{array}{ccc}
M_i & \xleftarrow{\ \mu_{i\,\varphi(\psi(i))}\ } & M_{\varphi(\psi(i))} \\
{\scriptstyle G_i}\nwarrow & & \swarrow{\scriptstyle F_{\psi(i)}} \\
& M'_{\psi(i)} &
\end{array}
\qquad\text{and}\qquad
\begin{array}{ccc}
M'_a & \xleftarrow{\ \mu'_{a\,\psi(\varphi(a))}\ } & M'_{\psi(\varphi(a))} \\
{\scriptstyle F_a}\nwarrow & & \swarrow{\scriptstyle G_{\varphi(a)}} \\
& M_{\varphi(a)} &
\end{array}
$$

commute for all $i, a\in\mathbb{N}$. A pair of such surjective submersions will be called an *equivalence transformation of smooth projective systems*.

Remark 3.2. In the definition of smooth projective systems and later in the one of smooth projective representations we use the partially ordered set $\mathbb{N}$ as index set. Obviously, $\mathbb{N}$ can be replaced there by any partially ordered set canonically isomorphic to $\mathbb{N}$ such as an infinite subset of $\mathbb{Z}$ bounded from below. We will silently use this observation in later applications for convenience of notation.

Example 3.3.

a) Let M be a manifold. Then (M_i, μ_{ij}) with $M_i := M$ and $\mu_{ij} := \mathrm{id}_M$ for $i\leq j$ is a smooth projective system which we call *constant* and which we denote shortly by (M, id_M).

b) Assume that for $i\leq j$ one has given surjective linear maps $\lambda_{ij}\colon V_j\to V_i$ between real finite dimensional vector spaces such that (SPS1) and (SPS2) are satisfied. Then (V_i, λ_{ij}) is a smooth projective system. For example, this situation arises in deformation quantization of symplectic manifolds when constructing the completed symmetric tensor algebra of a finite dimensional real vector space; see [34] for details. Of course, a simpler example is given by the canonical projections $\pi_{ij}\colon \mathbb{R}^j\to\mathbb{R}^i$ onto the first i coordinates, hence $(\mathbb{R}^i, \pi_{ij})$ is a (non-trivial) smooth projective system.

c) In the structure theory of topological groups [8, 26] one considers smooth projective systems $(\mathsf{G}_i, \eta_{ij})$ such that each G_j is a Lie group and the $\eta_{ij} \colon \mathsf{G}_j \to \mathsf{G}_i$ are continuous group homomorphisms. See Example 3.8(c) below for a precise description of the projective limits of such projective systems of Lie groups.

d) The tower of k-jets over a fiber bundle together with their canonical projections forms a smooth projective system (see Section 4.1).

Within the category of (smooth finite dimensional) manifolds, a projective limit of a smooth projective system obviously does in general not exist. In the following, we will enlarge the category of manifolds by the so-called profinite dimensional manifolds (and appropriate morphisms). The thus obtained category will contain projective limits of smooth projective systems.

Definition 3.4.

a) By a *smooth projective representation* of a commutative locally $\mathbb{R}$-ringed space $(M, \mathscr{C}_M^\infty)$ we understand a smooth projective system (M_i, μ_{ij}) together with a family of continuous maps $\mu_i \colon M \to M_i$, $i \in \mathbb{N}$, such that the following conditions hold true:

(PFM1) As a topological space, M together with the family of maps μ_i, $i \in \mathbb{N}$, is a projective limit of (M_i, μ_{ij}).

(PFM2) The section space $\mathscr{C}_M^\infty(U)$ of the structure sheaf over an open subset $U \subset M$ is given by the set of all $f \in \mathscr{C}(U)$ such that for every $p \in U$ there exists an $i \in \mathbb{N}$, an open $U_i \subset M_i$ and an $f_i \in \mathscr{C}^\infty(U_i)$ such that $p \in \mu_i^{-1}(U_i) \subset U$ and

$$f_{|\mu_i^{-1}(U_i)} = f_i \circ \mu_{i|\mu_i^{-1}(U_i)}$$

hold true.

We usually denote a smooth projective representation briefly as a family (M_i, μ_{ij}, μ_i).

b) A smooth projective representation (M_i, μ_{ij}, μ_i) of $(M, \mathscr{C}_M^\infty)$ is said to be *regular*, if each of the maps $\mu_{ij} \colon M_j \to M_i$ is a fiber bundle.

c) Two smooth projective representations (M_i, μ_{ij}, μ_i) and $(M'_a, \mu'_{ab}, \mu'_a)$ of $(M, \mathscr{C}_M^\infty)$ are called *equivalent*, if there is an equivalence transformation of smooth projective systems

$$(\varphi, F_a) \colon (M_i, \mu_{ij}) \longrightarrow (M'_a, \mu'_{ab}), \qquad (\psi, G_i) \colon (M'_a, \mu'_{ab}) \longrightarrow (M_i, \mu_{ij})$$

such that

$$\mu_i = G_i \circ \mu'_{\psi(i)} \qquad \text{and} \qquad \mu'_a = F_a \circ \mu_{\varphi(a)} \qquad \text{for all} \quad i, a \in \mathbb{N}.$$

In the following, we will sometimes call such a pair of surjective submersions an *equivalence transformation of smooth projective representations*. The equivalence class of a smooth projective system (M_i, μ_{ij}, μ_i) will be simply denoted by $[(M_i, \mu_{ij}, \mu_i)]$ and called a *pfd structure* on $(M, \mathscr{C}_M^\infty)$.

Proposition 3.5. *Let $(M, \mathscr{C}_M^\infty)$ be a commutative locally $\mathbb{R}$-ringed space with a smooth projective representation (M_i, μ_{ij}, μ_i). Assume further that (M'_a, μ'_{ab}) is a smooth projective system which is equivalent to (M_i, μ_{ij}). Then there are continuous maps $\mu'_a \colon M \to M'_a$, $a \in \mathbb{N}$, such that $(M'_a, \mu'_{ab}, \mu'_a)$ becomes a smooth projective representation of $(M, \mathscr{C}_M^\infty)$ which is equivalent to (M_i, μ_{ij}, μ_i).*

Proof. Choose an equivalence transformation of smooth projective systems

$$(\varphi, F_a)\colon\ (M_i, \mu_{ij}) \longrightarrow (M'_a, \mu'_{ab}), \qquad (\psi, G_i)\colon\ (M'_a, \mu'_{ab}) \longrightarrow (M_i, \mu_{ij}).$$

Put $\mu'_a := F_a \circ \mu_{\varphi(a)}$. Let us show first that M together with the family of continuous maps μ'_a, $a \in \mathbb{N}$ is a projective limit of (M'_a, μ'_{ab}). So assume that X is a topological space, and $h_a\colon X \to M'_a$, $a \in \mathbb{N}$ a family of continuous maps such that $h_a = \mu'_{ab} \circ h_b$ for $a \leq b$. Since M is a projective limit of (M_i, μ_{ij}), there exists a uniquely determined $h\colon X \to M$ such that $\mu_i \circ h = G_i \circ h_{\psi(i)}$ for all $i \in \mathbb{N}$. But then

$$\mu'_a \circ h = F_a \circ \mu_{\varphi(a)} \circ h = F_a \circ G_{\varphi(a)} \circ h_{\psi(\varphi(a))} = \mu'_{a\psi(\varphi(a))} \circ h_{\psi(\varphi(a))} = h_a.$$

Moreover, if $\widetilde{h}\colon X \to M$ is a continuous function such that $\mu'_a \circ \widetilde{h} = h_a$ for all $a \in \mathbb{N}$, one computes

$$\mu_i \circ \widetilde{h} = \mu_{i\varphi(\psi(i))} \circ \mu_{\varphi(\psi(i))} \circ \widetilde{h} = G_i \circ F_{\psi(i)} \circ \mu_{\varphi(\psi(i))} \circ \widetilde{h} = G_i \circ \mu'_{\psi(i)} \circ \widetilde{h} = G_i \circ h_{\psi(i)}.$$

Since M is a projective limit of (M_i, μ_{ij}), this entails $\widetilde{h} = h$. This proves that M is a projective limit of (M'_a, μ'_{ab}).

Next let us show that (PFM2) holds true with the μ_i replaced by the μ'_a. So let $U \subset M$ be open, $f \in \mathscr{C}^\infty_M(M)$, and $p \in U$. Choose $i \in \mathbb{N}$ such that there is an open $U_i \subset M_i$ and a smooth $f_i\colon U_i \to \mathbb{R}$ with $p \in \mu_i^{-1}(U_i) \subset U$ and $f_{|\mu_i^{-1}(U_i)} = f_i \circ \mu_{i|\mu_i^{-1}(U_i)}$. Put $a := \psi(i)$, $V_a := G_i^{-1}(U_i)$, and define $\widetilde{f}_a\colon V_a \to \mathbb{R}$ by $\widetilde{f}_a := f_i \circ G_{i|V_a}$. Then $\widetilde{f}_a$ is smooth, and

$$\widetilde{f}_a \circ \mu'_{a|\mu'_a{}^{-1}(V_a)} = f_i \circ G_i \circ F_{\psi(i)} \circ \mu_{\varphi(\psi(i))|\mu'_a{}^{-1}(V_a)}$$
$$= f_i \circ \mu_{i\varphi(\psi(i))} \circ \mu_{\varphi(\psi(i))|\mu'_a{}^{-1}(V_a)} = f_i \circ \mu_{i|\mu'_a{}^{-1}(V_a)} = f_{|\mu'_a{}^{-1}(V_a)},$$

where we have used that $\mu'_a{}^{-1}(V_a) = \mu_i^{-1}(U_i)$. Similarly one shows that a continuous $\widetilde{f}\colon U \to \mathbb{R}$ is an element of $\mathscr{C}^\infty_M(U)$, if for every $p \in U$ there is an $a \in \mathbb{N}$, an open $V_a \subset M'_a$, and a smooth function $\widetilde{f}_a\colon V_a \to \mathbb{R}$ such that $p \in \mu'_a{}^{-1}(V_a) \subset U$ and $\widetilde{f}_a \circ \mu'_{a|\mu'_a{}^{-1}(V_a)} = \widetilde{f}_{|\mu'_a{}^{-1}(V_a)}$.

Finally, it remains to prove that $\mu_i = G_i \circ \mu'_{\psi(i)}$ for all $i \in \mathbb{N}$, but this follows from

$$G_i \circ \mu'_{\psi(i)} = G_i \circ F_{\psi(i)} \circ \mu_{\varphi(\psi(i))} = \mu_{i\varphi(\psi(i))} \circ \mu_{\varphi(\psi(i))} = \mu_i.$$

This finishes the proof.   ∎

Remark 3.6. The preceding proposition entails that the structure sheaf of a commutative locally $\mathbb{R}$-ringed space $(M, \mathscr{C}^\infty_M)$ for which a smooth projective representation (M_i, μ_{ij}, μ_i) exists depends only on the equivalence class $[(M_i, \mu_{ij}, \mu_i)]$.

The latter remark justifies the following definition:

Definition 3.7.

a) By a *profinite dimensional manifold* we understand a commutative locally $\mathbb{R}$-ringed space $(M, \mathscr{C}^\infty_M)$ together with a pfd structure defined on it. The profinite dimensional manifold $(M, \mathscr{C}^\infty_M)$ is called *regular*, if there exists a regular smooth representation within the pfd structure on $(M, \mathscr{C}^\infty_M)$.

b) Assume that $(M, \mathscr{C}^\infty_M)$ and $(N, \mathscr{C}^\infty_N)$ are profinite dimensional manifolds. Then a continuous map $f\colon M \to N$ is said to be *smooth*, if the following condition holds true:

for every open $U \subset N$, and $g \in \mathscr{C}_N^\infty(U)$ one has

$$g \circ f_{|f^{-1}(U)} \in \mathscr{C}_M^\infty\big(f^{-1}(U)\big).$$

By definition, it is clear that the composition of smooth maps between profinite dimensional manifolds is smooth, hence profinite dimensional manifolds and the smooth maps between them as morphisms form a category, the isomorphisms of which can be safely called *diffeomorphisms*. All of this terminology is justified by the simple observation Example 3.8(a) below.

Example 3.8.

a) Given a manifold M, the constant smooth projective system (M, id_M) defines a smooth projective representation for the ringed space $(M, \mathscr{C}_M^\infty)$. Hence, every manifold is a profinite dimensional manifold in a natural way, and the category of manifolds a full subcategory of the category of profinite dimensional manifolds.

b) Assume that (M_i, μ_{ij}) is a smooth projective system. Let

$$M := \varprojlim_{i \in \mathbb{N}} M_i$$

together with the natural projections $\mu_i \colon M \to M_i$ denote the canonical projective limit of $\big(M_i, \mu_{ij}\big)$. Then, (PFM1) is fulfilled by assumption, and it is immediate that M carries a uniquely determined structure sheaf $\mathscr{C}_M^\infty$ which satisfies (PFM2). The locally ringed space $(M, \mathscr{C}_M^\infty)$ together with the pfd structure $[(M_i, \mu_{ij}, \mu_i)]$ then is a profinite dimensional manifold. This profinite dimensional manifold is even a projective limit of the projective system (M_i, μ_{ij}) within the category of profinite dimensional manifolds. We therefore write in this situation

$$\big(M, \mathscr{C}_M^\infty\big) = \varprojlim_{i \in \mathbb{N}} \big(M_i, \mathscr{C}_{M_i}^\infty\big)$$

and call $(M, \mathscr{C}_M^\infty)$ (together with $[(M_i, \mu_{ij}, \mu_i)]$) *the canonical smooth projective limit* of (M_i, μ_{ij}).

c) A locally compact Hausdorff topological group G is called *Lie projective*, if every neighbourhood of the identity contains a compact Lie normal subgroup, i.e., a normal subgroup $N \subset G$ such that G/N is a Lie group. One has the following structure theorem [8, Theorem 4.4], [26]. A locally compact metrizable group G is Lie projective, if and only if there is a smooth projective system $(\mathsf{G}_i, \eta_{ij})$ as in Example 3.3(c) together with continuous group homomorphisms $\eta_i \colon \mathsf{G} \to \mathsf{G}_i$, $i \in \mathbb{N}$ such that (G, η_i) is a projective limit of $(\mathsf{G}_i, \eta_{ij})$. Again, it follows that G carries a uniquely determined structure sheaf $\mathscr{C}_\mathsf{G}^\infty$ satisfying (PFM2). The locally ringed space $(\mathsf{G}, \mathscr{C}_\mathsf{G}^\infty)$ together with the pfd structure $[(\mathsf{G}_i, \eta_{ij}, \eta_i)]$ becomes a regular profinite dimensional manifold with a group structure such that all of its structure maps are smooth.

d) The space of infinite jets over a fiber bundle canonically is a profinite dimensional manifold (see Section 4.3).

Remark 3.9.

a) In the sequel, $(M, \mathscr{C}_M^\infty)$ or briefly M will always denote a profinite dimensional manifold. Moreover, (M_i, μ_{ij}, μ_i) always stands for a smooth projective representation defining the pfd structure on M. The sheaf of smooth functions on a profinite dimensional manifold will often briefly be denoted by $\mathscr{C}^\infty$, if no confusion can arise.

b) One of the intentions when constructing the category of profinite dimensional manifolds was that it should be a category with projective limits which extends the one of smooth manifolds and that it is minimal in a certain sense with respect to these properties. The category of profinite dimensional manifolds fulfills these requirements. That it extends the category of manifolds follows from Example 3.8(a). By a straightforward argument using the 'diagonal trick' for doubly projective limits one concludes that the category of profinite dimensional manifolds contains all projective limits. The minimality requirement is a direct consequence of the definition of profinite dimensional manifolds as abstract projective limits of manifolds.

c) The profinite dimensional manifolds defined in this paper coincide with the projective limits of manifolds from [1], but are in general not plb-manifolds in the sense of [14, Definition 3.1.2]. The latter have the property that they can be modelled locally on Fréchet spaces representable as projective limits of Banach spaces. A profinite dimensional manifold of infinite dimension, though, can in general not locally be modelled by open subsets of $\mathbb{R}^\infty$. In particular when the underlying profinite dimensional manifold is given as the manifold of formal solutions of a formally integrable PDE in the sense of Proposition and Definition 4.29 corresponding local charts with values $\mathbb{R}^\infty$ appear to exist only in particular cases. A more detailed study of this phenomenon is left for future work.

Let $N \subset M$ be a subset, and assume further that for some smooth projective representation (M_i, μ_{ij}, μ_i) of the pfd structure on M the following holds true:

(PFSM1) There is a stricly increasing sequence $(l_i)_{i\in\mathbb{N}}$ such that for every $i \in \mathbb{N}$ the set $N_i := \mu_{l_i}(N)$ is a submanifold of M_{l_i}.

(PFSM2) One has $N = \bigcap_{i\in\mathbb{N}} \mu_{l_i}^{-1}(N_i)$.

(PFSM3) The induced map

$$\nu_{ij} := \mu_{l_i l_j \,|N_j} : \ N_j \longrightarrow N_i$$

is a submersion for all $i, j \in \mathbb{N}$ with $j \geq i$.

Observe that the ν_{ij} are surjective by definition of the manifolds N_i and by $\nu_i = \nu_{ij} \circ \nu_j$, where we have put $\nu_i := \mu'_{l_i | N}$. In particular, (N_i, ν_{ij}) becomes a smooth projective system.

Proposition and Definition 3.10. *Let $N \subset M$ be a subset such that for some smooth projective representation (M_i, μ_{ij}, μ_i) of the pfd structure on M the axioms* (PFSM1) *to* (PFSM3) *are fulfilled. Then N carries in a natural way the structure of a profinite dimensional manifold such that its sheaf of smooth functions coincides with the sheaf $\mathscr{C}^\infty_{|N}$ of continuous functions on open subset of N which are locally restrictions of smooth functions on M. A smooth projective representation of N defining its natural pfd structure is given by the family (N_i, ν_{ij}, ν_i). From now on, such a subset $N \subset M$ will be called a* profinite dimensional submanifold *of M, and (M_i, μ_{ij}, μ_i) a smooth projective representation of M inducing the submanifold structure on N.*

Proof. We first show that N together with the maps ν_i is a (topological) projective limit of the projective system (N_i, ν_{ij}). Let $p_i \in N_i$, $i \in \mathbb{N}$ such that $\nu_{ij}(p_j) = p_i$ for all $j \geq i$. Since M together with the μ_i is a projective limit of (M_i, μ_{ij}), there exists an $p \in M$ such that $\mu_{l_i}(p) = p_i$ for all $i \in \mathbb{N}$. By axiom (PFSM2), $p \in N$, hence one concludes that N is a projective limit of the manifolds N_i.

Next, we show that $\mathscr{C}^\infty_{|N}$ coincides with the uniquely determined sheaf $\mathscr{C}^\infty_N$ satisfying axiom (PFM2). Since the canonical embeddings $N_i \hookrightarrow M_{l_i}$ are smooth by (PFSM1), the embedding

$N \hookrightarrow M$ is smooth as well, and $\mathscr{C}_{|N}^{\infty}$ is a subsheaf of the sheaf $\mathscr{C}_N^{\infty}$. It remains to prove that for every open $V \subset N$ a function $f \in \mathscr{C}_N^{\infty}(V)$ is locally the restriction of a smooth function on M. To show this let $p \in V$ and V_i an open subset of some N_i such that $p \in \nu_i^{-1}(V_i) \subset V$, and such that there is an $f_i \in \mathscr{C}^{\infty}(V_i)$ with $f_{|\nu_i^{-1}(V_i)} = f_i \circ \nu_{i|\nu_i^{-1}(V_i)}$. Since N_i is locally closed in M_{l_i}, we can assume after possibly shrinking V_i that there is an open $U_i \subset M_{l_i}$ with $V_i = N_i \cap U_i$ and such that $N_i \cap U_i$ is closed in U_i. Then there exists $F_i \in \mathscr{C}^{\infty}(U_i)$ such that $F_{i|V_i} = f_i$. Put $F := F_i \circ \mu_{l_i|\mu_{l_i}^{-1}(U_i)}$. Then $F \in \mathscr{C}^{\infty}(\mu_{l_i}^{-1}(U_i))$, and

$$f_{|\nu_i^{-1}(V_i)} = F_{|\nu_i^{-1}(V_i)},$$

which proves that $f \in \mathscr{C}_{|N}^{\infty}(V)$. The claim follows.   ∎

Example 3.11.

   a) Every open subset U of M is naturally a profinite dimensional submanifold since for each $i \in \mathbb{N}$ the set $U_i := \mu_i(U)$ is an open submanifold of M_i.

   b) Consider the profinite dimensional manifold

   $$\left(\mathbb{R}^{\infty}, \mathscr{C}_{\mathbb{R}^{\infty}}^{\infty}\right) := \varprojlim_{n \in \mathbb{N}} \left(\mathbb{R}^n, \mathscr{C}_{\mathbb{R}^n}^{\infty}\right),$$

   and let $\mathrm{B}^n(0)$ be the open unit ball in $\mathbb{R}^n$. The projective limit

   $$\left(\mathrm{B}^{\infty}(0), \mathscr{C}_{\mathrm{B}^{\infty}(0)}^{\infty}\right) := \varprojlim_{n \in \mathbb{N}} \left(\mathrm{B}^n(0), \mathscr{C}_{\mathrm{B}^n(0)}^{\infty}\right)$$

   then becomes a profinite dimensional submanifold of $\mathbb{R}^{\infty}$. Note that it is not locally closed in $\mathbb{R}^{\infty}$.

   c) The space of formal solutions of a formally integrable partial differential equation is a profinite dimensional submanifold of the space of infinite jets over the underlying fiber bundle (see Section 4.3).

We continue with:

Definition 3.12. Let $U \subset M$ be open. A smooth function $f \in \mathscr{C}^{\infty}(U)$ then is called *local*, if there is an open $U_i \subset M_i$ for some $i \in \mathbb{N}$ and a function $f_i \in \mathscr{C}^{\infty}(U_i)$ such that $U \subset \mu_i^{-1}(U_i)$ and $f = f_i \circ \mu_{i|U}$. We denote the space of local functions over U by $\mathscr{C}_{\mathrm{loc}}^{\infty}(U)$.

Remark 3.13.

   a) Observe that $\mathscr{C}_{\mathrm{loc}}^{\infty}$ forms a presheaf on M, which depends only on the pfd structure $[(M_i, \mu_{ij}, \mu_i)]$. Moreover, it is clear by construction that for every open $U \subset M$ and every representative (M_i, μ_{ij}, μ_i) of the pfd structure, $\mathscr{C}_{\mathrm{loc}}^{\infty}(U)$ together with the family of pull-back maps $\mu_i^*: \mathscr{C}^{\infty}(\mu_i(U)) \to \mathscr{C}_{\mathrm{loc}}^{\infty}(U)$ is an inductive limit of the injective system of linear spaces $(\mathscr{C}^{\infty}(\mu_i(U)), \mu_{ij}^*)_{i \in \mathbb{N}}$.

   b) $\mathscr{C}_{\mathrm{loc}}^{\infty}$ is in general not a sheaf unless M is a finite dimensional manifold. The sheaf associated to $\mathscr{C}_{\mathrm{loc}}^{\infty}$ naturally coincides with $\mathscr{C}^{\infty}$ since locally, every smooth function is local.

   c) By naming sections of $\mathscr{C}_{\mathrm{loc}}^{\infty}$ local functions we essentially follow Stasheff [45, Definition 1.1] and Barnich [6, Definition 1.1], where the authors consider jet bundles. Note that in [1], local functions are called cylindrical functions.

d) The representative $\mathcal{M} := (M_i, \mu_{ij}, \mu_j)$ leads to a particular filtration $\mathcal{F}_\bullet^{\mathcal{M}}$ of the presheaf of local functions by putting, for $l \in \mathbb{N}$,

$$\mathcal{F}_l^{\mathcal{M}}(\mathscr{C}_{\mathrm{loc}}^\infty) := \mu_l^* \mathscr{C}_{M_l}^\infty.$$

Observe that this filtration has the property that

$$\mathscr{C}_{\mathrm{loc}}^\infty = \bigcup_{l \in \mathbb{N}} \mathcal{F}_l^{\mathcal{M}}(\mathscr{C}_{\mathrm{loc}}^\infty).$$

3.2   Tangent bundles and vector fields

The tangent space at a point of a finite dimensional manifold can be defined as a set of equivalence classes of germs of smooth paths at that point or as the space of derivations on the stalk of the sheaf of smooth functions at that point. The definition via paths can not be immediately carried over to the profinite dimensional case, so we use the derivation approach.

Definition 3.14. Given a point p of the profinite dimensional manifold M, the *tangent space* of M at p is defined as the space of derivations on $\mathscr{C}_p^\infty$, the stalk of smooth functions at p, i.e., as the space

$$\mathrm{T}_p M := \mathrm{Der}\left(\mathscr{C}_p^\infty, \mathbb{R}\right).$$

Elements of $\mathrm{T}_p M$ will be called *tangent vectors* of M at p. The *tangent bundle* of M is the disjoint union

$$\mathrm{T}M := \bigcup_{p \in M} \mathrm{T}_p M,$$

and

$$\pi_{\mathrm{T}M} \colon \ \mathrm{T}M \longrightarrow M, \qquad \mathrm{T}_p M \ni Y \longmapsto p$$

the *canonical projection*.

Note that for every $i \in \mathbb{N}$ there is a canonical map $\mathrm{T}\mu_i \colon \mathrm{T}M \to \mathrm{T}M_i$ which maps a tangent vector $Y \in \mathrm{T}_p M$ to the tangent vector

$$Y_i \colon \ \mathscr{C}_{M_i, p_i}^\infty \to \mathbb{R}, \qquad [f_i]_{p_i} \mapsto Y\left([f_i \circ \mu_i]_p\right), \qquad \text{where} \quad p_i := \mu_i(p).$$

By construction, one has $\mathrm{T}\mu_{ij} \circ \mathrm{T}\mu_j = \mathrm{T}\mu_i$ for $i \leq j$. We give $\mathrm{T}M$ the coarsest topology such that all the maps $\mathrm{T}\mu_i$, $i \in \mathbb{N}$ are continuous. Now we record the following observation:

Lemma 3.15. *The topological space $\mathrm{T}M$ together with the maps $\mathrm{T}\mu_i$ is a projective limit of the projective system $(\mathrm{T}M_i, \mathrm{T}\mu_{ij})$.*

Proof. Assume that X is a topological space, and $(\Phi_i)_{i \in \mathbb{N}}$ a family of continuous maps $\Phi_i \colon X \to \mathrm{T}M_i$ such that $\mathrm{T}\mu_{ij} \circ \Phi_j = \Phi_i$ for all $i \leq j$. Since M is a projective limit of the projective system (M_i, μ_{ij}), there exists a uniquely determined continuous map $\varphi \colon X \to M$ such that $\pi_{\mathrm{T}M_i} \circ \Phi_i = \mu_i \circ \varphi$ for all $i \in \mathbb{N}$. Now let $x \in X$, and put $p := \varphi(x)$ and $p_i := \mu_i(p)$. Then, for every $i \in \mathbb{N}$, $\Phi_i(x)$ is a tangent vector of M_i with footpoint p_i. We now construct a derivation $\Phi(x) \in \mathrm{Der}(\mathscr{C}_p^\infty, \mathbb{R})$. Let $[f]_p \in \mathscr{C}_p^\infty$, i.e., let f be a smooth function defined on a neighborhood U of p, and $[f]_p$ its germ at p. Then there exists $i \in \mathbb{N}$, an open neighborhood $U_i \subset M_i$ of p_i and a smooth function $f_i \colon U_i \to \mathbb{R}$ such that

$$\mu_i^{-1}(U_i) \subset U \qquad \text{and} \qquad f_{|\mu_i^{-1}(U_i)} = f_i \circ \mu_{i|\mu_i^{-1}(U_i)}.$$

We now put

$$\Phi(x)\big([f]_p\big) := \Phi_i(x)\big([f_i]_{p_i}\big), \qquad \text{where} \quad p_i := \mu_i(p).$$

We have to show that $\Phi(x)$ is independent of the choices made, and that it is a derivation indeed. So let $f' \colon U' \to \mathbb{R}$ be another smooth function defining the germ $[f]_p$. Choose $j \in \mathbb{N}$, an open neighborhood $U'_j \subset M_j$ of p_j, and a smooth function $f'_j \colon U'_j \to \mathbb{R}$ such that

$$\mu_j^{-1}(U'_j) \subset U' \qquad \text{and} \qquad f_{|\mu_j^{-1}(U'_j)} = f'_j \circ \mu_{j|\mu_j^{-1}(U'_j)}.$$

Without loss of generality, we can assume $i \le j$. By assumption $[f]_p = [f']_p$, hence one concludes that

$$f_i \circ \mu_{ij|V_j} = f'_{j|V_j}$$

for some open neighborhood $V_j \subset M_j$ of $p_j := \mu_j(p)$. But this implies, using the assumption on the Φ_i that

$$\Phi_j(x)\big([f'_j]_{p_j}\big) = \mathrm{T}\mu_{ij}\Phi_j(x)\big([f_i]_{p_i}\big) = \Phi_i(x)\big([f_i]_{p_i}\big).$$

Hence, $\Phi(x)$ is well-defined, indeed.

Next, we show that $\Phi(x)$ is a derivation. So let $[f]_p, [g]_p \in \mathscr{C}_p^\infty$ be two germs of smooth functions at p. Then, after possibly shrinking the domains of f and g, one can find an $i \in \mathbb{N}$, an open neighborhood $U_i \subset M_i$ of p_i, and $f_i, g_i \in \mathscr{C}^\infty(U_i)$ such that

$$f_{|\mu_i^{-1}(U_i)} = f_i \circ \mu_{i|\mu_i^{-1}(U_i)} \qquad \text{and} \qquad g_{|\mu_i^{-1}(U_i)} = g_i \circ \mu_{i|\mu_i^{-1}(U_i)}.$$

Since $\Phi_i(x)$ acts as a derviation on $\mathscr{C}_{p_i}^\infty$, one checks

$$\begin{aligned}
\Phi(x)\big([f]_p[g]_p\big) &= \Phi_i(x)\big([f_i]_{p_i}[g_i]_{p_i}\big) = f_i(p_i)\Phi_i(x)\big([g_i]_{p_i}\big) + g_i(p_i)\Phi_i(x)\big([f_i]_{p_i}\big) \\
&= f(p)\Phi(x)\big([g]_p\big) + g(p)\Phi(x)\big([f]_p\big),
\end{aligned}$$

which means that $\Phi(x)$ is a derivation.

By construction, it is clear that

$$\mathrm{T}\mu_i\Phi(x) = \Phi_i(x) \qquad \text{for all} \quad i \in \mathbb{N}.$$

Let us verify that $\Phi(x)$ is uniquely determined by this property. So assume that $\Phi'(x)$ is another element of T_pM such that $\mathrm{T}\mu_i\Phi'(x) = \Phi_i(x)$ for all $i \in \mathbb{N}$. For $[f]_p \in \mathscr{C}_p^\infty$ of the form $f = f_i \circ \mu_{i|\mu_i^{-1}(U_i)}$ with $U_i \subset M_i$ an open neighborhood of p_i and $f_i \in \mathscr{C}^\infty(U_i)$ this assumption entails

$$\Phi(x)\big([f]_p\big) = \Phi_i(x)\big([f_i]_{p_i}\big) = \Phi'(x)\big([f]_p\big).$$

Since every germ $[f]_p$ is locally of the form $f_i \circ \mu_{i|\mu_i^{-1}(U_i)}$, we obtain $\Phi(x) = \Phi'(x)$.

Finally, we observe that $\Phi \colon X \to \mathrm{T}M$ is continuous, since all maps $\Phi_i = \mathrm{T}\mu_i\Phi$ are continuous, and $\mathrm{T}M$ carries the initial topology with respect to the maps $\mathrm{T}\mu_i$.

This concludes the proof that $\mathrm{T}M$ together with the maps $\mathrm{T}\mu_i$ is a projective limit of the projective system $(\mathrm{T}M_i, \mathrm{T}\mu_{ij})$. $\blacksquare$

Remark 3.16.

    a) If $p \in M$, $Y_p, Z_p \in \mathrm{T}_p M$, and $\lambda \in \mathbb{R}$, then the maps $Y_p + Z_p \colon \mathscr{C}_p^\infty \to \mathbb{R}$ and $\lambda Y_p \colon \mathscr{C}_p^\infty \to \mathbb{R}$ are derivations again. Hence $\mathrm{T}_p M$ becomes a topological vector space in a natural way and one has $\mathrm{T}_p M \cong \varprojlim_{i \in \mathbb{N}} \mathrm{T}_{\mu_i(p)} M_i$ canonically as topological vector spaces. In particular, this implies that $\pi_{\mathrm{T}M} \colon \mathrm{T}M \to M$ is a continuous family of vector spaces. Note that this family need not be locally trivial, in general.

    b) Denote by $\mathscr{P}_{M,p}^\infty$ the set of germs of smooth paths $\gamma \colon (\mathbb{R}, 0) \to (M, p)$. There is a canonical map $\mathscr{P}_{M,p}^\infty \to \mathrm{T}_p M$ which associates to each germ of a smooth path $\gamma \colon (\mathbb{R}, 0) \to (M, p)$ the derivation

$$\dot\gamma \colon \mathscr{C}_p^\infty \longrightarrow \mathbb{R}, \qquad [f]_p \longmapsto (f \circ \gamma)^{\cdot}(0).$$

Unlike in the finite dimensional case, this map need not be surjective, in general, as Example 3.18 below shows. But note the following result.

Proposition 3.17. *In case the profinite dimensional manifold M is regular, the "dot map"*

$$\mathscr{P}_{M,p}^\infty \longrightarrow \mathrm{T}_p M, \qquad [\gamma]_0 \longmapsto \dot\gamma(0)$$

is surjective for every $p \in M$.

Proof. We start with an auxiliary construction. Choose a smooth projective representation (M_i, μ_{ij}, μ_i) within the pfd structure on M such that all μ_{ij} are fiber bundles. Put $p_i := \mu_i(p)$ for every $i \in \mathbb{N}$. Then choose a relatively compact open neighborhood $U_0 \subset M_0$ of p_0 which is diffeomorphic to an open ball in some $\mathbb{R}^n$. In particular, U_0 is contractible, hence the fiber bundle $\mu_{01|\mu_{01}^{-1}(U_0)} \colon \mu_{01}^{-1}(U_0) \to U_0$ is trivial with typical fiber $F_1 := \mu_{01}^{-1}(p_0)$. Let $\Psi_0 \colon \mu_{01}^{-1}(U_0) \to U_0 \times F_1$ be a trivialization of that fiber bundle, and $D_1 \subset F_1$ an open neighborhood of p_1 which is diffeomorphic to an open ball in some Euclidean space. Put $U_1 := \Psi_0^{-1}(U_0 \times D_1)$. Then, U_1 is diffeomorphic to a ball in some Euclidean space, and $\mu_{01|U_1} \colon U_1 \to U_0$ is a trivial fiber bundle with fiber D_1. Assume now that we have constructed $U_0 \subset M_0, \ldots, U_j \subset M_j$ such that for all $i \leq j$ the following holds true:

    1) the set U_i is a relatively compact open neighborhood of p_i diffeomorphic to an open ball in some Euclidean space,

    2) for $i > 0$, the identity $\mu_{i-1i}(U_i) = U_{i-1}$ holds true,

    3) for $i > 0$, the restricted map $\mu_{i-1i|U_i} \colon U_i \to U_{i-1}$ is a trivial fiber bundle with fiber D_i diffeomorphic to an open ball in some Euclidean space.

Let us now construct U_{j+1} and D_{j+1}. To this end note first that $\mu_{jj+1|\mu_{jj+1}^{-1}(U_j)} \colon \mu_{jj+1}^{-1}(U_j) \to U_j$ is a trivial fiber bundle with typical fiber $F_j := \mu_{jj+1}^{-1}(p_j)$, since U_j is contractible. Choose a trivialization $\Psi_{j+1} \colon \mu_{jj+1|\mu_{jj+1}^{-1}(U_j)} \to U_j \times F_j$, and an open neighborhood $D_{j+1} \subset F_{j+1}$ of p_{j+1} which is diffeomorphic to an open ball in some Euclidean space. Put $U_{j+1} := \Psi_{j+1}^{-1}(U_j \times D_{j+1})$. Then, U_j is diffeomorphic to a ball in some Euclidean space, and $\mu_{jj+1|U_{j+1}} \colon U_{j+1} \to U_j$ is a trivial fiber bundle with fiber D_{j+1}. This finishes the induction step, and we obtain $U_i \subset M_i$ and D_i such that the three conditions above are satisfied.

    After these preliminaries, assume that $Z \in T_p M$ is a tangent vector. Let $Z_i := \mathrm{T}\mu_i(Z)$ for $i \in \mathbb{N}$. We now inductively construct smooth paths $\gamma_i \colon \mathbb{R} \to U_i$ such that

$$\gamma_i(0) = p_i, \qquad \dot\gamma_i(0) = Z_i, \qquad \text{and, if } i > 0, \qquad \mu_{i-1i} \circ \gamma_i = \gamma_{i-1}. \tag{3.1}$$

To start, choose a smooth path $\gamma_0 \colon \mathbb{R} \to U_0$ such that $\gamma_0(0) = p_0$, and $\dot{\gamma}_0(0) = Z_0$. Assume that we have constructed $\gamma_0, \ldots, \gamma_j$ such that (3.1) is satisfied for all $i \leq j$. Consider the trivial fiber bundle $\mu_{jj+1|U_{j+1}} \colon U_{j+1} \to U_j$, and let $\Psi_{j+1} \colon U_{j+1} \to U_j \times D_{j+1}$ be a trivialization. Then, $\mathrm{T}\Psi_{j+1}(Z_{j+1}) = (Z_j, Y_{j+1})$ for some tangent vector $Y_{j+1} \in \mathrm{T}_{p_{j+1}}D_{j+1}$. Choose a smooth path $\varrho_{j+1} \colon \mathbb{R} \to D_{j+1}$ such that $\varrho_{j+1}(0) = p_{j+1}$, and $\dot{\varrho}_{j+1}(0) = Y_{j+1}$. Put

$$\gamma_{j+1}(t) = \Psi_{j+1}^{-1}\big(\gamma_j(t), \varrho_{j+1}(t)\big) \qquad \text{for all} \quad t \in \mathbb{R}.$$

By construction, γ_{j+1} is a smooth path in U_{j+1} such that (3.1) is fulfilled for $i = j + 1$. This finishes the induction step, and we obtain a family of smooth paths γ_i with the desired properties.

Since M is the smooth projective limit of the M_i, there exists a uniquely determined smooth path $\gamma \colon \mathbb{R} \to M$ such that $\mu_i \circ \gamma = \gamma_i$ for all $i \in \mathbb{N}$. In particular, this entails $\gamma(0) = p$, and $\dot{\gamma}(0) = Z$, or in other words that Z is in the image of the map $\mathscr{P}_{M,p}^\infty \to \mathrm{T}_p M$. $\blacksquare$

Example 3.18. This example shows that there exist profinite dimensional manifolds having tangent vectors which can not be represented as the derivative of a smooth path. Denote by $rS^k \subset \mathbb{R}^{k+1}$ for $k \in \mathbb{N}^*$ the k-sphere of radius $r > 0$. Moreover, denote for $1 \leq i < j$ by $\mu_{ij} \colon \mathbb{R}^{j+1} \to \mathbb{R}^{i+1}$ the projection onto the first $i + 1$ coordinates. We use the same symbols for restrictions of μ_{ji} to open subsets. Now we define inductively a pfd system (M_i, μ_{ij}) with $M_i \subset \mathbb{R}^{i+1}$ open as follows:

$$M_0 := \mathbb{R}, \quad M_1 := \mathbb{R}^2 \setminus S^1, \quad M_2 := (M_1 \times \mathbb{R}) \setminus \tfrac{1}{2}S^2, \quad \ldots, \quad M_{i+1} := (M_i \times \mathbb{R}) \setminus \tfrac{1}{i+1}S^{i+1}.$$

Observe that all μ_{ij} are still surjective submersions when regarded as mappings from M_j to M_i. Next consider the point $p = (p_i)_{i \in \mathbb{N}} \in M := \varprojlim_{i \in \mathbb{N}} M_i$, where $p_i := 0 \in M_i$. Now let $Y \in \mathrm{T}_p M$ be the tangent vector represented by the family $(Y_i)_{i \in \mathbb{N}^*}$ of tangent vectors

$$Y_i \colon \mathscr{C}_{M_i,0}^\infty \to \mathbb{R}, \qquad [f]_0 \mapsto \frac{\partial f}{\partial x_1}(0),$$

where $(x_1, \ldots, x_i)$ are the canonical coordinates of $\mathbb{R}^i$. Assume that there is a smooth path $\gamma \colon (-\varepsilon, \varepsilon) \to M$ such that $\gamma(0) = p$ and $\dot{\gamma}(0) = Y$. Let $\gamma_i := \mu_i \circ \gamma$. Since $\dot{\gamma}_1(0) = 1$, one can achieve after possibly shrinking ε that $\dot{\gamma}_1(t) > \frac{1}{2}$ for all $t \in (-\varepsilon, \varepsilon)$. This implies by the mean value theorem that $|\gamma(t)| \geq \frac{1}{2}|t|$ for all $t \in (-\varepsilon, \varepsilon)$. Now choose $i \in \mathbb{N}^*$ such that $\frac{1}{i} < \frac{1}{4}\varepsilon$. Then $\gamma_i(0) = 0$ but $\gamma_i(\frac{1}{2}\varepsilon)$ has to be outside the connected component of 0 in M_i. This is a contradiction, so there does not exist a path γ with the claimed properties, and Y is not induced by a smooth path.

Let us define a structure sheaf $\mathscr{C}_{\mathrm{T}M}^\infty$ on $\mathrm{T}M$. To this end call a continuous map $f \in \mathscr{C}(U)$ defined on an open set $U \subset \mathrm{T}M$ *smooth*, if for every tangent vector $Z \in U$ there is an $i \in \mathbb{N}$, an open neighborhood $U_i \subset \mathrm{T}M_i$ of $Z_i := \mathrm{T}\mu_i(Z)$, and a smooth map $f_i \in \mathscr{C}^\infty(U_i)$ such that $(\mathrm{T}\mu_i)^{-1}(U_i) \subset U$ and $f_{|(\mathrm{T}\mu_i)^{-1}(U_i)} = f_i \circ (\mathrm{T}\mu_i)_{|(\mathrm{T}\mu_i)^{-1}(U_i)}$. The spaces

$$\mathscr{C}_{\mathrm{T}M}^\infty(U) := \big\{ f \in \mathscr{C}(U) \,|\, f \text{ is smooth} \big\}$$

for $U \subset \mathrm{T}M$ open then form the section spaces of a sheaf $\mathscr{C}_{\mathrm{T}M}^\infty$ which we call the *sheaf of smooth functions* on $\mathrm{T}M$. By construction, the family $(\mathrm{T}M_i, \mathrm{T}\mu_{ij}, \mathrm{T}\mu_i)$ now is a smooth projective representation of the locally ringed space $(\mathrm{T}M, \mathscr{C}_{\mathrm{T}M}^\infty)$, hence $(\mathrm{T}M, \mathscr{C}_{\mathrm{T}M}^\infty)$ becomes a profinite dimensional manifold. Since $\mu_i \circ \pi_{\mathrm{T}M} = \pi_{\mathrm{T}M_i} \circ \mathrm{T}\mu_i$ for all $i \in \mathbb{N}$, one immediately checks that the canonical map $\pi_{\mathrm{T}M} \colon \mathrm{T}M \to M$ is even a smooth map between profinite dimensional manifolds. With these preparations we can state:

Proposition and Definition 3.19. *The profinite dimensional manifold given by* $(\mathrm{T}M, \mathscr{C}^\infty_{\mathrm{T}M})$
and the pfd structure $[(\mathrm{T}M_i, \mathrm{T}\mu_{ij}, \mathrm{T}\mu_i)]$ *is called the* tangent bundle *of* M, *and* $\pi_{\mathrm{T}M}\colon \mathrm{T}M \to M$
its canonical projection. *The pfd structure* $[(\mathrm{T}M_i, \mathrm{T}\mu_{ij}, \mathrm{T}\mu_i)]$ *depends only on the equivalence*
class $[(M_i, \mu_{ij}, \mu_i)]$.

Proof. In order to check the last statement, consider a smooth projective representation
$(M'_a, \mu'_{ab}, \mu'_a)$ which is equivalent to (M_i, μ_{ij}, μ_i). Choose an equivalence transformation of
smooth projective representations

$$(\varphi, F_a)\colon\ (M_i, \mu_{ij}) \longrightarrow (M'_a, \mu'_{ab}), \qquad (\psi, G_i)\colon\ (M'_a, \mu'_{ab}) \longrightarrow (M_i, \mu_{ij}).$$

Then one obtains surjective submersions

$$(\varphi, \mathrm{T}F_a)\colon\ (\mathrm{T}M_i, \mathrm{T}\mu_{ij}) \longrightarrow (\mathrm{T}M'_a, \mathrm{T}\mu'_{ab}), \qquad (\psi, \mathrm{T}G_i)\colon\ (\mathrm{T}M'_a, \mathrm{T}\mu'_{ab}) \longrightarrow (\mathrm{T}M_i, \mathrm{T}\mu_{ij})$$

such that the following diagrams commute for all $i, a \in \mathbb{N}$:

Hence, $(\mathrm{T}M'_a, \mathrm{T}\mu'_{ab})$ is a smooth projective system which is equivalent to $(\mathrm{T}M_i, \mathrm{T}\mu_{ij})$. Now
recall that the map $\mathrm{T}\mu'_a\colon \mathrm{T}M \to \mathrm{T}M'_a$ is defined by $\mathrm{T}\mu'_a(Z_p) = Z_p \circ (\mu'_a)^*$, where $Z_p \in \mathrm{T}_pM$,
$p \in M$, and $(\mu'_a)^*$ denotes the pullback by μ'_a. One concludes that for all $i \in \mathbb{N}$

$$\mathrm{T}G_i \circ \mathrm{T}\mu'_{\psi(i)}(Z_p) = \mathrm{T}G_i\big(Z_p \circ (\mu'_{\psi(i)})^*\big) = Z_p \circ (\mu'_{\psi(i)})^* \circ G_i^* = Z_p \circ \mu_i^* = \mathrm{T}\mu_i(Z_p),$$

and likewise that $\mathrm{T}F_a \circ \mathrm{T}\mu_{\varphi(a)}(Y_p) = \mathrm{T}\mu'_a(Y_p)$ for all $a \in \mathbb{N}$. This entails that the smooth projec-
tive representations $(\mathrm{T}M_i, \mathrm{T}\mu_{ij}, \mathrm{T}\mu_i)$ and $(\mathrm{T}M'_a, \mathrm{T}\mu'_{ab}, \mathrm{T}\mu'_a)$ of the tangent bundle $(\mathrm{T}M, \mathscr{C}^\infty_{\mathrm{T}M})$
are equivalent, and the proof is finished. ∎

Remark 3.20.

a) By Example 3.8(c), the induced smooth projective system $(\mathrm{T}M_i, \mathrm{T}\mu_{ij})$ has the canonical
   smooth projective limit

   $$\big(\widetilde{\mathrm{T}}M, \mathscr{C}^\infty_{\widetilde{\mathrm{T}}M}\big) := \varprojlim_{i \in \mathbb{N}} \big(\mathrm{T}M_i, \mathscr{C}^\infty_{\mathrm{T}M_i}\big).$$

   Denote its canonical maps by $\widetilde{\mathrm{T}}\mu_i\colon \widetilde{\mathrm{T}}M \to \mathrm{T}M_i$. By the universal property of projective
   limits there exists a unique smooth map

   $$\tau\colon\ \mathrm{T}M \longrightarrow \widetilde{\mathrm{T}}M$$

   such that $\widetilde{\mathrm{T}}\mu_i \circ \tau = \mathrm{T}\mu_i$ for all $i \in \mathbb{N}$. By construction of the profinite dimensional manifold
   structure on the tangent bundle $\mathrm{T}M$, the map τ is even a linear diffeomorphism, and is
   in fact given by

   $$\mathrm{T}M \ni Y \longmapsto \big(\mathrm{T}\mu_i(Y)\big)_{i \in \mathbb{N}} \in \widetilde{\mathrm{T}}M.$$

b) As a generalization of the tangent bundle, one can define for every $k \in \mathbb{N}^* := \mathbb{N} \setminus \{0\}$ the tensor bundle $\mathrm{T}^{k,0}M$ of M. First, one puts for every $p \in M$

$$\mathrm{T}^{k,0}_p M := \widehat{\bigotimes}^k \mathrm{T}_p M,$$

where $\widehat{\otimes}$ denotes the completed projective tensor product, see Example A.1(a) or [25, 47]. The canonical maps $\mathrm{T}\mu_{p,i} := \mathrm{T}\mu_{i|\mathrm{T}_p M} \colon \mathrm{T}_p M \to \mathrm{T}_{p_i} M_i$, $p_i := \mu_i(p)$ induce continuous linear maps

$$\mathrm{T}^{k,0}\mu_{p,i} := \widehat{\bigotimes}^k \mathrm{T}\mu_{p,i} \colon \ \mathrm{T}^{k,0}_p M \longrightarrow \mathrm{T}^{k,0}_{p_i} M_i$$

by the universal property of the completed projective tensor product. Likewise, one constructs for $i \leq j$ the continuous linear maps

$$\mathrm{T}^{k,0}\mu_{p_j,ij} \colon \ \mathrm{T}^{k,0}_{p_j} M_j \longrightarrow \mathrm{T}^{k,0}_{p_i} M_i,$$

which turn $\left(\mathrm{T}^{k,0}_{p_i} M_i, \mathrm{T}^{k,0}\mu_{p_j,ij}\right)$ into a projective system of (finite dimensional) real vector spaces. By Theorem A.4, its projective limit within the category of locally convex topological Hausdorff spaces is given by $\mathrm{T}^{k,0}_p M$ together with the continuous linear maps $\mathrm{T}^{k,0}\mu_{p,i}$, that means we have

$$\mathrm{T}^{k,0}_p M = \varprojlim_{i \in \mathbb{N}} \mathrm{T}^{k,0}_{p_i} M_i.$$

Now define

$$\mathrm{T}^{k,0}M := \bigcup_{p \in M} \mathrm{T}^{k,0}_p M,$$

and give $\mathrm{T}^{k,0}M$ the coarsest topology such that all the canonical maps

$$\mathrm{T}^{k,0}\mu_i \colon \ \mathrm{T}^{k,0}M \longrightarrow \mathrm{T}^{k,0}M_i,$$
$$Z_1 \otimes \cdots \otimes Z_k \longmapsto \mathrm{T}\mu_i(Z_1) \otimes \cdots \otimes \mathrm{T}\mu_i(Z_k)$$

are continuous. By construction, $\mathrm{T}^{k,0}M$ together with the maps $\mathrm{T}^{k,0}\mu_i$ has to be a projective limit of the projective system $\left(\mathrm{T}^{k,0}M_i, \mathrm{T}^{k,0}\mu_{ij}\right)$. The sheaf of smooth functions $\mathscr{C}^\infty_{\mathrm{T}^{k,0}M}$ is uniquely determined by requiring axiom (PFM2) to hold true. One thus obtains a profinite dimensional manifold which depends only on the equivalence class of the smooth projective representation and which will be denoted by $\mathrm{T}^{k,0}M$ in the following. Moreover, $\mathrm{T}^{k,0}$ even becomes a functor on the category of profinite dimensional manifolds. If $(N, \mathscr{C}^\infty_N)$ is another profinite dimensional manifold and $f \colon M \to N$ a smooth map, then one naturally obtains the smooth map

$$\mathrm{T}^{k,0}f \colon \ \mathrm{T}^{k,0}M \longrightarrow \mathrm{T}^{k,0}N,$$
$$Z_1 \otimes \cdots \otimes Z_k \longmapsto \mathrm{T}f(Z_1) \otimes \cdots \otimes \mathrm{T}f(Z_k),$$

which satisfies $\pi_{\mathrm{T}^{k,0}N} \circ \mathrm{T}^{k,0}f = f \circ \pi_{\mathrm{T}^{k,0}M}$.

We continue with:

Definition 3.21. Let $U \subset M$ be open. Then a smooth section $V \colon U \to \mathrm{T}M$ of $\pi_{\mathrm{T}M} \colon \mathrm{T}M \to M$ is called a *smooth vector field* on M over U. The space of smooth vector fields over U will be denoted by $\mathscr{X}^\infty(U)$.

Assume that for $U \subset M$ open we are given a smooth vector field $V \colon U \to \mathrm{T}M$ and a smooth function $f \colon U \to \mathbb{R}$. We then define a function Vf over U by putting for $p \in U$

$$Vf(p) := V(p)\big([f]_p\big).$$

Lemma 3.22. *For every $V \in \mathscr{X}^\infty(U)$ and $f \in \mathscr{C}^\infty(U)$, the function Vf is smooth.*

Proof. Choose a point $p \in U$, and then an open $U_i \subset M_i$ and a function $f_i \in \mathscr{C}^\infty(U_i)$ for some appropriate $i \in \mathbb{N}$ such that $p \in \mu_i^{-1}(U_i) \subset U$ and

$$f_{|\mu_i^{-1}(U_i)} = f_i \circ \mu_{i|\mu_i^{-1}(U_i)}. \tag{3.2}$$

Consider $V_i \colon M \to \mathrm{T}M_i$, $V_i := \mathrm{T}\mu_i \circ V$. Since V_i takes values in a finite dimensional smooth manifold, there exists an integer $j_p \geq i$ (which we briefly denote by j, if no confusion can arise), an open $U_{pj} \subset M_j$ and a smooth vector field $V_p \colon U_{pj} \to \mathrm{T}M_i$ along μ_{ij} such that $p \in \mu_j^{-1}(U_{pj})$, $U_{pj} \subset \mu_{ij}^{-1}(U_i)$ and

$$\mathrm{T}\mu_i \circ V_{|\mu_j^{-1}(U_{pj})} = V_p \circ \mu_{j|\mu_j^{-1}(U_{pj})}.$$

Now define $g_{pj} \colon U_{pj} \to \mathbb{R}$ by

$$g_{pj}(q_j) := V_p(q_j)\big([f_i]_{\mu_{ij}(q_j)}\big) \qquad \text{for all} \quad q_j \in U_{pj}.$$

Then g_{pj} is smooth, hence $g_p := g_{pj} \circ \mu_{j|\mu_j^{-1}(U_{pj})}$ is an element of $\mathscr{C}^\infty\big(\mu_j^{-1}(U_{pj})\big)$. Now one checks for $q \in \mu_j^{-1}(U_{pj})$ that

$$g_p(q) = V_p(\mu_j(q))\big([f_i]_{\mu_i(q)}\big) = V_i(q)\big([f_i]_{\mu_i(q)}\big) = V(q)\big([f]_q\big)$$

by equation (3.2). Hence

$$g_p = (Vf)_{|\mu_j^{-1}(U_{pj})},$$

and Vf is smooth indeed. ∎

Proposition 3.23. *Every vector field $V \in \mathscr{X}^\infty(U)$ defined over an open subset $U \subset M$ induces a derivation*

$$\delta_V \colon \ \mathscr{C}^\infty(U) \longrightarrow \mathscr{C}^\infty(U), \qquad f \longmapsto Vf.$$

Proof. By construction, it is clear that the map

$$\mathscr{C}^\infty(U) \ni f \longmapsto Vf \in \mathscr{C}^\infty(U)$$

is $\mathbb{R}$-linear. It remains to check that δ_V is a derivation, or in other words that it satisfies Leibniz' rule. But this follows immediately by the definition of the action of V on $\mathscr{C}^\infty(U)$ and the fact that $V(p) \in \mathrm{Der}\big(\mathscr{C}_p^\infty, \mathbb{R}\big)$ for all $p \in U$. More precisely, one has, for $p \in U$ and $f, g \in \mathscr{C}^\infty(U)$,

$$V(fg)(p) = V(p)\big([fg]_p\big) = f(p)V(p)\big([g]_p\big) + g(p)V(p)\big([f]_p\big) = \big(fV(g) + gV(f)\big)(p).$$

This finishes the proof. ∎

Definition 3.24. Let $U \subset M$ be open. A smooth vector field $V \in \mathscr{X}^\infty(U)$ is called *local*, if for every $i \in \mathbb{N}$ there is an integer $m_i \geq i$ and a smooth vector field $V_{im_i} \colon \mu_{m_i}(U) \to \mathrm{T}M_i$ along μ_{im_i} such that

$$\mathrm{T}\mu_i \circ V = V_{im_i} \circ \mu_{m_i|U}. \tag{3.3}$$

The space of local vector fields over U will be denoted by $\mathscr{X}_{\mathrm{loc}}^\infty(U)$.

Remark 3.25.

a) Obviously, $\mathscr{X}^\infty$ is a sheaf of $\mathscr{C}^\infty$-modules on M, and $\mathscr{X}^\infty_{\mathrm{loc}}$ a presheaf of $\mathscr{C}^\infty_{\mathrm{loc}}$-modules. Note that $\mathscr{X}^\infty_{\mathrm{loc}}$ depends only on the pfd structure $[(M_i, \mu_{ij}, \mu_i)]$.

b) Let $V \in \mathscr{X}^\infty_{\mathrm{loc}}(U)$, and pick a representative (M_i, μ_{ij}, μ_i) of the underlying pfd structure. If $(m_i)_{i\in\mathbb{N}}$ is a sequence of integers such that (3.3) holds true, we sometimes say that V is of *type* $(m_0, m_1, m_2, \ldots)$ *with respect to the smooth projective representation* (M_i, μ_{ij}, μ_i). The notion of the type of a local vector field is known from jet bundle literature [2], where it makes perfect sense, since the profinite dimensional manifold of infinite jets has a distinguished representative of the underlying pfd structure, see Section 4.3.

Now we are in the position to prove the following structure theorem:

Theorem 3.26. *The map*

$$\delta\colon \ \mathscr{X}^\infty(M) \longrightarrow \mathrm{Der}\left(\mathscr{C}^\infty(M), \mathscr{C}^\infty(M)\right), \qquad V \longmapsto \delta_V$$

is a bijection. Moreover, for every $V \in \mathscr{X}^\infty(M)$, the derivation δ_V leaves the algebra $\mathscr{C}^\infty_{\mathrm{loc}}(M)$ of local functions on M invariant, if and only if one has $V \in \mathscr{X}^\infty_{\mathrm{loc}}(M)$.

Proof. *Surjectivity*: Assume that $D\colon \mathscr{C}^\infty(M) \to \mathscr{C}^\infty(M)$ is a derivation. Then one obtains for each $i \in \mathbb{N}$ and point $p \in M$ a linear map

$$D_{pi}\colon \ \mathscr{C}^\infty(M_i) \longrightarrow \mathbb{R}, \qquad f \longmapsto D(f \circ \mu_i)(p).$$

Note that for $f, f' \in \mathscr{C}^\infty(M_i)$

$$\begin{aligned} D_{pi}(ff') &= D\big((ff') \circ \mu_i\big)(p) = f \circ \mu_i(p) D(f' \circ \mu_i)(p) + f' \circ \mu_i(p) D(f \circ \mu_i)(p) \\ &= f \circ \mu_i(p) D_{pi}(f') + f' \circ \mu_i(p) D_{pi}(f), \end{aligned}$$

which entails that there is a tangent vector $V_{pi} \in \mathrm{T}_{\mu_i(p)} M_i$ such that $D_{pi} = V_{pi}$. Observe that for $j \geq i$ the relation

$$D_{pi}(f) = D(f \circ \mu_i)(p) = D(f \circ \mu_{ij} \circ \mu_j)(p) = D_{pj}(f \circ \mu_{ij})$$

holds true, which entails that $V_{pi} = \mathrm{T}\mu_{ij} \circ V_{pj}$. Hence, the sequence of tangent vectors $(V_{pi})_{i\in\mathbb{N}}$ defines an element V_p in

$$\mathrm{T}_p M \cong \varprojlim_{i\in\mathbb{N}} \mathrm{T}_{\mu_i(p)} M_i.$$

We thus obtain a section $V\colon M \to \mathrm{T}M$, $p \mapsto V_p$. Let us show that V is smooth. To this end, consider the composition

$$V_i := \mathrm{T}\mu_i \circ V\colon \ M \longrightarrow \mathrm{T}M_i.$$

By construction $V_i(p) = V_{pi}$ for all $p \in M$. It suffices to show that each of the maps V_i is smooth. To show this, choose a coordinate neighborhood $U_i \subset M_i$ of $\mu_i(p)$, and coordinates

$$\left(x^1, \ldots, x^k\right)\colon \ U_i \longrightarrow \mathbb{R}^k.$$

Then

$$\left(x^1 \circ \pi_{\mathrm{T}U_i}, \ldots, x^k \circ \pi_{\mathrm{T}U_i}, \mathrm{d}x^1, \ldots, \mathrm{d}x^k\right)\colon \ \mathrm{T}U_i \longrightarrow \mathbb{R}^{2k}$$

is a local coordinate system of $\mathrm{T}M_i$. The map V_i now is proven to be smooth, if $\mathrm{d}x^l \circ V_i$ is smooth for $1 \leq l \leq k$. But

$$\mathrm{d}x^l \circ V_{i|\mu_i^{-1}(U_i)} = D\big(x^l \circ \mu_{i|\mu_i^{-1}(U_i)}\big),$$

since for $q \in \mu_i^{-1}(U_i)$

$$\mathrm{d}x^l \circ V_{i|\mu_i^{-1}(U_i)}(q) = V_{qi}(q)\big([x^l]_{\mu_i(q)}\big) = D_{qi}\big(x^l\big) = D\big(x^l \circ \mu_{i|\mu_i^{-1}(U_i)}\big)(q).$$

Hence each V_i is smooth, and V is a smooth vector field on M which satisfies $\delta_V = D$. This proves surjectivity.

Injectivity: Assume that V is a smooth vector field on M such that $\delta_V = 0$. This means that $\delta_V f(p) = 0$ for all $f \in \mathscr{C}^\infty(M)$ and $p \in M$. Choose now a $i \in \mathbb{N}$ and let f_i be a smooth function on M_i. Put $f := f_i \circ \mu_i$ and $V_i = \mathrm{T}\mu_i \circ V$. Then, we have for all $p \in M$

$$V_i(p)\big([f_i]_{\mu_i(p)}\big) = \delta_V f(p) = 0,$$

which implies that $V_i(p) = 0$ for all $p \in M$. Since $V(p)$ is the projective limit of the $V_i(p)$, we obtain $V(p) = 0$ for all $p \in M$, hence $V = 0$. This finishes the proof that δ is bijective.

Local vector fields: Next, let us show that for a local vector field $V : M \to \mathrm{T}M$ the derivation δ_V maps local functions to local ones. To this end choose for every $i \in \mathbb{N}$ an integer $m_i \geq i$ such that there exists a smooth vector field $V_{im_i} : M_{m_i} \to \mathrm{T}M_i$ along μ_{im_i} which satisfies

$$\mathrm{T}\mu_i \circ V = V_{im_i} \circ \mu_{m_i}.$$

Now let f be a local function on M, which means that $f = f_i \circ \mu_i$ for some $i \in \mathbb{N}$ and $f_i \in \mathscr{C}^\infty(M_i)$. Define $g_{m_i} \in \mathscr{C}^\infty(M_{m_i})$ by $g_{m_i}(q) = V_{im_i}(q)\big([f_i]_{\mu_{im_i}(q)}\big)$ for all $q \in M_{m_i}$. Then, one obtains for $p \in M$

$$\delta_V f(p) = V_{im_i}(\mu_{m_i}(p))\big([f_i]_{\mu_i(p)}\big) = g_{m_i}(\mu_{m_i}(p)),$$

which means that $\delta_V f = g_{m_i} \circ \mu_{m_i}$ is local.

Invariance of $\mathscr{C}^\infty_{\mathrm{loc}}(M)$: Finally, we have to show that if δ_V for $V \in \mathscr{X}^\infty(M)$ leaves the space $\mathscr{C}^\infty_{\mathrm{loc}}(M)$ invariant, the vector field V has to be local. To this end fix $i \in \mathbb{N}$ and choose a proper embedding

$$\chi = (\chi_1, \ldots, \chi_N) : \ M_i \lhook\joinrel\longrightarrow \mathbb{R}^N.$$

Then $\chi_l \circ \mu_i \in \mathscr{C}^\infty_{\mathrm{loc}}(M)$ for $l = 1, \ldots, N$, hence there exist by assumption $j_1, \ldots, j_N \in \mathbb{N}$ and $g_{il} \in \mathscr{C}^\infty(M_{j_l})$ such that

$$\delta_V(\chi_l \circ \mu_i) = g_{il} \circ \mu_{j_l}.$$

After possibly increasing the j_l, we can assume that $m_i := j_1 = \cdots = j_N \geq i$. Denote by $z_l : \mathbb{R}^N \to \mathbb{R}$ the canonical projection onto the l-th coordinate, and define the vector field $\widetilde{V}_{im_i} : M_{m_i} \to \mathrm{T}\mathbb{R}^N$ along $\chi \circ \mu_{im_i}$ by

$$\widetilde{V}_{im_i}(q) := \sum_{l=1}^{N} g_{lj_l}(q) \frac{\partial}{\partial z_l}_{|\chi(\mu_{im_i}(q))} \qquad \text{for} \quad q \in M_{m_i}.$$

Since by construction

$$\widetilde{V}_{im_i}(\mu_{m_i}(p))\big([z_l]_{\chi(\mu_i(p))}\big) = g_{il}(\mu_{m_i}(p)) = (\mathrm{T}\mu_i \circ V)(p)\big([\chi_l]_{\mu_i(p)}\big)$$

for all $p \in M$, $\widetilde{V}_{im_i}(y)$ is in the image of $\mathrm{T}_q\chi$ for every $q \in M_{m_i}$, hence

$$V_{im_i} : \ M_{m_i} \longrightarrow \mathrm{T}M_i, \qquad q \longmapsto (\mathrm{T}_y\chi)^{-1}\big(\widetilde{V}_{im_i}(q)\big)$$

is well-defined and satisfies $\mathrm{T}\mu_i \circ V = V_{im_i} \circ \mu_{m_i}$. Therefore, V is a local vector field. ∎

The following result is an immediate consequence of Theorem 3.26.

Corollary 3.27. *For all $V, W \in \mathscr{X}^\infty(M)$, the map*

$$[V, W] \colon \mathscr{C}^\infty(M) \longrightarrow \mathscr{C}^\infty(M), \qquad f \longmapsto V(Wf) - W(Vf)$$

is a derivation on $\mathscr{C}^\infty(M)$. Its corresponding underlying vector field will be denoted by $[V, W]$ as well, and will be called the Lie bracket *of V and W. The Lie bracket of vector fields turns $\mathscr{X}^\infty(M)$ into a Lie algebra.*

Remark 3.28. Unlike in the finite dimensional case, smooth vector fields on a profinite dimensional manifold of infinite dimension need not be integrable. See [12, Section 3 & Concluding comments] and references therein for further information on this phenomenon and an integrability criterion in the particular case of the solution manifold of a formally integrable PDE.

3.3   Differential forms

In the finite dimensional case, differential forms are usually defined as smooth sections of alternating powers of the cotangent bundle. This approach can not directly be carried over to the profinite dimensional case, since there is no canonical construction of the cotangent bundle of a profinite dimensional manifold. The reason for this is that unlike for the tangent bundle functor, which is a covariant functor and transforms a smooth projective system (M_i, μ_{ij}) into a projective system $(TM_i, T\mu_{ij})$, forming the cotangent bundle is neither covariant nor contravariant functorial on the category of finite dimensional smooth manifolds and smooth maps. The way out is to define differential forms on profinite dimensional manifolds as continuous maps on a tensor product of the tangent bundle such that these maps are locally pull-backs of differential forms on the components of a smooth projective representation.

Definition 3.29. Let $k \in \mathbb{N}$ and $U \subset M$ open.

a) A continuous map

$$\omega \colon \ (\pi_{\mathrm{T}^{k,0}M})^{-1}(U) \longrightarrow \mathbb{R}$$

is called a *differential form of order k* or a *k-form* on M over U, if for every point $p \in U$ there is some $i \in \mathbb{N}$, an open subset $U_i \subset M_i$ with $p \in \mu_i^{-1}(U_i) \subset U$ and a k-form $\omega_i \in \Omega^k(U_i)$ such that

$$\omega_{|(\mu_i \circ \pi_{\mathrm{T}^{k,0}M})^{-1}(U_i)} = \omega_i \circ \mathrm{T}^{k,0}\mu_{i|(\mu_i \circ \pi_{\mathrm{T}^{k,0}M})^{-1}(U_i)}.$$

More precisely, this means that for all $y \in \mu_i^{-1}(U_i)$, and $V_1, \ldots, V_k \in \pi^{-1}(y) \subset \mathrm{T}M$ the relation

$$\omega(V_1 \otimes \cdots \otimes V_k) = \omega_i\big(\mathrm{T}\mu_i(V_1) \otimes \cdots \otimes \mathrm{T}\mu_i(V_k)\big)$$

holds true. In particular, a k-form ω over U is antisymmetric and k-multilinear in its arguments. The space of k-forms over U will be denoted by $\Omega^k(U)$.

b) A k-form $\omega \in \Omega^k(U)$ is called *local*, if there is an open $U_i \subset M_i$ for some $i \in \mathbb{N}$ and a k-form $\omega_i \in \Omega^k(U_i)$ such that $U \subset \mu_i^{-1}(U_i)$ and $\omega = (\mu_i^*\omega_i)_{|U}$, where here and from now on we use the notation $\mu_i^*\omega_i$ for the form $\omega_i \circ \mathrm{T}^{k,0}\mu_i$. The space of local k-forms over U will be denoted by $\Omega^k_{\mathrm{loc}}(U)$.

Remark 3.30.

a) By a straightforward argument one checks that the spaces $\Omega^k(U)$ and $\Omega^k_{\mathrm{loc}}(U)$ only depend on the pfd structure $[(M_i, \mu_{ij}, \mu_i)]$. Moreover, for every representative (M_i, μ_{ij}, μ_i) of the pfd structure, $\Omega^k_{\mathrm{loc}}(U)$ together with the family of pull-back maps $\mu_i^* \colon \Omega^k(\mu_i(U)) \to \Omega^k_{\mathrm{loc}}(U)$ is an inductive limit of the injective system of linear spaces $\left(\Omega^k(\mu_i(U)), \mu_{ij}^*\right)_{i \in \mathbb{N}}$.

b) By construction, it is clear that Ω^k forms a sheaf of $\mathscr{C}^\infty$-modules on M and Ω^k_{loc} a presheaf of $\mathscr{C}^\infty_{\mathrm{loc}}$-modules. Moreover, Ω^k coincides with the sheaf associated to Ω^k_{loc}.

c) The representative $\mathcal{M} := (M_i, \mu_{ij}, \mu_j)$ of the pfd structure on M leads to the particular filtration $\mathcal{F}^{\mathcal{M}}_\bullet$ of the presheaf Ω^k_{loc} of local k-forms on M by putting, for $l \in \mathbb{N}$,

$$\mathcal{F}^{\mathcal{M}}_l\left(\Omega^k_{\mathrm{loc}}\right) := \mu_l^* \Omega^k_{M_l}.$$

Observe that this filtration has the property that

$$\Omega^k_{\mathrm{loc}} = \bigcup_{l \in \mathbb{N}} \mathcal{F}^{\mathcal{M}}_l\left(\Omega^k_{\mathrm{loc}}\right).$$

Proposition and Definition 3.31.

a) *There exists a uniquely determined morphism of sheaves* $\mathrm{d} \colon \Omega^k \to \Omega^{k+1}$ *such that*

$$\mathrm{d}(\mu_i^* \omega_i) = \mu_i^*(\mathrm{d}\omega_i) \qquad \text{for all} \quad i \in \mathbb{N}, \quad U_i \subset M_i \text{ open}, \quad \omega_i \in \Omega^k(U_i).$$

The morphism d *is called the* exterior derivative, *fulfills* $\mathrm{d} \circ \mathrm{d} = 0$, *and maps* Ω^k_{loc} *to* $\Omega^{k+1}_{\mathrm{loc}}$.

b) *There exists a uniquely determined morphism of sheaves*

$$\wedge \colon \ \Omega^k \times \Omega^l \longrightarrow \Omega^{k+l},$$

called the wedge product, *such that for all* $i \in \mathbb{N}$, $U_i \subset M_i$ *open,* $\omega_i \in \Omega^k(U_i)$, *and* $\mu_i \in \Omega^l(U_i)$ *one has*

$$\mu_i^* \omega_i \wedge \mu_i^* \mu_i = \mu_i^*(\omega_i \wedge \mu_i).$$

The wedge product also leaves $\Omega^\bullet_{\mathrm{loc}}$ *invariant.*

c) *Given a vector field* $V \in \mathscr{X}^\infty(M)$, *there exists the* contraction with V *that means the sheaf morphism*

$$i_V \colon \ \Omega^k \longrightarrow \Omega^{k-1},$$

which is uniquely determined by the requirement that for all $\omega \in \Omega^k(U)$ *with* $U \subset M$ *open,* $p \in U$, *and* $W_1, \ldots, W_{k-1} \in \mathrm{T}_p M$ *the relation*

$$i_V(\omega)(W_1 \otimes \cdots \otimes W_{k-1}) = \omega\left(V(p) \otimes W_1 \otimes \cdots \otimes W_{k-1}\right)$$

holds true. If V *is a local vector field, contraction with* V *leaves* $\Omega^\bullet_{\mathrm{loc}}$ *invariant.*

Proof. Using the sheaf property of Ω^k one can reduce the claims to local statements which are immediately proved. ∎

4   Jet bundles and formal solutions of nonlinear PDEs

The aim of this section is to develop a precise geometric notion of formally integrable (systems of) partial differential equations, and to show that the formal solution spaces of these equations canonically become a profinite dimensional manifold in the sense of Section 3. Finally, we are going to give a criterion for the formal integrability of nonlinear scalar partial differential equations, and apply this result to a class of interacting relativistic scalar field theories that arise in theoretical physics.

We refer the reader to [9, 23, 27, 41] and also to [36, 42] for introductionary texts on jet bundles, where the latter two references have a strong focus on the highly nontrivial algorithmic aspects of this theory. A nice short overview is also included in the introduction of [49].

4.1   Finite order jet bundles

For the rest of the paper, we fix a fiber bundle $\pi\colon E \to X$. Moreover, F will denote the typical fiber of π and we set $m := \dim X$, $n := \dim F$.

Then one has $\dim E = m+n$ and the fibers $\pi^{-1}(p) \subset E$ become n-dimensional submanifolds, which are diffeomorphic to F. There are distinguished charts for E:

Definition 4.1. A manifold chart $(x, u)\colon W \to \mathbb{R}^m \times \mathbb{R}^n$ of E defined over some open $W \subset E$ is called a *fibered chart of* π, if for all $e, e' \in W$ with $\pi(e) = \pi(e')$ the equality $x(e) = x(e')$ holds true.

Remark 4.2.

a) Sometimes, fibered charts are called *adapted charts*.

b) Note that a fibered chart $(x, u)\colon W \to \mathbb{R}^m \times \mathbb{R}^n$ for π canonically gives rise to a well-defined manifold chart on X. It is given by

$$\tilde{x}\colon \ \pi(W) \longrightarrow \mathbb{R}^m, \qquad p \longmapsto x(e), \tag{4.1}$$

where $e \in W \cap \pi^{-1}(p)$ is arbitrary.

c) On the other hand, a manifold atlas for E that consists of fibered charts for π can be constructed from manifold charts for X and from the local triviality of E as follows: For an arbitrary $e \in E$, take a bundle chart $\phi\colon \pi^{-1}(U) \to U \times F$ around $\pi(e)$, that is, U is an open neighbourhood of $\pi(e)$ and $\phi\colon \pi^{-1}(U) \to U \times F$ is a diffeomorphism such that

$$
\begin{array}{ccc}
\pi^{-1}(U) & \xrightarrow{\ \phi\ } & U \times B \\
{\scriptstyle \pi}\downarrow & \swarrow{\scriptstyle \mathrm{pr}_1} & \\
U & &
\end{array}
\tag{4.2}
$$

commutes. Let $\tilde{x}\colon U \to \mathbb{R}^m$ be a manifold chart of X (here we assume that U is small enough), and let $\tilde{u}\colon B \to \mathbb{R}^n$ be a manifold chart of F. Then

$$(\tilde{x} \circ \pi, \tilde{u} \circ \mathrm{pr}_2 \circ \phi) = (\tilde{x} \circ \mathrm{pr}_1 \circ \phi, \tilde{u} \circ \mathrm{pr}_2 \circ \phi)\colon \ \pi^{-1}(U) \longrightarrow \mathbb{R}^m \times \mathbb{R}^n$$

is a fibered chart of π. Note here that by the commutativity of (4.2), the notation "$\tilde{x}$" is consistent with (4.1).

Let us introduce the following notation for multi-indices, which will be convenient in the following: For any $k_1, k_2 \in \mathbb{N}$ with $k_1 \leq k_2$ let $\mathbb{N}_{k_1,k_2}^m$ denote the set of all multi-indices $I \in \mathbb{N}^m$ such that

$$k_1 \leq |I| := \sum_{j=1}^m I_j \leq k_2$$

and let $\mathsf{F}(m, k_1, k_2)$ denote the linear space of all maps $\mathbb{N}_{k_1,k_2}^m \to \mathbb{R}$. For any $l \leq m$, $i_1, \ldots, i_l \in \{1, \ldots, m\}$, the symbol $1_{i_1 \ldots i_l} \in \mathbb{N}^m$ will denote the multi-index which has a 1 in its i_j's slot for $j = 1, \ldots, l$, and a 0 elsewhere.

Any $\psi \in \Gamma^\infty(p; \pi)$ allows the following local description: Choose a fibered chart $(x, u) \colon W \to \mathbb{R}^m \times \mathbb{R}^n$ of π with $W \cap \pi^{-1}(p) \neq \varnothing$. Then one has $x \circ \psi = \tilde{x}$ near p, so that ψ is determined by the coordinates $(u^1 \circ \psi, \ldots, u^n \circ \psi) = u \circ \psi$ near p. The special form of the following definition is motivated by the latter fact:

Definition 4.3. Let $p \in X$, $k \in \mathbb{N}$. Any two $\psi, \varphi \in \Gamma^\infty(p; \pi)$ are called *k-equivalent at p*, if for every fibered chart $(x, u) \colon W \to \mathbb{R}^m \times \mathbb{R}^n$ of π with $W \cap \pi^{-1}(p) \neq \varnothing$ one has

$$\frac{\partial^{|I|} (u^\alpha \circ \psi)}{\partial \tilde{x}^I}(p) = \frac{\partial^{|I|} (u^\alpha \circ \varphi)}{\partial \tilde{x}^I}(p) \tag{4.3}$$

for all $\alpha = 1, \ldots, n$ and all $I \in \mathbb{N}_{0,k}^m$. The corresponding equivalence class $\mathsf{j}_p^k \psi$ of ψ is called the *k-jet of ψ at p*.

Remark 4.4. In fact, it is enough to check (4.3) in *some* fibered chart. This can be proved by induction on k, using the multivariate version of Faà di Bruno's formula [13, Theorem 2.1] (details can be found in Lemma 6.2.1 in [41]).

Let us now come to several structures that can be defined via jets. Denoting by

$$\mathsf{J}^k(\pi) := \bigcup_{p \in X} \{\mathsf{j}_p^k \psi \mid \psi \in \Gamma^\infty(p; \pi)\}$$

the collection of all k-jets in π, we obtain the surjective maps

$$\pi_k \colon \mathsf{J}^k(\pi) \longrightarrow X, \qquad \mathsf{j}_p^k \psi \longmapsto p, \qquad \pi_{0,k} \colon \mathsf{J}^k(\pi) \longrightarrow E, \qquad \mathsf{j}_p^k \psi \longmapsto \psi(p).$$

Using these maps, one can give $\mathsf{J}^k(\pi)$ the structure of a finite dimensional manifold in a canonical way: For every fibered chart $(x, u) \colon W \to \mathbb{R}^m \times \mathbb{R}^n$ of π and every $I \in \mathbb{N}_{0,k}^m$, one defines the map

$$(x_k, u_{k,I}) \colon \pi_{0,k}^{-1}(W) \longrightarrow \mathbb{R}^m \times \mathbb{R}^{n \dim \mathsf{F}(m,0,k)},$$

$$\mathsf{j}_p^k \psi \longmapsto \left(\tilde{x}(p), \frac{\partial^{|I|}(u^1 \circ \psi)}{\partial \tilde{x}^I}(p), \ldots, \frac{\partial^{|I|}(u^n \circ \psi)}{\partial \tilde{x}^I}(p) \right). \tag{4.4}$$

The following result is checked straightforwardly (cf. [41]).

Proposition and Definition 4.5. *The maps* (4.4) *define an $m + n \dim \mathsf{F}(m, 0, k)$-dimensional manifold structure on $\mathsf{J}^k(\pi)$. In view of this fact, $\mathsf{J}^k(\pi)$ is called the k-jet manifold corresponding to π.*

For convenience, we set $J^0(\pi) := E$ and $j_p^0\psi := \psi(p)$ for any $\psi \in \Gamma^\infty(p;\pi)$, and $\pi_0 := \pi$. More generally, we have for any $k_1 \leq k_2$ the smooth surjective maps

$$\pi_{k_1,k_2}\colon\ J^{k_2}(\pi) \longrightarrow J^{k_1}(\pi), \qquad j_p^{k_2}\psi \longmapsto j_p^{k_1}\psi,$$

which satisfy $\pi_{k,k} = \mathrm{id}_{J^k(\pi)}$, and if one also has $k_2 \leq k_3$, then the following diagram commutes:

$$
\begin{array}{ccc}
J^{k_3}(\pi) & \xrightarrow{\ \pi_{k_2,k_3}\ } & J^{k_2}(\pi) \\
\Big\downarrow{\scriptstyle \pi_{k_1,k_3}} \quad {\scriptstyle \pi_{k_1,k_2}}\!\!\!\nearrow & & \Big\downarrow{\scriptstyle \pi_{k_2}} \\
J^{k_1}(\pi) & \xrightarrow[\ \pi_{k_1}\ ]{} & X
\end{array}
\tag{4.5}
$$

Let us collect all structures underlying the above maps. Let $(x,u)\colon W \to \mathbb{R}^m \times \mathbb{R}^n$ be a fibered chart of π. Then we set

$$(\pi_k, u_k)\colon\ \pi_{k,0}^{-1}(W) \longrightarrow \pi(W) \times \mathsf{F}(m,0,k)^n,$$

$$j_p^k\psi \longmapsto \left(p, \left\{ \frac{\partial^{|I|}(u \circ \psi)}{\partial \tilde{x}^I}(p) \right\}_{I \in \mathbb{N}_{0,k}^m} \right) \tag{4.6}$$

$$= \left(p, \left\{ \frac{\partial^{|I|}(u^1 \circ \psi)}{\partial \tilde{x}^I}(p) \right\}_{I \in \mathbb{N}_{0,k}^m}, \dots, \left\{ \frac{\partial^{|I|}(u^n \circ \psi)}{\partial \tilde{x}^I}(p) \right\}_{I \in \mathbb{N}_{0,k}^m} \right),$$

$$(\pi_{k_1,k_2}, u_{k_1,k_2})\colon\ \pi_{k_2,0}^{-1}(W) \longrightarrow \pi_{k_1,0}^{-1}(W) \times \mathsf{F}(m,k_1+1,k_2)^n,$$

$$j_p^{k_2}\psi \longmapsto \left(j_p^{k_1}\psi, \left\{ \frac{\partial^{|I|}(u \circ \psi)}{\partial \tilde{x}^I}(p) \right\}_{I \in \mathbb{N}_{k_1+1,k_2}^m} \right). \tag{4.7}$$

If π is a vector bundle, then, for every $p \in X$, the fiber $\pi_k^{-1}(p)$ canonically becomes a linear space through

$$c_1\left(j_p^k\psi\right) + c_2\left(j_p^k\varphi\right) := j_p^k(c_1\psi + c_2\varphi), \qquad c_j \in \mathbb{R}.$$

Furthermore, if $k_2 = k$, $k_1 = k-1$ and if $a \in J^{k-1}(\pi)$, then the fiber $\pi_{k-1,k}^{-1}(a)$ carries a canonical affine structure which is modelled on the linear space

$$\mathsf{S}^k\left(\mathrm{T}_{\pi_{k-1}(a)}^* X\right) \otimes \ker\left(\mathrm{T}\pi_{|\pi_{0,k-1}(a)}\right). \tag{4.8}$$

To see the latter fact, assume that $\pi_{0,k-1}(a) \in W$, let $j_{\pi_{k-1}(a)}^k\psi \in \pi_{k-1,k}^{-1}(a)$ and let v be an element of (4.8). Then v can be uniquely expanded as

$$v = \sum_{I \in \mathbb{N}_{k,k}^m} \sum_{\alpha=1}^n v_I^\alpha \mathrm{d}\tilde{x}_{I|\pi_{k-1}(a)} \otimes \frac{\partial}{\partial u^\alpha}_{|\pi_{0,k-1}(a)}, \qquad v_I^\alpha \in \mathbb{R},$$

where we have used the abbreviation

$$\mathrm{d}\tilde{x}_I := (\mathrm{d}\tilde{x}_1)^{\otimes I_1} \odot \cdots \odot (\mathrm{d}\tilde{x}_m)^{\otimes I_m},$$

so that one can define $j^k_{\pi_{k-1}(a)}\psi + v \in \pi^{-1}_{k-1,k}(a)$ to be the uniquely determined element whose image under (4.7) is given by

$$\left(j^{k-1}_{\pi_{k-1}(a)}\psi, \left\{ \frac{\partial^{|I|}(u \circ \psi)}{\partial \tilde{x}^I}\Big|_{\pi_{k-1}(a)} + v_I \right\}_{I \in \mathbb{N}^m_{k,k}} \right).$$

With these preparations, one has:

Lemma 4.6. *Let $k, k_1, k_2 \in \mathbb{N}$ with $k_1 \le k_2$. Then the following assertions hold.*

    a) *The maps (4.6) turn $\pi_k \colon J^k(\pi) \to X$ into a fiber bundle with typical fiber $F(m, 0, k)^n$. If π is a vector bundle, then so is π_k.*

    b) *The maps (4.7) turn $\pi_{k_1,k_2} \colon J^{k_2}(\pi) \to J^{k_1}(\pi)$ into a fiber bundle with typical fiber $F(m, l + 1, k)^n$, and $\pi_{k-1,k} \colon J^k(\pi) \to J^{k-1}(\pi)$ becomes an affine bundle, modelled on the vector bundle*

$$\pi^*_{k-1} S^k\big(\pi_{T^*X}\big) \otimes \pi^*_{k-1,0} V(\pi) \longrightarrow J^{k-1}(\pi).$$

Proof. The reader can find a detailed proof of Lemma 4.6 in Chapter 6 of [41]. ∎

We close this section with a simple observation about distinguished elements of $\Gamma^\infty(\pi_k)$. Let $U \subset X$ be an open subset for the moment. Then for any $\psi \in \Gamma^\infty(U; \pi)$, the map $U \to J^k(\pi)$, $p \mapsto j^k_p\psi$, defines an element of $\Gamma^\infty(U; \pi_k)$, called the *k-jet prolongation of ψ*. In fact, this construction induces a morphism of sheaves $j^k \colon \Gamma^\infty(\pi) \to \Gamma^\infty(\pi_k)$ (with values in the category of sets) such that

$$\pi_{k_1,k_2} \circ j^{k_2} = j^{k_1} \qquad \text{for} \quad k_1 \le k_2. \tag{4.9}$$

It should be noted that it is not possible to write an arbitrary element of $\Gamma^\infty(U; \pi_k)$ as $j^k_U\psi$ for some $\psi \in \Gamma^\infty(U; \pi)$. The elements of $\Gamma^\infty(U; \pi_k)$ having the latter property are called *projectable*. This notion is motivated by the following simple observation which follows readily from (4.9).

Lemma 4.7. *Let $U \subset X$ be an open subset and $\Psi \in \Gamma^\infty(U; \pi_k)$. Then the map $p \mapsto \pi_{0,k}(\Psi(p))$ defines an element of $\Gamma^\infty(U; \pi)$, and Ψ is projectable, if and only if one has $j^k(\pi_{0,k} \circ \Psi) = \Psi$.*

4.2 Partial differential equations

The aim of this section is to give a precise global definition of partial differential equations and the solutions thereof in the setting of arbitrary fiber bundles. We shall first consider the general (possibly nonlinear) situation in Section 4.2.1. Then, in Section 4.2.2, we are going to relate everything with the corresponding classical linear concepts.

Throughout this section, let $\underline{\pi} \colon \underline{E} \to X$ be a second fiber bundle, with typical fiber $\underline{F}$ and fiber dimension $\underline{n}$.

4.2.1 General facts

Definition 4.8.

    a) A subset $E \subset J^k(\pi)$ is called a *partial differential equation on π of order $\le k$*, if $\pi_{k|E} \colon E \to X$ is a fibered submanifold of π_k.

b) Let $\mathsf{E} \subset \mathsf{J}^k(\pi)$ be a partial differential equation on π of order $\leq k$. Some $\psi \in \Gamma^\infty(p;\pi)$ is called a *solution of* E *in* p, if $\mathsf{j}_p^k\psi \in \mathsf{E}$. For an open $U \subset X$, a section $\psi \in \Gamma^\infty(U;\pi)$ will simply be called a *solution* of E, if ψ is a solution of E in p for every $p \in U$, that is, if $\mathrm{im}(\mathsf{j}^k\psi) \subset \mathsf{E}$.

Remark 4.9.

a) The definition of a PDE we present here appears to go back to Goldschmidt [24] and is now widely used in the geometric PDE literature [32, 36, 37, 39, 40, 42]. One of its virtues is that it allows to clearly separate and globalize the notions "partial differential equation", "solution of a partial differential equation" and "partial differential operator".

b) By the Cartan–Kuranishi prolongation theorem [31], an analytic PDE either has no solutions or becomes involutive after finitely many prolongations, which means that it admits integral manifolds as stated in the Cartan–Kähler theorem [11]. In Goldschmidt's work, this observation is always in the background. Since we are mainly interested in the infinite prolongation or in other words the formal solution space of a PDE as in Proposition and Definition 4.29, Goldschmidt's approach fits perfectly for our purposes. For a slightly different general setup of non-linear PDEs of various orders (and their symbols) see [30].

c) There exist non-fibered submanifolds $E \subset \mathsf{J}^k(\pi)$ which one could refer to as "partial differential equations" as well. The problem with this more general concept of a PDE though is that then independent variables might not really be independent anymore, cf. [42, Section 2.3]. Moreover, only the fibered variant leads in general to profinite dimensional manifolds (cf. Proposition 4.29 below).

Definition 4.10. A morphism $h\colon \mathsf{J}^k(\pi) \to \underline{E}$ of fibered manifolds over X is called a *partial differential operator of order* $\leq k$ *from* π *to* $\underline{\pi}$.

Of course, the notion "operator" in Definition 4.10 is justified by the fact that as a morphism of fibered manifolds, any h as in Definition 4.10 induces the morphism of set theoretic sheaves

$$P^h := h \circ \mathsf{j}^k\colon \ \Gamma^\infty(\pi) \longrightarrow \Gamma^\infty(\underline{\pi}).$$

We define $\mathrm{D}^k(\pi,\underline{\pi})$ to be the set of all partial differential operators of order $\leq k$ from π to $\underline{\pi}$, and remark that the assignment $h \mapsto P^h$ induces an injection $P^\bullet$ of $\mathrm{D}^k(\pi,\underline{\pi})$ into the set theoretic sheaf morphisms $\Gamma^\infty(\pi) \to \Gamma^\infty(\underline{\pi})$. The connection between partial differential operators and partial differential equations is given in this abstract setting as follows: For every $h \in \mathrm{D}^k(\pi,\underline{\pi})$ and $O \in \Gamma^\infty(X;\underline{\pi})$, the set $\mathrm{Z}_O(h)$ is defined by $\mathrm{Z}_O(h) := h^{-1}(\mathrm{im}(O)) \subset \mathsf{J}^k(\pi)$, with the usual convention $\ker(h) := \mathrm{Z}_O(h)$ if π and $\underline{\pi}$ are vector bundles and h is linear. Observe that one has by definition

$$\mathrm{Z}_O(h) = \big\{a \in \mathsf{J}^k(\pi) \,|\, h(a) = O(\pi_k(a))\big\}.$$

The following fact is well-known:

Proposition 4.11. *If* $h \in \mathrm{D}^k(\pi,\underline{\pi})$ *has constant rank, and if* $O \in \Gamma^\infty(X;\underline{\pi})$ *fulfills* $\mathrm{im}(O) \subset \mathrm{im}(h)$*, then* $\mathrm{Z}_O(h) \subset \mathsf{J}^k(\pi)$ *is a partial differential equation.*

It is clear that with an open subset $U \subset X$, a section $\psi \in \Gamma^\infty(U;\pi)$ is a solution of $\mathrm{Z}_O(h)$ (in $p \in U$), if and only if one has $P_U^h(\psi) = O$ (in p).

Next, we explain how the affine structure of $\pi_{k-1,k}$ can be used to introduce the notion of "operator symbols of (possibly nonlinear) partial differential operators". To avoid any confusion, we remark that with "symbol" we will exclusively mean "principal symbol" in this paper.

To this end, note that the assignment

$$\mu_k^\pi\colon\ \pi_k^*\mathsf{S}^k\big(\pi_{\mathrm{T}^*X}\big)\otimes\pi_{0,k}^*\mathsf{V}(\pi)\longrightarrow\mathsf{V}(\pi_k),\qquad \mu_{k\mid a}^\pi(v):=\frac{\mathrm{d}}{\mathrm{d}t}[a+tv]_{|t=0},$$

for $a\in\mathsf{J}^k(\pi)$, $v\in\mathsf{S}^k\big(\mathrm{T}^*_{\pi_k(a)}X\big)\otimes\ker(\pi_{|\pi_{0,k}(a)})$, is a (mono)morphism of vector bundles over $\mathsf{J}^k(\pi)$. Note here that μ_k^π essentially extracts the pure k-th order part of vertical k-jets. Using the map μ_k^π, we can provide the following definition (see also [11]):

Definition 4.12. For every $h\in\mathrm{D}^k(\pi,\underline\pi)$, the morphism $\sigma(h)$ of vector bundles over h given by the composition

$$\pi_k^*\mathsf{S}^k\big(\pi_{\mathrm{T}^*X}\big)\otimes\pi_{k,0}^*\mathsf{V}(\pi)\overset{\sigma(h)}{\dashrightarrow}\mathsf{V}(\underline\pi)$$

with μ_k^π, h_V, and $\mathsf{V}(\pi_k)$ completing the diagram

is called the *operator symbol* of h.

Given a partial differential operator, one can use its symbol to check whether it defines a partial differential equation in the sense of Proposition 4.11 (see also Theorem 4.31 below):

Proposition 4.13. *Let $h\in\mathrm{D}^k(\pi,\underline\pi)$. If $\sigma(h)$ is surjective, then so is h. If $\sigma(h)$ is a submersion, then h is a submersion, too, which in particularly means that for every $O\in\Gamma^\infty(X;\underline\pi)$ with $\mathrm{im}(O)\subset\mathrm{im}(h)$ the set $\mathrm{Z}_O(h)\subset\mathsf{J}^k(\pi)$ is a partial differential equation.*

Proof. We have the following commuting diagrams

where the maps for the second diagram are given by the tangential maps corresponding to the first one. If $\sigma(h)=h_\mathsf{V}\circ\mu_k^\pi$ is surjective, then so is h_V and $\underline\pi^\mathsf{V}\circ h_\mathsf{V}$, so that the first assertion follows from the first diagramm. If $\sigma(h)$ is a submersion, then one can use the analogous argument for the second diagram to deduce that $\mathrm{T}h$ has full rank everywhere. ∎

4.2.2 Linear partial differential equations

We are now going to explain how the classical concepts of linear partial differential equations and partial differential operators fit into the general setting of Section 4.2.1. In fact, it will turn out that the notions "linear partial differential equation" and "linear partial differential operator" correspond to each other under natural assumptions in the following sense: $\ker(h)$ is a linear partial differential equation for every k-th order linear differential operator and, conversely, for every linear k-th order partial differential equation E there exists a (not uniquely determined) k-th order linear differential operator h with $E=\ker(h)$. Note that an analogous correspondence statement for the nonlinear case is only locally true [21]. Finally, the space of linear partial differential operators coincides with the space of classical linear partial differential operators (see Theorem 4.17 below). We begin with:

Definition 4.14. Let π and $\underline{\pi}$ be vector bundles.

a) A subset $\mathsf{E} \subset \mathsf{J}^k(\pi)$ is called a *linear partial differential equation on π of order $\leq k$*, if $\pi_{k|\mathsf{E}} \colon \mathsf{E} \to X$ is a sub-vector-bundle of π_k.

b) A morphism $h \colon \mathsf{J}^k(\pi) \to \underline{E}$ of vector bundles over X is called a *linear partial differential operator of order $\leq k$ from π to $\underline{\pi}$*.

Remark 4.15. Let π and $\underline{\pi}$ be vector bundles. Then $h \in \mathrm{D}^k(\pi, \underline{\pi})$ is linear, if and only if $P_X^h \colon \Gamma^\infty(X; \pi) \to \Gamma^\infty(X; \underline{\pi})$ is linear. We denote the linear space of linear partial differential operators by $\mathrm{D}^k_{\mathrm{lin}}(\pi, \underline{\pi}) \subset \mathrm{D}^k(\pi, \underline{\pi})$ and remark that if $h \in \mathrm{D}^k_{\mathrm{lin}}(\pi, \underline{\pi})$, then one has $h^{(l)} \in \mathrm{D}^{k+l}_{\mathrm{lin}}(\pi, \underline{\pi}_l)$ for all $l \in \mathbb{N}$.

Let us recall the definition of "classical" linear partial differential operators:

Definition 4.16. Let π and $\underline{\pi}$ be vector bundles. A *classical linear partial differential operator of order $\leq k$ from π to $\underline{\pi}$* is a morphism of sheaves

$$D \colon \ \Gamma^\infty(\pi) \longrightarrow \Gamma^\infty(\underline{\pi})$$

with the following property: For every manifold chart $\tilde{x} \colon U \to \mathbb{R}^m$ of X for which there are frames $e_1, \dots, e_n \in \Gamma^\infty(U; \pi)$ and $\underline{e}_1, \dots, \underline{e}_{\underline{n}} \in \Gamma^\infty(U; \underline{\pi})$ there exist (necessarily unique) functions $D_I^{\alpha, \beta} \in \mathscr{C}^\infty(U)$ for $\alpha = 1, \dots, n$, $\beta = 1, \dots, \underline{n}$, and $I \in \mathbb{N}_{0,k}^m$ such that one has for all $\psi^1, \dots, \psi^n \in \mathscr{C}^\infty(U)$

$$D_U \left(\sum_{j=1}^n \psi^\alpha e_\alpha \right) = \sum_{\beta=1}^{\underline{n}} \sum_{\alpha=1}^n \sum_{I \in \mathbb{N}_{0,k}^m} D_I^{\alpha, \beta} \frac{\partial^{|I|} \psi^\alpha}{\partial \tilde{x}^I} \underline{e}_\beta.$$

The linear space of classical partial differential operators will be denoted by $\mathrm{D}^k_{\mathrm{cl, lin}}(\pi, \underline{\pi})$.

Now one has:

Theorem 4.17. *Let π and $\underline{\pi}$ be vector bundles.*

a) *$P^\bullet$ induces the isomorphism of linear spaces*

$$P^\bullet_{\mathrm{lin}} \colon \ \mathrm{D}^k_{\mathrm{lin}}(\pi, \underline{\pi}) \longrightarrow \mathrm{D}^k_{\mathrm{cl, lin}}(\pi, \underline{\pi}), \qquad h \longmapsto P^h.$$

b) *If $h \in \mathrm{D}^k_{\mathrm{lin}}(\pi, \underline{\pi})$ has constant rank, then $\ker(h) \subset \mathsf{J}(\pi_k)$ is a linear partial differential equation. Conversely, if $\mathsf{E} \subset \mathsf{J}^k(\pi)$ is a linear partial differential equation, then there is a vector bundle $\underline{\underline{\pi}} \colon \underline{\underline{E}} \to X$ and an $\underline{h} \in \mathrm{D}^k_{\mathrm{lin}}(\pi, \underline{\underline{\pi}})$ with constant rank such that $\mathsf{E} = \ker(\underline{h})$.*

Proof. a) We first have to show that $P^\bullet_{\mathrm{lin}}$ is well-defined, which means that for any $h \in \mathrm{D}^k_{\mathrm{lin}}(\pi, \underline{\pi})$, P^h is in $\mathrm{D}^k_{\mathrm{cl, lin}}(\pi, \underline{\pi})$. It is then clear that $P^\bullet_{\mathrm{lin}}$ is a linear monomorphism. To this end, let $\tilde{x}$, e_α, $\underline{e}_\beta$, ψ^α be as in Definition 4.16 and let a_α be a basis for F. Then we have the vector bundle chart

$$\phi \colon \ \pi^{-1}(U) \to U \times F, \qquad \sum_{\alpha=1}^n v^\alpha e_\alpha(p) \longmapsto \left(p, \sum_{\alpha=1}^n v^\alpha a_\alpha \right), \qquad p \in U,$$

so that we get the fibered chart

$$(x, u) \colon \ \pi^{-1}(U) \longrightarrow \mathbb{R}^m \times \mathbb{R}^n, \qquad \sum_{\alpha=1}^n v^\alpha e_\alpha(p) \longmapsto \left(\tilde{x}(\pi(p)), \left(v^1, \dots, v^n \right) \right)$$

of π as in Remark 4.2(c). Then

$$(\pi_k, u_k)\colon\ \pi_k^{-1}(U) \longrightarrow U \times \mathsf{F}(m,0,k)^n$$

is a vector bundle chart of π_k by Lemma 4.6, and we get the frame $e_{I,\alpha} \in \Gamma(U;\pi_k)$, $\alpha = 1,\ldots,n$, $I \in \mathbb{N}_{0,k}^m$, given by

$$e_{I,\alpha} := (\pi_k, u_k)^{-1}(\bullet, \delta_{I,\alpha}).$$

Hereby, $\delta_{I,\alpha}\colon \mathbb{N}_{0,k}^m \to \mathbb{R}^n$ is defined by $\delta_{I,\alpha}(J) := 1_\alpha$, if $I = J$, and to be 0 elsewhere. Since h is a homomorphism of linear bundles over X, there are uniquely determined $h_I^{\alpha,\beta} \in \mathscr{C}^\infty(U)$ such that one has for all $\alpha = 1,\ldots,n$, $I \in \mathbb{N}_{0,k}^m$ and $\psi^{\alpha,I} \in \mathscr{C}^\infty(U)$

$$h\left(\sum_{\alpha=1}^{n}\sum_{I \in \mathbb{N}_{0,k}^m} \psi^{\alpha,I} e_{\alpha,I}\right) = \sum_{\beta=1}^{\underline{n}}\sum_{\alpha=1}^{n}\sum_{I \in \mathbb{N}_{0,k}^m} h_I^{\alpha,\beta}\psi^{\alpha,I}\underline{e}_\beta.$$

The proof of the asserted well-definedness of $P_{\mathrm{lin}}^\bullet$ is completed by observing that by the above construction of the frame $e_{I,\alpha}$ for $\Gamma^\infty(U;\pi_k)$ the following equality holds true:

$$\mathsf{j}^k\left(\sum_{\alpha=1}^{n}\psi^\alpha e_\alpha\right) = \sum_{\alpha=1}^{n}\sum_{I \in \mathbb{N}_{0,k}^m} \frac{\partial^{|I|}\psi^\alpha}{\partial \tilde{x}^I} e_{\alpha,I}.$$

In order to prove surjectivity of $P_{\mathrm{lin}}^\bullet$, let $D \in \mathrm{D}_{\mathrm{cl,lin}}^k(\pi,\underline{\pi})$, and let $\mathsf{j}_p^k\psi \in \mathsf{J}(\pi_k)$, with p from an open subset $U \subset X$. Then $h(\mathsf{j}_p^k\psi) := D_U\psi(p)$ gives rise to a well-defined element $h \in \mathrm{D}_{\mathrm{lin}}^k(\pi,\underline{\pi})$, which of course satisfies $P_{\mathrm{lin}}^h = D$.

b) The first fact is well-known. For the second assertion, we can simply take $\underline{E} \to X$ to be given by the quotient bundle $\mathsf{J}^k(\pi)/\mathsf{E} \to X$, and $\underline{h}$ to be given by the canonical projection $\mathsf{J}^k(\pi) \to \mathsf{J}^k(\pi)/\mathsf{E}$ (see [32, Proposition 3.10] for a more general statement). ∎

Finally, we explain in which sense the classical concept of linear operator symbols fits into the general setting of Section 4.2.1.

Let π and $\underline{\pi}$ be vector bundles for the moment. From the canonical identification of $\ker(\mathrm{T}\pi_{|e})$ with $\pi^{-1}(\pi(e))$, for $e \in E$, (and analogous ones for $\underline{\pi}$), we obtain canonical morphisms of vector bundles over the base map π (resp. $\underline{\pi}$)

$$\begin{aligned}
&\sigma^\pi\colon\ \mathsf{V}(\pi) \longrightarrow E,\\
&\sigma^\pi{}_{|\ker(\mathrm{T}\pi_{|e})}\colon\ \ker(\mathrm{T}\pi_{|e}) \longrightarrow \pi^{-1}(\pi(e)), \qquad e \in E, \qquad \text{and}
\end{aligned} \tag{4.10}$$

$$\begin{aligned}
&\sigma^{\underline{\pi}}\colon\ \mathsf{V}(\underline{\pi}) \longrightarrow \underline{E},\\
&\sigma^{\underline{\pi}}{}_{|\ker(\mathrm{T}\underline{\pi}_{|\underline{e}})}\colon\ \ker(\mathrm{T}\underline{\pi}_{|\underline{e}}) \longrightarrow \underline{\pi}^{-1}(\underline{\pi}(\underline{e})), \qquad \underline{e} \in \underline{E},
\end{aligned} \tag{4.11}$$

which both are fiberwise isomorphisms. It follows that for each $k \in \mathbb{N}^*$ the map

$$\begin{aligned}
&\sigma_k^\pi\colon\ \pi_k^* \mathsf{S}^k\big(\pi_{\mathrm{T}^*X}\big) \otimes \pi_{k,0}^*\mathsf{V}(\pi) \longrightarrow \mathsf{S}^k\big(\pi_{\mathrm{T}^*X}\big) \otimes E,\\
&\sigma_k^\pi(v \otimes w) := v \otimes \sigma^\pi(w),
\end{aligned} \tag{4.12}$$

where $v \otimes w \in \mathrm{Sym}^k\big(\mathrm{T}_{\pi_k(a)}^*X\big) \otimes \ker\big(\mathrm{T}\pi_{|\pi_{0,k}(a)}\big)$ for $a \in \mathsf{J}^k(\pi)$, is a morphism of vector bundles over the base map π_k and also acts by isomorphisms, fiberwise. Furthermore, for later reference, we record that there is a canonical (mono)morphism of vector bundles over X which is defined by

$$\begin{aligned}
&\mu_{k,\mathrm{lin}}^\pi\colon\ \mathsf{S}^k\big(\pi_{\mathrm{T}^*X}\big) \otimes E \longrightarrow \mathsf{J}^k(\pi),\\
&\mathrm{d}f_1(p) \odot \cdots \odot \mathrm{d}f_k(p) \otimes \psi(p) \longmapsto \mathsf{j}_p^k(f_1 \cdots f_k\psi).
\end{aligned}$$

Here, the f_j run through the elements of $\mathscr{C}^\infty(p; X)$ satisfying $f_j(p) = 0$, and $\psi \in \Gamma^\infty(p; \pi)$. Analogously to μ_k^π, the map $\mu_{k,\mathrm{lin}}^\pi$ also extracts the pure k-th order part of k-jets in an appropriate sense (taking into account the canonical isomorphisms (4.11) and (4.12)).

The following result recalls the classical definition of linear operator symbols and shows the naturality of Definition 4.12, in the sense that in the linear case, the linear operator symbol coincides with the operator symbol up to the canonical isomorphisms (4.11) and (4.12):

Proposition and Definition 4.18. *Let π and $\underline{\pi}$ be vector bundles and let $h \in \mathrm{D}_{\mathrm{lin}}^k(\pi, \underline{\pi})$.*

a) *There is a unique morphism of vector bundles over X*

$$\sigma_{\mathrm{lin}}(h)\colon\ \mathsf{S}^k(\pi_{\mathrm{T}^*X}) \otimes E \longrightarrow \underline{E}$$

with the following property: For every manifold chart $\tilde{x}\colon U \to \mathbb{R}^m$ of X for which there are frames $e_1, \dots, e_n \in \Gamma^\infty(U; \pi)$ and $\underline{e}_1, \dots, \underline{e}_n \in \Gamma^\infty(U; \underline{\pi})$, one has

$$\sigma_{\mathrm{lin}}(h)\left(\sum_{I \in \mathbb{N}_{k,k}^m} \sum_{\alpha=1}^n v_I^\alpha \mathrm{d}\tilde{x}_I \otimes e_\alpha \right) = \sum_{\beta=1}^{\underline{n}} \sum_{\alpha=1}^n \sum_{I \in \mathbb{N}_{k,k}^m} (P_{\mathrm{lin}}^h)_I^{\alpha,\beta} v_I^\alpha \underline{e}_\beta, \tag{4.13}$$

where $v_I^\alpha \in \mathscr{C}^\infty(U)$, and where we have used the notation from Definition 4.16 and Theorem 4.17. The morphism $\sigma_{\mathrm{lin}}(h)$ is called the linear operator symbol *of h.*

b) *The following diagram commutes*

$$
\begin{array}{ccc}
\pi_k^* \mathsf{S}^k(\pi_{\mathrm{T}^*X}) \otimes \pi_{k,0}^* \mathsf{V}(\pi) & \xrightarrow{\ \sigma(h)\ } & \mathsf{V}(\underline{\pi}) \\
{\scriptstyle \sigma_k^\pi} \downarrow & & \downarrow {\scriptstyle \sigma^{\underline{\pi}}} \\
\mathsf{S}^k(\pi_{\mathrm{T}^*X}) \otimes E & \xrightarrow{\ \sigma_{\mathrm{lin}}(h)\ } & \underline{E}
\end{array}
$$

Proof. a) Here, one only has to prove that the representation (4.13) does not depend on a particular choice of local data. In fact, the easiest way to see this, is to note that one can simply define $\sigma_{\mathrm{lin}}(h)$ by the diagram

$$
\begin{array}{ccc}
\mathsf{S}^k(\pi_{\mathrm{T}^*X}) \otimes E & \dashrightarrow{\ \sigma_{\mathrm{lin}}(h)\ } & \underline{E} \\
{\scriptstyle \mu_{k,\mathrm{lin}}^\pi} \searrow & & \nearrow {\scriptstyle h} \\
& \mathsf{J}^k(\pi) &
\end{array}
$$

To see that $\sigma_{\mathrm{lin}}(h)$ defined like this satisfies (4.13), let $\tilde{x}\colon U \to \mathbb{R}^m$ be a manifold chart of X such that there are frames $e_1, \dots, e_n \in \Gamma^\infty(U; \pi)$, $\underline{e}_1, \dots, \underline{e}_n \in \Gamma^\infty(U; \underline{\pi})$. Then, as in the proof of Theorem 4.17(a), picking a basis a_α for F, we get the corresponding frame $e_{I,\alpha} \in \Gamma^\infty(U; \pi_k)$, $\alpha = 1, \dots n, I \in \mathbb{N}_{0,k}^m$, and we denote the representation of h with respect to $e_{I,\alpha}$ and $\underline{e}_\beta$ by $h_I^{\alpha,\beta}$. Furthermore, by the proof of Theorem 4.17(a), we have $h_I^{\alpha,\beta} = (P_{\mathrm{lin}}^h)_I^{\alpha,\beta}$. Now one has

$$\mu_{k,\mathrm{lin}}^\pi\left(\sum_{I \in \mathbb{N}_{k,k}^m} \sum_{\alpha=1}^n v_I^\alpha \mathrm{d}\tilde{x}_I \otimes e_\alpha \right) = \sum_{I \in \mathbb{N}_{k,k}^m} \sum_{\alpha=1}^n v_I^\alpha e_{I,\alpha},$$

so that

$$h \circ \mu_{k,\mathrm{lin}}^{\pi}\left(\sum_{I \in \mathbb{N}_{k,k}^m}\sum_{\alpha=1}^{n} v_I^{\alpha}\mathrm{d}\tilde{x}_I \otimes e_{\alpha}\right) = \sum_{\beta=1}^{\underline{n}}\sum_{\alpha=1}^{n}\sum_{I \in \mathbb{N}_{k,k}^m} h_I^{\alpha,\beta} v_I^{\alpha}\underline{e}_{\beta},$$

which completes the proof of part a).

b) Let $a \in \mathsf{J}^k(\pi)$ be arbitrary, and let $\tilde{x}\colon U \to \mathbb{R}^m$ be a manifold chart of X around $\pi_k(a)$ such that there are frames $e_1,\ldots,e_n \in \Gamma^{\infty}(U;\pi)$, $\underline{e}_1,\ldots,\underline{e}_{\underline{n}} \in \Gamma^{\infty}(U;\underline{\pi})$. Then, again as in the proof of Theorem 4.17(a), picking a basis a_{α} for F and a basis $\underline{a}_{\beta}$ for $\underline{F}$, we get the corresponding adapted coordinates

$$(x,u)\colon\ \pi^{-1}(U) \longrightarrow \mathbb{R}^m \times \mathbb{R}^n, \qquad (\underline{x},\underline{u})\colon\ \underline{\pi}^{-1}(U) \longrightarrow \mathbb{R}^m \times \mathbb{R}^{\underline{n}}.$$

We can expand an arbitrary

$$v \in \mathsf{S}^k\big(\mathrm{T}_{\pi_k(a)}^* X\big) \otimes \ker(\mathrm{T}\pi_{|\pi_{0,k}(a)})$$

uniquely as

$$v = \sum_{I \in \mathbb{N}_{k,k}^m}\sum_{\alpha=1}^{n} v_I^{\alpha}\mathrm{d}\tilde{x}_{I\,|\,\pi_k(a)} \otimes \frac{\partial}{\partial u^{\alpha}}_{|\pi_{0,k}(a)}, \qquad v_I^{\alpha} \in \mathbb{R},$$

so that

$$\sigma^{\pi_k}(v) = \sum_{I \in \mathbb{N}_{k,k}^m}\sum_{\alpha=1}^{n} v_I^{\alpha}\mathrm{d}\tilde{x}_I \otimes e_{\alpha\,|\,\pi_k(a)},$$

and we arrive at

$$\sigma_{\mathrm{lin}}(h) \circ \sigma_{|v}^{\pi_k} = \sum_{\beta=1}^{\underline{n}}\sum_{\alpha=1}^{n}\sum_{I \in \mathbb{N}_{k,k}^m} (P_{\mathrm{lin}}^h)_I^{\alpha,\beta} v_I^{\alpha}\underline{e}_{\beta\,|\,\pi_k(a)}. \tag{4.14}$$

Let us now evaluate $\sigma^{\underline{\pi}} \circ \sigma(h) = \sigma^{\underline{\pi}} \circ h_{\mathsf{V}} \circ \mu_k^{\pi}$ in v: By Proposition 4.5 and Lemma 4.6 we have the frame

$$\frac{\partial}{\partial u_{I,\alpha}^k} \in \Gamma^{\infty}\big(\pi_k^{-1}(U);(\pi_k)^{\mathsf{V}}\big), \qquad I \in \mathbb{N}_{0,k}^m, \qquad \alpha = 1,\ldots,n.$$

As h is a linear morphism, the linear morphism h_{V} is represented with respect to the frames $\frac{\partial}{\partial u_{k,I}^{\alpha}}$ and $\frac{\partial}{\partial \underline{u}^{\beta}}$ precisely by the functions

$$h_I^{\alpha,\beta} \circ \big(\pi_{k\,|\,\pi_k^{-1}(U)}\big) \in \mathscr{C}^{\infty}\big(\pi_k^{-1}(U)\big),$$

where $h_I^{\alpha,\beta} \in \mathscr{C}^{\infty}(U)$ is the representation of h with respect to (x,u) and $(\underline{x},\underline{u})$ (cf. the proof of Theorem 4.17(a)). Thus, in view of

$$\mu_k^{\pi}(v) = \sum_{\alpha=1}^{n}\sum_{I \in \mathbb{N}_{k,k}^m} v_I^{\alpha}\frac{\partial}{\partial u_{k,I}^{\alpha}}_{|a},$$

we have

$$\sigma(h)_{|v} = \sum_{\beta=1}^{\underline{n}}\sum_{\alpha=1}^{n}\sum_{I \in \mathbb{N}_{k,k}^m} v_I^{\alpha} h_{I\,|\,\pi_k(a)}^{\alpha,\beta}\frac{\partial}{\partial \underline{u}^{\beta}}_{|\pi_{0,k}(a)},$$

so that

$$\sigma^{\underline{\pi}} \circ \sigma(h)|_v = \sum_{\beta=1}^{\underline{n}} \sum_{\alpha=1}^{n} \sum_{I \in \mathbb{N}_{k,k}^m} h_I^{\alpha,\beta} v_I^\alpha \underline{e}_\beta \,|\, \pi_k(a).$$

But this is equal to (4.14), in view of $(P_{\text{lin}}^h)_I^{\alpha,\beta} = h_I^{\alpha,\beta}$. The claim follows. ∎

4.3 The manifold of ∞-jets and formally integrable PDEs

Throughout Section 4.3, π will again be an arbitrary fiber bundle.

Finally, in this section we are going to make contact with the abstract theory on profinite dimensional manifolds from Section 3: We are going to prove that the space of "∞-jets" in π canonically becomes a profinite dimensional manifold (see Proposition 4.21), and that the space of "formal solutions" of a "formally integrable" partial differential equation on π canonically is a profinite dimensional submanifold of the latter (see Proposition 4.29).

We start by introducing the space of ∞-jets. In analogy to Definition 4.3, we have:

Definition 4.19. Let $p \in X$. Any two $\psi, \varphi \in \Gamma^\infty(p; \pi)$ are called *∞-equivalent at p*, if $\psi(p) = \varphi(p)$ and if for every fibered chart $(x, u) \colon W \to \mathbb{R}^m \times \mathbb{R}^n$ of π with $W \cap \pi^{-1}(p) \neq \varnothing$ one has

$$\frac{\partial^{|I|}(u^\alpha \circ \psi)}{\partial \tilde{x}^I}(p) = \frac{\partial^{|I|}(u^\alpha \circ \varphi)}{\partial \tilde{x}^I}(p)$$

for all $\alpha = 1, \ldots, n$ and all $I \in \mathbb{N}^m$ with $1 \leq |I| < \infty$. The corresponding equivalence class $\mathrm{j}_p^\infty \psi$ of ψ is called the *∞-jet of ψ at p*.

Remark 4.20. In view of Remark 4.4, ∞-equivalence also only has to be checked in *some* fibered chart.

It will be convenient in what follows to set $\mathsf{J}^{-1}(\pi) := X$, $\mathrm{j}_p^{-1}\psi_p := p$, and $\pi_{-1,0} := \pi$. We define

$$\mathsf{J}^\infty(\pi) := \bigcup_{p \in X} \{\mathrm{j}_p^\infty \psi \,|\, \psi \in \Gamma^\infty(p; \pi)\},$$

and obtain for every $i \in \mathbb{Z}_{\geq -1}$ a surjective map

$$\pi_{i,\infty} \colon \ \mathsf{J}^\infty(\pi) \longrightarrow \mathsf{J}^i(\pi), \qquad \mathrm{j}_p^\infty \psi \longmapsto \mathrm{j}_p^i \psi. \tag{4.15}$$

We equip $\mathsf{J}^\infty(\pi)$ with the initial topology with respect to the maps $\pi_{i,\infty}$, $i \in \mathbb{Z}_{\geq -1}$. Furthermore, we define $\mathscr{C}_\pi^\infty$ to be the sheaf on $\mathsf{J}^\infty(\pi)$, whose section space $\mathscr{C}_\pi^\infty(U)$ over an open $U \subset \mathsf{J}^\infty(\pi)$ is given by the set of all $f \in \mathscr{C}(U)$ such that for every $x \in U$ there is an $i \in \mathbb{Z}_{\geq -1}$, an open $U_i \subset \mathsf{J}^i(\pi)$ and an $f_i \in \mathscr{C}^\infty(U_i)$ with $x \in \pi_{i,\infty}^{-1}(U_i) \subset U$ and

$$f_{|\pi_{i,\infty}^{-1}(U_i)} = f_i \circ \pi_{i,\infty|\pi_{i,\infty}^{-1}(U_i)}.$$

In particular, $(\mathsf{J}^\infty(\pi), \mathscr{C}_\pi^\infty)$ becomes a locally $\mathbb{R}$-ringed space. Now observe that we have, in view of (4.5), a smooth projective system $(\mathsf{J}^i(\pi), \pi_{i,j})$, which graphically can be depicted by

$$\mathsf{J}^{-1}(\pi) \xleftarrow{\pi_{-1,0}} \mathsf{J}^0(\pi) \xleftarrow{\pi_{0,1}} \cdots \longleftarrow \mathsf{J}^i(\pi) \xleftarrow{\pi_{i,i+1}} \mathsf{J}^{i+1}(\pi) \longleftarrow \cdots, \tag{4.16}$$

together with a family of continuous maps

$$\pi_{i,\infty} \colon \ \mathsf{J}^\infty(\pi) \longrightarrow \mathsf{J}^i(\pi), \qquad i \in \mathbb{Z}_{\geq -1}.$$

These data have the following crucial property.

Proposition and Definition 4.21. *The family* $\big(\mathsf{J}^i(\pi), \pi_{i,j}, \pi_{i,\infty}\big)$ *is a smooth projective representation of* $(\mathsf{J}^\infty(\pi), \mathscr{C}_\pi^\infty)$. *In particular, when equipped with the corresponding pfd structure,* $(\mathsf{J}^\infty(\pi), \mathscr{C}_\pi^\infty)$ *canonically becomes a smooth profinite dimensional manifold, called the* manifold *of* ∞-*jets given by* π.

Proof. Let $\mathsf{J}^\infty(\pi)' := \varprojlim\limits_{i \in \mathbb{Z}_{\geq -1}} \mathsf{J}^i(\pi)$ denote the canonical projective limit of (4.16), that means let

$$\mathsf{J}^\infty(\pi)' = \Big\{ b = (b_{-1}, b_0, b_1, \dots) \in \prod_{i \in \mathbb{Z}_{\geq -1}} \mathsf{J}^i(\pi) \mid b_i = \pi_{i,j}(b_j) \text{ for all } i \leq j \Big\},$$

and let $\pi'_{i,\infty} \colon \mathsf{J}^\infty(\pi)' \to \mathsf{J}^i(\pi)$ denote the canonical projections. We are going to prove the existence of a homeomorphism Ξ such that the diagrams

$$(4.17)$$

commute for all $i \in \mathbb{Z}_{\geq -1}$. Then the universal property of $(\mathsf{J}^\infty(\pi)', \pi'_{i,\infty})$ will directly imply the same property for $(\mathsf{J}^\infty(\pi), \pi_{i,\infty})$, which is precisely (PFM1). As a consequence, (PFM2) is trivially satisfied by the definition of the structure sheaf $\mathscr{C}_\pi^\infty$.

We now simply define $\Xi(a)_j := \pi_{j,\infty}(a)$ for $a \in \mathsf{J}^\infty(\pi)$ and $j \in \mathbb{Z}_{\geq -1}$. Then it is obvious that Ξ is a well-defined injective map, and that the Ξ-diagram in (4.17) commutes. In particular, the continuity of Ξ is directly implied by that of the maps $\pi_{i,\infty}$. In order to see that Ξ is surjective and that Ξ^{-1} is continuous, let us recall that Borel's theorem states that for any map

$$t \colon \mathbb{N}^m = \bigcup_{j \in \mathbb{N}} \mathbb{N}_{0,j}^m \longrightarrow \mathbb{R}^n$$

there is a smooth function $\tilde{\psi} \colon \mathbb{R}^m \to \mathbb{R}^n$ such that $t_I = \partial_I \tilde{\psi}(0)/I!$ for all $I \in \mathbb{N}^m$. Let $b \in \mathsf{J}^\infty(\pi)'$ and let $\tilde{x} \colon U \to \mathbb{R}^m$ be a manifold chart of X around b_{-1} with $\tilde{x}(b_{-1}) = 0$. Choosing furthermore a bundle chart $\phi \colon \pi^{-1}(U) \to U \times F$ and a manifold chart $\tilde{u} \colon B \to \mathbb{R}^n$ of F, we get the fibered chart

$$(x, u) := (\tilde{x} \circ \pi, \tilde{u} \circ \mathrm{pr}_2 \circ \phi) \colon \pi^{-1}(U) \longrightarrow \mathbb{R}^m \times \mathbb{R}^n$$

of π by Remark 4.2(c). Moreover, the function $t \colon \mathbb{N}^m \to \mathbb{R}^n$, $t_I := u_{j,I}(b_j)/I!$, if $I \in \mathbb{N}_{0,j}^m$, is well-defined. Borel's theorem then produces a function $\tilde{\psi} \colon \mathbb{R}^m \to \mathbb{R}^n$ such that $t_I = \partial_I \tilde{\psi}(0)/I!$. It is clear that the section $\psi \in \Gamma_{b_{-1}}^\infty(\pi)$ defined by

$$\psi := \phi^{-1}\big(\bullet, \tilde{u}^{-1} \circ \tilde{\psi} \circ \tilde{x}\big)$$

satisfies $x_j(\mathsf{j}_{b_{-1}}^j \psi) = 0$, and $u_{j,I}(\mathsf{j}_{b_{-1}}^j \psi) = u_{j,I}(b_j)$ for all $j \in \mathbb{Z}_{\geq -1}$ and $I \in \mathbb{N}_{0,j}^m$, thus $\Xi(\mathsf{j}_{b_{-1}}^\infty \psi) = b$, and Ξ is surjective, indeed. Furthermore, by the construction of $\Xi^{-1}(b)$, it is also clear that the Ξ^{-1}-diagram in (4.17) commutes, so that the continuity of Ξ^{-1} trivially follows from that of the $\pi'_{i,\infty}$. This completes the proof. $\blacksquare$

Remark 4.22. An important additional structure on the profinite dimensional manifold $\mathsf{J}^\infty(\pi)$ is given by its *Cartan distribution*, a canonically given (involutive) distribution. This additional structure is the key to the so-called *secondary calculus*. We shall not touch this here and refer the interested reader to [48].

Next, we will prepare the notion of formal integrability. Let us first note the following simple result:

Proposition and Definition 4.23.

    a) *For every $h \in \mathrm{D}^k(\pi, \underline{\pi})$ there exists a unique $h^{(l)} \in \mathrm{D}^{k+l}(\pi, \underline{\pi}_l)$ such that the following diagram of set theoretic sheaf morphisms*

$$
\begin{array}{ccc}
\Gamma^\infty(\pi_{k+l}) & \dashrightarrow^{\ h^{(l)}\ } & \Gamma^\infty(\underline{\pi}_l) \\
\uparrow{\scriptstyle \mathrm{j}^{k+l}} & & \uparrow{\scriptstyle \mathrm{j}^l} \\
\Gamma^\infty(\pi) & \xrightarrow{\ h \circ \mathrm{j}^k\ } & \Gamma^\infty(\underline{\pi})
\end{array}
$$

    commutes. The partial differential operator $h^{(l)}$ is called the l-jet prolongation of h.

    b) *The partial differential operator*

$$
\iota^\pi_{l,k}\left(=\mathrm{id}^{(l)}_{\mathsf{J}^k(\pi)}\right)\colon\ \mathsf{J}^{k+l}(\pi) \longrightarrow \mathsf{J}^l(\pi_k), \qquad \mathrm{j}^{k+l}_p\psi \longmapsto \mathrm{j}^l_p(\mathrm{j}^k\psi)
$$

    is an embedding of manifolds.

Let $\mathsf{E} \subset \mathsf{J}^k(\pi)$ be an arbitrary partial differential equation for the moment. Since, by definition, the map

$$
\pi_{k|\mathsf{E}}\colon\ \mathsf{J}^k(\pi) \supset \mathsf{E} \longrightarrow X
$$

is again a fibered manifold, there exists for every $l \in \mathbb{N}$ an obvious well-defined map

$$
\iota_{l,\mathsf{E}}\colon\ \mathsf{J}^l(\pi_{k|\mathsf{E}}) \longrightarrow \mathsf{J}^l(\pi_k),
$$

which comes from considering a locally defined section in $\pi_{k|\mathsf{E}}$ as taking values in $\mathsf{J}^k(\pi)$.

Definition 4.24. Let $\mathsf{E} \subset \mathsf{J}^k(\pi)$ be a partial differential equation. Then the set

$$
\mathsf{E}^{(l)} := \begin{cases} \mathsf{E}, & \text{for } l = 0, \\ \iota^{\pi,-1}_{l,k}\left(\iota_{l,\mathsf{E}}\left(\mathsf{J}^l(\pi_{k\,|\,\mathsf{E}})\right)\right) \subset \mathsf{J}^{k+l}(\pi), & \text{for } l \in \mathbb{N}^*, \end{cases}
$$

is called the *l-jet prolongation* of E.

If the underlying partial differential equation is actually given by a partial differential operator, then there is an explicit description of the corresponding l-jet prolongation [24, p. 294]:

Proposition 4.25. *Let $h \in \mathrm{D}^k(\pi, \underline{\pi})$ with constant rank and let $O \in \Gamma^\infty(X; \underline{\pi})$ with $\mathrm{im}(O) \subset \mathrm{im}(h)$. Then one has, for every $l \in \mathbb{N}$,*

$$
\mathsf{Z}_O(h)^{(l)} = \mathsf{Z}_{\mathrm{j}^l O}\left(h^{(l)}\right) \subset \mathsf{J}^{k+l}(\pi).
$$

Let us note the simple fact that the following diagramm commutes, for every $r \in \mathbb{N}$,

$$
\begin{array}{ccc}
\mathsf{J}^{k+l+r}(\pi) & \xrightarrow{\ \iota^\pi_{l+r,k}\ } & \mathsf{J}^{l+r}(\pi_k) \\
\downarrow{\scriptstyle \pi_{k+l,k+l+r}} & & \downarrow{\scriptstyle (\pi_k)_{l+r,l}} \\
\mathsf{J}^{k+l}(\pi) & \xrightarrow{\ \iota^\pi_{l,k}\ } & \mathsf{J}^l(\pi_k)
\end{array}
$$

Applying this in the case $r = 1$ implies $\pi_{k+l,k+l+1}\big(\mathsf{E}^{(l+1)}\big) \subset \mathsf{E}^{(l)}$ for any partial differential equation $\mathsf{E} \subset \mathsf{J}^k(\pi)$ and every $l \in \mathbb{N}$, so that we obtain the maps

$$\mathsf{E}^{(l+1)} \longrightarrow \mathsf{E}^{(l)}, \qquad a \longmapsto \pi_{k+l,k+l+1}(a). \tag{4.18}$$

Now we have the tools to give

Definition 4.26. A partial differential equation $\mathsf{E} \subset \mathsf{J}^k(\pi)$ is called *formally integrable*, if $\mathsf{E}^{(l)}$ is a submanifold of $\mathsf{J}^{k+l}(\pi)$ and if (4.18) is a fibered manifold for every $l \in \mathbb{N}$.

Remark 4.27.

  a) Here, it should be noted that E itself can always be considered as a trivial formally integrable partial differential equation on π of order 0, where in this case one has $E^{(l)} = \mathsf{J}^l(\pi)$ for all $l \in \mathbb{N}$.

  b) Furthermore, there are abstract cohomological tests for partial differential equations to be formally integrable [24]. In fact, we will use such a test in the proof of Theorem 4.31 below; we refer the reader to [36] and particularly to [42] for the algorithmic aspects of these tests. Although it can become very involved to verify these test properties in particular examples, formal integrability could be shown for some particular equations of mathematical physics such as the Yang–Mills–Higgs equations [22, 23] or the Einstein's field equations [28]. In accordance with this, Theorem 4.31 below states that all reasonable (possibly nonlinear) scalar partial differential equations are formally integrable. But formal integrability need not always be given, as the example of a system of PDEs describing the isometries of a Riemannian metric shows, which is formally integrable if and only if the corresponding curvature is constant. See Pommaret [37, Introduction, p. 9] for more information.

An important purely analytic consequence of formal integrability is given by the highly nontrivial Theorem 4.28 below, which essentially states that if all underlying data are real analytic, then formal integrability implies the existence of local analytic solutions with prescribed finite order Taylor expansions. Theorem 4.28 goes back to Goldschmidt [24] and heavily relies on (cohomological) results by Spencer [43] and Ehrenpreis–Guillemin–Sternberg [17]. This result can also be regarded as a variant of Michael Artin's approximation theorem [3] or as an interpretation of the Cartan–Kähler theorem, cf. [35, Theorem 4.4.3] and [11].

Theorem 4.28. *Assume that X is real analytic, that π is a real analytic fiber bundle (then so is π_k), and that $\mathsf{E} \subset \mathsf{J}^k(\pi)$ is formally integrable such that in fact $\pi_{k\,|\,\mathsf{E}} \colon \mathsf{E} \to X$ is a real analytic fibered submanifold of π_k. Then, for every $l \in \mathbb{N}$ and $a \in \mathsf{E}^{(l)}$ there exists an open neighborhood $U \subset X$ of $\pi_{k+l}(a)$ and a real analytic solution $\psi \in \Gamma^\infty(U; \pi)$ of E such that $\mathsf{j}^{k+l}_{\pi_{k+l}(a)}\psi = a$.*

Proof. This result follows directly from Theorem 9.1 in [24] (in combination with Proposition 7.1 therein). ∎

Now let $\mathsf{E} \subset \mathsf{J}^k(\pi)$ be a formally integrable partial differential equation. Then we can define a subset $\mathsf{E}^{(\infty)} \subset \mathsf{J}^\infty(\pi)$ by $\mathsf{E}^{(\infty)} := \pi_{\infty,k}^{-1}(\mathsf{E})$. Inductively, one checks that the maps (4.15) restrict to surjective maps

$$\mathsf{E}^{(\infty)} \longrightarrow \mathsf{E}^{(i)}, \qquad a \longmapsto \pi_{k+i,\infty}(a), \qquad i \in \mathbb{N},$$
$$\mathsf{E}^{(\infty)} \longrightarrow X, \qquad a \longmapsto \pi_{k-1,\infty}(a),$$

so that

$$\mathsf{E}^{(\infty)} = \bigcap_{i \in \mathbb{N}} \pi_{\infty,k+i}^{-1}(\mathsf{E}).$$

In other words, this means that axioms (PFSM1) to (PFSM3) are satisfied for the subset $\mathsf{E}^{(\infty)} \subset \mathsf{J}^\infty(\pi)$ and the smooth projective representation $\big(\mathsf{J}^i(\pi), \pi_{i,j}, \pi_{i,\infty}\big)$. Hence, one readily obtains

Proposition and Definition 4.29. *Let* $\mathsf{E} \subset \mathsf{J}^k(\pi)$ *be a formally integrable partial differential equation. Then* $\big(\mathsf{J}^i(\pi), \pi_{i,j}, \pi_{i,\infty}\big)$ *induces on* $\mathsf{E}^{(\infty)}$ *the structure of a profinite dimensional submanifold of* $(\mathsf{J}^\infty(\pi), \mathscr{C}_\pi^\infty)$. *In view of this fact,* $\mathsf{E}^{(\infty)}$ *will be called the* manifold of formal solutions of E.

Remark 4.30. In the above situation, the Cartan distribution on $\mathsf{J}^\infty(\pi)$ restricts to a well-defined distribution on $\mathsf{E}^{(\infty)}$.

4.4   Scalar PDEs and interacting relativistic scalar fields

Let us first clarify that throughout Section 4.4, $\pi\colon X \times \mathbb{R} \to X$ will denote the canonical line bundle.

4.4.1   A criterion for formal integrability of scalar PDEs

We now come to the aforementioned result on formal integrability of PDEs. In order to keep the notation simple and in view of the applications that we have in mind, we restrict ourselves in this paper to scalar PDEs.

In the scalar situation, the sheaf of sections of π can be identified with the sheaf of smooth functions on X. Recall that the space of smooth functions defined near $p \in X$ is denoted by $\mathscr{C}^\infty(p; X)$. Likewise, the space of k-th order partial differential operators $\mathrm{D}^k(\pi, \pi)$ can be canonically identified as a linear space with $\mathscr{C}^\infty(\mathsf{J}^k(\pi))$. Given such an $h \in \mathrm{D}^k(\pi, \pi)$, the space of vector bundle morphisms

$$\pi_k^* \mathsf{S}^k(\pi_{\mathrm{T}^*X}) \otimes \pi_{k,0}^* \mathsf{V}(\pi) \longrightarrow \mathsf{V}(\pi_k)$$

over h can be identified canonically as a linear space (remember here the maps (4.11) and (4.12)) with the space of vector bundle morphisms $\pi_k^* \mathsf{S}^k(\pi_{\mathrm{T}^*X}) \to X \times \mathbb{R}$ over π_k. It follows that for every $a \in \mathsf{J}^k(\pi)$ the symbol $\sigma(h)$ induces a linear map

$$\sigma(h)_{|\mathsf{S}^k\left(\mathrm{T}^*_{\pi_k(a)}X\right)}\colon\ \mathsf{S}^k\big(\mathrm{T}^*_{\pi_k(a)}X\big) \longrightarrow \{\pi_k(a)\} \times \mathbb{R}.$$

With these preparations, we have:

Theorem 4.31. *Let* $h \in \mathscr{C}^\infty(\mathsf{J}^k(\pi))$, *and recall that*

$$\mathrm{Z}_0(h) := \{a \in \mathsf{J}^k(\pi) \,|\, h(a) = 0\}.$$

Assume furthermore that the following assumptions are satisfied:

(1) *One has* $\sigma(h)_{|a} \neq 0$ *for all* $a \in \mathsf{J}^k(\pi)$.

(2) *The map* $\mathrm{Z}_0(h)^{(1)} \to \mathrm{Z}_0(h)$, $a \mapsto \pi_{k,k+1}(a)$ *is surjective.*

Then $\mathrm{Z}_0(h) \subset \mathsf{J}^k(\pi)$ *is a formally integrable partial differential equation on* π.

Remark 4.32. Note that assumption (1) together with Proposition 4.13 implies that $\mathrm{Z}_0(h)$ indeed is a partial differential equation, as $\sigma(h)$ is surjective.

Proof of Theorem 4.31. The seemingly short proof that we are going to give actually combines two heavy machineries: The already mentioned abstract cohomological criterion for formal integrability of partial differential equations from [24], with a highly nontrivial reduction result for the cohomology of Cohen–Macaulay symbolic systems [29]. There seems to be no reasonable elementary proof of Theorem 4.31.

First observe that (1) is equivalent to

(1′) For all $a \in \mathsf{J}^k(\pi)$ there exists $v \in \mathrm{T}^*_{\pi_k(a)} X$ such that $\sigma(h)(v^{\otimes k}) \neq 0$.

Namely, if $\sigma_a(h)(v^{\otimes k}) = 0$ for all $v \in \mathrm{T}^*_{\pi_k(a)} X$, then $\sigma_a(h) = 0$ by polarization.

To prove our claim, let an arbitrary $a \in \mathsf{J}^k(\pi)$ be given and pick some $v \in \mathrm{T}^*_{\pi_k(a)} X$ with $\sigma(h)(v^{\otimes k}) \neq 0$. Then, in the terminology of [28],

$$V^* := \mathbb{C}v \subset \left(\mathrm{T}^*_{\pi_k(a)} X\right)_{\mathbb{C}}$$

is a one-dimensional noncharacteristic subspace corresponding to the Cohen–Macaulay symbolic system $\mathrm{g}(h;a)$ given by $\mathrm{Z}_0(h)$ over a. Thus we may apply Theorem A from [29] to deduce that all Spencer cohomology groups $\mathrm{H}^{i,j}(\mathrm{g}(h;a))$ except possibly $\mathrm{H}^{0,0}(\mathrm{g}(h;a))$ and $\mathrm{H}^{1,1}(\mathrm{g}(h;a))$ vanish. But now the result follows from combining (2), [24, Theorem 8.1] and [24, Proposition 7.1], once we have shown that the first prolongation

$$\bigcup_{a \in \mathrm{Z}_0(h)} \mathrm{g}(h;a)^{(1)} \longrightarrow \mathrm{Z}_0(h) \tag{4.19}$$

becomes a vector bundle. To prove the latter claim, note first that h induces the morphism of vector bundles

$$\sigma(h)^{(1)}: \ \pi_k^* \mathsf{S}^{k+1}(\pi_{\mathrm{T}^* X}) \longrightarrow \pi_k^* \mathrm{T}^* X,$$

which, for every $a \in \mathsf{J}^k(\pi)$, is given by

$$\sigma(h)^{(1)} \big|_{\mathsf{S}^{k+1}\left(\mathrm{T}^*_{\pi_k(a)} X\right)}: \ \mathsf{S}^{k+1}\left(\mathrm{T}^*_{\pi_k(a)} X\right) \longrightarrow \mathrm{T}^*_{\pi_k(a)} X,$$

$$v_1 \odot \cdots \odot v_{k+1} \longmapsto \underbrace{\sigma(h)(v_2 \odot \cdots \odot v_{k+1})}_{\in \mathbb{R}} v_1.$$

The assumption (1) immediately implies that the latter linear map is surjective (in fact for every $a \in \mathsf{J}^k(\pi)$), in particular, as one has [24]

$$\mathrm{g}(h;a)^{(1)} = \ker \left(\sigma(h)^{(1)} \big|_{\mathsf{S}^{k+1}\left(\mathrm{T}^*_{\pi_k(a)} X\right)} \right),$$

it follows that (4.19) is a vector bundle. ■

The assumption (2) from Theorem 4.31 is a technical regularity assumption (which can become tedious to check in applications), whereas the reader should notice that assumption (1) therein is essentially trivial and means nothing but that the underlying differential operator globally is a "genuine" k-th order operator.

4.4.2　Interacting relativistic scalar fields

As an application of Theorem 4.31, we will now consider evolution equations that correspond to (possibly nonlinearly!) interacting relativistic scalar fields on semi-Riemannian manifolds. To

this end, let $(X, \mathbf{g})$ be a smooth Lorentzian m-manifold. The corresponding d'Alembert operator will be written as

$$\Box_{\mathbf{g}} \colon \ \mathscr{C}^\infty(X) \longrightarrow \mathscr{C}^\infty(X).$$

With functions $F_1, F_2 \in \mathscr{C}^\infty(X)$, $K \in \mathscr{C}^\infty(\mathbb{R})$, we consider the partial differential operator $h_{\mathbf{g},F_1,F_2,K} \in \mathscr{C}^\infty(\mathsf{J}^2(\pi))$ given for $p \in X$, $\varphi \in \mathscr{C}^\infty(p; X)$ by

$$h_{\mathbf{g},F_1,F_2,K}\big(\mathsf{j}_p^2\varphi\big) := \Box_{\mathbf{g}}\varphi(p) + F_1(p)\varphi(p) + F_2(p)K(\varphi(p)).$$

What we have in mind here is:

Example 4.33. Let us assume that $m = 4$, that $(X, \mathbf{g})$ has a Lorentz signature, and that

$$F_1 = \alpha_1 \operatorname{scal}_{\mathbf{g}} + \alpha_2^2, \qquad F_2 \equiv 1, \qquad K = \alpha_3 \underline{K},$$

where $\alpha_1, \alpha_3 \in \mathbb{R}$, $\alpha_2 \geq 0$, $\operatorname{scal}_{\mathbf{g}} \in \mathscr{C}^\infty(X)$ denotes the scalar curvature of $\mathbf{g}$ and $\underline{K} \in \mathscr{C}^\infty(\mathbb{R})$. Then $\mathsf{Z}_0(h_{\mathbf{g},F_1,0,K}) \subset \mathsf{J}^2(\pi)$ describes the on-shell dynamics of a relativistic (real) scalar field with mass α_2, where $\underline{K}$ is the field self-interaction with coupling strength α_3, and where the number α_1 is an additional parameter, which is sometimes set equal to zero. For example, $\underline{K}(z) = z^3$ corresponds to what is called a φ^4-*perturbation* in the physics literature (since the corresponding potential in the Lagrange density which has $\mathsf{Z}_0(h_{\mathbf{g},F_1,0,K})$ as its Euler–Lagrange equation is given by $V(\varphi) = \varphi^4$). We refer the reader to [19] for the perturbative aspects of this equation in the flat φ^4 case.

Returning to the general situation, we can now prove the following result on scalar partial differential equations on semi-Riemannian manifolds:

Proposition 4.34. *In the above situation, the assumptions* (1) *and* (2) *from Theorem* 4.31 *are satisfied by* $h_{\mathbf{g},F_1,F_2,K}$. *In particular,* $\mathsf{Z}_0(h_{\mathbf{g},F_1,F_2,K}) \subset \mathsf{J}^2(\pi)$ *is formally integrable, and the corresponding space of formal solutions canonically becomes a profinite dimensional manifold via Proposition* 4.29.

Proof. In view of Theorem 4.28, we only have to prove that the assumptions (1), (2) from Theorem 4.31 are satisfied. To this end, we set $h := h_{\mathbf{g},F_1,F_2,K}$ and assume $F_2 = 0$. Firstly, in view of

$$\sigma(h)_{|\mathsf{S}^2\big(\mathrm{T}^*_{\pi_2(a)}X\big)}(v \odot v) = \mathbf{g}^*_{\pi_2(a)}(v, v) \qquad \text{for all} \quad a \in \mathsf{J}^2(\pi), \quad v \in \mathrm{T}^*_{\pi_2(a)}X,$$

assumption (1) is obviously satisfied and $\mathsf{Z}_0(h)$ indeed is a partial differential equation.

It remains to prove that the map $\mathsf{Z}_0(h)^{(1)} \to \mathsf{Z}_0(h)$, $a \mapsto \pi_{2,3}(a)$ is surjective. To see this, assume to be given $b \in \mathsf{Z}_0(h)$ and consider a $\mathbf{g}$-exponential manifold chart $\tilde{x} \colon U \to \mathbb{R}^m$ of X centered at $\pi_2(b)$. Then one gets the trivial fibered chart

$$(x, u) := (\tilde{x}, \mathrm{id}_\mathbb{R}) \colon \ U \times \mathbb{R} \longrightarrow \mathbb{R}^m \times \mathbb{R}$$

of π, and $b \in \mathsf{Z}_0(h)$ means nothing but $b \in \mathsf{J}^2(\pi)$ and

$$\sum_{i,j=1}^m \mathbf{g}^{ij}(\pi_2(b))u_{2,1_{ij}}(b) - \sum_{i,j,k=1}^m \mathbf{g}^{ij}(\pi_2(b))\Gamma_{ij}^k(\pi_2(b))u_{2,1_k}(b)$$
$$+ F_1(\pi_2(b))u_{2,(0,\dots,0)}(b) + K(u_{2,(0,\dots,0)}(b)) = 0,$$

where $\mathbf{g}^{ij}, \Gamma_{ij}^k \in \mathscr{C}^\infty(U)$ denote the components of the metric tensor and the Christoffel symbols of g with respect to $\tilde{x}$, respectively. Noting that Proposition 4.25 implies $Z_0(h)^{(1)} = Z_0(h^{(1)})$, one easily finds that some $a \in \mathsf{J}^3(\pi)$ is in $Z_0(h)^{(1)}$, if and only if

$$\sum_{i,j=1}^m \mathbf{g}^{ij}(\pi_3(a))u_{3,1_{ij}}(a) - \sum_{i,j,k=1}^m \mathbf{g}^{ij}(\pi_3(a))\Gamma_{ij}^k(\pi_3(a))u_{1_k}(a)$$
$$+ F_1(\pi_3(a))u_{3,(0,\ldots,0)}(a) + K\big(u_{3,(0,\ldots,0)}(a)\big) = 0,$$

and, for all $l = 1, \ldots, m$,

$$\sum_{i,j=1}^m \big(\partial_l \mathbf{g}^{ij}(\pi_3(a))u_{3,1_{ij}}(a) + \mathbf{g}^{ij}(\pi_3(a))u_{3,1_{ijl}}(a)\big)$$
$$- \sum_{i,j,k=1}^m \big(\partial_l \mathbf{g}^{ij}(\pi_3(a))\Gamma_{ij}^k(\pi_3(a))u_{3,1_k}(a) - \mathbf{g}^{ij}(\pi_3(a))\partial_l\Gamma_{ij}^k(\pi_3(a))u_{3,1_k}(a)\big)$$
$$- \sum_{i,j,k=1}^m \mathbf{g}^{ij}(\pi_3(a))\Gamma_{ij}^k(\pi_3(a))u_{3,1_{lk}}(a) + \partial_l F_1(\pi_3(a))u_{3,(0,\ldots,0)}(a)$$
$$+ F_1(\pi_3(a))u_{3,1_l}(a) + K'\big(u_{3,(0,\ldots,0)}(a)\big)u_{3,1_l}(a) = 0.$$

Here, we have used $\partial_l := \frac{\partial}{\partial \tilde{x}^l}$. Let us now assume that the signature of $\mathbf{g}$ is given by $(\varepsilon_1, \ldots, \varepsilon_m) = (1, -1, \ldots, -1)$. The general case can be treated with the same method. We define some $a \in \mathsf{J}^3(\pi)$ by requiring $\tilde{x}_3(a) := \tilde{x}(\pi_2(b))$, and, for $I \in \mathbb{N}_{0,3}^m$,

$$u_{3,I}(a) := \begin{cases} u_{2,I}(b), & \text{if } I \in \mathbb{N}_{0,2}^m, \\ -\sum\limits_{i,j=1}^m \partial_l \mathbf{g}^{ij}(\pi_2(b))u_{2,1_{ij}}(b) + \sum\limits_{i,j,k=1}^m \partial_l \mathbf{g}^{ij}(\pi_2(b))\Gamma_{ij}^k(\pi_2(b))u_{2,1_k}(b) \\ \quad + \sum\limits_{i,j,k=1}^m \mathbf{g}^{ij}(\pi_2(b))\partial_l\Gamma_{ij}^k(\pi_2(b))u_{2,1_k}(b) + \sum\limits_{i,j,k=1}^m \mathbf{g}^{ij}(\pi_2(b))\Gamma_{ij}^k(\pi_2(b))u_{2,1_{lk}}(b) \\ \quad - \partial_l F_1(\pi_2(b))u_{2,(0,\ldots,0)}(b) - F_1(\pi_2(b))u_{2,1_l}(b) \\ \quad - K'\big(u_{2,(0,\ldots,0)}(b)\big)u_{2,1_l}(b), & \text{if } I = 1_{11l} \text{ for some } l = 1, \ldots, m, \\ 0, & \text{else.} \end{cases}$$

Now we are almost done: Indeed, our construction of a directly gives $\pi_{2,3}(a) = b$, so $\pi_3(a) = \pi_2(b)$. Since we have

$$\mathbf{g}^{ij}(\pi_3(a)) = \mathbf{g}^{ij}(\pi_2(b)) = \begin{cases} \varepsilon_j, & \text{if } i = j, \\ 0, & \text{else,} \end{cases}$$

it follows immediately that $a \in Z_0(h)^{(1)}$, and the proof is complete, noting that F_2 has not played a role in the above argument. ∎

A   Two results on completed projective tensor products

Assume to be given two locally convex topological vector spaces V and W, and consider their algebraic tensor product $V \otimes W$. A topology τ on $V \otimes W$ is called *compatible* (in the sense of Grothendieck [25]) or a *tensor product topology*, if the following axioms hold true:

(TPT1) $V \otimes W$ equipped with τ is a locally convex topological vector space which will be denoted by $V \otimes_\tau W$.

(TPT2) The canonical map $V \times W \to V \otimes_\tau W$ is separately continuous.

(TPT3) For every equicontinuous subset A of the topological dual V' and every equicontinuous subset B of the topological dual W', the set $A \otimes B := \{\lambda \otimes \mu \mid \lambda \in A,\, \mu \in B\}$ is an equicontinuous subset of $(V \otimes_\tau W)'$.

If τ is a tensor product topology on $V \otimes W$, we denote by $V \widehat{\otimes}_\tau W$ the completion of $V \otimes_\tau W$.

Example A.1.

a) The *projective tensor product topology* is the finest locally convex vector space topology on $V \otimes W$ such that the canonical map $V \times W \to V \otimes W$ is continuous, cf. [25, 47]. The projective tensor product topology is denoted by π. It is generated by seminorms $p_A \otimes_\pi q_B$, where p_A, $A \in \mathcal{A}$ and q_B, $B \in \mathcal{B}$ each run through a family of seminorms generating the locally convex topology on V respectively W, and $p_A \otimes_\pi q_B$ is defined by

$$p_A \otimes_\pi q_B(z) := \inf \left\{ \sum_{l=1}^n p_A(v_l)\, q_B(w_l) \mid z = \sum_{l=1}^n v_l \otimes w_l \right\}.$$

The seminorm $p_A \otimes_\pi q_B$ is in particular a *cross seminorm*, i.e., it satisfies the relation

$$p_A \otimes_\pi q_B(v \otimes w) = p_A(v) q_B(w) \qquad \text{for all} \quad v \in V, \quad w \in W.$$

b) The *injective tensor product topology* on $V \otimes W$, denoted by ε, is the locally convex topology inherited from the canonical embedding $V \otimes W \hookrightarrow \mathcal{B}_s(V'_s \otimes W'_s)$, where $\mathcal{B}_s(V', W')$ denotes the space of separately continuous bilinear forms on the product $V' \times W'$ of the weak topological duals V' and W' endowed with the topology of uniform convergence on products of equicontinuous subsets of V' and W'. See [25] and [47, Section 43] for details.

Remark A.2.

a) By definition, the ε-topology on $V \otimes W$ is coarser than the π-topology. If V (or W) is a nuclear locally convex topological vector space, then these two topologies coincide, cf. [25, 47]. Since finite dimensional vector spaces over $\mathbb{R}$ are nuclear, this entails in particular that for finite dimensional V and W the natural vector space topology on $V \otimes W$ coincides with the (completed) π- and ε-topology.

b) The projective tensor product, the injective tensor product, and their completed versions are in fact functors, so it is clear what is meant by $f \otimes_\varepsilon g$, $f \widehat{\otimes}_\pi g$, and so on, where f and g denote continuous linear maps.

Theorem A.3. *Let $(V_i)_{i \in \mathbb{N}}$ and $(W_i)_{i \in \mathbb{N}}$ be two families of finite dimensional real vector spaces. Denote by V and W their respective product (within the category of locally convex topological vector spaces), i.e., let*

$$V := \prod_{i \in \mathbb{N}} V_i \qquad \text{and} \qquad W := \prod_{i \in \mathbb{N}} W_i.$$

Then V, W, and the completed projective tensor product $V \widehat{\otimes}_\pi W$ are nuclear Fréchet spaces. Moreover, one has the canonical isomorphism

$$V \widehat{\otimes}_\pi W \cong \prod_{(k,l) \in \mathbb{N} \times \mathbb{N}} V_k \otimes W_l. \tag{A.1}$$

Proof. Since each of the vector spaces V_i and W_i is a nuclear Fréchet space, and countable products of nuclear Fréchet are again nuclear Fréchet spaces by [47], the spaces V and W are nuclear Fréchet. Moreover, the same argument shows that $V \widehat{\otimes}_\pi W$ is nuclear Fréchet, if equation (A.1) holds true. So let us show equation (A.1). To this end recall first [10, Section 3.7] that there is a canonical injection

$$\iota \colon V \otimes W \lhook\joinrel\longrightarrow \prod_{(i,j) \in \mathbb{N} \times \mathbb{N}} V_i \otimes W_j,$$

$$(v_i)_{i \in \mathbb{N}} \otimes (w_j)_{j \in \mathbb{N}} \longmapsto (v_i \otimes w_j)_{(i,j) \in \mathbb{N} \times \mathbb{N}}.$$

Choose norms $p_i \colon V_i \to \mathbb{R}$ and $q_i \colon W_i \to \mathbb{R}$. The product topology on $\prod_{(i,j) \in \mathbb{N} \times \mathbb{N}} V_i \otimes W_j$ then is defined by the sequence of seminorms

$$r_{k,l} \colon \prod_{(i,j) \in \mathbb{N} \times \mathbb{N}} V_i \otimes W_j \longrightarrow \mathbb{R}, \qquad (z_{i,j})_{(i,j) \in \mathbb{N} \times \mathbb{N}} \longmapsto (p_k \otimes_\pi q_l)(z_{k,l}).$$

The product topology on V is generated by the seminorms $p_k^V \colon V \to \mathbb{R}$, $(v_i)_{i \in \mathbb{N}} \mapsto p_k(v_k)$, the topology on W by the seminorms $q_l^W \colon W \to \mathbb{R}$, $(w_i)_{i \in \mathbb{N}} \mapsto q_l(w_l)$. Hence, the π-topology on $V \otimes W$ is generated by the seminorms $p_k^V \otimes_\pi q_l^W$. But since these are cross seminorms, one obtains for $(v_i)_{i \in \mathbb{N}} \in V$ and $(w_i)_{i \in \mathbb{N}} \in W$ the equality

$$p_k^V \otimes_\pi q_l^W \big((v_i)_{i \in \mathbb{N}} \otimes (w_i)_{i \in \mathbb{N}} \big) = p_k^V \big((v_i)_{i \in \mathbb{N}} \big) q_l^W \big((w_i)_{i \in \mathbb{N}} \big) = p_k(v_k) q_l(w_l)$$
$$= p_k \otimes_\pi q_l (v_k \otimes w_l) = r_{k,l} \big((v_i \otimes w_j)_{(i,j) \in \mathbb{N} \times \mathbb{N}} \big).$$

This entails $p_k^V \otimes_\pi q_l^W = r_{k,l} \circ \iota$, or in other words that the π-topology on $V \otimes W$ coincides with the pull-back of the product topology on $\prod_{(i,j) \in \mathbb{N} \times \mathbb{N}} V_i \otimes W_j$ by the embedding ι. The claim now follows, if we can yet show that the image of ι is dense in its range. To prove this let $z = (z_{i,j})_{(i,j) \in \mathbb{N} \times \mathbb{N}}$ be an element of the product $\prod_{(i,j) \in \mathbb{N} \times \mathbb{N}} V_i \otimes W_j$. Choose representations

$$z_{i,j} = \sum_{l=1}^{n_{i,j}} v_{i,j,l} \otimes w_{i,j,l}, \qquad \text{where} \quad v_{i,j,l} \in V_i, \quad w_{i,j,l} \in V_j.$$

Put $v_{i,j,l} = 0$ and $w_{i,j,l} = 0$, if $l > n_{i,j}$. Let $\iota_i^V \colon V_i \to V$ the embedding of the i-th factor in V, i.e., the map which associates to $v_i \in V_i$ the family $(v_j)_{j \in \mathbb{N}}$, where $v_j := 0$, if $j \neq i$. Likewise, denote by $\iota_i^W \colon W_i \hookrightarrow W$ the embedding of the i-th factor in W. Then define for $n \in \mathbb{N}$

$$z_n := \sum_{i,j \leq n} \sum_{l \in \mathbb{N}} \iota_i^V(v_{i,j,l}) \otimes \iota_j^W(w_{i,j,l}),$$

and note that by construction the sum on the right side is finite. The sequence $(z_n)_{n \in \mathbb{N}}$ then is a family in $V \otimes_\pi W$. By construction, it is clear that $\lim_{n \to \infty} \iota(z_n) = z$. The proof is finished. $\blacksquare$

Theorem A.4. *Assume that V and W are projective limits of projective systems of finite dimensional real vector spaces (V_i, λ_{ij}) and (W_i, μ_{ij}), respectively. Denote by*

$$\lambda_i \colon V \longrightarrow V_i, \qquad \text{respectively by} \qquad \mu_i \colon W \longrightarrow W_i,$$

the corresponding canonical maps. The completed π-tensor product $V \widehat{\otimes}_\pi W$ together with the family of canonical maps

$$\lambda_i \widehat{\otimes}_\pi \mu_i \colon V \widehat{\otimes}_\pi W \longrightarrow V_i \otimes W_i$$

then is a projective limit of the projective system $(V_i \otimes W_i, \lambda_{ij} \otimes \mu_{ij})$ within the category of locally convex topological vector spaces. Moreover, both V and W are nuclear, hence $V \widehat{\otimes}_\pi W = V \widehat{\otimes}_\varepsilon W$.

Proof. First observe that $\left(V_i \otimes W_i, \lambda_{ij} \otimes \mu_{ij}\right)$ is a projective systems of finite dimensional real vector spaces, indeed. Next recall that projective limits of nuclear Fréchet spaces are nuclear by [47]. This proves the second claim. It remains to show the first one. To this end put $\widetilde{V}_0 := V_0$, $\widetilde{W}_0 := W_0$, and denote for every $i \in \mathbb{N}^*$ by $\widetilde{V}_i$ be the kernel of the map $\lambda_{i-1\,i}$ and by $\widetilde{W}_i$ the kernel of $\mu_{i-1\,i}$. Moreover, choose for every $i \in \mathbb{N}^*$ a splitting $f_i \colon V_{i-1} \to V_i$ of $\lambda_{i-1\,i}$, and a splitting $g_i \colon W_{i-1} \to W_i$ of $\mu_{i-1\,i}$. Put

$$\widetilde{V} := \prod_{i \in \mathbb{N}} \widetilde{V}_i \qquad \text{and} \qquad \widetilde{W} := \prod_{i \in \mathbb{N}} \widetilde{W}_i.$$

Let $\pi_i^{\widetilde{V}} \colon \widetilde{V} \to \widetilde{V}_i$ be the projection onto the i-th factor of $\widetilde{V}$, and $\pi_j^{\widetilde{W}} \colon \widetilde{W} \to \widetilde{W}_j$ the projection on the j-th factor of $\widetilde{W}$.

Now we inductively construct $\widetilde{\lambda}_i \colon \widetilde{V} \to V_i$ and $\widetilde{\mu}_i \colon \widetilde{W} \to W_i$. First, put $\widetilde{\lambda}_0 := \pi_0^{\widetilde{V}}$ and $\widetilde{\mu}_0 := \pi_0^{\widetilde{W}}$. Next, assume that we have constructed $\widetilde{\lambda}_0, \ldots, \widetilde{\lambda}_j$ and $\widetilde{\mu}_0, \ldots, \widetilde{\mu}_j$ such that for $i \leq k \leq j$

$$\widetilde{\lambda}_i = \lambda_{ik} \circ \widetilde{\lambda}_k \qquad \text{and} \qquad \widetilde{\mu}_i = \mu_{ik} \circ \widetilde{\mu}_k. \tag{A.2}$$

Then we define $\widetilde{\lambda}_{j+1} \colon \widetilde{V} \to V_{j+1}$ and $\widetilde{\mu}_{j+1} \colon \widetilde{W} \to W_{j+1}$ by

$$\widetilde{\lambda}_{j+1}(v) = \pi_{j+1}^{\widetilde{V}}(v) + f_{j+1}\widetilde{\lambda}_j(v) \qquad \text{and} \qquad \widetilde{\mu}_{j+1}(w) = \pi_{j+1}^{\widetilde{W}}(w) + g_{j+1}\widetilde{\lambda}_j(w),$$

where $v \in \widetilde{V}$, and $w \in \widetilde{W}$. By assumption on f_{j+1} and g_{j+1} one concludes that

$$\widetilde{\lambda}_j = \lambda_{j+1\,j} \circ \widetilde{\lambda}_{j+1} \qquad \text{and} \qquad \widetilde{\mu}_j = \mu_{j+1\,j} \circ \widetilde{\mu}_{j+1},$$

which entails that equation (A.2) holds true for $i \leq k \leq j+1$. We now claim that $\widetilde{V}$ together with the family $(\widetilde{\lambda}_i)$ is a projective limit of (V_i, λ_{ij}), and likewise for $\widetilde{W}$. We only need to prove the claim for $\widetilde{V}$. Let Z be a locally convex topological vector space, and $\nu_i \colon Z \to V_i$ a family of continuous linear maps such that $\nu_i = \lambda_{ij} \circ \nu_j$ for $i \leq j$. Put for every $z \in Z$

$$\widetilde{\nu}_0(z) := \nu_0(z) \qquad \text{and} \qquad \widetilde{\nu}_i(z) := \nu_i(z) - f_i(\nu_{i-1}(z))) \qquad \text{for} \quad i \in \mathbb{N}^*.$$

Then $\widetilde{\nu}_i(z) \in \widetilde{V}_i$ for all $i \in \mathbb{N}$, and

$$\nu \colon Z \longrightarrow \widetilde{V}, \qquad z \longmapsto \left(\widetilde{\nu}_i(z)\right)_{i \in \mathbb{N}}$$

is well-defined, linear, and continuous. Moreover, it follows by induction on $i \in \mathbb{N}$ that

$$\widetilde{\lambda}_i \nu = \nu_i.$$

For $i = 0$ this is clear, so assume that we have shown this for some $i \in \mathbb{N}$. Then, for $z \in Z$,

$$\widetilde{\lambda}_{i+1}\nu(z) = \nu_{i+1}(z) - f_{i+1}(\nu_i(z)) + f_{i+1}\widetilde{\lambda}_i(\nu(z)) = \nu_{i+1}(z),$$

which finishes the inductive argument. Assume that $\nu' \colon Z \to \widetilde{V}$ is another continuous linear map such that $\widetilde{\lambda}_i \nu' = \nu_i$ for all $i \in \mathbb{N}$. First, this entails that

$$\pi_0^{\widetilde{V}} \nu' = \widetilde{\lambda}_0 \nu' = \nu_0 = \widetilde{\nu}_0.$$

Assume that $\pi_i^{\widetilde{V}} \nu' = \widetilde{\nu}_i$ for some $i \in \mathbb{N}$. Then

$$\pi_{i+1}^{\widetilde{V}} \nu' = \widetilde{\lambda}_{i+1}\nu' - f_{i+1}\widetilde{\lambda}_i \nu' = \nu_{i+1} - f_{i+1}\nu_i = \widetilde{\nu}_{i+1}.$$

Hence, one obtains, for all $i \in \mathbb{N}$,

$$\pi_i^{\widetilde{V}} \nu' = \widetilde{\nu}_i = \pi_i^{\widetilde{V}} \nu,$$

which proves $\nu' = \nu$. So $\widetilde{V}$ is a projective limit of (V_i, λ_{ij}), and $\widetilde{W}$ a projective limit of (W_i, μ_{ij}). Moreover, $\widetilde{V}$ is canonically isomorphic to V, and $\widetilde{W}$ to W. The theorem is now proved, if we can show that $\widetilde{V} \otimes_\pi \widetilde{W}$ together with the family of canonical maps $\widetilde{\lambda}_i \widehat{\otimes}_\pi \widetilde{\mu}_i \colon \widetilde{V} \otimes_\pi \widetilde{W} \to V_i \otimes W_i$ is a projective limit of the projective system $(V_i \otimes W_i, \lambda_{ij} \otimes \mu_{ij})$. But this is clear, since by the preceding theorem,

$$\widetilde{V} \otimes_\pi \widetilde{W} \cong \prod_{(i,j)\in\mathbb{N}\times\mathbb{N}} \widetilde{V}_i \otimes \widetilde{W}_j \cong \varprojlim_{k\in\mathbb{N}} \prod_{\substack{(i,j)\in\mathbb{N}\times\mathbb{N}\\ i,j\leq k}} \widetilde{V}_i \otimes \widetilde{W}_j$$

and, for $k \in \mathbb{N}$,

$$\prod_{\substack{(i,j)\in\mathbb{N}\times\mathbb{N}\\ i,j\leq k}} \widetilde{V}_i \otimes \widetilde{W}_j \cong \prod_{i\leq k}\widetilde{V}_i \otimes \prod_{j\leq k}\widetilde{W}_j \cong V_k \otimes W_k. \qquad \blacksquare$$

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