The Geometry of Almost Einstein $(2, 3, 5)$ Distributions

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Abstract. We analyze the classic problem of existence of Einstein metrics in a given conformal structure for the class of conformal structures inducedf Nurowski's construction by (oriented) $(2, 3, 5)$ distributions. We characterize in two ways such conformal structures that admit an almost Einstein scale: First, they are precisely the oriented conformal structures **c** that are induced by at least two distinct oriented $(2, 3, 5)$ distributions; in this case there is a 1-parameter family of such distributions that induce **c**. Second, they are characterized by the existence of a holonomy reduction to $SU(1, 2)$, $SL(3, \mathbb{R})$, or a particular semidirect product $SL(2, \mathbb{R}) \ltimes Q_+$, according to the sign of the Einstein constant of the corresponding metric. Via the curved orbit decomposition formalism such a reduction partitions the underlying manifold into several submanifolds and endows each ith a geometric structure. This establishes novel links between $(2, 3, 5)$ distributions and many other geometries – several classical geometries among them – including: Sasaki–Einstein geometry and its paracomplex and null-complex analogues in dimension 5; Kähler–Einstein geometry and its paracomplex and null-complex analogues, Fefferman Lorentzian conformal structures, and para-Fefferman neutral conformal structures in dimension 4; CR geometry and the point geometry of second-order ordinary differential equations in dimension 3; and projective geometry in dimension 2. We describe a generalized Fefferman construction that builds from a 4-dimensional Kähler–Einstein or para-Kähler–Einstein structure a family of $(2, 3, 5)$ distributions that induce the same (Einstein) conformal structure. We exploit some of these links to construct new examples, establishing the existence of nonflat almost Einstein $(2, 3, 5)$ conformal structures for which the Einstein constant is positive and negative.

Key words: $(2, 3, 5)$ distribution; almost Einstein; conformal geometry; conformal Killing field; CR structure; curved orbit decomposition; Fefferman construction; G_2; holonomy reduction; Kähler–Einstein; Sasaki–Einstein; second-order ordinary differential equation

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1 Introduction

A (pseudo-)Riemannian metric g_{ab} is said to be *Einstein* if its Ricci curvature R_{ab} is a multiple of g_{ab}. The problem of determining whether a given conformal structure (locally) contains an Einstein metric has a rich history, and dates at least to Brinkmann's seminal investigations [9, 10] in the 1920s. Other significant contributions have been made by, among others, Hanntjes and Wrona [39], Sasaki [57], Wong [66], Yano [67], Schouten [58], Szekeres [63], Kozameh, Newman, and Tod [47], Bailey, Eastwood, and Gover [4], Fefferman and Graham [29], and Gover and Nurowski [36]. Developments in this topic in the last quarter century in particular have stimulated substantial development both within conformal geometry and far beyond it.

In the watershed article [4], Bailey, Eastwood, and Gover showed that the existence of such a metric in a conformal structure (here, and always in this article, of dimension $n \geq 3$) is governed by a second-order, conformally invariant linear differential operator $\Theta_0^{\mathcal{V}}$ that acts on sections of a natural line bundle $\mathcal{E}[1]$ (we denote by $\mathcal{E}[k]$ the kth power of $\mathcal{E}[1]$): Every conformal structure $(M, \mathbf{c})$ is equipped with a canonical bilinear form $\mathbf{g} \in \Gamma(S^2 T^* M \otimes \mathcal{E}[2])$, and a nowhere-vanishing section σ in the kernel of $\Theta_0^{\mathcal{V}}$ determines an Einstein metric $\sigma^{-2}\mathbf{g}$ in $\mathbf{c}$ and vice versa. Writing the differential equation $\Theta_0^{\mathcal{V}}(\sigma) = 0$ as a first-order system and prolonging once yields a closed system and hence determines a conformally invariant connection $\nabla^{\mathcal{V}}$ on a natural vector bundle $\mathcal{V}$, called the (standard) *tractor bundle*. (The conformal structure determines a parallel *tractor metric* $H \in \Gamma(S^2 \mathcal{V}^*)$.) By construction, this establishes a bijective correspondence between Einstein metrics in $\mathbf{c}$ and parallel sections of this bundle satisfying a genericity condition. This framework immediately suggests a natural relaxation of the Einstein condition: A section of the kernel of $\Theta_0^{\mathcal{V}}$ is called an *almost Einstein scale*, and it determines an Einstein metric on the complement of its zero locus and becomes singular along that locus. A conformal structure that admits a nonzero almost Einstein scale is itself said to be *almost Einstein*, and, somewhat abusively, the metric it determines is sometimes called an *almost Einstein metric* on the original manifold. The generalization to the almost Einstein setting has substantial geometric consequences: The zero locus itself inherits a geometric structure, which can be realized as a natural limiting geometric structure of the metric on the complement. This arrangement leads to the notion of conformal compactification, which has received substantial attention in its own right, including in the physics literature [62].

This article investigates the problem of existence of almost Einstein scales, as well as the geometric consequences of existence of such a scale, for a fascinating class of conformal structures that arise naturally from another geometric structure: A $(2, 3, 5)$ *distribution* is a 2-plane distribution $\mathbf{D}$ on a 5-manifold which is maximally nonintegrable in the sense that $[\mathbf{D}, [\mathbf{D}, \mathbf{D}]] = TM$. This geometric structure has attracted substantial interest, especially in the last few decades, for numerous reasons: $(2, 3, 5)$ distributions are deeply connected to the exceptional simple Lie algebra of type G_2 (in fact, the study of these distributions dates to 1893, when Cartan [20] and Engel [28] simultaneously realized that Lie algebra as the infinitesimal symmetry algebra of a distribution of this type), they are the subject of Cartan's most involved application of his celebrated method of equivalence [21], they comprise a first class of distributions with continuous local invariants, they arise naturally from a class of second-order Monge equations, they arise naturally from mechanical systems entailing one surface rolling on another without slipping or twisting [1, 2, 7], they can be used to construct pseudo-Riemannian metrics whose holonomy group is G_2 [38, 49], they are natural compactifying structures for indefinite-signature nearly Kähler geometries in dimension 6 [37], and they comprise an interesting example of a broad class of so-called *parabolic geometries* [18, Section 4.3.2]. (Here and henceforth, the symbol G_2 refers to the algebra automorphism group of the split octonions; this is a split real form of the complex simple Lie group of type G_2.) For our purposes their most important feature is the natural construction, due to Nurowski [52, Section 5], that associates to any $(2, 3, 5)$ distribution $(M, \mathbf{D})$ a conformal structure $\mathbf{c_D}$ of signature $(2, 3)$ on M, and it is these structures, which we call $(2, 3, 5)$ *conformal structures*, whose almost Einstein geometry we investigate. For expository convenience, we restrict ourselves to the oriented setting: A $(2, 3, 5)$ distribution $(M, \mathbf{D})$ is *oriented* iff $\mathbf{D} \to M$ is an oriented bundle; an orientation of $\mathbf{D}$ determines an orientation of M and vice versa.

The key ingredient in our analysis is that, like almost Einstein conformal structures, $(2, 3, 5)$ conformal structures can be characterized in terms of the holonomy group of the normal tractor connection, $\nabla^{\mathcal{V}}$ (for any oriented conformal structure of signature $(2, 3)$, $\nabla^{\mathcal{V}}$ has holonomy contained in $\mathrm{SO}(H) \cong \mathrm{SO}(3, 4)$): An oriented conformal structure $\mathbf{c}$ of signature $(2, 3)$ coincides with $\mathbf{c_D}$ for some $(2, 3, 5)$ distibution $\mathbf{D}$ iff the holonomy group of $\nabla^{\mathcal{V}}$ is contained inside G_2, or equivalently, iff there is a parallel tractor 3-form, that is, a section $\Phi \in \Gamma(\Lambda^3 \mathcal{V}^*)$, compatible

with the conformal structure in the sense that the pointwise stabilizer of Φ_p in $\mathrm{GL}(\mathcal{V}_p)$ (at any, equivalently, every point p) is isomorphic to G_2 and is contained inside $\mathrm{SO}(H_p)$ [41, 52]. While the construction $\mathbf{D} \rightsquigarrow \mathbf{c_D}$ depends at each point on the 4-jet of $\mathbf{D}$, the corresponding compatibility condition in the tractor setting is algebraic (pointwise), which reduces many of our considerations and arguments to properties of the algebra of G_2.

With these facts in hand, it is immediate that whether an oriented conformal structure of signature $(2, 3)$ is both $(2, 3, 5)$ and almost Einstein is characterized by the admission of both a compatible tractor 3-form Φ and a (nonzero) parallel tractor $\mathbb{S} \in \Gamma(\mathcal{V})$, which we may just as well frame as a reduction of the holonomy of $\nabla^{\mathcal{V}}$ to the 8-dimensional common stabilizer S in $\mathrm{SO}(3, 4)$ of a nonzero vector in the standard representation $\mathbb{V}$ of $\mathrm{SO}(3, 4)$ and a 3-form in $\Lambda^3 \mathbb{V}^*$ compatible with the conformal structure. The isomorphism type of S depends on the causality type of $\mathbb{S}$: If the vector is spacelike, then $S \cong \mathrm{SU}(1, 2)$; if it is timelike, then $S \cong \mathrm{SL}(3, \mathbb{R})$; if it is isotropic, then $S \cong \mathrm{SL}(2, \mathbb{R}) \ltimes Q_+$, where $Q_+ < \mathrm{G}_2$ is the connected, nilpotent subgroup of G_2 defined via Sections 2.3.4 and 2.3.6.[1]

Proposition A. *An oriented conformal structure of signature $(2, 3)$ is both $(2, 3, 5)$ and almost Einstein iff it admits a holonomy reduction to the common stabilizer S of a 3-form in $\Lambda^3 \mathbb{V}^*$ compatible with the conformal structure and a nonzero vector in $\mathbb{V}$.*

Throughout this article, S, $\mathrm{SU}(1, 2)$ $\mathrm{SL}(3, \mathbb{R})$, and $\mathrm{SL}(2, \mathbb{R}) \ltimes Q_+$ refer to the common stabilizer of the data described above – that is, to any subgroup in a particular conjugacy class in $\mathrm{SO}(3, 4)$ (and not just to a subgroup of $\mathrm{SO}(3, 4)$ of the respective isomorphism types).

Given an almost Einstein $(2, 3, 5)$ conformal structure, algebraically combining Φ and $\mathbb{S}$ yields other parallel tractor objects. The simplest of these is the contraction $\mathbb{K} := -\mathbb{S} \lrcorner \Phi$, which we may identify with a skew endomorphism of the standard tractor bundle, $\mathcal{V}$. Underlying this endomorphism is a conformal Killing field ξ of the induced conformal structure $\mathbf{c_D}$ that does not preserve any distribution that induces that structure. Thus, if ξ is complete (or alternatively, if we content ourselves with a suitable local statement) the images of $\mathbf{D}$ under the flow of ξ comprise a 1-parameter family of distinct $(2, 3, 5)$ distributions that all induce the same conformal structure. This suggests – and connects with the problem of existence of an almost Einstein scale – a natural question that we call the *conformal isometry problem* for $(2, 3, 5)$ distributions: Given a $(2, 3, 5)$ distribution $(M, \mathbf{D})$, what are the $(2, 3, 5)$ distributions $\mathbf{D}'$ on M that induce the same conformal structure, that is, for which $\mathbf{c_{D'}} = \mathbf{c_D}$? Put another way, what are the fibers of the map $\mathbf{D} \rightsquigarrow \mathbf{c_D}$? By our previous observation, working in the tractor setting essentially reduces this to an algebraic problem, which we resolve in Proposition 4.1 (and which extends to the split real form of G_2 an analogous result of Bryant [12, Remark 4] for the compact real form). Translating this result to the tractor setting and then reinterpreting it in terms of the underlying data gives a complete description of all $(2, 3, 5)$ distributions $\mathbf{D}'$ that induce the conformal structure $\mathbf{c_D}$. In order to formulate it, we note first that, given a fixed oriented conformal structure $\mathbf{c}$ of signature $(2, 3)$, underlying any compatible parallel tractor 3-form, and hence corresponding to a $(2, 3, 5)$ distribution $\mathbf{D}$, is a conformally weighted 2-form $\phi \in \Gamma(\Lambda^2 T^* M \otimes \mathcal{E}[3])$, which in particular is a solution to the conformally invariant *conformal Killing 2-form equation*. The weighted 2-forms that arise this way are called *generic*. This solution turns out to satisfy $\phi \wedge \phi = 0$ – so it is locally decomposable – but vanishes nowhere. Hence, it defines an oriented 2-plane distribution, and this distribution is $\mathbf{D}$.

Theorem B. *Fix an oriented $(2, 3, 5)$ distribution $(M, \mathbf{D})$, denote by ϕ the corresponding generic conformal Killing 2-form, by $\Phi \in \Gamma(\Lambda^3 \mathcal{V}^*)$ the corresponding parallel tractor 3-form, and by $H_\Phi \in \Gamma(S^2 \mathcal{V}^*)$ the parallel tractor metric associated to $\mathbf{c_D}$.*

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[1] Cf. [45, Corollary 2.4].

1. *Suppose $(M, \mathbf{c_D})$ admits the nonzero almost Einstein scale $\sigma \in \Gamma(\mathcal{E}[1])$, and denote by $\mathbb{S} \in \Gamma(\mathcal{V})$ the corresponding parallel tractor; by rescaling, we may assume that $\varepsilon := -H_\Phi(\mathbb{S}, \mathbb{S}) \in \{-1, 0, +1\}$. Then, for any $(\bar{A}, B) \in \mathbb{R}^2$ such that $-\varepsilon \bar{A}^2 + 2\bar{A} + B^2 = 0$ (there is a 1-parameter family of such pairs) the weighted 2-form*

$$\phi'_{ab} := \phi_{ab} + \bar{A}\left[\tfrac{1}{5}\sigma^2\left(\tfrac{1}{3}\phi_{ab,c}{}^c + \tfrac{2}{3}\phi_{c[a,b]}{}^c + \tfrac{1}{2}\phi_{c[a,}{}^c{}_{b]} + 4\mathsf{P}^c{}_{[a}\phi_{b]c}\right) - \sigma\sigma^{,c}\phi_{[ca,b]} \qquad (1.1)$$
$$- \tfrac{1}{2}\sigma\sigma_{,[a}\phi_{b]c,}{}^c - \tfrac{1}{5}\sigma\sigma_{,c}{}^c\phi_{ab} + 3\sigma^{,c}\sigma_{,[c}\phi_{ab]}\right] + B[-\tfrac{1}{4}\sigma\phi^{cd},{}_d\phi_{[ab,c]} + \tfrac{3}{4}\sigma^{,c}\phi_{[ab}\phi_{c]d,}{}^d]$$

   is a generic conformal Killing 2-form, and the oriented $(2, 3, 5)$ distribution $\mathbf{D}'$ it determines induces the same oriented conformal structure that $\mathbf{D}$ does, that is $\mathbf{c_{D'}} = \mathbf{c_D}$.

2. *Conversely, all conformally isometric oriented $(2, 3, 5)$ distributions arise this way: If an oriented $(2, 3, 5)$ distribution $\mathbf{D}'$ satisfies $\mathbf{c_{D'}} = \mathbf{c_D}$ (this condition is equality of oriented conformal structures), then there is an almost Einstein scale σ of $\mathbf{c_D}$ (we may assume that the corresponding parallel tractor $\mathbb{S}$ satisfies $\varepsilon := -H_\Phi(\mathbb{S}, \mathbb{S}) \in \{-1, 0, +1\}$) and $(\bar{A}, B) \in \mathbb{R}^2$ satisfying $-\varepsilon \bar{A}^2 + 2\bar{A} + B^2 = 0$ such that the normal conformal Killing 2-form ϕ' corresponding to $\mathbf{D}'$ is given by (1.1).*

Herein, a comma $_,$ denotes the covariant derivative with respect to (any) representative $g \in \mathbf{c_D}$, and P_{ab} denotes the Schouten tensor (2.19) of g. We say that the distributions in the 1-parameter family $\mathcal{D}$ determined by $\mathbf{D}$ and σ as in the theorem are *related by σ*. Theorem B is proved in Section 4.2.

Different signs of the Einstein constant of the Einstein metric $\sigma^{-2}\mathbf{g}$, or equivalently, different causality types of the corresponding parallel tractor $\mathbb{S}$, determine families of distributions with different qualitative behaviors. Section 4.2.1 gives simple parameterizations of the 1-parameter families of conformally isometric distributions, and Section 4.2.3 gives an explicit algorithm for recovering an almost Einstein scale σ of $\mathbf{c_D}$ relating $\mathbf{D}$ and $\mathbf{D}'$ whose existence is guaranteed by Part (2) of Theorem B.

An immediate corollary of Theorem B is a natural geometric characterization of almost Einstein oriented $(2, 3, 5)$ distributions:

Theorem C. *The conformal structure $\mathbf{c_D}$ induced by an oriented $(2, 3, 5)$ distribution $(M, \mathbf{D})$ is almost Einstein iff there is a distribution $(M, \mathbf{D}')$, $\mathbf{D}' \neq \mathbf{D}$ such that $\mathbf{c_D} = \mathbf{c_{D'}}$.*

Now, fix an oriented conformal structure $\mathbf{c}$ of signature $(2, 3)$ and a nonzero almost Einstein scale σ of $\mathbf{c}$. The conformal Killing field ξ of $\mathbf{c}$ determined together by σ and a choice of distribution $\mathbf{D}$ in the 1-parameter family $\mathcal{D}$ of oriented $(2, 3, 5)$ distributions inducing $\mathbf{c}$ and related by σ turns out not to depend on the choice of $\mathbf{D}$ (Proposition 4.6), and we can ask for all of the geometric objects that (like ξ) are determined by σ and $\mathcal{D}$. The almost Einstein scale σ alone partitions M into three subsets, M_+, Σ, M_-, according to the sign $+$, 0, $-$ of σ at each point. By construction, σ determines an Einstein metric on the complement $M - \Sigma = M_+ \cup M_-$. If the Einstein metric determined by σ is not Ricci-flat (that is, if the parallel tractor $\mathbb{S}$ corresponding to σ is nonisotropic), the boundary Σ itself inherits a conformal structure $\mathbf{c}_\Sigma$ that is suitably compatible with and that can be regarded as a natural compactifying structure for $(M_\pm, g_\pm)$ along $\partial M_\pm = \Sigma$ [33]. Something similar but more involved occurs in the Ricci-flat case.

This decomposition of M according to the geometry of the object σ – equivalently, the holonomy reduction of $\nabla^\mathcal{V}$ determined by the parallel standard tractor $\mathbb{S}$ – along with descriptions of the geometry induced on each subset in the decomposition, is formalized by the theory of *curved orbit decompositions* [17]. Here, the involved geometric structures are encoded as *Cartan geometries* (Section 2.3.1) of an appropriate type, which are geometric structures modeled on appropriate homogeneous spaces G/P endowed with G-invariant geometric structures, and the decomposition of M in the presence of a holonomy reduction to a group $H \leq G$ is a natural

generalization of the H-orbit decomposition of G/P; the subsets in the decomposition are accordingly termed the *curved orbits* of the reduction. The curved orbits are parameterized by the intersections of H and P up to conjugacy in G, and H together with these intersections determine the respective geometric structures on each curved orbit.

Section 5 carries out this decomposition for the Cartan geometry canonically associated to $(M, \mathbf{c})$ determined by σ and $\mathcal{D}$, that is, by a holonomy reduction to the group S. Besides elucidating the geometry of almost Einstein $(2,3,5)$ conformal structures for its own sake, this serves three purposes: First, this documents an example of a curved orbit decomposition for which the decomposition is relatively involved. Second, and more importantly, we will see that several classical geometries occur in the curved orbit decompositions, establishing novel and nonobvious links between $(2,3,5)$ distributions and those structures. Third, we can then exploit these connections to give new methods for construction of almost Einstein $(2,3,5)$ conformal structures from classical geometries, and using these we produce, for the first time, examples both with negative (Example 6.1) and positive (Example 6.2) Einstein constants.

Different signs of the Einstein constant (equivalently, different causality types of the parallel tractor $\mathbb{S}$ corresponding to σ) lead to qualitatively different curved orbit decompositions, so we treat them separately. We say that an almost Einstein scale is *Ricci-negative*, *-positive*, or *-flat* if the Einstein constant of the Einstein metric it determines is negative, positive, or zero, respectively. See also Appendix A, which summarizes the results here and records geometric characterizations of the curved orbits.

In the Ricci-negative case, the decomposition of a manifold into submanifolds is the same as that determined by σ alone, but the family $\mathcal{D}$ determines additional structure on each closed orbit. (Herein, for readability we often suppress notation denoting restriction to a curved orbit.)

Theorem $\mathbf{D}_-$. *Let $(M, \mathbf{c})$ be an oriented conformal structure of signature $(2,3)$. A holonomy reduction of $\mathbf{c}$ to $\mathrm{SU}(1,2)$ determines a 1-parameter family $\mathcal{D}$ of oriented $(2,3,5)$ distributions related by a Ricci-negative almost Einstein scale such that $\mathbf{c} = \mathbf{c_D}$ for all $\mathbf{D} \in \mathcal{D}$, as well as a decomposition $M = M_5^+ \cup M_5^- \cup M_4$:*

- *(Section 5.4) The orbits $M_5^\pm$ are open, and $M_5 := M_5^+ \cup M_5^-$ is equipped with a Ricci-negative Einstein metric $g := \sigma^{-2}\mathbf{g}|_{M_5}$. The pair $(-g, \xi)$ is a Sasaki structure (see Section 5.4.3) on M_5. Locally, M_5 fibers along the integral curves of ξ, and the leaf space L_4 inherits a Kähler–Einstein structure $(\hat{g}, \hat{K})$.*

- *(Section 5.5.3) The orbit M_4 is a smooth hypersurface, and inherits a Fefferman conformal structure $\mathbf{c_S}$, which has signature $(1,3)$: Locally, $\mathbf{c_S}$ arises from the classical Fefferman construction [16, 30, 51], which (in this dimension) canonically associates to any 3-dimensional CR structure $(L_3, \mathbf{H}, \mathbf{J})$ a conformal structure on a circle bundle over L_3. Again in the local setting, the fibers of the fibration $M_4 \to L_3$ are the integral curves of ξ.*

The Ricci-positive case is similar to the Ricci-negative case but entails 2-dimensional curved orbits that have no analogue there.

Theorem $\mathbf{D}_+$. *Let $(M, \mathbf{c})$ be an oriented conformal structure of signature $(2,3)$. A holonomy reduction of $\mathbf{c}$ to $\mathrm{SL}(3, \mathbb{R})$ determines a 1-parameter family $\mathcal{D}$ of oriented $(2,3,5)$ distributions related by a Ricci-positive almost Einstein scale such that $\mathbf{c} = \mathbf{c_D}$ for all $\mathbf{D} \in \mathcal{D}$, as well as a decomposition $M = M_5^+ \cup M_5^- \cup M_4 \cup M_2^+ \cup M_2^-$:*

- *(Section 5.4) The orbits $M_5^\pm$ are open, and $M_5 := M_5^+ \cup M_5^-$ is equipped with a Ricci-positive Einstein metric $g := \sigma^{-2}\mathbf{g}|_{M_5}$. The pair $(-g, \xi)$ is a para-Sasaki structure (see Section 5.4.3) on M_5. Locally, M_5 fibers along the integral curves of ξ, and the leaf space L_4 inherits a para-Kähler–Einstein structure $(\hat{g}, \hat{K})$.*

- (*Section* 5.5.4) *The orbit M_4 is a smooth hypersurface, and inherits a para-Fefferman conformal structure $\mathbf{c_S}$, which has signature $(2, 2)$: Locally, $\mathbf{c_S}$ arises from the paracomplex analogue of the classical Fefferman construction, which (in this dimension) canonically associates to any Legendrean contact structure $(L_3, \mathbf{H}_+ \oplus \mathbf{H}_-)$ – or, locally equivalently, a point equivalence class of second-order ODEs $\ddot{y} = F(x, y, \dot{y})$ – a conformal structure on a $\mathrm{SO}(1, 1)$-bundle over L_3. Again in the local setting, the fibers of the fibration $M_4 \to L_3$ are the integral curves of ξ.*

- (*Section* 5.6.1) *The orbits $M_2^\pm$ are 2-dimensional and inherit oriented projective structures.*

The descriptions of the geometric structures in the above two cases are complete in the sense that any other geometric data determined by the holonomy reduction to S can be recovered from the indicated data. We do not claim the same for the descriptions in the Ricci-flat case, which we can view as a sort of degenerate analogue of the other two cases.

Theorem $\mathbf{D}_0$. *Let $(M, \mathbf{c})$ be an oriented conformal structure of signature $(2, 3)$. A holonomy reduction of $\mathbf{c}$ to $\mathrm{SL}(2, \mathbb{R}) \ltimes Q_+$ determines a 1-parameter family $\mathcal{D}$ of oriented $(2, 3, 5)$ distributions related by a Ricci-flat almost Einstein scale such that $\mathbf{c} = \mathbf{c_D}$ for all $\mathbf{D} \in \mathcal{D}$, as well as a decomposition $M = M_5^+ \cup M_5^- \cup M_4 \cup M_2 \cup M_0^+ \cup M_0^-$.*

- (*Section* 5.4) *The orbits $M_5^\pm$ are open, and $M_5 := M_5^+ \cup M_5^-$ is equipped with a Ricci-flat metric $g := \sigma^{-2}\mathbf{g}|_{M_5}$. The pair $(-g, \xi)$ is a null-Sasaki structure (see Section 5.4.3) on M_5. Locally, M_5 fibers along the integral curves of ξ, and the leaf space L_4 inherits a null-Kähler–Einstein structure $(\hat{g}, \hat{K})$.*

- (*Section* 5.5.5) *The orbit M_4 is a smooth hypersurface, and it locally fibers over a 3-manifold $\widetilde{L}$ that carries a conformal structure $\mathbf{c}_{\widetilde{L}}$ of signature $(1, 2)$ and isotropic line field.*

- (*Section* 5.6.2) *The orbit M_2 has dimension 2 and is equipped with a preferred line field.*

- (–) *The orbits $M_0^\pm$ consist of isolated points and so carry no intrinsic geometry.*

The statements of these theorems involve an expository choice that entails some subtle consequences: In each case, the holonomy reduction determines an almost Einstein scale σ and 1-parameter family $\mathcal{D}$ of oriented $(2, 3, 5)$ distributions, but this reduction does not distinguish a distribution within this family. Alternatively, we could specify for $\mathbf{c}$ an almost Einstein scale and a distribution $\mathbf{D}$ such that $\mathbf{c} = \mathbf{c_D}$. Such a specification determines a holonomy reduction to S as above, but the choice of a preferred $\mathbf{D}$ is additional data, and this is reflected in the induced geometries on the curved orbits.

Proposition 5.18 gives a partial converse to the statements in Theorems D_- and D_+ about the ε-Sasaki–Einstein structures $(-g, \xi)$ induced on the open orbits by the corresponding holonomy reductions: Any ε-Sasaki–Einstein structure $(-g, \xi)$ (here restricting to $\varepsilon = \pm 1$) determines around each point a 1-parameter family of oriented $(2, 3, 5)$ distributions related by the almost Einstein scale for $[g]$ corresponding to g, and by construction the ε-Sasaki structure is the one induced by the corresponding holonomy reduction.

We also briefly present a generalized Fefferman construction that essentially inverts the projection $M_5 \to L_4$ along the leaf space fibration in the non-Ricci-flat cases, in a way that emphasizes the role of almost Einstein $(2, 3, 5)$ conformal structures (see Section 5.4.6). In particular, any non-Ricci-flat ε-Kähler–Einstein metric of signature $(2, 2)$ gives rise to a 1-parameter family of $(2, 3, 5)$ distributions. We treat this construction in more detail in an article currently in preparation [56].

As mentioned above we have for convenience formulated our results for oriented $(2, 3, 5)$ distributions and conformal structures, but all the results herein have analogues for unoriented distributions and many of our considerations are anyway local. Alternatively one could further

restrict attention to space- and time-oriented conformal structures (see Remark 4.11) or work with conformal spin structures, the latter of which would connect the considerations here more closely with those in [42].

Finally, we mention briefly one aspect of this geometry we do not discuss here, but which will be taken up in a shorter article currently in preparation: One can construct for any oriented $(2,3,5)$ distribution $\mathbf{D}$ an invariant second-order linear differential operator that acts on sections of $\mathcal{E}[1] \cong \Lambda^2 \mathbf{D}$ closely related to $\Theta_0^{\mathcal{V}}$ [55]. Its kernel can again be interpreted as the space of almost Einstein scales of $\mathbf{c_D}$, but it is a simpler object than $\Theta_0^{\mathcal{V}}$, enough so that one can use it to construct new explicit examples of almost Einstein $(2,3,5)$ distributions. Among other things, the existence of such an operator emphasizes that almost Einstein geometry of the induced conformal structure is a fundamental feature of the geometry of $(2,3,5)$ distributions.

For simplicity of statements of results, we assume that all given manifolds are connected; we do not include this hypothesis explicitly in our statements of results.

We use both index-free and Penrose index notation throughout, according to convenience.

2 Preliminaries

2.1 ε-complex structures

The ε-complex numbers, $\varepsilon \in \{-1, 0, +1\}$, is the ring $\mathbb{C}_\varepsilon$ generated over $\mathbb{R}$ by the generator i_ε, which satisfies precisely the relations generated by $i_\varepsilon^2 = \varepsilon$. An ε-complex structure on a real vector space $\mathbb{W}$ (necessarily of even dimension, say, $2m$) is an endomorphism $\mathbb{K} \in \operatorname{End}(\mathbb{W})$ such that $\mathbb{K}^2 = \varepsilon \operatorname{id}_{\mathbb{W}}$;[2] if $\varepsilon = +1$, we further require that the (± 1)-eigenspaces both have dimension m. This identifies $\mathbb{W}$ with $\mathbb{C}_\varepsilon^m$ (as a free $\mathbb{C}_\varepsilon$-module) so that the action of $\mathbb{K}$ coincides with multiplication by i_ε, and the pair $(\mathbb{W}, \mathbb{K})$ is an ε-complex vector space.

One specializes the names of structures to particular values of ε by omitting (-1)-, replacing $(+1)$- with the prefix *para-*, and replacing 0- with the modifier *null-*. See [59, Section 1].

2.2 The group $\mathbf{G_2}$

2.2.1 Split cross products in dimension 7

The geometry studied in this article depends critically on the algebraic features of a so-called (split) cross product $\times$ on a 7-dimensional real vector space $\mathbb{V}$. One can realize this explicitly using the algebra $\widetilde{\mathbb{O}}$ of split octonions; we follow [37, Section 2], and see also [54]. This is a composition $(\mathbb{R}\text{-})$algebra and so is equipped with a unit 1 and a nondegenerate quadratic form N multiplicative in the sense that $N(xy) = N(x)N(y)$ for all $x, y \in \widetilde{\mathbb{O}}$. In particular $N(1) = 1$, and polarizing N yields a nondegenerate symmetric bilinear form, which turns out for $\widetilde{\mathbb{O}}$ to have signature $(4, 4)$. So, the 7-dimensional vector subspace $\mathbb{I} = \langle 1 \rangle^\perp$ of *imaginary split octonions* inherits a nondegenerate symmetric bilinear form H of signature $(3, 4)$, as well as a map $\times \colon \mathbb{I} \times \mathbb{I} \to \mathbb{I}$ defined by

$$x \times y := xy + H(x, y)1;$$

this is just the orthogonal projection of xy onto $\mathbb{I}$. This map is a (binary) cross product in the sense of [11], that is, it satisfies

$$H(x \times y, x) = 0 \qquad \text{and} \qquad H(x \times y, x \times y) = H(x, x)H(y, y) - H(x, y)^2 \tag{2.1}$$

for all $x, y \in \mathbb{I}$.

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[2]In the case $\varepsilon = 0$, some references require additionally that a null-complex structure satisfy $\operatorname{rank} \mathbb{K} = m$ [27] this anyway holds for the null-complex structures that appear in this article.

Definition 2.1. We say that a bilinear map $\times \colon \mathbb{V} \times \mathbb{V} \to \mathbb{V}$ on a 7-dimensional real vector space $\mathbb{V}$ is a *split cross product* iff there is a linear isomorphism $A \colon \mathbb{I} \to \mathbb{V}$ such that $A(x \times y) = A(x) \times A(y)$.

A split cross product $\times$ determines a bilinear form

$$H_\times(x, y) := -\tfrac{1}{6} \operatorname{tr}(x \times (y \times \cdot));$$

of signature $(3, 4)$ on the underlying vector space. For the split cross product $\times$ on $\mathbb{I}$, $H_\times = H$. We say that a cross product $\times$ is *compatible* with a bilinear form iff $\times$ induces H, that is, iff $H = H_\times$. It follows from the alternativity identity $(xx)y = x(xy)$ satisfied by the split octonions that

$$x \times (x \times y) := -H_\times(x, x)y + H_\times(x, y)x. \tag{2.2}$$

By (2.1), $\times$ is totally $H_\times$-skew, so lowering its upper index with $H_\times$ yields a 3-form $\Phi \in \Lambda^3 \mathbb{V}^*$:

$$\Phi(x, y, z) := H_\times(x \times y, z).$$

A 3-form is said to be *split-generic* iff it arises this way, and such forms comprise an open $\operatorname{GL}(\mathbb{V})$-orbit under the standard action on $\Lambda^3 \mathbb{V}^*$. One can recover from any split-generic 3-form the split cross product $\times$ that induces it.

A split cross product $\times$ also determines a nonzero volume form $\epsilon_\times \in \Lambda^7 \mathbb{V}^*$ for $H_\times$:

$$(\epsilon_\times)_{ABCDEFG} := \tfrac{1}{42} \Phi_{K[AB} \Phi^K{}_{CD} \Phi_{EFG]}.$$

Thus, $\times$ determines an orientation $[\epsilon_\times]$ on $\mathbb{V}$ and Hodge star operators $* \colon \Lambda^k \mathbb{V}^* \to \Lambda^{7-k} \mathbb{V}^*$.

If $\mathbb{V}$ is a vector space endowed with a bilinear form H of signature (p, q) and an orientation Ω, the subgroup of $\operatorname{GL}(\mathbb{V})$ preserving the pair (H, Ω) is $\operatorname{SO}(p, q)$, so we refer to such a pair as a $\operatorname{SO}(p, q)$-structure on $\mathbb{V}$. We say that a cross product $\times$ on $\mathbb{V}$ is *compatible* with an $\operatorname{SO}(p, q)$-structure (H, Ω) iff $\times$ induces H and Ω, that is, iff $H = H_\times$ and $\Omega = [\epsilon_\times]$.

A split-generic 3-form Φ satisfies various contraction identities, including [12, equations (2.8) and (2.9)]:

$$\Phi^E{}_{AB} \Phi_{ECD} = (*_\Phi \Phi)_{ABCD} + H_{AC} H_{BD} - H_{AD} H_{BC}, \tag{2.3}$$

$$\Phi^F{}_{AB} (*_\Phi \Phi)_{FCDE} = 3(H_{A[C} \Phi_{DE]B} - H_{B[C} \Phi_{DE]A}). \tag{2.4}$$

2.2.2   The group $\mathbf{G_2}$

The (algebra) automorphism group of $\widetilde{\mathbb{O}}$ is a connected, split real form of the complex Lie group of type G_2, and so we denote it by G_2. One can recover the algebra structure of $\widetilde{\mathbb{O}}$ from $(\mathbb{I}, \times)$, so G_2 is also the automorphism group of $\times$, and equivalently, the stabilizer subgroup in $\operatorname{GL}(\mathbb{V})$ of a split-generic 3-form on a 7-dimensional real vector space $\mathbb{V}$. For much more about G_2, see [45].

The action of G_2 on $\mathbb{V}$ defines the smallest nontrivial, irreducible representation of G_2, which is sometimes called the *standard representation*. This action stabilizes a unique split cross product, or equivalently, a unique split-generic 3-form (up to a positive multiplicative constant). Thus, by a G_2-*structure* on a 7-dimensional real vector space $\mathbb{V}$ we mean either (1) a representation of G_2 on $\mathbb{V}$ isomorphic to the standard one, or, (2) slightly abusively (on account of the above multiplicative ambiguity), a split cross product $\times$ on $\mathbb{V}$, or equivalently, a split-generic 3-form Φ on $\mathbb{V}$.

Since a split cross product $\times$ on a vector space $\mathbb{V}$ determines algebraically both a bilinear form $H_\times$ and an orientation $[\epsilon_\times]$ on $\mathbb{V}$, the induced actions of G_2 preserve both, defining a natural embedding

$$\operatorname{G}_2 \hookrightarrow \operatorname{SO}(H_\times) \cong \operatorname{SO}(3, 4).$$

Moreover, $H_\times$ realizes $\mathbb{V}$ as the standard representation of $SO(3,4)$, and its restriction to G_2 is the standard representation $\times$ defines. Like $SO(3,4)$, the G_2-action on the ray projectivization of $\mathbb{V}$ has exactly three orbits, namely the sets of spacelike, isotropic, and timelike rays [65, Theorem 3.1].

It is convenient for our purposes to use the G_2-structure on $\mathbb{V}$ defined in a basis (E_a) (with dual basis, say, (e^a)) via the 3-form

$$\Phi := -e^{147} + \sqrt{2}e^{156} + \sqrt{2}e^{237} + e^{245} + e^{346}, \tag{2.5}$$

where $e^{a_1 \cdots a_k} := e^{a_1} \wedge \cdots \wedge e^{a_k}$ (cf. [41, equation (23)]). With respect to the basis (E_a), the induced bilinear form H_Φ has matrix representation

$$[H_\Phi] = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & I_2 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & I_2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{2.6}$$

where I_2 denotes the 2×2 identity matrix, and the induced volume form is

$$\epsilon_\Phi = -e^{1234567}.$$

Let $\mathfrak{g}_2$ denote the Lie algebra of G_2. Differentiating the inclusion $G_2 \hookrightarrow GL(\mathbb{V})$ yields a Lie algebra representation $\mathfrak{g}_2 \hookrightarrow \mathfrak{gl}(\mathbb{V}) \cong \mathrm{End}(\mathbb{V})$, and with respect to the basis (E_a) its elements are precisely those of the form

$$\begin{pmatrix} \operatorname{tr} A & Z & s & W^\top & 0 \\ X & A & \sqrt{2}JZ^\top & \frac{s}{\sqrt{2}}J & -W \\ r & -\sqrt{2}X^\top J & 0 & -\sqrt{2}ZJ & s \\ Y^\top & -\frac{r}{\sqrt{2}}J & \sqrt{2}JX & -A^\top & -Z^\top \\ 0 & -Y & r & -X^\top & -\operatorname{tr} A \end{pmatrix}, \tag{2.7}$$

where $A \in \mathfrak{gl}(2,\mathbb{R})$, $W, X \in \mathbb{R}^2$, $Y, Z \in (\mathbb{R}^2)^*$, $r, s \in \mathbb{R}$, and $J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

2.2.3 Some G_2 representation theory

Fix a 7-dimensional real vector space $\mathbb{V}$ and a G_2-structure $\Phi \in \Lambda^3\mathbb{V}^*$. We briefly record the decompositions of the G_2-representations $\Lambda^2\mathbb{V}^*$, $\Lambda^3\mathbb{V}^*$, and $S^2\mathbb{V}^*$ into irreducible subrepresentations; we use and extend the notation of [12, Section 2.6]. In each case, Λ_l^k denotes the irreducible subrepresentation of $\Lambda^k\mathbb{V}^*$ of dimension l, which is unique up to isomorphism.

The representation $\Lambda^2\mathbb{V}^* \cong \mathfrak{so}(H_\Phi)$ decomposes into irreducible subrepresentations as

$$\Lambda^2\mathbb{V}^* = \Lambda_7^2 \oplus \Lambda_{14}^2;$$

$\Lambda_7^2 \cong \mathbb{V}$ and Λ_{14}^2 is isomorphic to the adjoint representation $\mathfrak{g}_2$. Define the map $\iota_7^2 \colon \mathbb{V} \to \Lambda^2\mathbb{V}^*$ by

$$\iota_7^2 \colon \quad \mathbb{S}^A \mapsto \mathbb{S}^C \Phi_{CAB}; \tag{2.8}$$

by raising an index we can view ι_7^2 as the map $\mathbb{V} \to \mathfrak{so}(H_\Phi)$ given by $\mathbb{S} \mapsto (\mathbb{T} \mapsto \mathbb{T} \times \mathbb{S})$. It is evidently nontrivial, so by Schur's lemma its (isomorphic) image is Λ_7^2.

Conversely, consider the map $\pi_7^2 \colon \Lambda^2\mathbb{V}^* \to \mathbb{V}$ defined by

$$\pi_7^2 \colon \quad \mathbb{A}_{AB} \mapsto \tfrac{1}{6}\mathbb{A}_{BC}\Phi^{BCA}. \tag{2.9}$$

Raising indices gives a map $\Lambda^2 \mathbb{V} \to \mathbb{V}$, which up to the multiplicative constant, is the descent of $\times \colon \mathbb{V} \times \mathbb{V} \to \mathbb{V}$ via the wedge product. In particular it is nontrivial, so it has kernel Λ^2_{14} and restricts to an isomorphism $\pi^2_7|_{\Lambda^2_7} \colon \Lambda^2_7 \to \mathbb{V}$; we have chosen the coefficient so that $\pi^2_7 \circ \iota^2_7 = \mathrm{id}_\mathbb{V}$ and $\iota^2_7 \circ \pi^2_7|_{\Lambda^2_7} = \mathrm{id}_{\Lambda^2_7}$. Since G_2 is the stabilizer subgroup in $\mathrm{SO}(H_\Phi)$ of Φ, $\mathfrak{g}_2$ is the annihilator in $\mathfrak{so}(H_\Phi) \cong \Lambda^2 \mathbb{V}^*$ of Φ. Expanding $\mathrm{id}_\mathbb{V} - \iota^2_7 \circ \pi^2_7$ using (2.8) and (2.9) and applying (2.3) gives that under this identification, the corresponding projection $\pi^2_{14} \colon \mathfrak{so}(\mathbb{V}) \cong \Lambda^2 \mathbb{V}^* \to \mathfrak{g}_2$ is

$$\pi^2_{14} \colon \ \mathbb{A}^A{}_B \mapsto \tfrac{2}{3} \mathbb{A}^A{}_B - \tfrac{1}{6} (*_\Phi \Phi)_D{}^{EA}{}_B \mathbb{A}^D{}_E.$$

The representation $\Lambda^3 \mathbb{V}^*$ decomposes into irreducible subrepresentations as

$$\Lambda^3 \mathbb{V}^* = \Lambda^3_1 \oplus \Lambda^3_7 \oplus \Lambda^3_{27}.$$

Here, Λ^3_1 is just the trivial representation spanned by Φ, and the map $\pi^3_1 \colon \Lambda^3 \mathbb{V}^* \to \mathbb{R}$ defined by

$$\pi^3_1 \colon \ \Psi_{ABC} \mapsto \tfrac{1}{42} \Phi^{ABC} \Psi_{ABC} \tag{2.10}$$

is a left inverse for the map $\mathbb{R} \xrightarrow{\cong} \Lambda^3_1 \hookrightarrow \Lambda^3 \mathbb{V}^*$ defined by $a \mapsto a\Phi$.

The map $\iota^3_7 \colon \mathbb{V} \to \Lambda^3 \mathbb{V}^*$ defined by

$$\iota^3_7 \colon \ \mathbb{S}^A \mapsto -\mathbb{S}^D (*_\Phi \Phi)_{DABC} = [*_\Phi(\mathbb{S} \wedge \Phi)]_{ABC}.$$

is nonzero, so it defines an isomorphism $\mathbb{V} \cong \Lambda^3_7$. The map $\pi^3_7 \colon \Lambda^3 \mathbb{V}^* \to \mathbb{V}$ defined by

$$\pi^3_7 \colon \ \Psi_{ABC} \mapsto \tfrac{1}{24} (*_\Phi \Phi)^{BCDA} \Psi_{BCD} = \tfrac{1}{4} [*_\Phi (\Phi \wedge \Psi)]^A \tag{2.11}$$

is scaled so that $\pi^3_7 \circ \iota^3_7 = \mathrm{id}_\mathbb{V}$ and $\iota^3_7 \circ \pi^3_7 = \mathrm{id}_{\Lambda^3_7}$.

The G_2-representation $S^2 \mathbb{V}^*$ decomposes into irreducible modules as $\mathbb{R} \oplus S^2_\circ \mathbb{V}^*$, namely, into its H_Φ-trace and H_Φ-tracefree components, respectively. The linear map $i \colon S^2 \mathbb{V}^* \to \Lambda^3 \mathbb{V}^*$ defined by

$$i \colon \ \mathbb{A}_{AB} \mapsto 6\Phi^D{}_{[AB} \mathbb{A}_{C]D}. \tag{2.12}$$

satisfies $i(H_\Phi) = 6\Phi$ and is nonzero on $S^2_\circ \mathbb{V}^*$, so $i|_{S^2_\circ}$ is an isomorphism $S^2_\circ \xrightarrow{\cong} \Lambda^3_{27}$.

The projection $\pi^3_{27} \colon \Lambda^3 \mathbb{V}^* \to S^2_\circ \mathbb{V}^*$

$$\pi^3_{27} \colon \ \Psi_{ABC} \mapsto -\tfrac{1}{8} *_\Phi [(\cdot \lrcorner \Phi) \wedge (\cdot \lrcorner \Phi) \wedge \Psi]_{AB} - \tfrac{3}{4} \pi^3_1(\Psi) H_{AB} \tag{2.13}$$

is scaled so that $\pi^3_{27} \circ i|_{S^2_\circ \mathbb{V}^*} = \mathrm{id}_{S^2_\circ \mathbb{V}^*}$ and $i \circ \pi^3_{27}|_{\Lambda^3_{27}} = \mathrm{id}_{\Lambda^3_{27}}$.

2.3 Cartan and parabolic geometry

2.3.1 Cartan geometry

In this subsubsection we follow [18, 61]. Given a P-principal bundle $\pi \colon \mathcal{G} \to M$, we denote the (right) action of $\mathcal{G} \times P \to \mathcal{G}$ by $R^p(u) = u \cdot p$ for $u \in \mathcal{G}, p \in P$. For each $V \in \mathfrak{p}$, the corresponding *fundamental vector field* $\eta_V \in \Gamma(T\mathcal{G})$ is $(\eta_V)_u := \partial_t|_0 [u \cdot \exp(tV)]$.

Definition 2.2. For a Lie group G and a closed subgroup P (with respective Lie algebras $\mathfrak{g}$ and $\mathfrak{p}$), a *Cartan geometry* of type (G, P) on a manifold M is a pair $(\mathcal{G} \to M, \omega)$, where $\mathcal{G} \to M$ is a P-principal bundle and ω is a *Cartan connection*, that is, a section of $T^* \mathcal{G} \otimes \mathfrak{g}$ satisfying

1) (right equivariance) $\omega_{u \cdot p}(T_u R^p \cdot \eta) = \mathrm{Ad}(p^{-1})(\omega_u(\eta))$ for all $u \in \mathcal{G}, p \in P, \eta \in T_u \mathcal{G}$,

2) (reproduction of fundamental vector fields) $\omega(\eta_V) = V$ for all $V \in \mathfrak{p}$, and

3) (absolute parallelism) $\omega_u \colon T_u \mathcal{G} \to \mathfrak{g}$ is an isomorphism for all $u \in \mathcal{G}$.

The *(flat) model* of Cartan geometry of type (G, P) is the pair $(G \to G/P, \omega_{\mathrm{MC}})$, where ω_{MC} is the Maurer–Cartan form on G defined by $(\omega_{\mathrm{MC}})_u := T_u L_{u^{-1}}$ (here $L_{u^{-1}} \colon G \to G$ denotes left

multiplication by u^{-1}). This form satisfies the identity $d\omega_{\mathrm{MC}} + \frac{1}{2}[\omega_{\mathrm{MC}}, \omega_{\mathrm{MC}}] = 0$. We define the *curvature (form)* of a Cartan geometry $(\mathcal{G}, \omega)$ to be the section $\Omega := d\omega + \frac{1}{2}[\omega, \omega] \in \Gamma(\Lambda^2 T^* \mathcal{G} \otimes \mathfrak{g})$, and say that $(\mathcal{G}, \omega)$ is *flat* iff $\Omega = 0$. This is the case iff around any point $u \in \mathcal{G}$ there is a local bundle isomorphism between G and $\mathcal{G}$ that pulls back ω to ω_{MC}.

One can show that the curvature Ω of any Cartan geometry $(\mathcal{G} \to M, \omega)$ is horizontal (it is annihilated by vertical vector fields), so invoking the absolute parallelism and passing to the quotient defines an equivalent object $\kappa \colon \mathcal{G} \to \Lambda^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}$, which we also call the *curvature*.

2.3.2   Holonomy

Given any Cartan geometry $(\mathcal{G} \to M, \omega)$ of type (G, P), we can extend the Cartan connection ω to a unique principal connection $\hat{\omega}$ on $\hat{\mathcal{G}} := \mathcal{G} \times_P G$ characterized by (1) G-equivariance and (2) $\iota^* \hat{\omega} = \omega$, where $\iota \colon \mathcal{G} \hookrightarrow \hat{\mathcal{G}}$ is the natural inclusion $u \mapsto [u, e]$. Then, to any point $\hat{u} \in \hat{\mathcal{G}}$ we can associate the holonomy group $\mathrm{Hol}_{\hat{u}}(\hat{\omega}) \leq G$. Different choices of $\hat{u}$ lead to conjugate subgroups of G, so the conjugacy class $\mathrm{Hol}(\hat{\omega})$ thereof is independent of $\hat{u}$, and we define the *holonomy* of ω (or just as well, of $(\mathcal{G} \to M, \omega)$) to be this class.

2.3.3   Tractor geometry

Fix a pair (G, P) as in Section 2.3.1, denote the Lie algebra of G by $\mathfrak{g}$, and fix a G-representation $\mathbb{U}$. Then, for any Cartan geometry $(\pi \colon \mathcal{G} \to M, \omega)$ of type (G, P), we can form the associated *tractor bundle* $\mathcal{U} := \mathcal{G} \times_P \mathbb{U} \to M$, which we can also view as the associated bundle $\hat{\mathcal{G}} \times_G \mathbb{U} \to M$. Then, the principal connection $\hat{\omega}$ on $\hat{\mathcal{G}}$ determined by ω induces a vector bundle connection $\nabla^{\mathcal{U}}$ on $\mathcal{U}$.

Of distinguished importance is the *adjoint tractor bundle* $\mathcal{A} := \mathcal{G} \times_P \mathfrak{g}$. The canonical map $\Pi_0^{\mathcal{A}} \colon \mathcal{A} \to TM$ defined by $(u, V) \mapsto T_u \pi \cdot \omega_u^{-1}(V)$ descends to a natural isomorphism $\mathcal{G} \times_P (\mathfrak{g}/\mathfrak{p}) \overset{\cong}{\to} TM$, and via this identification $\Pi_0^{\mathcal{A}}$ is the bundle map associated to the canonical projection $\mathfrak{g} \to \mathfrak{g}/\mathfrak{p}$.

Since the curvature Ω of $(\mathcal{G}, \omega)$ is also P-equivariant, we may regard it as a section $K \in \Gamma(\Lambda^2 T^* M \otimes \mathcal{A})$, and again we call it the *curvature* of $(\mathcal{G} \to M, \omega)$.

2.3.4   Parabolic geometry

In this article we will mostly (but not exclusively) work with geometries that can be realized as a special class of Cartan geometries that enjoy additional properties, most importantly suitable normalization conditions on ω that guarantee (subject to a usually satisfied cohomological condition) a correspondence between Cartan geometries satisfying those conditions and geometric structures on the underlying manifold. We say that a Cartan geometry $(\mathcal{G} \to M, \omega)$ of type (G, P) is a *parabolic geometry* iff G is semisimple and P is a parabolic subgroup. For a detailed survey of parabolic geometry, including details of the below, see the standard reference [18].

Recall that a parabolic subgroup $P < G$ determines a so-called $|k|$-grading on the Lie algebra $\mathfrak{g}$ of G: This is a vector space decomposition $\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{+k}$ compatible with the Lie bracket in the sense that $[\mathfrak{g}_a, \mathfrak{g}_b] \subseteq \mathfrak{g}_{a+b}$ and minimal in the sense that none of the summands $\mathfrak{g}_a$, $a = -k, \ldots, k$, is zero. The grading induces a P-invariant filtration $(\mathfrak{g}^a)$ of $\mathfrak{g}$, where $\mathfrak{g}^a := \mathfrak{g}_a \oplus \cdots \oplus \mathfrak{g}_{+k}$. In particular, $\mathfrak{p} = \mathfrak{g}^0 = \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_{+k}$. We denote by $G_0 < P$ the subgroup of elements $p \in P$ for which $\mathrm{Ad}(g)$ preserves the grading $(\mathfrak{g}_a)$ of $\mathfrak{g}$, and by $P_+ < P$ the subgroup of elements $p \in P$ for which $\mathrm{Ad}(p) \in \mathrm{End}(\mathfrak{g})$ have homogeneity of degree > 0 with respect to the filtration $(\mathfrak{g}^a)$; in particular, the Lie algebra of P_+ is $\mathfrak{p}_+ = \mathfrak{g}^{+1} = \mathfrak{g}_{+1} \oplus \cdots \oplus \mathfrak{g}_{+k}$.

Since $\mathfrak{g}$ is semisimple, its Killing form is nondegenerate, and it induces a P-equivariant identification $(\mathfrak{g}/\mathfrak{p})^* \leftrightarrow \mathfrak{p}_+$. Via this identification, for any G-representation $\mathbb{U}$ we may identify

the Lie algebra homology $H_\bullet(\mathfrak{p}_+, \mathbb{U})$ with the chain complex

$$\cdots \to \Lambda^{i+1}(\mathfrak{g}/\mathfrak{p})^* \otimes \mathbb{U} \xrightarrow{\partial^*} \Lambda^i(\mathfrak{g}/\mathfrak{p})^* \otimes \mathbb{U} \to \cdots .$$

The *Kostant codifferential* ∂^* is P-equivariant, so it induces bundle maps $\partial^* \colon \Lambda^{i+1}T^*M \otimes \mathcal{U} \to \Lambda^i T^*M \otimes \mathcal{U}$ between the associated bundles.

The normalization conditions for a parabolic geometry $(\mathcal{G} \to M, \omega)$ are that

1) (normality) the curvature κ satisfies $\partial^*\kappa = 0$, and

2) (regularity) the curvature κ satisfies $\kappa(u)(\mathfrak{g}^i, \mathfrak{g}^j) \subseteq \mathfrak{g}^{i+j+1}$ for all $u \in \mathcal{G}$ and all i, j.

Finally, tractor bundles associated to parabolic geometries inherit additional natural structure: Given a G-representation $\mathbb{U}$, P determines a natural filtration $(\mathbb{U}^a)$ of $\mathbb{U}$ by successive action of the nilpotent Lie subalgebra $\mathfrak{p}_+ < \mathfrak{g}$, namely

$$\mathbb{U} \supseteq \mathfrak{p}_+ \cdot \mathbb{U} \supseteq \mathfrak{p}_+ \cdot (\mathfrak{p}_+ \cdot \mathbb{U}) \supseteq \cdots \supseteq \{0\}. \tag{2.14}$$

Since the filtration $(\mathbb{U}^a)$ of $\mathbb{U}$ is P-invariant, it determines a bundle filtration $(\mathcal{U}^a)$ of the tractor bundle $\mathcal{U} = \mathcal{G} \times_P \mathbb{U}$.

For the adjoint representation $\mathfrak{g}$ itself, this filtration (appropriately indexed) is just $(\mathfrak{g}^a)$, and the images of the filtrands $\mathcal{A} = \mathcal{G} \times_P \mathfrak{g}^{-k} \supsetneq \cdots \supsetneq \mathcal{G} \times_P \mathfrak{g}^{-1}$ under the projection $\Pi_0^{\mathcal{A}}$ comprise a canonical filtration $TM = T^{-k}M \supsetneq \cdots \supsetneq T^{-1}M$ of the tangent bundle.

2.3.5   Oriented conformal structures

The group $\mathrm{SO}(p+1, q+1)$, $p+q \geq 3$, acts transitively on the space of isotropic rays in the standard representation $\mathbb{V}$, and the stabilizer subgroup $\bar{P}$ of such a ray is parabolic. There is an equivalence of categories between regular, normal parabolic geometries of type $(\mathrm{SO}(p+1, q+1), \bar{P})$ and oriented conformal structures of signature (p, q) [18, Section 4.1.2].

Definition 2.3. A *conformal structure* $(M, \mathbf{c})$ is an equivalence class $\mathbf{c}$ of metrics on M, where we declare two metrics to be equivalent if one is a positive, smooth multiple of the other. The signature of $\mathbf{c}$ is the signature of any (equivalently, every) $g \in \mathbf{c}$, and we say that $(M, \mathbf{c})$ is *oriented* iff M is oriented. The *conformal holonomy* of an oriented conformal structure $\mathbf{c}$ is $\mathrm{Hol}(\mathbf{c}) := \mathrm{Hol}(\omega)$, where ω is the normal Cartan connection corresponding to $\mathbf{c}$.

We can choose a basis of $\mathbb{V}$ for which the nondegenerate, symmetric bilinear form H preserved by $\mathrm{SO}(p+1, q+1)$ has block matrix representation

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & \Sigma & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{2.15}$$

(With respect to the basis (E_a), the matrix representation $[H_\Phi]$ (2.6) of the bilinear form H_Φ determined by the explicit expression (2.5) for Φ has the form (2.15).) The Lie algebra $\mathfrak{so}(p+1, q+1)$ consists of exactly the elements

$$\begin{pmatrix} b & Z & 0 \\ X & B & -\Sigma^{-1}Z^\top \\ 0 & -X^\top\Sigma & -b \end{pmatrix},$$

where $B \in \mathfrak{so}(\Sigma)$, $X \in \mathbb{R}^{p+q}$, $Z \in (\mathbb{R}^{p+q})^*$. The first element of the basis is isotropic, and if we take choose the preferred isotropic ray in $\mathbb{V}$ to be the one determined by that element, the corresponding Lie algebra grading on $\mathfrak{so}(p+1, q+1)$ is the one defined by the labeling

$$\begin{pmatrix} \mathfrak{g}_0 & \mathfrak{g}_{+1} & 0 \\ \mathfrak{g}_{-1} & \mathfrak{g}_0 & \mathfrak{g}_{+1} \\ 0 & \mathfrak{g}_{-1} & \mathfrak{g}_0 \end{pmatrix}. \tag{2.16}$$

Since the grading on $\mathfrak{so}(p+1, q+1)$ induced by $\bar{P}$ has the form $\mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{+1}$, any parabolic geometry of this type is regular. The normality condition coincides with Cartan's normalization condition for what is now called a Cartan geometry of this type [18, Section 4.1.2].

2.3.6 Oriented $(2,3,5)$ distributions

The group G_2 acts transitively on the space of H_Φ-isotropic rays in $\mathbb{V}$, and the stabilizer subgroup Q of such a ray is parabolic [54]. The subgroup Q is the intersection of G_2 with the stabilizer subgroup $\bar{P} < SO(3,4)$ of the preferred isotropic ray in Section 2.3.5. In particular, the first basis element is isotropic, and if we again choose the preferred isotropic ray to be the one determined by that element, the corresponding Lie algebra grading on $\mathfrak{g}_2$ is the one defined by the block decomposition (2.7) and the labeling

$$\begin{pmatrix} \mathfrak{g}_0 & \mathfrak{g}_{+1} & \mathfrak{g}_{+2} & \mathfrak{g}_{+3} & 0 \\ \mathfrak{g}_{-1} & \mathfrak{g}_0 & \mathfrak{g}_{+1} & \mathfrak{g}_{+2} & \mathfrak{g}_{+3} \\ \mathfrak{g}_{-2} & \mathfrak{g}_{-1} & 0 & \mathfrak{g}_{+1} & \mathfrak{g}_{+2} \\ \mathfrak{g}_{-3} & \mathfrak{g}_{-2} & \mathfrak{g}_{-1} & \mathfrak{g}_0 & \mathfrak{g}_{+1} \\ 0 & \mathfrak{g}_{-3} & \mathfrak{g}_{-2} & \mathfrak{g}_{-1} & \mathfrak{g}_0 \end{pmatrix}.$$

There is an equivalence of categories between regular, normal parabolic geometries of type (G_2, Q) and so-called oriented $(2,3,5)$ distributions [18, Section 4.3.2].

On a manifold M, define the bracket of distributions $\mathbf{E}, \mathbf{F} \subseteq TM$ to be the set $[\mathbf{E}, \mathbf{F}] := \{[\alpha, \beta]_x : x \in M; \alpha \in \Gamma(\mathbf{E}), \beta \in \Gamma(\mathbf{F})\} \subseteq TM$.

Definition 2.4. A $(2,3,5)$ *distribution* is a 2-plane distribution $\mathbf{D}$ on a 5-manifold M that is maximally nonintegrable in the sense that (1) $[\mathbf{D}, \mathbf{D}]$ is a 3-plane distribution, and (2) $[\mathbf{D}, [\mathbf{D}, \mathbf{D}]] = TM$. A $(2,3,5)$ distribution is *oriented* iff the bundle $\mathbf{D} \to M$ is oriented.

An orientation of $\mathbf{D}$ determines an orientation of M and vice versa. The appropriate restrictions of the Lie bracket of vector fields descend to natural vector bundle isomorphisms $\mathcal{L} \colon \Lambda^2 \mathbf{D} \xrightarrow{\cong} [\mathbf{D}, \mathbf{D}]/\mathbf{D}$ and $\mathcal{L} \colon \mathbf{D} \otimes ([\mathbf{D}, \mathbf{D}]/\mathbf{D}) \xrightarrow{\cong} TM/[\mathbf{D}, \mathbf{D}]$; these are components of the *Levi bracket*.

For a regular, normal parabolic geometry $(\mathcal{G}, \omega)$ of type (G_2, Q), the underlying $(2,3,5)$ distribution $\mathbf{D}$ is $T^{-1}M = \mathcal{G} \times_Q (\mathfrak{g}^{-1}/\mathfrak{q})$, and $[\mathbf{D}, \mathbf{D}]$ is $T^{-2}M = \mathcal{G} \times_Q (\mathfrak{g}^{-2}/\mathfrak{q})$.

2.4 Conformal geometry

In this subsection we partly follow [4].

2.4.1 Conformal density bundles

A conformal structure $(M, \mathbf{c})$ of signature (p, q) (denote $n := p+q$), determines a family of natural (*conformal*) *density bundles* on M: Denote by $\mathcal{E}[1]$ the positive $(2n)$th root of the canonically oriented line bundle $(\Lambda^n TM)^2$, and its respective wth integer powers by $\mathcal{E}[w]$; $\mathcal{E} := \mathcal{E}[0]$ is the trivial bundle with fiber $\mathbb{R}$, and there are natural identifications $\mathcal{E}[w] \otimes \mathcal{E}[w'] \cong \mathcal{E}[w+w']$. Given

any vector bundle $B \to M$, we denote $B[w] := B \otimes \mathcal{E}[w]$, and refer to the sections of $B[w]$ as *sections of B of conformal weight w*.

We may view $\mathbf{c}$ itself as the canonical *conformal metric*, $\mathbf{g}_{ab} \in \Gamma(S^2 T^* M[2])$. Contraction with $\mathbf{g}_{ab}$ determines an isomorphism $TM \to T^* M[2]$, which we may use to raise and lower indices of objects on the tangent bundle at the cost of an adjustment of conformal weight. By construction, the Levi-Civita connection ∇^g of any metric $g \in \mathbf{c}$ preserves $\mathbf{g}_{ab}$ and its inverse, $\mathbf{g}^{ab} \in \Gamma(S^2 TM[-2])$.

We call a nowhere zero section $\tau \in \Gamma(\mathcal{E}[1])$ a *scale* of $\mathbf{c}$. A scale determines *trivializations* $B[w] \overset{\cong}{\to} B$, $b \mapsto \underline{b} := \tau^{-w} b$, of all conformally weighted bundles, and in particular a representative metric $\tau^{-2} \mathbf{g} \in \mathbf{c}$.

2.4.2   Conformal tractor calculus

For an oriented conformal structure $(M, \mathbf{c})$ of signature (p, q), $n := p + q \geq 3$, the tractor bundle $\mathcal{V}$ associated to the standard representation $\mathbb{V}$ of $\mathrm{SO}(p+1, q+1)$ is the *standard tractor bundle*. It inherits from the normal parabolic geometry corresponding to $\mathbf{c}$ a vector bundle connection $\nabla^{\mathcal{V}}$. The $\mathrm{SO}(p+1, q+1)$-action preserves a canonical nondegenerate, symmetric bilinear form $H \in S^2 \mathbb{V}^*$ and a volume form $\epsilon \in \Lambda^{n+2} \mathbb{V}^*$; these respectively induce on $\mathcal{V}$ a parallel *tractor metric* $H \in \Gamma(S^2 \mathcal{V}^*)$ and parallel volume form $\epsilon \in \Gamma(\Lambda^{n+2} \mathcal{V}^*)$.

Consulting the block structure (2.16) of $\bar{\mathfrak{p}}_+ < \mathfrak{so}(p+1, q+1)$ gives that the filtration (2.14) of the standard representation $\mathbb{V}$ of $\mathrm{SO}(p+1, q+1)$ determined by $\bar{P}$ is

$$\left\{ \begin{pmatrix} * \\ * \\ * \end{pmatrix} \right\} \supset \left\{ \begin{pmatrix} * \\ * \\ 0 \end{pmatrix} \right\} \supset \left\{ \begin{pmatrix} * \\ 0 \\ 0 \end{pmatrix} \right\} \supset \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}. \tag{2.17}$$

We may identify the composition series of the corresponding filtration of $\mathcal{V}$ as

$$\mathcal{V} \cong \mathcal{E}[1] \oplus\!\!\!+ \, TM[-1] \oplus\!\!\!+ \, \mathcal{E}[-1].$$

We denote elements and sections of $\mathcal{V}$ using uppercase Latin indices, $A, B, C, \ldots$, as $\mathbb{S}^A \in \Gamma(\mathcal{V})$, and those of the dual bundle $\mathcal{V}^*$ with lower indices, as $\mathbb{S}_A \in \Gamma(\mathcal{V}^*)$; we freely raise and lower indices using H. The bundle inclusion $\mathcal{E}[-1] \hookrightarrow \mathcal{V}$ determines a canonical section $X^A \in \Gamma(\mathcal{V}[1])$.

Any scale τ determines an identification of $\mathcal{V}$ with the associated graded bundle determined by the above filtration, that is, an isomorphism $\mathcal{V} \cong \mathcal{E}[1] \oplus TM[-1] \oplus \mathcal{E}[-1]$ [4]. So, τ also determines (non-invariant, that is, scale-dependent) inclusions $TM[-1] \hookrightarrow \mathcal{V}$ and $\mathcal{E}[1] \hookrightarrow \mathcal{V}$, which we can respectively regard as sections $Z^A{}_a \in \Gamma(\mathcal{V} \otimes T^* M[1])$ and $Y^A \in \Gamma(\mathcal{V}[-1])$. So, for any choice of τ we can decompose a section $\mathbb{S} \in \Gamma(\mathcal{V})$ uniquely as $\mathbb{S}^A \overset{\tau}{=} \sigma Y^A + \mu^a Z^A{}_a + \rho X^A$, where the notation $\overset{\tau}{=}$ indicates that Y^A and $Z^A{}_a$ are the inclusions determined by τ. Reusing the notation of the filtration of $\mathbb{V}$ we write

$$\mathbb{S}^A \overset{\tau}{=} \begin{pmatrix} \rho \\ \mu^a \\ \sigma \end{pmatrix}. \tag{2.18}$$

With respect to any scale τ, the tractor metric has the form (cf. (2.15))

$$H_{AB} \overset{\tau}{=} \begin{pmatrix} 0 & 0 & 1 \\ 0 & \mathbf{g}_{ab} & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

In particular, the filtration (2.17) of $\mathcal{V}$ is $\mathcal{V} \supset \langle X \rangle^{\perp} \supset \langle X \rangle \supset \{0\}$.

The normal tractor connection $\nabla^{\mathcal{V}}$ on $\mathcal{V}$ is [4]

$$\nabla^{\mathcal{V}}_b \begin{pmatrix} \rho \\ \mu^a \\ \sigma \end{pmatrix} \overset{\tau}{=} \begin{pmatrix} \rho_{,b} - \mathsf{P}_{bc}\mu^c \\ \mu^a{}_{,b} + \mathsf{P}^a{}_b\sigma + \delta^a{}_b\rho \\ \sigma_{,b} - \mu_b \end{pmatrix} \in \Gamma\left(\begin{pmatrix} \mathcal{E}[-1] \\ TM[-1] \\ \mathcal{E}[1] \end{pmatrix} \otimes T^*M \right).$$

The subscript $_{,b}$ denotes the covariant derivative with respect to $g := \tau^{-2}\mathbf{g}$, and P_{ab} is the *Schouten tensor* of g, which is a particular trace adjustment of the Ricci tensor R_{ab}:

$$\mathsf{P}_{ab} := \frac{1}{n-2}\left(R_{ab} - \frac{1}{2(n-1)}R^c{}_c g_{ab} \right). \tag{2.19}$$

A section $\mathbb{A}_{A_1\cdots A_k}$ of the tractor bundle $\Lambda^k\mathcal{V}^*$ associated to the alternating representation $\Lambda^k\mathbb{V}^*$ decomposes uniquely as

$$\mathbb{A}_{A_1\cdots A_k} \overset{\tau}{=} k\phi_{a_2\cdots a_k}Y_{[A_1}Z_{A_2}{}^{a_2}\cdots Z_{A_k]}{}^{a_k} + \chi_{a_1\cdots a_k}Z_{[A_1}{}^{a_1}\cdots Z_{A_k]}{}^{a_k}$$
$$+ k(k-1)\theta_{a_3\cdots a_k}Y_{[A_1}X_{A_2}Z_{A_3}{}^{a_3}\cdots Z_{A_k]}{}^{a_k} + k\psi_{a_2\cdots a_k}X_{[A_1}Z_{A_2}{}^{a_2}\cdots Z_{A_k]}{}^{a_k},$$

which we write more compactly as

$$\mathbb{A}_{A_1\cdots A_k} \overset{\tau}{=} \begin{pmatrix} \psi_{a_2\cdots a_k} \\ \chi_{a_1\cdots a_k} \mid \theta_{a_3\cdots a_k} \\ \phi_{a_2\cdots a_k} \end{pmatrix} \in \Gamma\left(\begin{matrix} \Lambda^{k-1}T^*M[k-2] \\ \Lambda^k T^*M[k] \quad \mid \Lambda^{k-2}T^*M[k-2] \\ \Lambda^{k-1}T^*M[k] \end{matrix} \right).$$

The tractor connection $\nabla^{\mathcal{V}}$ induces a connection on $\Lambda^k\mathcal{V}^*$, and we denote this connection again by $\nabla^{\mathcal{V}}$.

In the special case $k = 2$, raising an index using H gives $\Lambda^2\mathbb{V}^* \cong \mathfrak{so}(p+1,q+1)$, so we can identify $\Lambda^2\mathcal{V}^* \cong \mathcal{A}$. Any section $\mathbb{A}^A{}_B \in \Gamma(\mathcal{A})$ decomposes uniquely as

$$\mathbb{A}^A{}_B = \xi^a\left(Y^A Z_{Ba} - Z^A{}_a Y_B\right) + \zeta^a{}_b Z^A{}_a Z_B{}^b$$
$$+ \alpha\left(Y^A X_B - X^A Y_B\right) + \nu_b\left(X^A Z_B{}^b - Z^{Ab}X_B\right),$$

which we write as

$$\mathbb{A}^A{}_B \overset{\tau}{=} \begin{pmatrix} \nu_b \\ \zeta^a{}_b \mid \alpha \\ \xi^b \end{pmatrix} \in \Gamma\left(\begin{matrix} T^*M \\ \mathrm{End}_{\mathrm{skew}}(TM) \mid \quad \mathcal{E} \\ TM \end{matrix} \right).$$

Finally, $\bar{\mathfrak{p}}_+$ annihilates $\Lambda^{n+2}\mathbb{V}^* \cong \mathbb{R}$, yielding a natural bundle isomorphism $\Lambda^{n+2}\mathcal{V}^* \cong \Lambda^n T^*M[n]$. This identifies the tractor volume form ϵ with the conformal volume form $\epsilon_{\mathbf{g}}$ of $\mathbf{g}$.

2.4.3 Canonical quotients of conformal tractor bundles

For any irreducible $\mathrm{SO}(p+1,q+1)$-representation $\mathbb{U}$, the canonical Lie algebra cohomology quotient map $\mathbb{U} \mapsto H_0 := H_0(\mathfrak{p}_+, \mathbb{U}) = \mathbb{U}/(\bar{\mathfrak{p}}_+ \cdot \mathbb{U})$ is $\bar{P}$-invariant and so induces a canonical bundle quotient map $\Pi_0^{\mathcal{U}} : \mathcal{U} \to \mathcal{H}_0$ between the corresponding associated $\bar{P}$-bundles. (We reuse the notation $\Pi_0^{\mathcal{U}}$ for the induced map $\Gamma(\mathcal{U}) \to \Gamma(\mathcal{H}_0)$ on sections.) Given a section $\mathbb{A} \in \Gamma(\mathcal{U})$, its image $\Pi_0^{\mathcal{U}}(\mathbb{A}) \in \Gamma(\mathcal{H}_0)$ is its *projecting part*.

For the standard representation $\mathbb{V}$ this quotient map is $\Pi_0^{\mathcal{V}} : \mathcal{V} \to \mathcal{E}[1]$,

$$\Pi_0^{\mathcal{V}} : \begin{pmatrix} * \\ * \\ \sigma \end{pmatrix} \mapsto \sigma.$$

For the alternating representation $\Lambda^k \mathbb{V}^*$, the quotient map is $\Pi_0^{\Lambda^k \mathcal{V}^*} : \Lambda^k \mathcal{V}^* \to \Lambda^{k-1} T^* M[k]$,

$$
\Pi_0^{\Lambda^k \mathcal{V}^*} : \quad \begin{pmatrix} * \\ * & | & * \\ \phi_{a_2 \cdots a_k} \end{pmatrix} \mapsto \phi_{a_2 \cdots a_k} \tag{2.20}
$$

For the adjoint representation $\mathfrak{so}(p+1, q+1)$, the quotient map coincides with the map $\Pi_0^{\mathcal{A}} : \mathcal{A} \to TM$ defined in Section 2.3.3; in a splitting, it is

$$
\Pi_0^{\mathcal{A}} : \quad \begin{pmatrix} * \\ * & | & * \\ \xi^a \end{pmatrix} \mapsto \xi^a.
$$

2.4.4 Conformal BGG splitting operators

Conversely, for each irreducible $SO(p+1, q+1)$-representation $\mathbb{U}$ there is a canonical differential *BGG splitting operator* $L_0^{\mathcal{U}} : \Gamma(\mathcal{H}_0) \to \Gamma(\mathcal{U})$ characterized by the properties (1) $\Pi_0^{\mathcal{U}} \circ L_0^{\mathcal{U}} = \mathrm{id}_{\mathcal{H}_0}$ and (2) $\partial^* \circ \nabla^{\mathcal{U}} \circ L_0^{\mathcal{U}} = 0$ [14, 19]. The only property of the operators $L_0^{\mathcal{U}}$ we need here follows immediately from this characterization: If $\mathbb{A} \in \Gamma(\mathcal{U})$ is $\nabla^{\mathcal{U}}$-parallel, then $L_0^{\mathcal{U}}(\Pi_0^{\mathcal{U}}(\mathbb{A})) = \mathbb{A}$.

2.5 Almost Einstein scales

The BGG splitting operator $L_0^{\mathcal{V}} : \Gamma(\mathcal{E}[1]) \to \Gamma(\mathcal{V})$ corresponding to the standard representation is [40, equation (114)]

$$
L_0^{\mathcal{V}} : \quad \sigma \mapsto \begin{pmatrix} -\frac{1}{n}(\sigma_{,b}{}^b + \mathsf{P}^b{}_b \sigma) \\ \sigma^{,a} \\ \sigma \end{pmatrix}. \tag{2.21}
$$

Computing gives

$$
\nabla_b^{\mathcal{V}} L_0^{\mathcal{V}}(\sigma)^A \overset{\tau}{=} \begin{pmatrix} * \\ (\sigma_{,ab} + \mathsf{P}_{ab}\sigma)_\circ \\ 0 \end{pmatrix} \in \Gamma \left(\begin{pmatrix} \mathcal{E}[-1] \\ T^* M[1] \\ \mathcal{E}[1] \end{pmatrix} \otimes T^* M \right), \tag{2.22}
$$

where $(T_{ab})_\circ$ denotes the tracefree part $T_{ab} - \frac{1}{n} T^c{}_c \mathbf{g}_{ab}$ of the (possibly weighted) covariant 2-tensor T_{ab}, and where $*$ is some third-order differential expression in σ. Since the bottom component of $\nabla^{\mathcal{V}} L_0^{\mathcal{V}}(\sigma)$ is zero, the middle component, regarded as a (second-order) linear differential operator $\Theta_0^{\mathcal{V}} : \Gamma(\mathcal{E}[1]) \to \Gamma(S^2 T^* M[1])$,

$$
\Theta_0^{\mathcal{V}} : \quad \sigma \mapsto (\sigma_{,ab} + \mathsf{P}_{ab}\sigma)_\circ,
$$

is conformally invariant. The operator $\Theta_0^{\mathcal{V}}$ is the *first BGG operator* [19] associated to the standard representation $\mathbb{V}$ for (oriented) conformal geometry.

We can readily interpret a solution $\sigma \in \ker \Theta_0^{\mathcal{V}}$ geometrically: If we restrict to the complement $M - \Sigma$ of the zero locus $\Sigma := \{x \in M : \sigma_x = 0\}$, we can work in the scale of the solution σ itself: We have $\sigma \overset{\sigma}{=} 1$ and hence $0 = \Theta_0^{\mathcal{V}}(\sigma) = \mathsf{P}_\circ$. This says simply says that the Schouten tensor, P, of $g := \sigma^{-2} \mathbf{g}|_{M-\Sigma}$ is a multiple of g, and hence so is its Ricci tensor, that is, that g is Einstein. This motivates the following definition [32]:

Definition 2.5. An *almost Einstein scale*[3] of an (oriented) conformal structure of dimension $n \geq 3$ is a solution $\sigma \in \Gamma(\mathcal{E}[1])$ of the operator $\Theta_0^{\mathcal{V}}$. A conformal structure is *almost Einstein* if it admits a nonzero almost Einstein scale.

---

[3]Our terminology follows that of the literature on almost Einstein scales, but this consistency entails a mild perversity, namely that, since they may vanish, almost Einstein scales need not be scales.

We denote the set $\ker \Theta_0^{\mathcal{V}}$ of almost Einstein scales of a given conformal structure $\mathbf{c}$ by $\mathrm{aEs}(\mathbf{c})$. Since $\Theta_0^{\mathcal{V}}$ is linear, $\mathrm{aEs}(\mathbf{c})$ is a vector subspace of $\Gamma(\mathcal{E}[1])$.

The vanishing of the component $*$ in (2.22) turns out to be a differential consequence of the vanishing of the middle component, $\Theta_0^{\mathcal{V}}(\sigma)$. So, $\nabla^{\mathcal{V}}$ is a prolongation connection for the operator $\Theta_0^{\mathcal{V}}$:

Theorem 2.6 ([4, Section 2]). *For any conformal structure $(M, \mathbf{c})$, $\dim M \geq 3$, the restrictions of $L_0^{\mathcal{V}} \colon \Gamma(\mathcal{E}[1]) \to \Gamma(\mathcal{V})$ and $\Pi_0^{\mathcal{V}} \colon \Gamma(\mathcal{V}) \to \Gamma(\mathcal{E}[1])$ comprise a natural bijective correspondence between almost Einstein scales and parallel standard tractors:*

$$\mathrm{aEs}(\mathbf{c}) \underset{\Pi_0^{\mathcal{V}}}{\overset{L_0^{\mathcal{V}}}{\rightleftarrows}} \left\{ \nabla^{\mathcal{V}}\text{-parallel sections of } \mathcal{V} \right\}.$$

In particular, if σ is an almost Einstein scale and vanishes on some nonempty open set, then $\sigma = 0$. In fact, the zero locus Σ of σ turns out to be a smooth hypersurface [17]; see Example 5.2.

We define the *Einstein constant* of an almost Einstein scale σ to be

$$\lambda := -\tfrac{1}{2} H(L_0^{\mathcal{V}}(\sigma), L_0^{\mathcal{V}}(\sigma)) = \tfrac{1}{n}\sigma(\sigma_{,a}{}^a + \mathsf{P}^a{}_a \sigma) - \tfrac{1}{2}\sigma_{,a}\sigma^{,a}. \tag{2.23}$$

This definition is motivated by the following computation: On $M - \Sigma$ the Schouten tensor of the representative metric $g := \sigma^{-2}\mathbf{g}|_{M-\Sigma} \in \mathbf{c}|_{M-\Sigma}$ determined by the scale $\sigma|_{M-\Sigma}$ is $\mathsf{P} = \lambda g$. Thus, the Ricci tensor of g is $R_{ab} = 2(n-1)\lambda g_{ab}$, so we say that σ (or the metric g it induces) is *Ricci-negative*, *-flat*, or *-positive* respectively iff $\lambda < 0$, $\lambda = 0$, or $\lambda > 0$.[4]

2.6 Conformal Killing fields and $(k-1)$-forms

The BGG splitting operator $L_0^{\Lambda^k \mathcal{V}^*} \colon \Gamma(\Lambda^{k-1}T^*M[k]) \to \Gamma(\Lambda^k \mathcal{V}^*)$ determined by the alternating representation $\Lambda^k \mathbb{V}^*$, $1 < k < n+1$, is [40, equation (134)]

$$L_0^{\Lambda^k \mathcal{V}^*} \colon \ \phi_{a_2 \cdots a_k} \mapsto \left(\begin{array}{c} \left(\begin{array}{c} \tfrac{1}{n}\big[-\tfrac{1}{k}\phi_{a_2 \dots a_k, b}{}^b + \tfrac{k-1}{k}\phi_{b[a_3 \cdots a_k, a_2]}{}^b + \tfrac{k-1}{n-k+2}\phi_{b[a_3 \cdots a_k,}{}^b{}_{a_2]} \\ +2(k-1)\mathsf{P}^b{}_{[a_2}\phi_{|b|a_3 \cdots a_k]} - \mathsf{P}^b{}_b \phi_{a_2 \cdots a_k} \big] \end{array} \right) \\ \phi_{[a_2 \cdots a_k, a_1]} \quad \Big| \quad -\tfrac{1}{n-k+2}\phi_{b[a_3 \cdots a_k a_2],}{}^b \\ \phi_{a_2 \cdots a_k} \end{array} \right). \tag{2.24}$$

Proceeding as in Section 2.5, we find that

$$\nabla_b^{\mathcal{V}} L_0^{\Lambda^k \mathcal{V}^*}(\phi)_{A_1 \cdots A_k} = \left(\begin{array}{ccc} & * & \\ * & \Big| & * \\ \phi_{a_2 \cdots a_k, b} - \phi_{[a_2 \cdots a_k, b]} - \tfrac{k-1}{n-k+2}\mathbf{g}_{b[a_2}\phi_{|c|a_3 \cdots a_k],}{}^c & & \end{array} \right),$$

where each $*$ denotes some differential expression in σ. The bottom component defines an invariant conformal differential operator $\Theta_0^{\Lambda^k \mathcal{V}^*} \colon \Gamma(\Lambda^{k-1}T^*M[k]) \to \Gamma(\Lambda^{k-1}T^*M \odot T^*M[k])$ (here $\odot$ denotes the Cartan product) and elements of its kernel are called *conformal Killing $(k-1)$-forms* [60]. Unlike in the case of almost Einstein scales, vanishing of $\Theta_0^{\Lambda^k \mathcal{V}^*}(\phi)$ does not in general imply the vanishing of the remaining components $*$; if they do vanish, that is, if $\nabla^{\mathcal{V}} L_0^{\Lambda^k \mathcal{V}^*}(\phi) = 0$, ϕ is called a *normal* conformal Killing $(k-1)$-form [40, Section 6.2], [50].

The BGG splitting operator $L_0^{\mathcal{A}} \colon \Gamma(TM) \to \Gamma(\mathcal{A})$ for the adjoint representation $\mathfrak{so}(p+1, q+1)$ is [40, equation (119)]

$$L_0^{\Lambda^k \mathcal{V}^*} \colon \ \xi^b \mapsto \left(\begin{array}{c} \tfrac{1}{n}\left(-\tfrac{1}{2}\xi_{b,c}{}^c + \tfrac{1}{2}\xi^c{}_{,bc} + \tfrac{1}{n}\xi^c{}_{,cb} + 2\mathsf{P}_{bc}\xi^c - \mathsf{P}^c{}_c \xi_b \right) \\ \tfrac{1}{2}(-\xi^a{}_{,b} + \xi_b{}^{,a}) \quad \Big| \quad -\tfrac{1}{n}\xi^c{}_{,c} \\ \xi^b \end{array} \right).$$

---

[4]The definition here of *Einstein constant* is consistent with some of the literature on almost Einstein conformal structures, but elsewhere this term is sometimes used for the quantity $2(n-1)\lambda$.

So viewed, $\Theta_0^{\mathcal{A}}$ is the map $\Gamma(TM) \to \Gamma(S_\circ^2 T^*M[2])$, $\xi^a \mapsto (\xi_{(a,b)})_\circ = (\mathcal{L}_\xi \mathbf{g})_{ab}$. Thus, the solutions of $\ker \Theta_0^{\mathcal{A}}$ are precisely the vector fields whose flow preserves $\mathbf{c}$, and so these are called *conformal Killing fields*. If $\nabla^{\mathcal{V}} L_0^{\mathcal{A}}(\xi) = 0$, we say ξ is a *normal* conformal Killing field.

2.7 $(2, 3, 5)$ conformal structures

About a decade ago, Nurowski observed the following:

Theorem 2.7. *A* $(2, 3, 5)$ *distribution* $(M, \mathbf{D})$ *canonically determines a conformal structure* $\mathbf{c_D}$ *of signature* $(2, 3)$ *on* M.

This construction has since been recognized as a special case of a *Fefferman construction*, so named because it likewise generalizes a classical construction of Fefferman that canonically assigns to any nondegenerate hypersurface-type CR structure on a manifold N a conformal structure on a natural circle bundle over N [30]. In fact, this latter construction arises in our setting, too; see Section 5.5.3.

We use the following terminology:

Definition 2.8. A conformal structure $\mathbf{c}$ is a $(2, 3, 5)$ *conformal structure* iff $\mathbf{c} = \mathbf{c_D}$ for some $(2, 3, 5)$ distribution $\mathbf{D}$.

An oriented $(2, 3, 5)$ distribution $\mathbf{D}$ determines an orientation of TM, and hence $\mathbf{c_D}$ is oriented (henceforth, that symbol refers to an oriented conformal structure).

Because we will need some of the ingredients anyway, we briefly sketch a construction of $\mathbf{c_D}$ using the framework of parabolic geometry: Fix an oriented $(2, 3, 5)$ distribution $(M, \mathbf{D})$, and per Section 2.3.6 let $(\mathcal{G} \to M, \omega)$ be the corresponding regular, normal parabolic geometry of type (G_2, Q). Form the extended bundle $\bar{\mathcal{G}} := \mathcal{G} \times_Q \bar{P}$, and let $\bar{\omega}$ denote the Cartan connection equivariantly extending ω to $\bar{\mathcal{G}}$. By construction $(\bar{\mathcal{G}}, \bar{\omega})$ is a parabolic geometry of type $(\mathrm{SO}(3, 4), \bar{P})$ (for which $\bar{\omega}$ turns out to be normal, see [41, Proposition 4]), and hence defines an oriented conformal structure on M.

For any $(2, 3, 5)$ distribution $(M, \mathbf{D})$ and for any representation $\mathbb{U}$ of $\mathrm{SO}(p+1, q+1)$, we may identify the associated tractor bundle $\mathcal{G} \times_Q \mathbb{U}$ (here regarding $\mathbb{U}$ as a Q-representation) with the conformal tractor bundle $\bar{\mathcal{G}} \times_{\bar{P}} \mathbb{U}$, and so denote both of these bundles by $\mathcal{U}$. Since $\bar{\omega}$ is itself normal, the (normal) tractor connections that ω and $\bar{\omega}$ induce on $\mathcal{U}$ coincide.

2.7.1   Holonomy characterization of oriented $(2, 3, 5)$ conformal structures

An oriented $(2, 3, 5)$-distribution $\mathbf{D}$ corresponds to a regular, normal parabolic geometry $(\mathcal{G}, \omega)$ of type (G_2, Q). In particular, this determines on the tractor bundle $\mathcal{V} = \mathcal{G} \times_Q \mathbb{V}$ a G_2-structure $\Phi \in \Gamma(\Lambda^3 \mathcal{V}^*)$ parallel with respect to the induced normal connection on $\mathcal{V}$, and again we may identify $\mathcal{V}$ and the normal connection thereon with the standard conformal tractor bundle $\bar{\mathcal{G}} \times_{\bar{P}} \mathbb{V}$ of $\mathbf{c_D}$ and the normal conformal tractor connection. The G_2-structure determines fiberwise a bilinear form $H_\Phi \in \Gamma(S^2 \mathcal{V}^*)$. Since this construction is algebraic, H_Φ is parallel, and by construction it coincides with the conformal tractor metric on $\mathcal{V}$ determined by $\mathbf{c_D}$.

Conversely, if an oriented, signature $(2, 3)$ conformal structure $\mathbf{c}$ admits a parallel tractor G_2-structure Φ whose restriction to each fiber $\mathcal{V}_x$ is compatible with the restriction H_x of the tractor metric (in which case we simply say that Φ is compatible with H), the distribution $\mathbf{D}$ underlying Φ satisfies $\mathbf{c} = \mathbf{c_D}$. This recovers a correspondence stated in the original work of Nurowski [52] and worked out in detail in [41]:

Theorem 2.9. *An oriented conformal structure* $(M, \mathbf{c})$ *(necessarily of signature* $(2, 3)$*) is induced by some* $(2, 3, 5)$ *distribution* $\mathbf{D}$ *(that is,* $\mathbf{c} = \mathbf{c_D}$*) iff the normal conformal tractor connection admits a holonomy reduction to* G_2*, or equivalently, iff* $\mathbf{c}$ *admits a parallel tractor* G_2*-structure* Φ *compatible with the tractor metric* H.

2.7.2 The conformal tractor decomposition of the tractor G_2-structure

Fix an oriented $(2,3,5)$ distribution $(M, \mathbf{D})$, let $\Phi \in \Gamma(\Lambda^3 \mathcal{V}^*)$ denote the corresponding parallel tractor G_2-structure, and denote its components with respect to any scale τ of the induced conformal structure $\mathbf{c_D}$ according to

$$\Phi_{ABC} \stackrel{\tau}{=} \begin{pmatrix} \psi_{bc} \\ \chi_{abc} \mid \theta_c \\ \phi_{bc} \end{pmatrix} \in \Gamma \begin{pmatrix} \Lambda^2 T^* M[1] \\ \Lambda^3 T^* M[3] \mid T^* M[1] \\ \Lambda^2 T^* M[3] \end{pmatrix}. \tag{2.25}$$

In the language of Section 2.6, $\phi = \Pi_0^{\Lambda^3 \mathcal{V}^*}(\Phi)$ is a normal conformal Killing 2-form, and $\Phi = L_0^{\Lambda^3 \mathcal{V}^*}(\phi)$. An argument analogous to that in the proof of Proposition 2.10(5) below shows that ϕ is locally decomposable, and Proposition 2.10(8) shows that it vanishes nowhere, so the (weighted) bivector field $\phi^{ab} \in \Gamma(\Lambda^2 TM[-1])$ determines a 2-plane distribution on M, and this is precisely $\mathbf{D}$ [41].

We collect for later some useful geometric facts about $\mathbf{D}$ and encode them in algebraic identities in the tractor components ϕ, χ, θ, ψ. Parts (1) and (2) of the Proposition 2.10 are well-known features of $(2,3,5)$ distributions.

Proposition 2.10. *Let $(M, \mathbf{D})$ be an oriented $(2,3,5)$ distribution, let $\Phi \in \Gamma(\Lambda^3 \mathcal{V}^*)$ denote the corresponding parallel tractor G_2-structure, and denote its components with respect to an arbitrary scale τ as in (2.25). Then:*

1. *The distribution $\mathbf{D}$ is totally $\mathbf{c_D}$-isotropic; equivalently, $\phi^{ac} \phi_{cb} = 0$.*

2. *The annihilator of ϕ_{ab} (in TM) is $[\mathbf{D}, \mathbf{D}]$, and hence $\mathbf{D}^\perp = [\mathbf{D}, \mathbf{D}]$ (here, $\mathbf{D}^\perp$ is the subbundle of TM orthogonal to $\mathbf{D}$ with respect to $\mathbf{c_D}$); equivalently, $\phi^{bc} \chi_{bca} = 0$.*

3. *The weighted vector field $\theta^b \in \Gamma(TM[-1])$ is a section of $[\mathbf{D}, \mathbf{D}][-1]$, or equivalently, the line field $\mathbf{L}$ that θ determines (which depends on τ) is orthogonal to $\mathbf{D}$; equivalently, $\theta^b \phi_{ba} = 0$.*

4. *The weighted vector field θ^b satisfies $\theta_b \theta^b = -1$. In particular, the line field $\mathbf{L}$ is timelike.*

5. *Like ϕ, the weighted 2-form ψ is locally decomposable, that is, $(\psi \wedge \psi)_{abcd} = 6\psi_{[ab}\psi_{cd]} = 0$. Since (by equation (2.26)) it vanishes nowhere, it determines a 2-plane distribution $\mathbf{E}$ (which depends on τ).*

6. *The distribution $\mathbf{E}$ is totally $\mathbf{c_D}$-isotropic; equivalently, $\psi^{ac}\psi_{cb} = 0$.*

7. *The line field $\mathbf{L}$ is orthogonal to $\mathbf{E}$; equivalently, $\theta^b \psi_{ba} = 0$.*

8. *The (weighted) conformal volume form $\epsilon_{\mathbf{g}} \in \Gamma(\Lambda^5 T^* M[5])$ satisfies*

$$(\epsilon_{\mathbf{g}})_{abcde} \stackrel{\tau}{=} \tfrac{1}{2}(\phi \wedge \theta \wedge \psi)_{abcde} = 15\phi_{[ab}\theta_c \psi_{de]}. \tag{2.26}$$

In particular, (8) implies that $\mathbf{D}$, $\mathbf{L}$, and $\mathbf{E}$ are pairwise transverse and hence span TM. Moreover, (2) and (3) imply that $\mathbf{D} \oplus \mathbf{L} = [\mathbf{D}, \mathbf{D}]$ and so $\mathbf{D} \oplus \mathbf{L} \oplus \mathbf{E}$ is a splitting of the canonical filtration $\mathbf{D} \subset [\mathbf{D}, \mathbf{D}] \subset TM$.[5]

It is possible to give abstract proofs of the identities in Proposition 2.10, but it is much faster to use frames of the standard tractor bundle suitably adapted to the parallel tractor G_2-structure Φ.

---

[5]The splitting $\mathbf{D} \oplus \mathbf{L} \oplus \mathbf{E}$ determined by τ is a special case of a general feature of parabolic geometry, in which a choice of Weyl structure yields a splitting of the canonical filtration of the tangent bundle of the underlying structure [18, Section 5.1].

Proof of Proposition 2.10. Call a local frame (E_a) of $\mathcal{V}$ *adapted* to Φ iff (1) E_1 is a local section of the line subbundle $\langle X \rangle$ determined by X, and (2) the representation of Φ in the dual coframe (e^a) is given by (2.5); it follows from [65, Theorem 3.1] that such a local frame exists in some neighborhood of any point in M.

Any adapted local frame determines a (local) choice of scale: Since $X \in \Gamma(\mathcal{V}[1])$, we have $\tau := e^7(X) \in \mathcal{E}[1]$, and by construction it vanishes nowhere. Then, since $\langle X \rangle^\perp = \langle E_1, \dots, E_6 \rangle$, the (weighted) vector fields $F_a := E_a + \langle E_1 \rangle$, $a = 2, \dots, 6$ comprise a frame of $\langle X \rangle^\perp / \langle X \rangle$ which by Section 2.4.2 is canonically isomorphic to $TM[-1]$. Trivializing these frame fields (by multiplying by τ) yields a local frame $(\underline{F}_2, \dots, \underline{F}_6)$ of TM; denote the dual coframe by $(f^2, \dots, f^6)$. One can read immediately from (2.5) that in an adapted local frame, (the trivialized) components of Φ are

$$\underline{\phi} \stackrel{\tau}{=} \sqrt{2} f^5 \wedge f^6, \qquad \underline{\chi} \stackrel{\tau}{=} f^2 \wedge f^4 \wedge f^5 + f^3 \wedge f^4 \wedge f^6, \qquad \underline{\theta} \stackrel{\tau}{=} f^4, \qquad \underline{\psi} \stackrel{\tau}{=} \sqrt{2} f^2 \wedge f^3,$$

and consulting the form of equation (2.6) gives that the (trivialized) conformal metric is

$$\mathbf{g} \stackrel{\tau}{=} f^2 f^5 + f^3 f^6 - \left(f^4 \right)^2.$$

In an adapted frame, ϵ_Φ is given by $-e^1 \wedge \cdots \wedge e^7$, so the (trivialized) conformal volume form is $\underline{\epsilon}_{\mathbf{g}} = f^2 \wedge \cdots \wedge f^6$.

All of the identities follow immediately from computing in this frame. For example, to compute (1), we see that raising indices gives $\underline{\phi}^{\sharp\sharp} = \sqrt{2} \underline{F}_2 \wedge \underline{F}_3$, and that contracting an index of this bivector field with $\underline{\phi} = \sqrt{2} f^5 \wedge f^6$ yields 0.

It remains to show that the geometric assertions are equivalent to the corresponding identities; these are nearly immediate for all but the first two parts. For both parts, pick a local frame (α, β) of $\mathbf{D}$ around an arbitrary point; by scaling we may assume that $\underline{\phi}^{\sharp\sharp} = \alpha \wedge \beta$.

1. The identity implies that the trace over the second and third indices of the tensor product $\underline{\phi}^{\sharp\sharp} \otimes \underline{\phi} = (\alpha \wedge \beta) \otimes (\alpha^\flat \wedge \beta^\flat)$ is zero, or, expanding, that

   $$0 = -\mathbf{g}(\alpha, \alpha) \beta \otimes \beta^\flat + \mathbf{g}(\alpha, \beta) \alpha \otimes \beta^\flat + \mathbf{g}(\alpha, \beta) \beta \otimes \alpha^\flat - \mathbf{g}(\beta, \beta) \beta \otimes \beta^\flat.$$

   Since α, β are linearly independent, the four coefficients on the right-hand side vanish separately, but up to sign these are the components of the restriction of $\mathbf{c_D}$ to $\mathbf{D}$ in the given frame.

2. By Part (1), $\phi(\alpha, \cdot) = \phi(\beta, \cdot) = 0$. Any local section $\eta \in \Gamma([\mathbf{D}, \mathbf{D}])$ can be written as $\eta = A\alpha + B\beta + C[\alpha, \beta]$ for some smooth functions A, B, C, giving $\phi(\eta, \gamma) = C\phi([\alpha, \beta], \gamma)$. The invariant formula for the exterior derivative of a 2-form then gives $\phi([\alpha, \beta], \gamma) = -d\phi(\alpha, \beta, \gamma)$. Now, in the chosen scale, $d\phi = \chi$ so

   $$-[d\phi(\alpha, \beta, \cdot)]_a = -\chi_{bca} \alpha^b \beta^c = -\tfrac{1}{2} \chi_{bca} \cdot 2\alpha^{[b} \beta^{c]} = -\tfrac{1}{2} \chi_{bca} \phi^{bc}. \qquad \blacksquare$$

Since the tractor Hodge star operator $*_\Phi$ is algebraic, $*_\Phi \Phi \in \Gamma(\Lambda^4 \mathcal{V}^*)$ is parallel. We can express its components with respect to a scale τ in terms of those of Φ and the weighted Hodge star operators $* : \Lambda^l T^* M[w] \to \Lambda^{5-l} T^* M[7-w]$ determined by $\mathbf{g}$ [50]:

$$(*_\Phi \Phi)_{ABCD} \stackrel{\tau}{=} \begin{pmatrix} (*\psi)_{fgh} \\ -(*\theta)_{efgh} \mid (*\chi)_{gh} \\ -(*\phi)_{fgh} \end{pmatrix} \in \Gamma \begin{pmatrix} \Lambda^3 T^* M[2] \\ \Lambda^4 T^* M[4] \mid \Lambda^2 T^* M[2] \\ \Lambda^3 T^* M[4] \end{pmatrix}. \tag{2.27}$$

Computing in an adapted frame as in the proof of Proposition 2.10 yields some useful identities relating the components of Φ and their images under $*$:

$$(*\phi)_{fgh} = 3\phi_{[fg}\theta_{h]}, \qquad (*\chi)_{gh} = \theta^i \chi_{igh},$$
$$(*\theta)_{efgh} = -3\phi_{[ef}\psi_{gh]}, \qquad (*\psi)_{fgh} = 3\psi_{[fg}\theta_{h]}. \tag{2.28}$$

3 The global geometry of almost Einstein $(2,3,5)$ distributions

In this section we investigate the global geometry of $(2,3,5)$ distributions $(M, \mathbf{D})$ that induce almost Einstein conformal structures $\mathbf{c_D}$; naturally, we call such distributions themselves *almost Einstein*.

Almost Einstein $(2,3,5)$ distributions are special among $(2,3,5)$ conformal structures: In a sense that can be made precise [38, Theorem 1.2, Proposition 5.1], for a generic $(2,3,5)$ distribution $\mathbf{D}$ the holonomy of $\mathbf{c_D}$ is equal to G_2 and hence $\mathbf{c_D}$ admits no nonzero almost Einstein scales.

Via the identification of the standard tractor bundles of $\mathbf{D}$ and $\mathbf{c_D}$, Theorem 2.6 gives that an oriented $(2,3,5)$ distribution is almost Einstein iff its standard tractor bundle $\mathcal{V}$ admits a nonzero parallel standard tractor $\mathbb{S} \in \Gamma(\mathcal{V})$, or equivalently, iff it admits a holonomy reduction from G_2 to the stabilizer subgroup S of a nonzero vector in the standard representation $\mathbb{V}$.

3.1 Distinguishing a vector in the standard representation $\mathbb{V}$ of G_2

In this subsection, let $\mathbb{V}$ denote the standard representation of G_2 and $\Phi \in \Lambda^3 \mathbb{V}^*$ the corresponding 3-form. We establish some of the algebraic consequences of fixing a nonzero vector $\mathbb{S} \in \mathbb{V}$.

3.1.1 Stabilizer subgroups

Recall from the introduction that the stabilizer group in G_2 of $\mathbb{S} \in \mathbb{V}$ is as follows:

Proposition 3.1. *The stabilizer subgroup of a nonzero vector $\mathbb{S}$ in the standard representation $\mathbb{V}$ of G_2 is isomorphic to:*

1) $SU(1, 2)$, *if $\mathbb{S}$ is spacelike,*

2) $SL(3, \mathbb{R})$, *if $\mathbb{S}$ is timelike, and*

3) $SL(2, \mathbb{R}) \ltimes Q_+$, *where $Q_+ < G_2$ is the connected, nilpotent subgroup of G_2 defined via Sections 2.3.4 and 2.3.6, if $\mathbb{S}$ is isotropic.*

3.1.2 An ε-Hermitian structure

Contracting a nonzero vector $\mathbb{S} \in \mathbb{V}$ with Φ determines an endomorphism:

$$\mathbb{K}^A{}_B := -\iota_7^2(\mathbb{S})^A{}_B = -\mathbb{S}^C \Phi_C{}^A{}_B \in \mathfrak{so}(3, 4). \tag{3.1}$$

We can identify $\mathbb{K}$ with the map $\mathbb{T} \mapsto \mathbb{S} \times \mathbb{T}$, so if we scale $\mathbb{S}$ so that $\varepsilon := -H_\Phi(\mathbb{S}, \mathbb{S}) \in \{-1, 0, 1\}$, identity (2.2) becomes

$$\mathbb{K}^2 = \varepsilon \mathrm{id}_{\mathbb{V}} + \mathbb{S} \otimes \mathbb{S}^\flat. \tag{3.2}$$

By skewness, $H_{AC} \mathbb{S}^A \mathbb{K}^C{}_B = -\mathbb{S}^A \mathbb{S}^D \mathbb{K}_{DAB} = 0$, so the image of $\mathbb{K}$ is contained in $\mathbb{W} := \langle \mathbb{S} \rangle^\perp$, and hence we can regard $\mathbb{K}|_{\mathbb{W}}$ as an endomorphism of $\mathbb{W}$, which by abuse of notation we also denote $\mathbb{K}$. Restricting (3.2) to $\mathbb{W}$ gives that this latter endomorphism is an ε-complex structure on that bundle: $\mathbb{K}^2 = \varepsilon \mathrm{id}_{\mathbb{W}}$. Thus, $(H_\Phi|_{\mathbb{W}}, \mathbb{K})$ is an *ε-Hermitian structure on $\mathbb{W}$*: this is a pair $(g, \mathbb{K})$, where $g \in S^2 \mathbb{W}^*$ is a symmetric, nondegenerate, bilinear form and $\mathbb{K}$ is an ε-complex structure on $\mathbb{W}$ compatible in the sense that $g(\cdot, \mathbb{K} \cdot)$ is skew-symmetric. If $\mathbb{K}$ is complex, g has signature $(2p, 2q)$ for some integers p, q; if $\mathbb{K}$ is paracomplex, g has signature (m, m).

3.1.3   Induced splittings and filtrations

If $\mathbb{S}$ is nonisotropic, it determines an orthogonal decomposition $\mathbb{V} = \mathbb{W} \oplus \langle \mathbb{S} \rangle$. If $\mathbb{S}$ is isotropic, it determines a filtration $(\mathbb{V}^a_{\mathbb{S}})$ [37, Proposition 2.5]:

$$
\begin{array}{ccccccccccccc}
-2 & & -1 & & 0 & & +1 & & +2 & & +3 & \\
\mathbb{V} & \supset & \mathbb{W} & \supset & \operatorname{im} \mathbb{K} & \supset & \ker \mathbb{K} & \supset & \langle \mathbb{S} \rangle & \supset & \{0\} & \\
7 & & 6 & & 4 & & 3 & & 1 & & 0 &
\end{array}
\tag{3.3}
$$

The number above each filtrand is its filtration index a (which are canonical only up to addition of a given integer to each index) and the number below its dimension. Moreover, $\operatorname{im} \mathbb{K} = (\ker \mathbb{K})^\perp$ (so $\ker \mathbb{K}$ is totally isotropic). If we take Q to be the stabilizer subgroup of the ray spanned by $\mathbb{S}$, then the filtration is Q-invariant, and checking the (representation-theoretic) weights of $\mathbb{V}$ as a Q-representation shows that it coincides with the filtration (2.14) determined by Q. The map $\mathbb{K}$ satisfies $\mathbb{K}(\mathbb{V}^a_{\mathbb{S}}) = \mathbb{V}^{a+2}_{\mathbb{S}}$, where we set $\mathbb{V}^a_{\mathbb{S}} = 0$ for all $a > 2$.

3.1.4   The family of stabilized 3-forms

For nonzero $\mathbb{S} \in \mathbb{V}$, elementary linear algebra gives that the subspace of 3-forms in $\Lambda^3 \mathbb{V}^*$ fixed by the stabilizer subgroup S of $\mathbb{S}$ has dimension 3 and contains

$$
\Phi_I := \mathbb{S} \lrcorner (\mathbb{S}^\flat \wedge \Phi) \in \Lambda^3_1 \oplus \Lambda^3_{27},
\tag{3.4}
$$

$$
\Phi_J := -\iota^3_7(\mathbb{S}) = -*_\Phi(\mathbb{S}^\flat \wedge \Phi) = \mathbb{S} \lrcorner *_\Phi \Phi \in \Lambda^3_7,
\tag{3.5}
$$

$$
\Phi_K := \mathbb{S}^\flat \wedge (\mathbb{S} \lrcorner \Phi) \in \Lambda^3_1 \oplus \Lambda^3_{27}.
\tag{3.6}
$$

The containment $\Phi_K \in \Lambda^3_1 \oplus \Lambda^3_{27}$ follows from the fact that $\Phi_K = \frac{1}{2}i(\mathbb{S}^\flat \circ \mathbb{S}^\flat)$, where i is the G_2-invariant map defined in (2.12). The containment $\Phi_I \in \Lambda^3_1 \oplus \Lambda^3_{27}$ follows from that containment, the identity

$$
\Phi_I + \Phi_K = \mathbb{S} \lrcorner (\mathbb{S}^\flat \wedge \Phi) + \mathbb{S}^\flat \wedge (\mathbb{S} \lrcorner \Phi) = H_\Phi(\mathbb{S}, \mathbb{S})\Phi,
\tag{3.7}
$$

and the fact that $\Phi \in \Lambda^3_1$. It follows immediately from the definitions that

$$
\mathbb{S} \lrcorner \Phi_I = \mathbb{S} \lrcorner \Phi_J = 0 \qquad \text{and} \qquad \mathbb{S} \lrcorner \Phi_K = H_\Phi(\mathbb{S}, \mathbb{S}) \mathbb{S} \lrcorner \Phi.
\tag{3.8}
$$

Since $\mathbb{S}$ annihilates Φ_I but not Φ, the containments in (3.4), (3.5) show that $\{\Phi, \Phi_I, \Phi_J\}$ is a basis of the subspace of stabilized 3-forms. If $H_\Phi(\mathbb{S}, \mathbb{S}) \neq 0$, then (3.7) implies that $\{\Phi_I, \Phi_J, \Phi_K\}$ is also a basis of that space. If $H_\Phi(\mathbb{S}, \mathbb{S}) = 0$ then $\Phi_K = -\Phi_I$. It is convenient to abuse notation and denote by Φ_I, Φ_J the pullbacks to $\mathbb{W}$ of the 3-forms of the same names via the inclusion $\mathbb{W} \hookrightarrow \mathbb{V}$.

For nonisotropic $\mathbb{S}$, define $\mathbb{W}^{1,0} \subset \mathbb{W} \otimes_\mathbb{R} \mathbb{C}_\varepsilon$ to be the $(+i_\varepsilon)$-eigenspace of (the extension of) $\mathbb{K}$, and an *ε-complex volume form* to be an element of $\Lambda^m_{\mathbb{C}_\varepsilon} := \Lambda^m(\mathbb{W}^{1,0})^*$.

Proposition 3.2. *Suppose* $\varepsilon := -H_\Phi(\mathbb{S}, \mathbb{S}) \in \{\pm 1\}$. *For each* (A, B) *such that* $A^2 - \varepsilon B^2 = 1$,

$$
\Psi_{(A,B)} := [A\Phi_I + \varepsilon B\Phi_J] + i_\varepsilon[B\Phi_I + A\Phi_J] \in \Gamma\left(\Lambda^3_{\mathbb{C}_\varepsilon} \mathbb{W}\right)
$$

is an ε-complex volume form for the ε-Hermitian structure $(H_\Phi|_\mathbb{W}, \mathbb{K})$ *on* $\mathbb{W}$.

Proposition 3.3. *Suppose* $\mathbb{V}'$ *is a 7-dimensional real vector space and* $H \in S^2(\mathbb{V}')^*$ *is a symmetric bilinear form of signature* $(3, 4)$. *Now, fix a vector* $\mathbb{S} \in \mathbb{V}'$ *such that* $-\varepsilon := H(\mathbb{S}, \mathbb{S}) \in \{\pm 1\}$, *denote* $\mathbb{W} := \langle \mathbb{S} \rangle^\perp$, *fix an ε-complex structure* $\mathbb{K} \in \operatorname{End}(\mathbb{W})$ *such that* $(H|_\mathbb{W}, \mathbb{K})$ *is a Hermitian structure on* $\mathbb{W}$, *and fix a compatible ε-complex volume form* $\Psi \in \Lambda^3_{\mathbb{C}_\varepsilon} \mathbb{W}^*$ *satisfying the normalization condition*

$$
\Psi \wedge \bar{\Psi} = -\frac{4}{3} i_\varepsilon \mathbb{K} \wedge \mathbb{K} \wedge \mathbb{K}.
$$

Then, the 3-form

$$\operatorname{Re}\Psi + \varepsilon\mathbb{S}^\flat \wedge \mathbb{K} \in \Lambda^3(\mathbb{V}')^*$$

is a G_2-structure on $\mathbb{V}'$ compatible with H. Here, $\operatorname{Re}\Psi$ and $\mathbb{K}$ are regarded as objects on $\mathbb{V}'$ via the decomposition $\mathbb{V}' = \mathbb{W} \oplus \langle \mathbb{S} \rangle$.

This proposition can be derived, for example, from [23, Proposition 1.12], since, using the terminology of the article, $(\operatorname{Re}\Psi, \mathbb{K})$ is a compatible and normalized pair of stable forms.

3.2 The canonical conformal Killing field ξ

For this subsection, fix an oriented $(2,3,5)$ distribution $\mathbf{D}$, let $\Phi \in \Gamma(\Lambda^3 \mathcal{V}^*)$ denote the corresponding parallel tractor G_2-structure, and denote its components with respect to an arbitrary scale τ as in (2.25); in particular, $\phi := \Pi_0^{\Lambda^3 \mathcal{V}^*}(\Phi)$ is the underlying normal conformal Killing 2-form. Also, fix a nonzero almost Einstein scale $\sigma \in \Gamma(\mathcal{E}[1])$ of $\mathbf{c_D}$, denote the corresponding parallel standard tractor by $\mathbb{S} := L_0^{\mathcal{V}}(\sigma)$, and denote its components with respect to τ as in (2.18). By scaling, we assume that $-\varepsilon := H_\Phi(\mathbb{S}, \mathbb{S}) \in \{-1, 0, +1\}$.

We view the adjoint tractor

$$\mathbb{K}^A{}_B := -\mathbb{S}^C \Phi_C{}^A{}_B \in \Gamma(\mathcal{A}).$$

as a bundle endomorphism of $\mathcal{V}$ (cf. (3.1)), and computing gives that the components of $\mathbb{K}$ with respect to τ are

$$\mathbb{K}^A{}_B \overset{\tau}{=} \begin{pmatrix} & \mu^c \psi_{ca} - \rho\theta_a & \\ -\sigma\psi^a{}_b - \mu_c\chi^{ca}{}_b - \rho\phi^a{}_b \mid & \mu^c\theta_c & \\ \sigma\theta^a + \mu_c\phi^{ca} & & \end{pmatrix}. \tag{3.9}$$

We denote the projecting part of $\mathbb{K}^A{}_B$ by

$$\xi^a := \Pi_0^{\mathcal{A}}(\mathbb{K})^a = \sigma\theta^a + \mu_b\phi^{ba} \in \Gamma(TM); \tag{3.10}$$

because $\mathbb{K}$ is parallel, ξ is a normal conformal Killing field for $\mathbf{c_D}$. By (3.2) $\mathbb{K}$ is not identically zero and hence neither is ξ. This immediately gives a simple geometric obstruction – nonexistence of a conformal Killing field – for the existence of an almost Einstein scale for an oriented $(2,3,5)$ conformal structure.

By construction, $\xi = \iota_7(\sigma)$, where ι_7 is the manifestly invariant differential operator $\iota_7 := \Pi_0^{\mathcal{A}} \circ (-\iota_7^2) \circ L_0^{\mathcal{V}} : \Gamma(\mathcal{E}[1]) \to \Gamma(TM)$. Here, ι_7^2 is the bundle map $\mathcal{V} \to \Lambda^2 \mathcal{V}^*$ associated to the algebraic map (2.8) of the same name, and we have implicitly raised an index with H_Φ. Computing gives $\xi^a = \iota_7(\sigma)^a = -\phi^{ab}\sigma_{,b} + \frac{1}{4}\phi^{ab}{}_{,b}\sigma$.[6]

Proposition 3.4. *Given an oriented $(2,3,5)$ distribution $\mathbf{D}$, let ϕ denote the corresponding normal conformal Killing 2-form, and suppose the induced conformal class $\mathbf{c_D}$ admits an almost Einstein scale σ. The corresponding vector field $\xi := \iota_7(\sigma)$ is a section of $[\mathbf{D}, \mathbf{D}]$.*

Proof. By (3.10), $\phi_{ba}\xi^b = \phi_{ba}(\sigma\theta^b + \mu_c\phi^{cb}) = \sigma\phi_{ba}\theta^b + \mu_c\phi^{cb}\phi_{bc}$, but the first and second term vanish respectively by Proposition 2.10(1),(3). Thus, $\xi \in \ker\phi$, which by Part (2) of that proposition is $[\mathbf{D}, \mathbf{D}]$. $\blacksquare$

---

[6]This formula corrects a sign error in [41, equation (41)], and (3.11) below corrects a corresponding sign error in equation (40) of that reference.

On the set $M_\xi := \{x \in M : \xi_x \neq 0\}$, ξ spans a canonical line field

$$\mathbf{L} := \langle \xi \rangle|_{M_\xi},$$

and by Proposition 3.4, $\mathbf{L}$ is a subbundle of $[\mathbf{D}, \mathbf{D}]|_{M_\xi}$. Henceforth we often suppress the restriction notation $|_{M_\xi}$. We will see in Proposition 5.8 that $\mathbf{L}$ coincides with the line field of the same name determined via Proposition 2.10 by the preferred scale σ (on the complement of its zero locus).

3.3 Characterization of conformal Killing fields induced by almost Einstein scales

Hammerl and Sagerschnig showed that for any oriented $(2, 3, 5)$ distribution $\mathbf{D}$, the Lie algebra $\mathfrak{aut}(\mathbf{c_D})$ of conformal Killing fields of the induced conformal structure $\mathbf{c_D}$ admits a natural (vector space) decomposition, corresponding to the G_2-module decomposition $\mathfrak{so}(3, 4) \cong \mathfrak{g}_2 \oplus \mathbb{V}$ into irreducible submodules, that encodes features of the geometry of the underlying distribution.

Given an oriented $(2, 3, 5)$ distribution M, a vector field $\eta \in \Gamma(TM)$ is an *infinitesimal symmetry* of $\mathbf{D}$ iff $\mathbf{D}$ is invariant under the flow of η, and the infinitesimal symmetries of $\mathbf{D}$ comprise a Lie algebra $\mathfrak{aut}(\mathbf{D})$ under the usual Lie bracket of vector fields. The construction $\mathbf{D} \rightsquigarrow \mathbf{c_D}$ is functorial, so $\mathfrak{aut}(\mathbf{D}) \subseteq \mathfrak{aut}(\mathbf{c_D})$.

By construction, the map π_7 given in (3.12) below is a left inverse for ι_7, so in particular ι_7 is injective.

Theorem 3.5 ([41, Theorem B]). *If $(M, \mathbf{D})$ is an oriented $(2, 3, 5)$ distribution, the Lie algebra $\mathfrak{aut}(\mathbf{c_D})$ of conformal Killing fields of the induced conformal structure $\mathbf{c_D}$ admits a natural (vector space) decomposition*

$$\mathfrak{aut}(\mathbf{c_D}) = \mathfrak{aut}(\mathbf{D}) \oplus \iota_7(\mathrm{aEs}(\mathbf{c_D})) \tag{3.11}$$

and hence an isomorphism $\mathfrak{aut}(\mathbf{c_D}) \cong \mathfrak{aut}(\mathbf{D}) \oplus \mathrm{aEs}(\mathbf{c_D})$.

The projection $\mathfrak{aut}(\mathbf{c_D}) \to \mathrm{aEs}(\mathbf{c_D})$ is (the restriction of) the invariant differential operator $\pi_7 := \Pi_0^{\mathcal{V}} \circ (-\pi_7^2) \circ L_0^{\mathcal{A}} \colon \Gamma(TM) \to \Gamma(\mathcal{E}[1])$, which is given by

$$\pi_7 \colon \eta^a \mapsto \tfrac{1}{6}\phi^{ab}\eta_{a,b} - \tfrac{1}{12}\phi_{ab},{}^b\eta^a. \tag{3.12}$$

The canonical projection $\mathfrak{aut}(\mathbf{c_D}) \to \mathfrak{aut}(\mathbf{D})$ is (the restriction of) the invariant differential operator $\pi_{14} := \mathrm{id}_{\Gamma(TM)} - \iota_7 \circ \pi_7 \colon \Gamma(TM) \to \Gamma(TM)$.

The map π_7^2 is the bundle map $\mathcal{A} \cong \Lambda^2 \mathcal{V}^* \to \mathcal{V}^*$ associated to the algebraic map (2.9) of the same name.

The conformal Killing fields in the distinguished subspace $\iota_7(\mathrm{aEs}(\mathbf{c_D}))$, that is, those corresponding to almost Einstein scales, admit a simple geometric characterization:

Proposition 3.6. *Let $(M, \mathbf{D})$ be an oriented $(2, 3, 5)$ distribution. Then, a conformal Killing field of $\mathbf{c_D}$ is in the subspace $\iota_7(\mathrm{aEs}(\mathbf{c_D}))$ iff it is a section of $[\mathbf{D}, \mathbf{D}]$.*

Hence, the indicated restrictions of ι_7 and π_7 comprise a natural bijective correspondence

$$\mathrm{aEs}(\mathbf{c_D}) \underset{\pi_7}{\overset{\iota_7}{\rightleftarrows}} \mathfrak{aut}(\mathbf{c_D}) \cap \Gamma([\mathbf{D}, \mathbf{D}]).$$

Proof. Let q_{-3} denote the canonical projection $TM \to TM/[\mathbf{D}, \mathbf{D}]$ and the map on sections it induces. It follows from a general fact about infinitesimal symmetries of parabolic geometries [15] that an infinitesimal symmetry ξ of $\mathbf{D}$ can be recovered from its image $q_{-3}(\xi) \in \Gamma(TM/[\mathbf{D}, \mathbf{D}])$ via a natural linear differential operator $\Gamma(TM/[\mathbf{D}, \mathbf{D}]) \to \Gamma(TM)$; in particular, if $q_{-3}(\xi) = 0$ then $\xi = 0$, so $\mathfrak{aut}(\mathbf{D})$ intersects trivially with $\ker q_{-3}$. On the other hand, Proposition 3.4 gives that the image of ι_7 is contained in $\ker q_{-3} = [\mathbf{D}, \mathbf{D}]$. The claim now follows from the decomposition in Theorem 3.5. ∎

3.4 The weighted endomorphisms I, J, K

Since they are algebraic combinations of parallel tractors, the 3-forms $\Phi_I, \Phi_J, \Phi_K \in \Gamma(\Lambda^3 \mathcal{V}^*)$ respectively defined pointwise by (3.4), (3.5), (3.6) are themselves parallel. Thus, their respective projecting parts, $I_{ab} := \Pi_0^{\Lambda^3 \mathcal{V}^*}(\Phi_I)_{ab}$, $J_{ab} := \Pi_0^{\Lambda^3 \mathcal{V}^*}(\Phi_J)_{ab}$, $K_{ab} := \Pi_0^{\Lambda^3 \mathcal{V}^*}(\Phi_K)_{ab}$, are normal conformal Killing 2-forms.

The definitions of Φ_I, Φ_J, Φ_K, together with (2.27) and (2.28) give

$$I_{ab} = -\sigma^2 \psi_{ab} - \sigma\mu^c \chi_{cab} - 2\sigma\mu_{[a}\theta_{b]} + \sigma\rho\phi_{ab} + 3\mu^c \mu_{[c}\phi_{ab]}m, \tag{3.13}$$

$$J_{ab} = -\sigma\theta^c \chi_{cab} + 3\mu^c \phi_{[ca}\theta_{b]}, \tag{3.14}$$

$$K_{ab} = \sigma^2 \psi_{ab} + \sigma\mu^c \chi_{cab} + 2\sigma\mu_{[a}\theta_{b]} + \sigma\rho\phi_{ab} - 2\mu^c \mu_{[a}\phi_{b]c}. \tag{3.15}$$

Using the splitting operators $L_0^{\mathcal{V}}$ (2.21) and $L_0^{\Lambda^3 \mathcal{V}^*}$ (2.24), we can write these normal conformal Killing 2-forms as differential expressions in ϕ and σ:

$$I_{ab} = \tfrac{1}{5}\sigma^2 \left(\tfrac{1}{3}\phi_{ab,c}{}^c + \tfrac{2}{3}\phi_{c[a,b]}{}^c + \tfrac{1}{2}\phi_{c[a,}{}^c{}_{b]} + 4\mathsf{P}^c{}_{[a}\phi_{b]c} \right)$$
$$\qquad - \sigma\sigma^{,c}\phi_{[ca,b]} - \tfrac{1}{2}\sigma\sigma_{,[a}\phi_{b]c,}{}^c - \tfrac{1}{5}\sigma\sigma_{,c}{}^c\phi_{ab} + 3\sigma^{,c}\sigma_{,[c}\phi_{ab]}, \tag{3.16}$$

$$J_{ab} = -\tfrac{1}{4}\sigma\phi^{cd,}{}_d\phi_{[ab,c]} + \tfrac{3}{4}\sigma^{,c}\phi_{[ab}\phi_{c]d,}{}^d, \tag{3.17}$$

$$K_{ab} = -\tfrac{1}{5}\sigma^2 \left(\tfrac{1}{3}\phi_{ab,c}{}^c + \tfrac{2}{3}\phi_{c[a,b]}{}^c + \tfrac{1}{2}\phi_{c[a,}{}^c{}_{b]} + 4\mathsf{P}^c{}_{[a}\phi_{b]c} + 2\mathsf{P}^c{}_c\phi_{ab} \right)$$
$$\qquad + \sigma\sigma^{,c}\phi_{[ab,c]} + \tfrac{1}{2}\sigma\sigma_{,[a}\phi_{b]c,}{}^c - \tfrac{1}{5}\sigma\sigma_{,c}{}^c\phi_{ab} - 2\sigma^{,c}\sigma_{,[a}\phi_{b]c}. \tag{3.18}$$

Raising indices gives weighted **g**-skew endomorphisms $I^a{}_b, J^a{}_b, K^a{}_b \in \Gamma(\mathrm{End}_{\mathrm{skew}}(TM)[1])$.

3.5 The (local) leaf space

Let L denote the space of integral curves of ξ in $M_\xi := \{x \in M : \xi_x \neq 0\}$, and denote by $\pi_L : M_\xi \to L$ the projection that maps a point to the integral curve through it. Since ξ vanishes nowhere, around any point in M_ξ there is a neighborhood such that the restriction of π_L thereto is a trivial fibration over a smooth 4-manifold; henceforth in this subsection, we will assume we have replaced M_ξ with such a neighborhood.

3.5.1 Descent of the canonical objects

Some of the objects we have already constructed on M descend to L via the projection π_L. One can determine which do by computing the Lie derivatives of the various tensorial objects with respect to the generating vector field ξ, but again it turns out to be much more efficient to compute derivatives in the tractor setting. Since any conformal tractor bundle $\mathcal{U} \to M$ is a natural bundle in the category of conformal manifolds, one may pull back any section $\mathbb{A} \in \Gamma(\mathcal{U})$ by the flow Ξ_t of ξ and define the Lie derivative $\mathcal{L}_\xi \mathbb{A}$ to be $\mathcal{L}_\xi \mathbb{A} := \partial_t|_0 \Xi_t^* \mathbb{A}$ [46]. Since the tractor projection $\Pi_0^{\mathcal{U}}$ is associated to a canonical vector space projection, it commutes with the Lie derivative. We exploit the following identity:

Lemma 3.7 ([24, Appendix A.3]). *Let $(M, \mathbf{c})$ be a conformal structure of signature (p, q), $p + q \geq 3$, let $\mathbb{U}$ be a $\mathrm{SO}(p+1, q+1)$-representation, and denote by $\mathcal{U} \to M$ the tractor bundle it induces. If $\xi \in \Gamma(TM)$ is a conformal Killing field for $\mathbf{c}$, and $\mathbb{A} \in \Gamma(\mathcal{U})$, then $\mathcal{L}_\xi \mathbb{A} = \nabla_\xi \mathbb{A} - L_0^{\mathcal{A}}(\xi) \cdot \mathbb{A}$, where $\cdot$ denotes the action on sections induced by the action $\mathfrak{so}(p+1, q+1) \times \mathbb{U} \to \mathbb{U}$. In particular, if $\mathbb{A}$ is parallel, then*

$$\mathcal{L}_\xi \mathbb{A} = -L_0^{\mathcal{A}}(\xi) \cdot \mathbb{A}. \tag{3.19}$$

Proposition 3.8. *Suppose* $(M, \mathbf{D})$ *is an oriented* $(2, 3, 5)$ *distribution and let* $\phi \in \Gamma(\Lambda^2 T^* M[3])$ *denote the corresponding normal conformal Killing 2-form. Suppose moreover that the conformal structure* $\mathbf{c_D}$ *induced by* $\mathbf{D}$ *admits a nonzero almost Einstein scale* $\sigma \in \Gamma(\mathcal{E}[1])$, *let* $\xi \in \Gamma(TM)$ *denote the corresponding normal conformal Killing field, and let* I, J, K *denote the normal conformal Killing 2-forms defined in Section* 3.4. *Then,*

$$\mathcal{L}_\xi \sigma = 0, \qquad (\mathcal{L}_\xi \phi)_{ab} = 3 J_{ab}, \qquad (\mathcal{L}_\xi I)_{ab} = -3\varepsilon J_{ab},$$
$$(\mathcal{L}_\xi J)_{ab} = 3 I_{ab}, \qquad (\mathcal{L}_\xi K)_{ab} = 0. \tag{3.20}$$

As usual, we scale σ *so that* $\varepsilon := -H_\Phi(\mathbb{S}, \mathbb{S}) \in \{-1, 0, +1\}$, *where* $\mathbb{S}^A := L_0^{\mathcal{V}}(\sigma)^A$ *is the parallel standard tractor corresponding to* σ.

In particular, σ *and* K *descend via* $\pi_L \colon M_\xi \to N$ *to well-defined objects* $\hat{\sigma}$ *and* $\hat{K}$, *but* ϕ *and* J *do not descend, and when* $\varepsilon \neq 0$ *neither does* I *(recall that when* $\varepsilon = 0$, $I = -K$).

Proof. As usual, denote $\mathbb{K} := L_0^{\mathcal{A}}(\xi)$, $\Phi := L_0^{\Lambda^3 \mathcal{V}^*}(\phi)$, $\Phi_I := L_0^{\Lambda^3 \mathcal{V}^*}(I)$, $\Phi_J := L_0^{\Lambda^3 \mathcal{V}^*}(J)$, and $\Phi_K := L_0^{\Lambda^3 \mathcal{V}^*}(K)$. Since ξ is a conformal Killing field, by (3.19) $(\mathcal{L}_\xi \mathbb{S})^A = -(L_0^{\mathcal{A}}(\xi) \cdot \mathbb{S})^A = -\mathbb{K}^A{}_B \mathbb{S}^B = -\mathbb{S}^C \Phi_C{}^A{}_B \mathbb{S}^B = 0$. Applying $\Pi_0^{\mathcal{V}}$ yields $\mathcal{L}_\xi \sigma = \mathcal{L}_\xi \Pi_0^{\mathcal{V}}(\mathbb{S}) = \Pi_0^{\mathcal{V}}(\mathcal{L}_\xi \mathbb{S}) = \Pi_0^{\mathcal{V}}(0) = 0$.

The proofs for J and ϕ are similar, and use the identities (2.3), (2.4).

By definition, $\mathcal{L}_\xi \Phi_K = \mathcal{L}_\xi[\mathbb{S}^\flat \wedge (\mathbb{S} \lrcorner \Phi)]$, and since $\mathcal{L}_\xi \mathbb{S} = 0$, we have $\mathcal{L}_\xi \Phi_K = \mathbb{S}^\flat \wedge (\mathbb{S} \lrcorner \mathcal{L}_\xi \Phi) = \mathbb{S}^\flat \wedge [\mathbb{S} \lrcorner (3\Phi_J)]$, but by (3.5) this is $3\mathbb{S}^\flat \wedge [\mathbb{S} \lrcorner (\mathbb{S} \lrcorner *_\Phi \Phi)]$, which is zero by symmetry. Applying $\Pi_0^{\Lambda^3 \mathcal{V}^*}$ gives $\mathcal{L}_\xi K = 0$.

Finally, (3.7) gives $\mathcal{L}_\xi \Phi_I = \mathcal{L}_\xi(-\varepsilon \Phi - \Phi_K) = -\varepsilon \mathcal{L}_\xi \Phi - \mathcal{L}_\xi \Phi_K = -\varepsilon(3\Phi_J) - (0) = -3\varepsilon \Phi_J$, and applying $\Pi_0^{\Lambda^3 \mathcal{V}^*}$ gives $\mathcal{L}_\xi I = -3\varepsilon J$. ∎

4 The conformal isometry problem

In this section we consider the problem of determining when two distributions $(M, \mathbf{D})$ and $(M, \mathbf{D}')$ induce the same oriented conformal structure; we say two such distributions are *conformally isometric*. This problem turns out to be intimately related to existence of a nonzero almost Einstein scale for $\mathbf{c_D}$.

Approaching this question at the level of underlying structures is prima facie difficult: The value of the conformal structure $\mathbf{c_D}$ at a point $x \in M$ induced by a $(2, 3, 5)$ distribution $(M, \mathbf{D})$ depends on the 4-jet of $\mathbf{D}$ at x [52, equation (54)] (or, essentially equivalently, multiple prolongations and normalizations), so analyzing directly the dependence of $\mathbf{c_D}$ on $\mathbf{D}$ involves apprehending high-order differential expressions that turn out to be cumbersome.

We have seen that in the tractor bundle setting, however, this construction is essentially algebraic: At each point, the parallel tractor G_2-structure $\Phi \in \Gamma(\Lambda^3 \mathcal{V}^*)$ determined by an oriented $(2, 3, 5)$ distribution $(M, \mathbf{D})$ determines the parallel tractor bilinear form $H_\Phi \in \Gamma(S^2 \mathcal{V}^*)$ and orientation $[\epsilon_\Phi]$ canonically associated to the oriented conformal structure $\mathbf{c_D}$. So, the problem of determining the distributions $(M, \mathbf{D}')$ such that $\mathbf{c_{D'}} = \mathbf{c_D}$ amounts to the corresponding algebraic problem of identifying for a G_2-structure Φ on a 7-dimensional real vector space $\mathbb{V}$ the G_2-structures Φ' on $\mathbb{V}$ such that $(H_{\Phi'}, [\epsilon_{\Phi'}]) = (H_\Phi, [\epsilon_\Phi])$. We solve this algebraic problem in Section 4.1 and then transfer the result to the setting of parallel sections of conformal tractor bundles to resolve the conformal isometry problem in Section 4.2.

4.1 The space of G_2-structures compatible with an $SO(3, 4)$-structure

In this subsection, which consists entirely of linear algebra, we characterize explicitly the space of G_2-structures compatible with a given $SO(3, 4)$-structure on a 7-dimensional real vector space $\mathbb{V}$, or more precisely, the $SO(3, 4)$-structure determined by a reference G_2-structure Φ. This characterization is essentially equivalent to that in [12, Remark 4] for the analogous inclusion of

the compact real form of G_2 into $SO(7, \mathbb{R})$. The following proposition can be readily verified by computing in an adapted frame. (Computer assistance proved particularly useful in this verification.)

Proposition 4.1. *Let $\mathbb{V}$ be a 7-dimensional real vector space and fix a G_2-structure $\Phi \in \Lambda^3\mathbb{V}^*$.*

1. *Fix a nonzero vector $\mathbb{S} \in \mathbb{V}$; by rescaling we may assume that $\varepsilon := -H_\Phi(\mathbb{S}, \mathbb{S}) \in \{-1, 0, +1\}$. For any $(\bar{A}, B) \in \mathbb{R}^2$ such that $-\varepsilon\bar{A}^2 + 2\bar{A} + B^2 = 0$ (there is a 1-parameter family of such pairs) the 3-form*

$$\Phi' := \Phi + \bar{A}\Phi_I + B\Phi_J \in \Lambda^3\mathbb{V}^* \tag{4.1}$$

   is a G_2-structure compatible with the $SO(3,4)$-structure $(H_\Phi, [\epsilon_\Phi])$, that is, $(H_{\Phi'}, [\epsilon_{\Phi'}]) = (H_\Phi, [\epsilon_\Phi])$.

2. *Conversely, all compatible G_2-structures arise this way: If a G_2-structure Φ' on $\mathbb{V}$ satisfies $(H_\Phi, [\epsilon_\Phi]) = (H_{\Phi'}, [\epsilon_{\Phi'}])$, there is a vector $\mathbb{S} \in \mathbb{V}$ (we may assume that $\varepsilon := -H_\Phi(\mathbb{S}, \mathbb{S}) \in \{-1, 0, +1\}$) and $(\bar{A}, B) \in \mathbb{R}^2$ satisfying $-\varepsilon\bar{A}^2 + 2\bar{A} + B^2 = 0$ such that Φ' is given by (4.1).*

Remark 4.2. Let $\mathbb{V}$ be the standard representation of $SO(3, 4)$, and let S denote the intersection of (a copy of) G_2 in $SO(3, 4)$ and the stabilizer subgroup in $SO(3, 4)$ of a nonzero vector $\mathbb{S} \in \mathbb{V}$. By Section 3.1.4 the space of 3-forms in $\Lambda^3\mathbb{V}^*$ stabilized by S is $\langle \Phi, \Phi_I, \Phi_J \rangle$. By Proposition 4.1, this determines a 1-parameter family of copies of G_2 containing S and contained in $SO(3, 4)$, or equivalently, a 1-parameter family $\mathcal{F}$ of G_2-structures Φ' compatible with the $SO(3, 4)$-structure, but S does not distinguish a G_2-structure in this family.

Remark 4.3. We may identify the space of G_2-structures that induce a particular $SO(3, 4)$-structure with the homogeneous space $SO(3, 4)/G_2$. Since $SO(3, 4)$ has two components but G_2 is connected, this homogeneous space has two components; the G_2-structures in one determine the opposite space and time orientations as those in the other. We can identify one component with the projectivization $\mathbb{P}(\mathbb{S}^{3,4})$ of the cone of spacelike elements in $\mathbb{R}^{4,4}$ and the other as the projectivization $\mathbb{P}(\mathbb{S}^{4,3})$ of the cone of timelike elements, and the homogeneous space $SO(3, 4)/G_2$ (the union of these projectivizations) as the complement of the neutral null quadric in $\mathbb{P}(\mathbb{R}^{4,4})$ [45, Theorem 2.1].

Henceforth denote by $\mathcal{F}[\Phi; \mathbb{S}]$ the 1-parameter family of G_2-structures compatible with the $SO(3, 4)$-structure $(H_\Phi, [\epsilon_\Phi])$ defined by Proposition 4.1. By construction, if $\Phi' \in \mathcal{F}[\Phi; \mathbb{S}]$, then $\mathcal{F}[\Phi'; \mathbb{S}] = \mathcal{F}[\Phi; \mathbb{S}]$.

Proposition 4.4. *For any G_2-structure $\Phi' \in \mathcal{F}[\Phi; \mathbb{S}]$, the endomorphism $(\mathbb{K}')^A{}_B := -\mathbb{S}^C(\Phi')_C{}^A{}_B$ coincides with $\mathbb{K}^A{}_B := -\mathbb{S}^C\Phi_C{}^A{}_B$.*

Proof. Proposition 4.1 gives that $\Phi' = \Phi + \bar{A}\Phi_I + B\Phi_J$ for some constants $\bar{A}, B$, so (3.8) gives that $\mathbb{K}' := -\mathbb{S} \lrcorner \Phi' = -\mathbb{S} \lrcorner (\Phi + \bar{A}\Phi_I + B\Phi_J) = -\mathbb{S} \lrcorner \Phi = \mathbb{K}$. ∎

We can readily parameterize the families $\mathcal{F}[\Phi; \mathbb{S}]$ of G_2-structures. It is convenient henceforth to split cases according to the causality type of $\mathbb{S}$, that is, according to ε. If $\varepsilon \neq 0$, then $\Phi = -\varepsilon(\Phi_I + \Phi_K)$, so in terms of $A := \bar{A} - \varepsilon$ and B, $\Phi' = A\Phi_I + B\Phi_J - \varepsilon\Phi_K$ and the condition on the coefficients is $A^2 - \varepsilon B^2 = 1$.

If $\varepsilon = -1$, then $A^2 + B^2 = 1$, and so we can parameterize $\mathcal{F}[\Phi; \mathbb{S}]$ by

$$\Phi_\upsilon := (\cos\upsilon)\Phi_I + (\sin\upsilon)\Phi_J + \Phi_K. \tag{4.2}$$

The parameterization descends to a bijection $\mathbb{R}/2\pi\mathbb{Z} \cong \mathbb{S}^1 \leftrightarrow \mathcal{F}[\Phi; \mathbb{S}]$, and $\Phi_0 = \Phi$.

If $\varepsilon = +1$, then $A^2 - B^2 = 1$, and so we can parameterize $\mathcal{F}[\Phi; \mathbb{S}]$ by

$$\Phi_t^{\mp} := (\mp \cosh t)\Phi_I + (\sinh t)\Phi_J - \Phi_K \tag{4.3}$$

and $\Phi_0^- = \Phi$.

If $\varepsilon = 0$, the compatibility conditions simplify to $2\bar{A} + B^2 = 0$, so we can parameterize $\mathcal{F}[\Phi; \mathbb{S}]$ by

$$\Phi_s := \Phi - \tfrac{1}{2}s^2\Phi_I + s\Phi_J \tag{4.4}$$

and $\Phi_0 = \Phi$.

Each of the above parameterizations Φ_u satisfies $\frac{d}{du}\big|_0 \Phi_u = \Phi_J$, and the parameterizations in the latter two cases are bijective. In the nonisotropic cases, we can encode the compatible G_2-structures efficiently in terms of the ε-complex volume forms in Proposition 3.2.

Proposition 4.5. *Suppose $\mathbb{S}$ is nonisotropic, and let $\Psi_{(A,B)} \in \Gamma(\Lambda^3_{\mathbb{C}_\varepsilon}\mathbb{W})$, $A^2 - \varepsilon B^2 = 1$, denote the corresponding 1-parameter family of ε-complex volume forms defined pointwise in Proposition 3.2. Then, $\mathcal{F}[\Phi; \mathbb{S}]$ consists of the G_2-structures*

$$\Phi_{(A,B)} := \Pi_{\mathbb{W}}^* \operatorname{Re}\Psi_{(A,B)} + \varepsilon\Phi_K,$$

$A^2 - \varepsilon B^2 = 1$. (Here, $\Pi_{\mathbb{W}}$ is the orthogonal projection $\mathbb{V} \to \mathbb{W}$.)

Proof. This follows immediately from the appearance of the condition $A^2 - \varepsilon B^2 = 1$ in the discussion after the proof of Proposition 4.4 and the form of $\Psi_{(A,B)}$. ∎

4.2 Conformally isometric $(2,3,5)$ distributions

We now transfer the results of Proposition 4.1 to the level of parallel sections of conformal tractor bundles, thereby proving Theorem B:

Proof of Theorem B. The oriented $(2,3,5)$ distributions $\mathbf{D}'$ conformally isometric to $\mathbf{D}$ are precisely those for which the corresponding parallel tractor 3-form Φ is compatible with the parallel tractor metric H_Φ and orientation $[\epsilon_\Phi]$. Transferring the content of Proposition 4.1 to the tractor setting gives that these are precisely the 3-forms $\Phi' := \Phi + \bar{A}\Phi_I + B\Phi_J \in \Gamma(\Lambda^3\mathcal{V}^*)$, and applying $\Pi_0^{\Lambda^3\mathcal{V}^*}$ (2.20) gives that the underlying normal conformal Killing 2-forms are $\phi' := \phi + \bar{A}I + BJ \in \Gamma(\Lambda^2 T^*M[3])$. Substituting for I, J respectively using (3.16) and (3.17) yields the formula (1.1). ∎

We reuse the notation $\mathcal{F}[\Phi; \mathbb{S}]$ for the 1-parameter family of parallel tractor G_2-structures defined pointwise by (4.1) by a parallel tractor 3-form Φ and a parallel, nonzero standard tractor $\mathbb{S}$. By analogy, we denote by $\mathcal{D}[\mathbf{D}; \sigma]$ the 1-parameter family of conformally isometric oriented $(2,3,5)$ distributions determined by $\mathbf{D}$ and σ as in Theorem B; we say that the distributions in the family are *related by* σ. Again, if $\mathbf{D}' \in \mathcal{D}[\mathbf{D}; \sigma]$, then by construction $\mathcal{D}[\mathbf{D}'; \sigma] = \mathcal{D}[\mathbf{D}; \sigma]$. Henceforth, $\mathcal{D}$ denotes a family $\mathcal{D}[\mathbf{D}; \sigma]$ for some $\mathbf{D}$ and σ.

Proposition 4.6. *Let $\mathbf{D}$ be an oriented $(2,3,5)$ distribution and σ an almost Einstein scale of $\mathbf{c_D}$. For any $\mathbf{D}' \in \mathcal{D}[\mathbf{D}; \sigma]$, the conformal Killing fields ξ and ξ' respectively determined by $(\mathbf{D}, \sigma)$ and $(\mathbf{D}', \sigma)$ coincide. In particular, ξ, the line $\mathbf{L} \subset TM|_{M_\xi}$ its restriction spans, and its orthogonal hyperplane field $\mathbf{C} := \mathbf{L}^\perp \subset TM|_{M_\xi}$ depend only on the family $\mathcal{D}[\mathbf{D}; \sigma]$ and not on $\mathbf{D}$.*

Proof. Let Φ, Φ' denote the parallel tractor G_2-structures corresponding respectively to $\mathbf{D}, \mathbf{D}'$. Translating Proposition 4.4 to the tractor bundle setting gives that $\mathbb{K}' := -L_0^{\mathcal{V}}(\sigma) \lrcorner \Phi'$ and $\mathbb{K} := -L_0^{\mathcal{V}}(\sigma) \lrcorner \Phi$ coincide, and hence $\xi' = \Pi_0^{\mathcal{A}}(\mathbb{K}') = \Pi_0^{\mathcal{A}}(\mathbb{K}) = \xi$. ∎

Corollary 4.7. *Let $\mathcal{D}$ be a 1-parameter family of conformally isometric oriented $(2,3,5)$ distributions related by an almost Einstein scale. Every distribution $\mathbf{D} \in \mathcal{D}$ satisfies $\mathbf{D} \subset \mathbf{C}$.*

4.2.1   Parameterizations of conformally isometric distributions

Now, given an oriented $(2,3,5)$ distribution $\mathbf{D}$ and a nonzero almost Einstein scale σ of $\mathbf{c_D}$, we can explicitly parameterize the family $\mathcal{D}[\mathbf{D};\sigma]$ they determine by passing to the projecting parts $\phi' := \Pi_0^{\Lambda^3 \mathcal{V}^*}(\Phi')$ of the corresponding parallel tractor G$_2$-structures $\Phi' \in \mathcal{F}[\Phi;\mathbb{S}]$. To do so, it is convenient to split cases according to the sign of the Einstein constant (2.23) of σ. As usual we denote by $\Phi \in \Gamma(\Lambda^3 \mathcal{V}^*)$ the parallel tractor G$_2$-structure corresponding to $\mathbf{D}$ and scale σ (by a constant) so that $\mathbb{S} := L_0^{\mathcal{V}}(\sigma)$ satisfies $\varepsilon := -H_\Phi(\mathbb{S},\mathbb{S}) \in \{-1,0,+1\}$.

For $\varepsilon = -1$, $\mathcal{D}[\mathbf{D};\mathbb{S}]$ consists of the distributions $\mathbf{D}_\upsilon$ corresponding to the normal conformal Killing 2-forms

$$\phi_\upsilon := \Pi_0^{\Lambda^3 \mathcal{V}^*}(\Phi_\upsilon) = (\cos \upsilon)I + (\sin \upsilon)J + K. \tag{4.5}$$

As for the corresponding family $\mathcal{F}[\Phi;\mathbb{S}]$ of parallel tractor G$_2$-structures, this parameterization descends to a bijection $\mathbb{R}/2\pi\mathbb{Z} \cong \mathbb{S}^1 \leftrightarrow \mathcal{D}[\mathbf{D};\sigma]$.

For $\varepsilon = +1$, $\mathcal{D}[\mathbf{D};\mathbb{S}]$ consists of the distributions $\mathbf{D}_t^\mp$ corresponding to

$$\phi_t^\mp := \Pi_0^{\Lambda^3 \mathcal{V}^*}(\Phi_t^\mp) = (\mp \cosh t)I + (\sinh t)J - K. \tag{4.6}$$

Each value of the parameter $(\pm, t)$ corresponds to a distinct distribution.

For $\varepsilon = 0$, $\mathcal{D}[\mathbf{D};\mathbb{S}]$ consists of the distributions $\mathbf{D}_\upsilon$ corresponding to

$$\phi_s := \Pi_0^{\Lambda^3 \mathcal{V}^*}(\Phi_s) = \phi - \tfrac{1}{2}s^2 I + sJ. \tag{4.7}$$

Each value of the parameter s corresponds to a distinct distribution.

These parameterizations are distinguished: Locally they agree (up to an overall constant) with the flow of the distinguished conformal Killing field ξ determined by $\mathbf{D}$ and σ.

Proposition 4.8. *Let $\mathbf{D}$ be an oriented $(2,3,5)$ distribution and σ an almost Einstein scale of $\mathbf{c_D}$. Denote $\xi := \iota_7(\sigma)$ and denote its flow by $\Xi_\bullet$. Then, for each $x \in M$ there is a neighborhood U of x and an interval T containing 0 such that:*

1) $(T\Xi_{\upsilon/3}) \cdot \mathbf{D}|_U = \mathbf{D}_\upsilon|_U$ *for all $\upsilon \in T$, if σ is Ricci-negative,*

2) $(T\Xi_{t/3}) \cdot \mathbf{D}|_U = \mathbf{D}_t^-|_U$ *and* $(T\Xi_{t/3}) \cdot \mathbf{D}_0^+|_U = \mathbf{D}_t^+|_U$ *for all $t \in T$, if σ is Ricci-positive, and*

3) $(T\Xi_{s/3}) \cdot \mathbf{D}|_U = \mathbf{D}_s|_U$ *for all $s \in T$, if σ is Ricci-flat.*

Proof. In the Ricci-negative case this follows immediately from the facts that the normal conformal Killing 2-form ϕ corresponding to $\mathbf{D}$ satisfies $\mathcal{L}_\xi \phi = 3J$ (3.20) and that the 1-parameter family of normal conformal Killing 2-forms ϕ_υ corresponding to the distributions $\mathbf{D}_\upsilon$ satisfy $\frac{d}{d\upsilon}\big|_0 \phi_\upsilon = J$. The other cases are analogous. ∎

4.2.2   Additional induced distributions

An almost Einstein scale for an oriented $(2,3,5)$ distribution $(M,\mathbf{D})$ naturally determines one or more additional 2-plane distributions on M, depending on the sign of the Einstein constant.

Definition 4.9. We say that two oriented $(2,3,5)$ distributions $\mathbf{D}, \mathbf{D}'$ in a given 1-parameter family $\mathcal{D}$ of conformally isometric oriented $(2,3,5)$ distributions are *antipodal* iff (1) they are distinct, and (2) their respective corresponding parallel tractor G$_2$-structures, Φ, Φ', together satisfy $\Phi \wedge \Phi' = 0$.

This condition is visibly symmetric in $\mathbf{D}, \mathbf{D}'$. Note that rearranging (2.11) and passing to the tractor bundle setting gives that $\Phi \wedge \Phi' = 4 *_\Phi \pi_7^3(\Phi')$, where π_7^3 denotes the bundle map $\Lambda^3 \mathcal{V}^* \to \mathcal{V}$ associated to the algebraic map of the same name in that equation.

Proposition 4.10. *Let $\mathcal{D}$ be a 1-parameter family of conformally isometric oriented $(2,3,5)$ distributions related by an almost Einstein scale and fix $\mathbf{D} \in \mathcal{D}$.*

- *If the almost Einstein scale determining $\mathcal{D}$ is non-Ricci-flat, there is precisely one distribution $\mathbf{E}$ antipodal to $\mathbf{D}$.*

- *If the almost Einstein scale determining $\mathcal{D}$ is Ricci-flat, there are no distributions antipodal to $\mathbf{D}$.*

Proof. Let Φ denote the parallel tractor G_2-structure corresponding to $\mathbf{D}$ and let $\mathbb{S}$ the parallel standard tractor corresponding to the almost Einstein scale. Any $\mathbf{D}' \in \mathcal{D}$ corresponds to a compatible parallel tractor G_2-structure $\Phi' \in \mathcal{F}[\Phi; \mathbb{S}]$ and by Proposition 4.1 we can write $\Phi' = \Phi + \bar{A}\Phi_I + B\Phi_J$, so $\Phi \wedge \Phi' = 4*_\Phi \pi_7^3(\Phi + \bar{A}\Phi_I + B\Phi_J)$. Since $\Phi \in \Lambda_1^3$, $\Phi_I \in \Lambda_7^3$, and $\Phi_J \in \Lambda_1^3 \oplus \Lambda_{27}^3$ (see (3.4), (3.5)), where $\Lambda_\bullet^3 \subset \Lambda^3 \mathcal{V}^*$ denote the subbundles associated to the G_2-representations $\Lambda_\bullet^3 \subset \Lambda^3 \mathbb{V}^*$, Schur's Lemma implies that $\pi_7^3(\Phi) = \pi_7^3(\Phi_I) = 0$. On the other hand, $\pi_7^3(\Phi_J) = \mathbb{S} \neq 0$, so $\Phi \wedge \Phi' = 4B*_\Phi \mathbb{S}$, which is zero iff $B = 0$.

If $\varepsilon = -1$, then in the parameterization $\{\Phi_\upsilon\}$ (4.2), the coefficient of Φ_J is $\sin \upsilon$, and this vanishes only for $\Phi_0 = \Phi = \Phi_I + \Phi_K$ and $\Phi_\pi = -\Phi_I + \Phi_K$, corresponding to the distributions $\mathbf{D} = \mathbf{D}_0$ and $\mathbf{E} := \mathbf{D}_\pi$.

If $\varepsilon = +1$, then in the parameterization $\{\Phi_t^\mp\}$ (4.3), the coefficient is $\sinh t$, and this vanishes only for $\Phi_0^+ = \Phi_0 = \Phi_I - \Phi_K$ and $\Phi_0^- = -\Phi_I - \Phi_K$, corresponding to the distributions $\mathbf{D} = \mathbf{D}_0^-$ and $\mathbf{E} := \mathbf{D}_0^+$.

Finally, if $\varepsilon = 0$, then in the parameterization $\{\Phi_s\}$ (4.4), the coefficient is s, and this vanishes only for $\Phi_0 = \Phi$ itself, corresponding to $\mathbf{D} = \mathbf{D}_0$. ∎

Though in the Ricci-flat case there are no antipodal distributions (see Proposition 4.10), there is in that case a suitable replacement: Given an oriented $(2,3,5)$ distribution $(M, \mathbf{D})$ and an almost Einstein scale σ of $\mathbf{c_D}$, the family $2s^{-2}\phi_s$ converges to the normal conformal Killing 2-form $\phi_\infty = -I = K$ as $s \to \pm\infty$. By continuity, ϕ_∞ is decomposable and hence defines on the set where ϕ_∞ does not vanish a distinguished 2-plane distribution $\mathbf{E}$ called the *null-complementary distribution* (for $\mathbf{D}$ and σ), and the distribution so defined is the same for every $\mathbf{D} \in \mathcal{D}[\mathbf{D}; \sigma]$. (This is analogous to the notion of antipodal distribution in that both antipodal and null-complementary distributions are spanned by the decomposable conformal Killing form $-I - \varepsilon K$.) Corollary 5.7 gives a precise description of the set on which ϕ_∞ does not vanish and hence on which $\mathbf{E}$ is defined; this set turns out to be the complement of a set that (if nonempty) has codimension ≥ 3. Corollary 5.17 below shows that $\mathbf{E}$ is integrable (and hence not a $(2,3,5)$ distribution).

Remark 4.11. Since G_2 is connected, it is contained in the connected component $SO_+(3,4)$ of the identity of $SO(3,4)$, and hence a G_2-structure determines space- and time-orientations on the underlying vector space. If we replace $SO(3,4)$ with $SO_+(3,4)$ in the description of the construction $\mathbf{c} \rightsquigarrow \mathbf{c_D}$ in Section 2.7, the construction assigns to an oriented $(2,3,5)$ distribution $\mathbf{D}$ the (oriented) conformal structure $\mathbf{c_D}$ along with space and time orientations.

Suppose $\mathbf{c_D}$ admits a nonzero almost Einstein scale σ. If σ is Ricci-negative or Ricci-flat, then the family $\mathcal{D} := \mathcal{D}[\mathbf{D}; \sigma]$ (parameterized respectively as in (4.5) or (4.7)) is connected, so the space and time orientations of $\mathbf{c_D}$ determined by the distributions in $\mathcal{D}$ all coincide.

If instead σ is Ricci-positive, then $\mathcal{D}$ consists of two connected components. Again, by connectness, the distributions $\mathbf{D}_t^-$, which comprise the component containing $\mathbf{D}_0^- = \mathbf{D}$ (in the notation of Section 4.2.1) all determine the same space and time orientations of $\mathbf{c_D}$, but the distributions $\mathbf{D}_t^+$, which comprise the other component and which include the antipodal distribution $\mathbf{D}_0^+ = \mathbf{E}$, determine the space and time orientations opposite those determined by $\mathbf{D}$.

In the case that σ is Ricci-positive, $\mathbf{D}$ and σ determine two additional distinguished distributions: In the notation of Section 4.2.1, the family $(\operatorname{sech} t)\phi_t^{\mp}$ converges to the normal conformal Killing 2-form $\mp I \pm' J$ as $t \to \pm'\infty$. By continuity $\phi_{\mp\infty} := \pm I + J$ are decomposable and hence determine distributions $\mathbf{D}_{\mp\infty}$ on the sets where they respectively do not vanish. Proposition 5.28 below describes precisely these sets (their complements, if nonempty, have codimension 3). By construction $\mathbf{D}_{\mp\infty}$ depend only on the family $\mathcal{D}[\mathbf{D};\sigma]$ and not $\mathbf{D}$ itself. Computing in an adapted frame shows that the corresponding parallel tractor 3-forms $\Phi_{\mp\infty} := L_0^{\Lambda^3 \mathcal{V}^*}(\phi_{\mp\infty})$ are not generic (they both annihilate $L_0^{\mathcal{V}}(\sigma)$, that is, they are not G_2-structures, and hence the distributions $\mathbf{D}_{\mp\infty}$ are not $(2,3,5)$ distributions).

4.2.3   Recovering the Einstein scale relating conformally isometric distributions

Given two distinct, oriented $(2,3,5)$ distributions $\mathbf{D}, \mathbf{D}'$ for which $\mathbf{c_D} = \mathbf{c_{D'}}$, we can reconstruct explicitly an almost Einstein scale $\sigma \in \Gamma(\mathcal{E}[1])$ of $\mathbf{c_D}$ for which $\mathbf{D}' \in \mathcal{D}[\mathbf{D};\sigma]$. If we require that the corresponding parallel standard tractor $\mathbb{S} := L_0^{\mathcal{V}}(\sigma)$ satisfies $\varepsilon := -H_\Phi(\mathbb{S}, \mathbb{S}) \in \{-1, 0, +1\}$, then $\mathbf{D}$ and $\mathbf{D}'$ together determine ε. If $\varepsilon \in \{\pm 1\}$, σ is determined up to sign. If $\varepsilon = 0$, then we may choose σ so that $\mathbf{D}' = \mathbf{D}_1$, where the right-hand side refers to the parameterization in Section 4.2.1, and this additional condition determines σ. The maps $\pi_1^3 \colon \Lambda^3 \mathcal{V}^* \to \mathcal{E}$, $\pi_7^3 \colon \Lambda^3 \mathcal{V}^* \to \mathcal{V}$, $\pi_{27}^3 \colon \Lambda^3 \mathcal{V}^* \to S_\circ^2 \mathcal{V}^*$ denote the bundle maps respectively associated to (2.10), (2.11), (2.13).

Algorithm 4.12. *Input: Fix distinct, oriented $(2,3,5)$ distributions $\mathbf{D}$, $\mathbf{D}'$ on M such that $\mathbf{c_D} = \mathbf{c_{D'}}$.*

Let Φ, Φ' denote the parallel tractor G_2-structures respectively corresponding to $\mathbf{D}$, $\mathbf{D}'$, and define $\mathbb{T} := \pi_7^3(\Phi')$. In each case, $\sigma := \Pi_0^{\mathcal{V}}(\mathbb{S})$.

- *If $H_\Phi(\mathbb{T}, \mathbb{T}) < 0$, set $s := \sqrt{-H_\Phi(\mathbb{T}, \mathbb{T})}$, so that $\mathbb{S} := s^{-1}\mathbb{T}$ satisfies $-H_\Phi(\mathbb{S}, \mathbb{S}) = +1$. Then, $\phi' = \phi_{\operatorname{arsinh} s} = \mp\sqrt{s^2 + 1}I + sJ - K$, where $\mp$ is the negative of the sign of $\pi_1^3(\Phi')$.*

- *If $H_\Phi(\mathbb{T}, \mathbb{T}) > 0$, set $s := -\sqrt{H_\Phi(\mathbb{T}, \mathbb{T})}$, so that $\mathbb{S} := s^{-1}\mathbb{T}$ satisfies $-H_\Phi(\mathbb{S}, \mathbb{S}) = -1$. Then, $\phi' = \phi_v = cI + sJ + K$, where $c := \frac{1}{4}[7\pi_1^3(\Phi') - 3]$ and v is an angle that satisfies $\cos v = c$, $\sin v = s$.*

- *If $H_\Phi(\mathbb{T}, \mathbb{T}) = 0$ but $\mathbb{T} \neq 0$, set $\mathbb{S} := -\frac{1}{4}\mathbb{T}$, giving $\phi' = \phi_1 = \phi - \frac{1}{2}I + J$.*

- *If $\mathbb{T} = 0$, then (by definition) the distributions are antipodal. Now, $\pi_{27}^3(\Phi') + \frac{1}{7}H_\Phi = \pm\mathbb{S}^\flat \otimes \mathbb{S}^\flat$ for a unique choice of $\pm$ and a parallel tractor $\mathbb{S} \in \Gamma(\mathcal{V})$ determined up to sign. If the equality holds for the sign $+$, then $\varepsilon = -1$ and $\phi' = \phi_\pi$. If the equality holds for $-$, then $\varepsilon = 1$ and $\phi' = \phi_0^+$.*

Since all of the involved tractor objects are parallel, the reconstruction problem is equivalent to the algebraic one recovering a normalized vector $\mathbb{S}$ from G_2-structures Φ, Φ' on a 7-dimensional real vector space inducing the same $\mathrm{SO}(3,4)$-structure such that $\Phi' \in \mathcal{F}[\Phi; \mathbb{S}]$. One can thus verify the algorithm by computing in an adapted basis.

5   The curved orbit decomposition

In this section, we treat the curved orbit decomposition of an oriented $(2,3,5)$ distribution determined by an almost Einstein scale, that is of a parabolic geometry of type (G_2, Q) to the stabilizer S of a nonzero ray in the standard representation $\mathbb{V}$ of G_2.

In Section 5.1 we briefly review the general theory of curved orbit decompositions and the decomposition of an oriented conformal manifold determined by an almost Einstein scale. In Section 5.2 we determine the orbit decomposition of the flat model. In Section 5.3 we state and prove geometric characterizations of the curved orbits, both in terms of tractor data and in

terms of data on the base manifold. In the remaining subsections we elaborate on the induced geometry determined on each of the curved orbits, which among other things yields proofs of Theorems D_-, D_+, and D_0.

5.1   The general theory of curved orbit decompositions

Here we follow [17]. If the holonomy $\mathrm{Hol}(\omega)$ of a Cartan geometry $(\mathcal{G}, \omega)$ is a proper subgroup of G, the principal connection $\hat{\omega}$ extending ω (see Section 2.3.2) can be reduced: If $H \leq G$ is a closed subgroup that contains any group in the conjugacy class $\mathrm{Hol}(\omega)$, $\hat{\mathcal{G}} := \mathcal{G} \times_P G$ admits a reduction $j \colon \mathcal{H} \to \hat{\mathcal{G}}$ of structure group to H, and $j^*\hat{\omega}$ is a principal connection on $\mathcal{H}$. Such a reduction can be viewed equivalently as a section of the associated fiber bundle $\hat{\mathcal{G}}/H := \hat{\mathcal{G}} \times_G (G/H)$. We henceforth work with an abstract G-homogeneous space $\mathcal{O}$ instead of G/H, which makes some exposition more convenient, and we call the corresponding G-equivariant section $s \colon \hat{\mathcal{G}} \to \mathcal{O}$ a *holonomy reduction of type $\mathcal{O}$*. Note that we can identify $\hat{\mathcal{G}} \times_G \mathcal{O}$ with $\mathcal{G} \times_P \mathcal{O}$.

Given a Cartan geometry $(\mathcal{G} \to M, \omega)$ of type (G, P) and a holonomy reduction thereof of type $\mathcal{O}$ corresponding to a section $s \colon \hat{\mathcal{G}} \to \mathcal{O}$, we define for each $x \in M$ the *P-type of x* (*with respect to s*) to be the P-orbit $s(\mathcal{G}_x) \subseteq \mathcal{O}$. This partitions M by P-type into a disjoint union $\bigcup_{a \in P \backslash \mathcal{O}} M_a$ of so-called *curved orbits* parameterized by the space $P \backslash \mathcal{O}$ of P-orbits of $\mathcal{O}$. By construction, the P-type decomposition of the flat model G/P coincides with the decomposition of G/P into H-orbits (for any particular choice of conjugacy class representative H). Put another way, the P-types correspond to the possible intersections of H and P in G up to conjugacy.

The central result of the theory of curved orbit decompositions is that each curved orbit inherits from the Cartan connection ω and the holonomy reduction s an appropriate Cartan geometry: We need some notation to state the result: Given a G-homogeneous space $\mathcal{O}$ and elements $x, x' \in \mathcal{O}$, we have $x' = g \cdot x$ for some $g \in G$, and their respective stabilizer subgroups G_x, $G_{x'}$ are related by $G_{x'} = gG_x g^{-1}$. If x, x' are in the same P-orbit, we can choose $g \in P$, and if we denote $P_x := G_x \cap P$, we likewise have $P_{x'} = gP_x g^{-1}$. Thus, as groups endowed with subgroups, $(G_x, P_x) \cong (G_{x'}, P_{x'})$. Given an orbit $a \in P \backslash \mathcal{O}$, we denote by (H, P_a) an abstract representative of the isomorphism class of groups so endowed.

Theorem 5.1 ([17, Theorem 2.6]). *Let $(\mathcal{G} \to M, \omega)$ be a parabolic (more generally, Cartan) geometry of type (G, P) with a holonomy reduction of type $\mathcal{O}$. Then, for each orbit $a \in P \backslash \mathcal{O}$, there is a principal bundle embedding $j_a \colon \mathcal{G}_a \hookrightarrow \mathcal{G}|_{M_a}$, and $(\mathcal{G}_a \to M_a, \omega_a)$ is a Cartan geometry of type (H_a, P_a) on the curved orbit M_a, where $\omega_a := j_a^*\omega$.*

Informally, since each P-type corresponds to an intersection of H and P up to conjugacy in G, for each such intersection $H \cap P$ (up to conjugacy) the induced Cartan geometry on the corresponding curved orbit has type $(H, H \cap P)$.

Example 5.2 (almost Einstein scales, [17, Theorem 3.5]). Given a conformal structure $(M, \mathbf{c})$ of signature (p, q), $n := p + q \geq 4$, by Theorem 2.6 a nonzero almost Einstein scale $\sigma \in \Gamma(\mathcal{E}[1])$ corresponds to a nonzero parallel standard tractor $\mathbb{S} := L_0^{\mathcal{V}}(\sigma)$ and hence determines a holonomy reduction to the stabilizer subgroup $\bar{S}$ of a nonzero vector in the standard representation $\mathbb{V}$ of $\mathrm{SO}(p+1, q+1)$; the conjugacy class of $\bar{S}$ depends on the causality type of $\mathbb{S}$.

If $\mathbb{S}$ is nonisotropic, there are three curved orbits, characterized by the sign of σ. The union of the open orbits is the complement $M - \Sigma$ of the zero locus $\Sigma := \{x \in M : \sigma_x = 0\}$, and the reduced Cartan geometries on these orbits are equivalent to the non-Ricci-flat Einstein metric $\sigma^{-2}\mathbf{g}$ of signature (p, q). If $\mathbb{S}$ is spacelike (timelike) the reduced Cartan geometry on the hypersurface curved orbit Σ is a normal parabolic geometry of type $(\mathrm{SO}(p, q+1), \bar{P})$ $((\mathrm{SO}(p+1, q), \bar{P}))$, which corresponds to an oriented conformal structure $\mathbf{c}_\Sigma$ of signature $(p-1, q)$ $((p, q-1))$.

If $\mathbb{S}$ is isotropic, then again there are two open orbits, and on the union $M - \Sigma$ of these, the reduced Cartan geometry is equivalent to the Ricci-flat metric $\sigma^{-2}\mathbf{g}$ of signature (p,q). In this case, Σ decomposes into three curved orbits: $\{x \in M : \sigma_x = 0, (\nabla\sigma)_x \neq 0\}$, $M_0^+ := \{x \in M : \sigma_x = 0, (\nabla\sigma)_x = 0, (\Delta\sigma)_x < 0\}$, and $M_0^- := \{x \in M : \sigma_x = 0, (\nabla\sigma) = 0, (\Delta\sigma)_x > 0\}$; here $\Delta\sigma$ denotes the Laplacian $\sigma_{,a}{}^a$. The curved orbits $M_0^\pm$ are discrete, but the formermost curved orbit is a hypersurface that naturally (locally) fibers by the integral curves of the line field $\mathbf{S}$ spanned by $\sigma^{,a}$, and the (local) leaf space thereof inherits a conformal structure of signature $(p-1, q-1)$.

Example 5.3 ($(2,3,5)$ conformal structures)**.** By Theorem 2.9, an oriented conformal structure $(M, \mathbf{c})$ of signature $(2,3)$ is induced by a $(2,3,5)$ distribution iff it admits a holonomy reduction to G_2. Since G_2 acts transitively on the flat model $\mathrm{SO}(3,4)/\bar{P} \cong \mathbb{S}^2 \times \mathbb{S}^3$, the holonomy reduction to G_2 determines only a single curved orbit.

5.2 The orbit decomposition of the flat model $\mathcal{M}$

In this subsection, we determine the orbits and stabilizer subgroups of the action of S on the flat model $\mathcal{M} := \mathrm{G}_2 / Q \cong \mathbb{S}^2 \times \mathbb{S}^3$, which by Theorem 5.1 determines the curved orbit decomposition of a parabolic geometry of type (G_2, Q).

Remark 5.4. Alternatively, as in the statements of Theorems D_-, D_+, and D_0, we could fix a conformal structure $\mathbf{c}$, that is, a normal parabolic geometry of type $(\mathrm{SO}(3,4), \bar{P})$ (Section 2.3.5) equipped with a holonomy reduction to the intersection S of a copy of G_2 in $\mathrm{SO}(3,4)$ and the stabilizer of a nonzero vector $\mathbb{S} \in \mathbb{V}$ (where we now temporarily view $\mathbb{V}$ as the standard representation of $\mathrm{SO}(3,4)$). By Remark 4.2, this determines a 1-parameter family $\mathcal{F} \subset \Lambda^3 \mathbb{V}^*$ of compatible G_2-structures but does not distinguish an element of this family. Transferring this statement to the setting of tractor bundles and then translating it into the setting of a tangent bundle, such a holonomy reduction determines a 1-parameter family $\mathcal{D}$ of conformally isometric oriented $(2,3,5)$ distributions for which $\mathbf{c} = \mathbf{c_D}$ for all $\mathbf{D} \in \mathcal{D}$, but does not distinguish a distribution among them.

As usual by scaling assume $\mathbb{S}$ satisfies $\varepsilon := -H_\Phi(\mathbb{S}, \mathbb{S}) \in \{-1, 0, +1\}$ and denote $\mathbb{W} := \langle \mathbb{S} \rangle^\perp$. The parabolic subgroup Q preserving any ray $x \in \mathcal{M}$ (spanned by the isotropic weighted vector $X \in \mathbb{V}[1]$) preserves the filtration $(\mathbb{V}_X^a)$ of $\mathbb{V}$ determined via (3.3) by X: Explicitly this is

$$
\begin{array}{ccccccc}
-2 & -1 & 0 & +1 & +2 & +3 & \\
\mathbb{V} & \supset \langle X \rangle^\perp & \supset \operatorname{im}(X \times \cdot) & \supset \ker(X \times \cdot) & \supset \langle X \rangle & \supset \{0\}
\end{array}
\tag{5.1}
$$

Here, $X \times \cdot$ is the map $-X^C \Phi_C{}^A{}_B \in \mathrm{End}(\mathbb{V})[1]$, and by the comments after (3.3) it satisfies $X \times \mathbb{V}_X^a = \mathbb{V}_X^{a+2}$. Since Q preserves this filtration, the corresponding set differences $\{x \in \mathcal{M} : \mathbb{S}_x \in \mathbb{V}_X^a - \mathbb{V}_X^{a+1}\}$ are each unions of curved orbits.

5.2.1 Ricci-negative case

In this case, which corresponds to $\mathbb{S}$ spacelike ($\varepsilon = -1$), $S \cong \mathrm{SU}(1,2)$ (Proposition 3.1), and $\mathbb{W}$ inherits a complex structure $\mathbb{K}$ (3.1) and a complex volume form Ψ (Proposition 3.2). Since $\ker X$ is isotropic and H_Φ has signature $(3,4)$, $\operatorname{im} X = (\ker X)^\perp$ is negative-semidefinite, so the only unions of orbits determined by the filtrations are $\{x : \mathbb{S} \in \mathbb{V} - \langle X \rangle^\perp\}$ and $\{x : \mathbb{S} \in \langle X \rangle^\perp - \operatorname{im} X\}$.

Pick a nonzero vector $\underline{X} \in \mathbb{V}$ in the ray determined by $X \in \mathbb{V}[1]$. Since $\mathbb{S}$ is nonisotropic, $\mathbb{V}$ decomposes (as an $\mathrm{SU}(1,2)$-module) as $\mathbb{W} \oplus \langle \mathbb{S} \rangle$, and with respect to this decomposition, $\underline{X}$ decomposes as $\underline{w} + \underline{\sigma}\mathbb{S} \in \mathbb{W} \oplus \langle \mathbb{S} \rangle$, where $\underline{\sigma} := H_\Phi(\underline{X}, \mathbb{S})$; so, $0 = H_\Phi(\underline{X}, \underline{X}) = H_\Phi(\underline{w}, \underline{w}) + \underline{\sigma}^2$.

If $\underline{\sigma} > 0$, then $\underline{\sigma}^{-1}\underline{w} \in \mathbb{W}$ satisfies $H_\Phi(\underline{\sigma}^{-1}\underline{w}, \underline{\sigma}^{-1}\underline{w}) = -1$. But the set of vectors $\underline{w}_0 \in \mathbb{W}$ satisfying $H(\underline{w}_0, \underline{w}_0) = -1$ is just the sphere $\mathbb{S}^{2,3}$, and $\mathrm{SU}(1,2)$ acts transitively on this space,

and hence on the 5-dimensional space $\mathcal{M}_5^+ \cong \mathbb{S}^{2,3}$ of rays in $\mathcal{M}$ it subtends. The isotropy group of the ray spanned by $\underline{w}_0 + \mathbb{S}$ preserves the appropriate restrictions of H, $\mathbb{K}$, Ψ to the four-dimensional subspace $\langle \underline{w}_0, \mathbb{K}\underline{w}_0 \rangle^\perp \subset \mathbb{W}$, so that space is neutral Hermitian and admits a complex volume form, and hence the isotropy subgroup is contained in $\mathrm{SU}(1,1) \cong \mathrm{SL}(2,\mathbb{R})$. On the other hand, the isotropy subgroup has dimension $\dim S - \dim \mathcal{M}_5^+ = 8 - 5 = 3$, so it must coincide with $\mathrm{SU}(1,1)$.

If $\underline{\sigma} = 0$, then $\underline{w} = \underline{X} \in \mathbb{W}$ is isotropic. The set of such vectors is the intersection of the null cone of H with $\mathbb{W}$. Again, $\mathrm{SU}(1,2)$ acts transitively on the ray projectivization $\mathcal{M}_4 \cong \mathbb{S}^3 \times \mathbb{S}^1$ of this space. By construction, the isotropy subgroup P_-, which (since $\dim \mathcal{M}_4 = 4$) has dimension four, is contained in the 5-dimensional stabilizer subgroup $P_{\mathrm{SU}(1,2)}$ in $\mathrm{SU}(1,2)$ of the complex line $\langle \underline{X}, \mathbb{K}\underline{X} \rangle \subset \mathbb{W}$ generated by $\underline{X}$; this latter group is (up to connectedness) the only parabolic subgroup of $\mathrm{SU}(1,2)$.

The case $\underline{\sigma} < 0$ is essentially identical to the case $\underline{\sigma} > 0$, and we denote the correponding orbit $\mathcal{M}_5^- \cong \mathbb{S}^{2,3}$.

5.2.2 Ricci-positive case

In this case, which corresponds to $\mathbb{S}$ timelike ($\varepsilon := +1$), $S \cong \mathrm{SL}(3,\mathbb{R})$, and $\mathbb{W}$ inherits a paracomplex structure $\mathbb{K}$ and a paracomplex volume form Ψ. This case is similar to the Ricci-negative one, and we omit details that are similar thereto. Since all of the vectors in $\operatorname{im} X - \ker X$ are timelike, however, that set difference corresponds to a union of orbits that has no analogue for the other causality types of $\mathbb{S}$.

Similarly to the Ricci-negative case, $\mathbb{V}$ decomposes (as an $\mathrm{SL}(3,\mathbb{R})$-module) as $\mathbb{W} \oplus \langle \mathbb{S} \rangle$, we can decompose $\underline{X} = \underline{w} + \underline{\sigma}\mathbb{S} \in \mathbb{W} \oplus \langle \mathbb{S} \rangle$, where $\sigma = H_\Phi(\underline{X}, \mathbb{S})$, and we have $0 = H_\Phi(\underline{X}, \underline{X}) = H_\Phi(\underline{w}, \underline{w}) - \underline{\sigma}^2$.

If $\underline{\sigma} > 0$, then the resulting curved orbit is $M_5^+ \cong \mathbb{S}^{2,3}$, and the stabilizer subgroup is isomorphic to $\mathrm{SL}(2,\mathbb{R})$.

If $\underline{\sigma} = 0$, then $\underline{w} = \underline{X} \in \mathbb{W}$ is isotropic. As mentioned at the beginning of this subsubsection, unlike in the Ricci-negative case, this subcase entails more than one orbit. To see this, let $\mathbb{E}$ denote the $(+1)$-eigenspace of $\mathbb{K}$. Using (the restriction of) H_Φ we may identify the -1-eigenspace with $\mathbb{E}^*$, and so we can write $\underline{X}$ as $(0, \underline{e}, \underline{\beta}) \in \langle \mathbb{S} \rangle \oplus \mathbb{E} \oplus \mathbb{E}^*$. Since $\underline{X}$ is isotropic, $0 = \frac{1}{2}H_\Phi(\underline{X}, \underline{X}) = \beta(\underline{e})$, and we can identify the ray projectivization of the set of such triples with $\mathbb{S}^2 \times \mathbb{S}^2$. Now, the action of S preserves whether each of the components $\underline{e}$, $\underline{\beta}$ is zero, giving three cases.

One can readily compute that S acts transitively on pairs $(\underline{e}, \underline{\beta})$ of nonzero elements with isotropy group $P_+ \cong \mathbb{R}_+ \ltimes \mathbb{R}^3$, which is characterized by its restriction to $\mathbb{E}$, and which in turn is given in a convenient basis by

$$\left\{ \begin{pmatrix} a & b & c \\ 0 & 1 & d \\ 0 & 0 & a^{-1} \end{pmatrix} : a \in \mathbb{R}_+; \, b, c, d \in \mathbb{R} \right\}.$$

So, the corresponding orbit $\mathcal{M}_4$ has dimension 4, and it follows from the remaining two cases that $\mathcal{M}_4 \cong \mathbb{S}^2 \times (\mathbb{S}^2 - \{\pm *\}) \cong \mathbb{S}^2 \times \mathbb{S}^1 \times \mathbb{R}$ for some point $* \in \mathbb{S}^2$. By construction, P_+ is contained in the 5-dimensional stabilizer subgroup P_{12} in $\mathrm{SL}(3,\mathbb{R})$ of the paracomplex line $\langle \underline{X}, \mathbb{K}\underline{X} \rangle \subset \mathbb{W}$ generated by $\underline{X}$, and we may identify P_{12} with the subgroup of the stabilizer subgroup in $\mathrm{SL}(3,\mathbb{R})$ of a complete flag in $\mathbb{W}$ that preserves either (equivalently, both of) the rays of the 1-dimensional subspace in the flag.

In the case that $\underline{e} \neq 0$ but $\underline{\beta} = 0$, S again acts transitively, and this time the isotropy subgroup is isomorphic to the (parabolic) stabilizer subgroup $P_1 := \mathrm{GL}(2,\mathbb{R}) \ltimes (\mathbb{R}^2)^*$ of a ray in $\mathbb{E}$, which is the first parabolic subgroup in $\mathrm{SL}(3,\mathbb{R})$, so the corresponding orbit is $\mathcal{M}_2^+ \cong \mathbb{S}^2$.

The dual case $\underline{e} = 0$, $\underline{\beta} \neq 0$ is similar: S acts transitively, and we can identify the isotropy subgroup with the (parabolic) stabilizer subgroup $P_2 := \mathrm{GL}(2,\mathbb{R}) \ltimes \mathbb{R}^2 \cong P_1$ of a dual ray in $\mathbb{E}^*$, so the corresponding orbit is $\mathcal{M}_2^- \cong \mathbb{S}^2$.

Finally, the case $\underline{\sigma} < 0$ is essentially identical to the case $\underline{\sigma} > 0$ and we denote the corresponding orbit by $\mathcal{M}_5^- \cong \mathbb{S}^{2,3}$.

5.2.3 Ricci-flat case

In this case, which corresponds to $\mathbb{S}$ isotropic ($\varepsilon = 0$), $S \cong \mathrm{SL}(2,\mathbb{R}) \ltimes Q_+$ and $\mathbb{W}$ inherits a endomorphism $\mathbb{K}$ whose square is zero. Since $\mathbb{S}$ is isotropic, it determines a filtration $(\mathbb{V}_\mathbb{S}^a)$ of $\mathbb{V}$. By symmetry, we may identify the sets $\{x \in \mathcal{M} : \mathbb{S}_x \in \mathbb{V}_X^a - \mathbb{V}_X^{a+1}\}$ that occur in this case with $\{x \in \mathcal{M} : X_x \in \mathbb{V} - \langle \mathbb{S} \rangle^\perp\}$, $\{x \in \mathcal{M} : X_x \in \langle \mathbb{S} \rangle^\perp - \mathrm{im}\,\mathbb{K}\}$, $\{x \in \mathcal{M} : X_x \in \ker \mathbb{K} - \langle \mathbb{S} \rangle\}$, and $\{x \in \mathcal{M} : X_x \in \langle \mathbb{S} \rangle - \{0\}\}$. (The difference $\{x \in \mathcal{M} : X_x \in \mathrm{im}\,\mathbb{K} - \ker \mathbb{K}\}$ does not occur here, as every vector in $\mathrm{im}\,\mathbb{K} - \ker \mathbb{K}$ is timelike.)

If $\underline{\sigma} > 0$, S acts transitively on the 5-dimensional space of rays. Computing directly shows that the isotropy subgroup is conjugate to the Levi factor $\mathrm{SL}(2,\mathbb{R}) < \mathrm{G}_2$, so we may identify $\mathcal{M}_5^+ \cong (\mathrm{SL}(2,\mathbb{R}) \ltimes Q_+)/\mathrm{SL}(2,\mathbb{R}) \cong Q_+ \cong \mathbb{R}^5$.

If $\underline{\sigma} = 0$, we see there are several possibilities. Since every vector in $\mathrm{im}\,\mathbb{K} - \ker \mathbb{K}$ is timelike, $\{x \in \mathcal{M} : X \in \langle \mathbb{S} \rangle^\perp - \mathrm{im}\,\mathbb{K}\}$ is the set of points $x \in \mathcal{M}$ such that $H_\Phi(X, \mathbb{S}) = 0$ but $X \times \mathbb{S} \neq 0$. Again, S acts transitively on the 4-dimensional space $\mathcal{M}_4$ of rays. In this case, computing gives that the isotropy subgroup is isomorphic to $\mathbb{R} \ltimes \mathbb{R}^3$.

Next, we consider the set of points $\{x \in \mathcal{M} : X \in \ker \mathbb{K} - \langle \mathbb{S} \rangle\}$. Since $\ker \mathbb{K}$ is totally isotropic, $\ker \mathbb{K} - \langle X \rangle$ is the set difference of a 3-dimensional affine space and a linear subspace, so the corresponding orbit $\mathcal{M}_2$ of rays is a twice-punctured 2-sphere. Again, computing directly gives that S acts transitively on this space, and the stabilizer subgroup is a certain 6-dimensional solvable group.

When $X \in \langle \mathbb{S} \rangle$, either $\mathbb{S}$ is in the ray determined by X or its opposite, and these correspond respectively to 0-dimensional orbits $\mathcal{M}_0^+$ and $\mathcal{M}_0^-$.

Finally, the case $\underline{\sigma} < 0$ is again essentially identical to the case $\underline{\sigma} > 0$, and again we denote the corresponding orbit $\mathcal{M}_5^- \cong \mathbb{R}^5$.

5.3 Characterizations of the curved orbits

In this subsection we give geometric characterizations of the curved orbits $M_\bullet$ determined by the holonomy reduction to S.

For the rest of this section, let $\mathbf{D}$ be an oriented $(2,3,5)$ distribution, denote the corresponding parallel tractor G_2-structure by Φ, and denote its components with respect to a scale τ by ϕ, χ, θ, ψ as in (2.25). Also fix an almost Einstein scale σ, denote the corresponding parallel tractor by $\mathbb{S} := L_0^{\mathcal{V}}(\sigma)$, and denote its components with respect to τ by σ, μ, ρ as in (2.18). On the zero locus $\Sigma := \{x \in M : \sigma_x = 0\}$ of σ, μ is invariant (that is, independent of the choice of scale τ), so on the set where moreover $\mu \neq 0$, μ determines a line field $\mathbf{S}$. See also Appendix A.

Proposition 5.5. *The curved orbits are characterized (separately) by the following conditions on tractorial and tangent data. The bullets $\bullet$ indicate which curved orbits occur for each causality type. For the curved orbits $M_2^\pm$, $(*)$ indicates the following: On $M_2^+ \cup M_2^-$ we have $\mu_x \in [\mathbf{D}, \mathbf{D}]_x[-1] - \mathbf{D}_x[-1]$, so projecting to $([\mathbf{D}, \mathbf{D}]_x / \mathbf{D}_x)[-1]$ gives a nonzero element and via the Levi bracket we can regard this as an element of $\Lambda^2 \mathbf{D}_x[-1]$. Then, $x \in M_2^-$ (M_2^+) iff this (weighted) bivector is oriented (anti-oriented). Also, $\Delta \sigma := \sigma_{,a}{}^a$.*

M_a	ε			tractor condition	tangent condition
	-1	0	$+1$		
$M_5^\pm$	•	•	•	$\pm H_\Phi(X, \mathbb{S}) > 0$	$\pm\sigma > 0$
M_4	•	•	•	$H_\Phi(X, \mathbb{S}) = 0$ $(X \times \mathbb{S}) \wedge X \neq 0$	$\sigma = 0$ $\xi \neq 0$
$M_2^\pm$			•	$X \times \mathbb{S} = \pm X$	$\xi = 0$ $(*)$
M_2		•		$X \times \mathbb{S} = 0$ $X \wedge \mathbb{S} \neq 0$	$\mathbf{S} \subset \mathbf{D}$
$M_0^\pm$		•		$\mathbb{S} \in (\pm X \cdot \mathbb{R}_+)[-1]$	$\sigma = 0$ $\nabla\sigma = 0$ $\mp\Delta\sigma > 0$

Proof. The characterizations of $M_5^\pm$ are immediate from the descriptions in Section 5.2.

Passing to the tractor setting, a point $x \in M$ is in M_4 if $\mathbb{S}_x \in \langle\mathbb{S}_x\rangle^\perp - \mathrm{im}(X_x \times \cdot)$ (for readability, in this proof we herein sometimes suppress the subscript $_x$). By the discussion after (5.1), $X \times (\mathrm{im}\, X) = \langle X\rangle$, so $\mathbb{S} \in \mathrm{im}(X \times \cdot)$ iff $(X \times \mathbb{S}) \wedge X = 0$, yielding the tractor characterization of M_4. Next, $x \in M_2^\pm$ if $\mathbb{S} \in \mathrm{im}(X \times \cdot) - \ker(X \times \cdot)$, so this curved orbit is characterized by $X \times \mathbb{S} \in \langle X\rangle - \{0\}$; in fact, since $X \times \mathbb{S} = -\mathbb{S} \times X = \mathbb{K}(X)$, X is an eigenvalue of $\mathbb{K}$, and we acutally have $X \times \mathbb{S} = \pm X$. Finally, $x \in M_0^\pm$ iff $\mathbb{S} \in \langle X\rangle$, that is, if $X \wedge \mathbb{S} = 0$, so $x \in M_2 = \ker X - \langle X\rangle$ iff $X \times \mathbb{S} = 0$ but $X \wedge \mathbb{S} \neq 0$.

In the splitting determined by a scale τ,

$$(X \times \mathbb{S})^A = \mathbb{K}^A{}_B X^B \stackrel{\tau}{=} \begin{pmatrix} \alpha \\ \xi^a \\ 0 \end{pmatrix} = \begin{pmatrix} \mu^c\theta_c \\ \sigma\theta^a + \mu_c\phi^{ca} \\ 0 \end{pmatrix} \in \Gamma\begin{pmatrix} \mathcal{E} \\ TM \\ \mathcal{E}[2] \end{pmatrix}.$$

So, the only nonzero component of $(X \times \mathbb{S}) \wedge X$ is ξ^a, which together with the tractor characterization gives the tangent bundle characterization of M_4. Since $\sigma = 0$ on M_4, we have on that curved orbit that $0 \neq \xi^a = \mu_c\phi^{ca}$, so $\mu^b \notin [\mathbf{D}, \mathbf{D}]$ and hence $\mathbf{S} \cap [\mathbf{D}, \mathbf{D}] = \{0\}$ (including this one, the assertions about the components of $\mathbb{S}$ and Φ in this proof all follow from Proposition 2.10).

For $M_2^\pm$, comparing the components of $X \times \mathbb{S} = \pm X$ gives $\xi^a = 0$ and $\alpha = \mu^c\theta_c = \pm 1$. Together with $\sigma = 0$ the first condition gives $\phi^{ab}\mu_b = 0$, which is equivalent to $\mu^b \in [\mathbf{D}, \mathbf{D}][-1]$; on the other hand, $\mu^c\theta_c \neq 0$ gives that $\mu^b \notin \mathbf{D}[-1]$, so μ^b projects to a nonzero element of $([\mathbf{D}, \mathbf{D}]/\mathbf{D})[-1] \cong \Lambda^2\mathbf{D}[-1]$ (the isomorphism is the one given by the Levi bracket). Since $\theta^c \in [\mathbf{D}, \mathbf{D}][-1] - \mathbf{D}[-1]$, it also determines a nonzero (and by construction, oriented) element of $\Lambda^2\mathbf{D}[-1]$. Thus, since $\theta^c\theta_c = -1$, we have $\mu^c\theta = -1$ (and hence $x \in M_2^-$) iff the element of $\Lambda^2\mathbf{D}[-1]$ determined by μ is oriented. (The above gives $\mathbf{S} \subset [\mathbf{D}, \mathbf{D}]$ and $\mathbb{S} \cap \mathbf{D} = \{0\}$.)

For M_2, if $\mathbb{S} \in \ker(X \times \cdot)$ we have $\sigma = 0$, $\xi^a = 0$, and $\alpha = 0$, so by the argument in the previous case we have $\mathbf{S} \subset \mathbf{D}$. Since $\mathbb{S} \notin \langle X\rangle$, we have $\mathbb{S} \wedge X \neq 0$, which is equivalent to $\mu^a = 0$, and by (2.21) $\mu = \nabla\sigma$.

Finally, if $\mathbb{S} \in \langle X\rangle$, which by the previous case comprises the points where $\mu = 0$. Again using $L_0^\mathcal{V}$ (and that $\sigma = 0$) gives that $\pm\rho = \mp\frac{1}{5}\Delta\sigma$, and so $\pm\rho > 0$ (and hence $x \in M_0^\pm$) iff $\mp\Delta\sigma > 0$. ∎

Corollary 5.6. *Let $\mathcal{D}$ be a 1-parameter family of conformally isometric oriented $(2, 3, 5)$ distributions related by the almost Einstein scale σ, and let ξ denote the corresponding conformal Killing field. Then:*

1. *The set $M_\xi := \{x \in M : \xi_x \neq 0\}$ is the union of the open and hypersurface curved orbits:*

   $$M_\xi = M_5 \cup M_4 \cup M_5^-.$$

   In particular, (a) $M_\xi \supseteq M_5^+ \cup M_5^-$, (b) the complement $M - M_\xi$ (if nonempty) has codimension 3, and (c) if σ is Ricci-negative then $M_\xi = M$.

2. *The curved orbits $M_5^\pm$ and M_4 are preserved by the flow of ξ.*

3. *If $x \in M_5^+ \cup M_5^-$, then for every distribution $\mathbf{D} \in \mathcal{D}$, $\mathbf{L}_x \subset [\mathbf{D}, \mathbf{D}]_x$ and $\mathbf{L}_x$ is transverse to $\mathbf{D}_x$. In particular, $\mathbf{L}_x$ is timelike.*

4. *If $x \in M_4$, then $\mathbf{L}_x \subset \mathbf{D}_x$ for every distribution $\mathbf{D} \in \mathcal{D}$. In particular, $\mathbf{L}_x$ is isotropic.*

Proof. Only (2) is not immediate: It follows from the characterizations $M_5^\pm = \{\pm\sigma > 0\}$ and $M_4 = \{\sigma = 0\} \cap M_\xi$ together with the fact (3.20) that the flow of ξ preserves σ. ∎

Corollary 5.7. *Let $\mathcal{D}$ be a 1-parameter family of conformally isometric oriented $(2,3,5)$ distributions related by the almost Ricci-flat scale σ. Then, the limiting distribution $\mathbf{D}_\infty$ vanishes precisely on $M_2 \cup M_0^+ \cup M_0^-$, which (if nonempty) has codimension 3, and so the null-complementary distribution $\mathbf{E}$ it determines is defined precisely on $M_\xi = M_5^+ \cup M_4 \cup M_5^-$.*

Proof. On $M_5^+ \cup M_5^-$, (5.4) below gives $\phi_\infty = I \stackrel{\sigma}{=} -\psi$, which vanishes nowhere by Proposition 2.10(8) (here, I is the 2-form determined by any $\mathbf{D} \in \mathcal{D}$; in the Ricci-flat case, it does not depend on that choice). On $M - (M_5^+ \cup M_5^-)$, (5.10) below gives that $I = \xi^\flat \wedge \mu^\flat$. In the proof of Proposition 5.5 we saw that on M_4, $\xi \in \mathbf{D}$ and $\mu \notin [\mathbf{D}, \mathbf{D}] \supset \mathbf{D}$, so $\xi^\flat$ and $\mu^\flat$ are linearly independent there. On $M_2 \cup M_0^+ \cup M_0^-$, that proposition gives $\xi = 0$, and hence $I = 0$ there. ∎

5.4 The open curved orbits $M_5^\pm$

In this section, we fix an oriented $(2,3,5)$ distribution $\mathbf{D}$ and a nonzero almost Einstein scale σ of $\mathbf{c_D}$ and restrict our attention to the union

$$M_5 := M_5^+ \cup M_5^- = \{x \in M : \sigma_x \neq 0\} = M - \Sigma$$

determined by the corresponding holonomy reduction; in particular M_5 is open, and moreover, by the discussion after Theorem 2.6, it is dense in M. Since σ is nowhere zero on M_5, we can work in the scale $\sigma|_{M_5}$ itself, in which many earlier formulae simplify. (Henceforth in this subsection, we suppress the restriction notation $|_{M_5}$.)

As usual, by rescaling we may assume that the parallel tractor $\mathbb{S}$ corresponding to σ satisfies $\varepsilon := -H_\Phi(\mathbb{S}, \mathbb{S}) \in \{-1, 0, +1\}$. Then, from the discussion after Theorem 2.6, $g_{ab} := \sigma^{-2}\mathbf{g}_{ab}$ is almost Einstein and has Schouten tensor $\mathsf{P}_{ab} = \frac{1}{2}\varepsilon g_{ab}$, or equivalently, Ricci tensor $R_{ab} = 4\varepsilon g_{ab}$, and hence scalar curvature $R = 20\varepsilon$. In the scale σ, the components of the parallel tractor $\mathbb{S}$ itself are

$$\sigma \stackrel{\sigma}{=} 1, \qquad \mu^e \stackrel{\sigma}{=} 0, \qquad \rho \stackrel{\sigma}{=} -\tfrac{1}{2}\varepsilon, \tag{5.2}$$

and substituting in (3.9) gives that

$$\mathbb{K}^A{}_B := -\mathbb{S}^C \Phi_C{}^A{}_B \stackrel{\sigma}{=} \begin{pmatrix} & \tfrac{1}{2}\varepsilon\underline{\theta}_b & \\ \tfrac{1}{2}(\varepsilon\underline{\phi}_{ab} + \bar{\underline{\phi}}_{ab}) \,| & & 0 \\ & \underline{\theta}^b & \end{pmatrix}. \tag{5.3}$$

We have introduced the 2-form $\bar{\phi}_{ab} := -2\psi_{ab} \in \Gamma(\Lambda^2 T^*M[1])$ for notational convenience. Note that $\xi^b \stackrel{\sigma}{=} \underline{\theta}^b$.

Substituting (5.2) in (3.13), (3.14), (3.15) gives that I, J, K simplify to

$$\underline{I}_{ab} = \tfrac{1}{2}\big(-\varepsilon\underline{\phi}_{ab} + \bar{\underline{\phi}}_{ab}\big), \qquad \underline{J}_{ab} = \xi^c \underline{\chi}_{cab} \qquad \underline{K}_{ab} = \tfrac{1}{2}\big(-\varepsilon\underline{\phi}_{ab} - \bar{\underline{\phi}}_{ab}\big). \tag{5.4}$$

The endomorphism component of $\mathbb{K}$ in the scale σ coincides with $-\underline{K}^a{}_b$.

5.4.1 The canonical splitting

The components of canonical tractor objects on M_5 in the splitting determined by σ are themselves canonical (and so are, just as well, their trivializations); in particular this includes the components χ_{abc}, θ_c, ψ_{bc} of Φ_{ABC}. Moreover, via Proposition 2.10, $\mathbf{D}$ and the scale σ together determine a canonical splitting of the canonical filtration $\mathbf{D} \subset [\mathbf{D}, \mathbf{D}] \subset TM_5$:

$$TM_5 = \mathbf{D} \oplus \mathbf{L} \oplus \mathbf{E}. \tag{5.5}$$

The fact that $\xi = \underline{\theta}$ gives the following:

Proposition 5.8. *The line field $\mathbf{L}$ in the splitting* (5.5) *determined by σ coincides with* (*the restriction to M_5 of*) *the line field of the same name spanned by ξ in Section* 3.2.

The splitting (5.5) entails isomorphisms $\mathbf{L} \cong [\mathbf{D}, \mathbf{D}]/\mathbf{D}$ and $\mathbf{E} \cong TM_5/[\mathbf{D}, \mathbf{D}]$, and so the components $\Lambda^2 \mathbf{D} \overset{\cong}{\to} [\mathbf{D}, \mathbf{D}]$ and $\mathbf{D} \otimes ([\mathbf{D}, \mathbf{D}]/\mathbf{D}) \overset{\cong}{\to} TM_5/[\mathbf{D}, \mathbf{D}]$ of the Levi bracket $\mathcal{L}$ give rise to isomorphisms $\Lambda^2 \mathbf{D} \overset{\cong}{\to} \mathbf{L}$ and $\mathbf{D} \otimes \mathbf{L} \overset{\cong}{\to} \mathbf{E}$. The preferred nonvanishing section $\xi = \underline{\theta} \in \Gamma(\mathbf{L})$ thus yields isomorphisms $\Lambda^2 \mathbf{D} \overset{\cong}{\to} \mathbb{R}$ (equivalently, a volume form on $\mathbf{D}$) and $\mathbf{D} \overset{\cong}{\to} \mathbf{E}$. We may use the volume form to identify $\mathbf{D} \cong \Lambda^2 \mathbf{D} \otimes \mathbf{D}^* \cong \mathbf{D}^*$ and dualize the previous isomorphism to yield a bilinear pairing $\mathbf{D} \times \mathbf{E} \to \mathbb{R}$.

This splitting is closely connected with the notions of antipodal and null-complementary distributions introduced in Section 4.2.2.

Theorem 5.9. *The canonical distribution $\mathbf{E}$ spanned by the tractor component ψ in the splitting $\mathbf{D} \oplus \mathbf{L} \oplus \mathbf{E}$ determined by σ* (*equivalently, the distribution determined by $\bar{\phi}$*), *is* (*the restriction to M_5 of*) *the distribution antipodal or null-complementary to $\mathbf{D}$.*

Proof. From Section 4.2.2 the antipodal or null-complementary distribution is spanned by $I - K$, and substituting using (5.4) gives that this is $\bar{\phi}$. ∎

Remark 5.10. Together $\mathbf{D}$ and $\mathbf{L}$ comprise the underlying data of another parabolic geometry on M_5, namely of type $(\mathrm{SL}(4, \mathbb{R}), P_{12})$, where P_{12} is the stabilizer subgroup of a partial flag in $\mathbb{R}^4$ of signature $(1, 2)$ under the action induced by the standard action. The underlying structure for a regular, normal geometry of this type is a *generalized path geometry* in dimension 5, which consists of a 5-manifold M_5, a line field $\mathbf{L} \subset TM_5$, and a 2-plane distribution $\mathbf{D} \subset TM$ such that (1) $\mathbf{L} \cap \mathbf{D} = \{0\}$, (2) $[\mathbf{D}, \mathbf{D}] \subseteq \mathbf{D} \oplus \mathbf{L}$, and (3) if $\eta \in \Gamma(\mathbf{D})$, $\xi' \in \Gamma(\mathbf{L})$, and $x \in M$ together satisfy $[\eta, \xi']_x = 0$, then $\eta_x = 0$ or $\xi'_x = 0$ [18, Section 4.3.3] (in our case, these conditions follow from the properties of the Levi bracket $\mathcal{L}$). In this dimension, this geometry is sometimes called XXO geometry, in reference to the marked Dynkin diagram that encodes it. The restriction of $\mathbf{L}$ to $M_\xi - M_5 = M_4$ is contained in $\mathbf{D}|_{M_4}$, so M_5 is the largest set on which this construction yields a generalized path geometry.

5.4.2 The canonical hyperplane distribution

Denote by $\mathbf{C} \subset TM_\xi$ the hyperplane distribution orthogonal to $\mathbf{L} := \langle \xi \rangle |_{M_\xi}$.

Proposition 5.11. *Let $\mathcal{D}$ be a family of conformally isometric oriented $(2, 3, 5)$ distributions related by an almost Einstein scale σ. Then, on M_5:*

1. *The pullback $g_{\mathbf{C}}$ of the metric $g := \sigma^{-2} \mathbf{g}$ to the hyperplane distribution $\mathbf{C}$ has neutral signature.*

2. *The hyperplane distribution $\mathbf{C}$ is a contact distribution iff σ is not Ricci-flat.*

3. *If we fix $\mathbf{D} \in \mathcal{D}$, then $\mathbf{C} = \mathbf{D} \oplus \mathbf{E}$, where $\mathbf{E}$ is the distribution antipodal or null-complementary to $\mathbf{D}$.*

4. *The canonical pairing $\mathbf{D} \times \mathbf{E} \to \mathbb{R}$ is nondegenerate, and the bilinear form it induces on $\mathbf{C}$ via the direct sum decomposition $\mathbf{C} = \mathbf{D} \oplus \mathbf{E}$ is $g_{\mathbf{C}}$.*

Proof. (1) The conformal class has signature $(2,3)$, and Corollary 5.6(3) gives that $\mathbf{L} = \mathbf{C}^\perp$ is timelike.

(2) In the scale σ, $\xi \overset{\sigma}{=} \underline{\theta}$ and $d\xi^\flat \overset{\sigma}{=} \underline{K}^\flat = -\frac{1}{2}(\varepsilon\underline{\phi} + \bar{\underline{\phi}})$. The decomposability of ϕ and $\bar{\phi}$ (the latter follows from Proposition 2.10(5)) implies $\underline{\xi}^\flat \wedge d\xi^\flat \wedge d\xi^\flat \overset{\sigma}{=} -\frac{1}{2}\varepsilon\underline{\phi} \wedge \underline{\theta} \wedge \bar{\underline{\phi}}$. By Proposition 2.10(8), $\phi \wedge \theta \wedge \bar{\phi}$ is a nonzero multiple of the conformal volume form $\epsilon_{\mathbf{g}}$ and so vanishes nowhere iff $\varepsilon \neq 0$.

(3) By Theorem 5.9, $\mathbf{D}$ and $\mathbf{E}$ are transverse, and by Corollary 4.7 they are both contained in $\mathbf{C}$, so the claim follows from counting dimensions.

(4) This follows from computing in an adapted frame. ∎

Computing in an adapted frame gives the following pointwise description:

Proposition 5.12. *Let $\mathcal{D}$ be a 1-parameter family of conformally isometric oriented $(2,3,5)$ distributions related by an almost Einstein scale σ. Then, for any $x \in M_5 = \{x \in M : \sigma_x \neq 0\}$, the family $\mathcal{D}_x := \{\mathbf{D}_x : \mathbf{D} \in \mathcal{D}\}$ is precisely the set of totally isotropic 2-planes in $\mathbf{C}_x$ self-dual with respect to the (weighted) bilinear form $\mathbf{g}_{\mathbf{C}}$ and the orientation determined by $\epsilon_{\mathbf{C}}$.*

Computing in an adapted frame shows that the images of $I, J, K \in \Gamma(\mathrm{End}(TM)[1])$ are contained in $\mathbf{C}[1]$, so they restrict sections of $\mathrm{End}(\mathbf{C})[1]$, which by mild abuse of notation we denote $I^\alpha{}_\beta$, $J^\alpha{}_\beta$, $K^\alpha{}_\beta$ (here and henceforth, we use lowercase Greek indices $\alpha, \beta, \gamma, \ldots$ for tensorial objects on M). It also gives that these maps satisfy, for example, $I^\alpha{}_\gamma I^\gamma{}_\beta = -\varepsilon\sigma^2\delta^\alpha{}_\beta \in \Gamma(\mathrm{End}(\mathbf{C})[2])$ and Proposition 5.13 below. In the scale σ, this and the remaining equations become:

$$\underline{I}^\alpha{}_\gamma\underline{I}^\gamma{}_\beta \overset{\sigma}{=} -\varepsilon\delta^a{}_\beta, \qquad \underline{J}^\alpha{}_\gamma\underline{K}^\gamma{}_\beta = -\underline{K}^\alpha{}_\gamma\underline{J}^\gamma{}_\beta \overset{\sigma}{=} \underline{I}^\alpha{}_\beta,$$
$$\underline{J}^\alpha{}_\gamma\underline{J}^\gamma{}_\beta \overset{\sigma}{=} \delta^a{}_\beta, \qquad \underline{K}^\alpha{}_\gamma\underline{I}^\gamma{}_\beta = -\underline{I}^\alpha{}_\gamma\underline{K}^\gamma{}_\beta \overset{\sigma}{=} -\varepsilon\underline{J}^\alpha{}_\beta,$$
$$\underline{K}^\alpha{}_\gamma\underline{K}^\gamma{}_\beta \overset{\sigma}{=} \varepsilon\delta^a{}_\beta, \qquad \underline{I}^\alpha{}_\gamma\underline{J}^\gamma{}_\beta = -\underline{J}^\alpha{}_\gamma\underline{I}^\gamma{}_\beta \overset{\sigma}{=} -\underline{K}^\alpha{}_\beta. \tag{5.6}$$

Proposition 5.13. *Let $(M, \mathbf{D})$ be an oriented $(2,3,5)$ distribution and σ an almost Einstein scale σ for $\mathbf{c}_{\mathbf{D}}$, and let $\mathbf{E}$ be the distribution antipodal or null-complementary to $\mathbf{D}$ determined by σ. Then, on M_5:*

1. *$\underline{I} \in \mathrm{End}(C)$ is an almost $(-\varepsilon)$-complex structure, and $-\underline{I}|_{\mathbf{D}}$ is the isomorphism $\mathbf{D} \overset{\cong}{\to} \mathbf{E}$ determined by σ introduced at the beignning of the subsection. If σ is Ricci-flat, then $\underline{I}|_{\mathbf{E}} = 0$, and if σ is not Ricci-flat, then $\underline{I}|_{\mathbf{E}}$ is an isomorphism $\mathbf{E} \overset{\cong}{\to} \mathbf{D}$.*

2. *$\underline{J} \in \mathrm{End}(\mathbf{C})$ is an almost paracomplex structure, and its eigenspaces are $\mathbf{D}$ and $\mathbf{E}$: $\underline{J}|_{\mathbf{D}} = \mathrm{id}_{\mathbf{D}}$, $\underline{J}|_{\mathbf{E}} = -\mathrm{id}_{\mathbf{E}}$.*

3. *$\underline{K} \in \mathrm{End}(\mathbf{C})$ is an almost ε-complex structure, $\underline{K}|_{\mathbf{D}} = -\underline{I}|_{\mathbf{D}}$, and $\underline{K}|_{\mathbf{E}} = \underline{I}|_{\mathbf{E}}$. If $\varepsilon = +1$, the (∓ 1)-eigenspaces of $\underline{K}$ are the limiting 2-plane distributions $\mathbf{D}_{\mp}$ defined in Section 4.2.2.*

In the non-Ricci-flat case, we can identify the pointwise $\mathrm{U}(1,1)$-structure on M_5 as follows:

Proposition 5.14. *Let $\mathcal{D}$ be a 1-parameter family of conformally isometric oriented $(2,3,5)$ distributions related by a non-Ricci-flat almost Einstein scale σ.*

1. *For any $\mathbf{D} \in \mathcal{D}$, the endomorphisms $\underline{I}$, $\underline{J}$, $\underline{K}$ determine an almost split-quaternionic structure on (the restriction to M_5 of) $\mathbf{C}$, that is, an injective ring homomorphism $\widetilde{\mathbb{H}} \hookrightarrow \mathrm{End}(\mathbf{C}_x)$ for each $x \in M_5$, where $\widetilde{\mathbb{H}}$ is the ring of split quaternions.*

2. *The almost split-quaternionic structure in (1) depends only on $\mathcal{D}$.*

Proof. (1) This follows immediately from the identities (5.6). (2) This follows from computing in an adapted frame. ∎

5.4.3 The ε-Sasaki structure

The union M_5 of the open orbits turns out also to inherit an ε-Sasaki structure, the odd-dimensional analogue of a Kähler structure.

Definition 5.15. For $\varepsilon \in \{-1, 0, +1\}$, an ε-*Sasaki structure* on a (necessarily odd-dimensional) manifold M is a pair (h, ξ), where $h \in \Gamma(S^2 T^* M)$ is a pseudo-Riemannian metric on M and $\xi \in \Gamma(TM)$ is a vector field on M such that

1) $h_{ab}\xi^a\xi^b = 1$,

2) $\xi_{(a,b)} = 0$ (or equivalently, $(\mathcal{L}_\xi h)_{ab} = 0$, that is, ξ is a Killing field for h), and

3) $\xi^a{}_{,bc} = \varepsilon(\xi^a h_{bc} - \delta^a{}_c \xi_b)$.

An ε-Sasaki–Einstein structure is an ε-Sasaki structure (h, ξ) for which h is Einstein.

It follows quickly from the definitions that the restriction of $\xi^a{}_b$ is an almost ε-complex structure on the subbundle $\langle \xi \rangle^\perp$, and that if $\varepsilon = +1$, the (± 1)-eigenbundles of this restriction (which have equal, constant rank) are integrable and totally isotropic.

Theorem 5.16. *Let $\mathcal{D}$ be a 1-parameter family of conformally isometric oriented $(2, 3, 5)$ distributions related by an almost Einstein scale σ. On $M_5 := \{\sigma \neq 0\}$, the signature-$(3, 2)$ Einstein metric*

$$-g = -\sigma^{-2}\mathbf{c_D}$$

and the Killing field ξ together comprise an ε-Sasaki–Einstein structure.

Proof. Substituting in (3.2) the components of $\mathbb{K}$ in the scale σ (5.3), using that the endomorphism component of $\mathbb{K}$ in that scale is $-\underline{K}^a{}_b$, and simplifying leaves

$$-\xi_c\xi^c \overset{\sigma}{=} 1, \qquad \underline{K}^a{}_c\xi^c \overset{\sigma}{=} 0, \qquad \underline{K}^a{}_c\underline{K}^c{}_b \overset{\sigma}{=} \varepsilon(\delta^a{}_b + \xi^a\xi_b), \tag{5.7}$$

together with $\xi^c K^b{}_c = 0$, but this last equation follows from the second equation and the g-skewness of $\underline{K}$. Similarly, expanding the left-hand side of $\nabla\mathbb{K} = 0$ using (5.3) with respect to the scale σ, eliminating duplicated equations, and rearranging gives

$$\xi^b{}_{,c} \overset{\sigma}{=} \underline{K}^b{}_c, \qquad \underline{K}^a{}_{b,c} \overset{\sigma}{=} -\varepsilon(\xi^a g_{bc} - \xi_b\delta^a{}_c). \tag{5.8}$$

1. Rearranging the first equation in (5.7) gives $(-g_{ab})\xi^a\xi^b = 1$.

2. Since $\underline{K}^a{}_c$ is g-skew, symmetrizing the first equation in (5.8) with g_{ab} gives $\xi_{(b,c)} = \underline{K}_{(bc)}$ = 0; equivalently, ξ is a Killing field for g and hence for $-g$.

3. This follows immediately from substituting the first equation in (5.8) into the second. (Here indices are raised and lowered with g and not with the candidate Sasaki metric $-g$.) ∎

Corollary 5.17. *Let $\mathbf{D}$ be an oriented $(2, 3, 5)$ distribution and σ a nonzero Ricci-flat almost Einstein scale for $\mathbf{c_D}$. The null-complementary distribution $\mathbf{E}$ they determine is integrable.*

Proof. Since $\varepsilon = 0$, the second equation of (5.8) says that $\underline{K}$ is parallel. By Section 4.2.2, $\underline{K}$ is decomposable and (as a decomposable bivector field) spans $\mathbf{E}|_{M_5}$, so that distribution is ∇-parallel and hence integrable. But M_5 is dense in M, and integrability is a closed condition, so $\mathbf{E}$ is integrable (on all of M). ∎

Proposition 5.18. *Let (g, ξ) be an oriented ε-Sasaki–Einstein structure (with $\varepsilon = \pm 1$) of signature $(2, 3)$ on a 5-manifold M. Then, locally, there is a canonical 1-parameter family $\mathcal{D}$ of oriented $(2, 3, 5)$ distributions related by an almost Einstein scale for which the associated conformal structure is $\mathbf{c} = [-g]$.*

Proof. Set $\mathbf{c} = [-g]$, let $\sigma \in \Gamma(\mathcal{E}[1])$ be the unique section such that $-g = \sigma^{-2}\mathbf{g}$, and $\mathbb{S} := L_0^{\mathcal{V}}(\sigma)$ the corresponding parallel tractor. Define the adjoint tractor $\mathbb{K} := L_0^{\mathcal{A}}(\xi)$. It is known that a Sasaki–Einstein metric g satisfies $\mathsf{P}_{ab} = \frac{1}{2}\varepsilon g_{ab}$, and so by (2.23) $\mathbb{S}$ satisfies $H(\mathbb{S},\mathbb{S}) = -\varepsilon$, where H is the tractor metric determined by $\mathbf{c}$. Thus, the proof of Theorem 5.16 gives

$$\nabla^{\mathcal{V}}\mathbb{K} = 0, \qquad \mathbb{K}^2 = \varepsilon\,\mathrm{id} + \mathbb{S} \otimes \mathbb{S}^{\flat}, \qquad \text{and} \qquad \mathbb{S}\lrcorner\,\mathbb{K} = 0.$$

Transferring the content of Section 3.1.2 to the tractor bundle setting then shows that the parallel subbundle $\mathcal{W} := \langle\mathbb{S}\rangle^{\perp} \subset \mathcal{V}$ inherits a parallel almost ε-Hermitian structure.

Denote the curvature of the normal tractor connection by $\Omega_{ab}{}^C{}_D \in \Gamma(\Lambda^2 T^*M \otimes \mathrm{End}(\mathcal{V}))$. The curvature of the induced connection on the bundle $\Lambda_{\mathbb{C}_\varepsilon}^3 \mathcal{W}$ of ε-complex volume forms on $\mathcal{W}$ is given by $\Omega_{ab}{}^C{}_C + \varepsilon i_\varepsilon \Omega_{ab}{}^C{}_D \mathbb{K}^D{}_C$. Now $\Omega_{ab}{}^C{}_C = 0$ by skew-symmetry, and, since $\mathbb{K}$ is parallel, $\Omega_{ab}{}^C{}_D \mathbb{K}^D{}_C = 0$ by [16, Proposition 2.1]. Thus, the induced connection on $\Lambda_{\mathbb{C}_\varepsilon}^3 \mathcal{W}$ is flat, so it admits local parallel sections. Let Ψ be such a (local) parallel section normalized so that $\Psi \wedge \bar{\Psi} = -\frac{4}{3} i_\varepsilon \mathbb{K} \wedge \mathbb{K} \wedge \mathbb{K}$. Denote $\mathrm{Re}\,\Psi$ the pullback to $\mathcal{V}$ of the real part of Ψ. Then, by Proposition 3.3, the parallel tractor 3-form

$$\Phi = \mathrm{Re}\Psi + \varepsilon\,\mathbb{S}^{\flat} \wedge \mathbb{K} \in \Gamma(\Lambda^3 \mathcal{V})$$

defines a parallel G_2-structure on $\mathcal{V}$ compatible with H. By the discussion before Proposition 2.10, its projecting slot defines a $(2,3,5)$ distribution with associated conformal structure $\mathbf{c} = [-g]$. Finally, parallel sections of $\Lambda_{\mathbb{C}_\epsilon}^3 \mathcal{W}$ satisfying $\Psi \wedge \bar{\Psi} = -\frac{4}{3} i_\varepsilon \mathbb{K} \wedge \mathbb{K} \wedge \mathbb{K}$ are parametrized by $\{z \in \mathbb{C}_\varepsilon \colon z\bar{z} = 1\}$ (that is, $\mathbb{S}^1$ if $\varepsilon = -1$ and $\mathrm{SO}(1,1)$ if $\varepsilon = 1$). $\blacksquare$

5.4.4 Projective geometry

On the complement M_5 of the zero locus Σ of σ, we may canonically identify (the restriction of) the parallel subbundle $\mathcal{W} := \langle\mathbb{S}\rangle^{\perp}$ with the *projective* tractor bundle of the projective structure $[\nabla^g]$, where g is the Einstein metric $\sigma^{-2}\mathbf{g}$, and the connection $\nabla^{\mathcal{W}}$ that $\nabla^{\mathcal{V}}$ induces on $\mathcal{W}$ with the normal projective tractor connection [34, Section 8].

This compatibility determines a holonomy reduction of the latter connection to S, and one can analyze separately the consequences of this projective reduction. For example, if σ is non-Ricci-flat, then lowering an index of the parallel complex structure $\mathbb{K} \in \Gamma(\mathrm{End}(\mathcal{W}))$ with $H|_{\mathcal{W}}$ yields a parallel symplectic form on $\mathcal{W}$. A holonomy reduction of the normal projective tractor connection on a $(2m+1)$-dimensional projective manifold M to the stabilizer $\mathrm{Sp}(2m+2,\mathbb{R})$ of a symplectic form on a $(2m+2)$-dimensional real vector space determines precisely a torsion-free contact projective structure [18, Section 4.2.6] on M suitably compatible with the projective structure [31].

This also leads to an alternative proof that the open curved orbits inherit a Sasaki–Einstein structure in the Ricci-negative case: The holonomy of $\nabla^{\mathcal{W}}$ is reduced to $\mathrm{SU}(1,2)$, but [3, Section 4.2.2] identifies $\mathfrak{su}(p',q')$ as the Lie algebra to which the projective holonomy connection determined by an Sasaki–Einstein structure is reduced.

The upcoming article [35] discusses the consequences of a holonomy reduction of (the normal projective tractor connection of) a $(2m+1)$-dimensional projective structure to the special unitary group $\mathrm{SU}(p',q')$, $p' + q' = m + 1$.

5.4.5 The open leaf space L_4

As in Section 3.5, we assume that we have replaced M by an open subset so that π_L is a locally trivial fibration over a smooth 4-manifold. Define $L_4 := \pi_L(M_5)$: By Corollary 5.6(2) M_5 is a union of π_L-fibers, so $L_3 := \pi_L(M_4) = L - L_4$ is a hypersurface.

Since $\xi|_{M_5}$ is a nonisotropic Killing field, $-g := -\sigma^{-2}\mathbf{c_D}|_{M_5}$ descends to a metric $\hat{g}$ on L_4 (henceforth in this subsection we sometimes suppress the restriction notation $|_{M_5}$). By Proposition 3.8 $\mathcal{L}_\xi\sigma = 0$ and $\mathcal{L}_\xi K = 0$, so the trivialization $\underline{K} \in \Gamma(\mathrm{End}(TM_5))$ is invariant under the flow of ξ. Since it annihilates ξ, it descends to an endormorphism field we denote $\hat{K} \in \mathrm{End}(TL_4)$. Then, Proposition 5.6 implies $\hat{K}^2 = -\varepsilon\,\mathrm{id}_{TL_4}$, that is, K is an almost ε-complex structure on L_4. This yields a specialization to our setting of a well-known result in Sasaki geometry.

Theorem 5.19. *The triple $(L_4, \hat{g}, \hat{K})$ is an ε-Kähler–Einstein structure with $\hat{R} = -6\varepsilon\hat{g}$.*

Proof. Since $\underline{K}^a{}_b\xi^b = 0$, the g-skewness of $\underline{K}$ implies the $\hat{g}$-skewness of $\hat{K}$. Thus, $\hat{g}$ and $\hat{K}$ together comprise an almost Kähler structure on L_4; the integrability of $\hat{K}$ is proved, for example, in [5], so they in fact consistute a Kähler structure. Since $\pi_L|_{L_4}$ is a (pseudo-)Riemannian submersion, we can relate the curvatures of g and $\hat{g}$ via the O'Neill formula, which gives that $\hat{g}$ is Einstein and determines the Einstein constant. $\blacksquare$

5.4.6   The ε-Kähler–Einstein Fefferman construction

The well known construction of Sasaki–Einstein structures from Kähler–Einstein structures immediately generalizes to the ε-Kähler–Einstein setting; see, for example, [44] (in this subsubsection, we restrict to $\varepsilon \in \{\pm 1\}$). Here we briefly describe the passage from ε-Kähler–Einstein structures to almost Einstein $(2, 3, 5)$ conformal structures as a generalized Fefferman construction [18, Section 4.5] between the respective Cartan geometries. Further details will be discussed in an article in preparation [56].

An ε-Kähler structure $(\hat{g}, \hat{K})$ of signature $(2, 2)$ on a manifold L_4 can be equivalently encoded in a torsion-free Cartan geometry $(\mathcal{S} \to L_4, \omega)$ of type (S, A), where $(S, A) = (\mathrm{SU}(1, 2), \mathrm{U}(1, 1))$ if $\varepsilon = -1$ and $(S, A) = (\mathrm{SL}(3, \mathbb{R}), \mathrm{GL}(2, \mathbb{R}))$ if $\varepsilon = 1$, see, for example, [17] for the Kähler case. We realize A within S as block diagonal matrices $\begin{pmatrix} \det A^{-1} & 0 \\ 0 & A \end{pmatrix}$. The action of A preserves the decomposition $\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{m} = \begin{pmatrix} \mathfrak{a} & \mathfrak{m} \\ \mathfrak{m} & \mathfrak{a} \end{pmatrix}$ and is given on $\mathfrak{m} \subset \mathfrak{s}$ by $X \mapsto \det(A)AX$; in particular it preserves an ε-Hermitian structure (unique up to multiples) on $\mathfrak{m}$ and we fix a (standard) choice. The $\mathfrak{m}$-part θ of a Cartan connection $\hat{\omega}$ of type (S, A) determines an isomorphism $TL_4 \cong \mathcal{S} \times_A \mathfrak{m}$ and (via this isomorphism) an ε-Hermitian structure on TL_4. The $\mathfrak{a}$-part γ of the Cartan connection defines a linear connection ∇ preserving this ε-Hermitian structure. If ω is torsion-free then ∇ is torsion-free, and thus the ε-Hermitian structure is ε-Kähler. Conversely, given an ε-Kähler structure, the Cartan bundle $\mathcal{S} \to L_4$ is the reduction of structure group of the frame bundle to $A \subset \mathrm{SO}(2, 2)$ defined by the parallel ε-Hermitian structure and the (reductive) Cartan connection $\hat{\omega} \in \Omega^1(\mathcal{S}, \mathfrak{s})$ is given by the sum $\hat{\omega} = \gamma + \theta$ of the pullback of the Levi-Civita connection form $\gamma \in \Omega^1(\mathcal{S}, \mathfrak{a})$ and the soldering form $\theta \in \Omega^1(\mathcal{S}, \mathfrak{m})$.

For the construction we first build the *correspondence space*

$$\mathcal{C}L_4 := \mathcal{S}/A_0 \cong \mathcal{S} \times_A (A/A_0),$$

where $A_0 = \mathrm{SU}(1, 1)$ if $\varepsilon = -1$ and $A_0 = \mathrm{SL}(2, \mathbb{R})$ if $\varepsilon = 1$. Then, $\mathcal{C}L_4 \to L_4$ is an $\mathbb{S}^1$-bundle if $\varepsilon = -1$ and an $\mathrm{SO}(1, 1)$-bundle if $\varepsilon = 1$. We can view $\hat{\omega} \in \Omega^1(\mathcal{S}, \mathfrak{s})$ as a Cartan connection on the A_0-principal bundle $\mathcal{S} \to \mathcal{C}L_4$. Next we fix inclusions

$$S \hookrightarrow \mathrm{G}_2 \hookrightarrow \mathrm{SO}(3, 4),$$

such that S stabilizes a vector $\mathbb{S}$ satisfying $H(\mathbb{S}, \mathbb{S}) = -\varepsilon$ in the standard representation $\mathbb{V}$ of G_2 (here H is the bilinear form the representation determines on $\mathbb{V}$), the S-orbit in $\mathrm{G}_2/Q \cong \mathrm{SO}(3, 4)/\bar{P}$ is open and $A_0 = S \cap Q = S \cap \bar{P}$. Consider the extended Cartan bundles $\mathcal{G} = \mathcal{S} \times_{A_0} Q$

and $\bar{\mathcal{G}} = \mathcal{S} \times_{A_0} \bar{P}$. There exist unique Cartan connections $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g}_2)$ and $\bar{\omega} \in \Omega^1(\bar{\mathcal{G}}, \mathfrak{so}(3,4))$ extending $\hat{\omega}$ [18]. Thus one obtains Cartan geometries of type (G_2, Q) and $(SO(3,4), \bar{P})$, respectively, on $\mathcal{C}L_4$ that are non-flat whenever one applies the construction to a torsion-free non-flat Cartan connection ω of type (S, A).

Proposition 5.20. *Let $(\hat{g}, \hat{K})$ be an ε-Kähler–Einstein structure, $\varepsilon \in \{\pm 1\}$, of signature $(2,2)$ on L_4 such that $\hat{R}_{ab} = -6\varepsilon\hat{g}_{ab}$. Then the induced conformal structure $\mathbf{c} := \hat{g}$ on the correspondence space $\mathcal{C}L_4$ is a $(2,3,5)$ conformal structure equipped with a parallel standard tractor $\mathbb{S}$, $H(\mathbb{S}, \mathbb{S}) = -\varepsilon$, which corresponds to a non-Ricci-flat Einstein metric in $\mathbf{c}$. (Here H is the canonical tractor metric determined by $\mathbf{c}$.)*

Conversely, locally, all $(2,3,5)$ conformal structures containing non-Ricci-flat Einstein metrics arise via this construction from ε-Kähler–Einstein structures.

Proof. We first show that the conformal Cartan geometry $(\bar{\mathcal{G}} \to \mathcal{C}L_4, \bar{\omega})$ on the correspondence space is normal if and only if the ε-Kähler structure on L_4 is Einstein with $\hat{R}_{ab} = -6\varepsilon\hat{g}_{ab}$. The curvature of the Cartan connection $\hat{\omega} = \gamma + \theta$ is given by

$$\hat{\Omega} = \mathrm{d}\gamma + \tfrac{1}{2}[\gamma, \gamma] + \tfrac{1}{2}[\theta, \theta] \in \Omega^2(\mathcal{S}, \mathfrak{a}).$$

Computing the Lie bracket $[[X, Y], Z]$ for $X, Y, Z \in \mathfrak{m}$, and interpreting the curvature as a tensor field on L_4 shows that it can be expressed as

$$\hat{\Omega}_{ij}{}^k{}_l = \hat{R}_{ij}{}^k{}_l + \varepsilon\hat{g}_{jl}\delta^k{}_i - \varepsilon\hat{g}_{il}\delta^k{}_j + \hat{K}_{jl}\hat{K}^k{}_i - \hat{K}_{il}\hat{K}^k{}_j - 2\,\hat{K}_{ij}\hat{K}^k{}_l,$$

where $\hat{g}_{ij}$ denotes the metric, $\hat{R}_{ij}{}^k{}_l$ its Riemannian curvature tensor and $\hat{K}_{ij} = \hat{g}_{ik}\hat{K}^k{}_j$ the Kähler form. Tracing over i and k shows that

$$\hat{\Omega}_{kj}{}^k{}_l = \hat{R}_{kj}{}^k{}_l + 6\varepsilon\hat{g}_{jl}.$$

Thus $\hat{\Omega}_{kj}{}^k{}_l = 0$ if and only if $\hat{R}_{ab} = -6\varepsilon\hat{g}_{ab}$. Further, for an ε-Kähler–Einstein structure

$$\hat{R}_{ij}{}^k{}_l\hat{K}^l{}_k = 2\,\hat{K}^l{}_i\hat{R}_{kl}{}^k{}_j$$

holds. If $\hat{R}_{kj}{}^k{}_l = -6\varepsilon\hat{g}_{jl}$, this implies that $\hat{\Omega}_{ij}{}^k{}_l\hat{K}^l{}_k = 0$. This precisely means that the Cartan curvature takes values in the subalgebra $\mathfrak{a}_0 \subset \mathfrak{a}$ of matrices with vanishing complex trace. Since $\mathfrak{a}_0 \subset \bar{\mathfrak{p}}$, the resulting conformal Cartan connection $\bar{\omega}$ is torsion-free. Vanishing of the Ricci-type contraction of $\hat{\Omega}$, i.e., $\hat{\Omega}_{kj}{}^k{}_l = 0$, then further implies that that the conformal Cartan connection is normal. Conversely, normality of the conformal Cartan connection implies $\hat{\Omega}_{kj}{}^k{}_l = 0$ and thus $\hat{R}_{ab} = -6\varepsilon\hat{g}_{ab}$.

By construction and normality of the conformal Cartan geometry $(\bar{\mathcal{G}} \to \mathcal{C}L_4, \bar{\omega})$, the induced conformal structure on $\mathcal{C}L_4$ is a $(2,3,5)$ conformal structure that admits a parallel standard tractor $\mathbb{S}$, $H(\mathbb{S}, \mathbb{S}) = -\varepsilon$, with underlying nowhere-vanishing Einstein scale σ. The vertical bundle $V\mathcal{C}L_4$ for $\mathcal{C}L_4 \to L_4$ corresponds to the subspace $\mathfrak{a}/\mathfrak{a}_0 \subset \mathfrak{s}/\mathfrak{a}_0$, i.e., $V\mathcal{C}L_4 = \mathcal{S} \times_{A_0} (\mathfrak{a}/\mathfrak{a}_0)$. Since this is the unique A_0-invariant 1-dimensional subspace in $\mathfrak{s}/\mathfrak{a}_0$, the vertical bundle coincides with the subbundle $\mathbf{L}$ spanned by the Killing field ξ.

We now prove the converse. Let $(\mathcal{G}_5 \to M_5, \omega_5)$ the Cartan geometry of type (S, A_0) on M_5 determined (according to Theorem 5.1) by the holonomy reduction coresponding to a parallel standard tractor $\mathbb{S}$, $H(\mathbb{S}, \mathbb{S}) = -\varepsilon$, of a $(2,3,5)$ conformal structure. Restrict to an open subset so that $\pi_L \colon M_5 \to L_4$ is a fibration over the leaf space determined by the corresponding Killing field ξ. Since ξ is a normal conformal Killing field, it inserts trivially into the curvature of ω_5. Since ξ spans VM_5, by [18, Theorem 1.5.14] this guarantees that on a sufficiently small leaf space L_4 one obtains a Cartan geometry of type (S, A) such that the restriction of $(\mathcal{G}_5 \to M_5, \omega_5)$ is locally isomorphic to the canonical geometry on the correspondence space over L_4. Normality of the conformal Cartan connection implies that the Cartan geometry of type (S, A) is torsion-free and the corresponding ε-Kähler metric is non-Ricci-flat Einstein. ∎

Remark 5.21. It is interesting to note the following geometric interpretation of the correspondence spaces: If $\varepsilon = -1$, the bundle $\mathcal{C}L_4 \to L_4$ can be identified with the twistor bundle $\mathbb{T}L_4 \to L_4$ whose fiber over a point $x \in L_4$ comprises all self-dual totally isotropic 2-planes in $T_x \mathcal{C}L_4$. If $\varepsilon = 1$, $\mathcal{C}L_4 \to L_4$ can be identified with the subbundle of the twistor bundle whose fiber over a point $x \in L_4$ comprises all self-dual 2-planes in $T_x \mathcal{C}L_4$ except the eigenspaces of the endomorphism $\hat{K}_x$. The total space $\mathcal{C}L_4$ carries a tautological rank 2-distribution obtained by lifting each self-dual totally isotropic 2-plane horizontally to its point in the fiber, and it was observed [2, 6] that, provided the self-dual Weyl tensor of the metric on L_4 vanishes nowhere, this distribution is $(2, 3, 5)$ almost everywhere. This suggests a relation of the present work to the An–Nurowski twistor construction (and recent work of Bor and Nurowski).

5.5   The hypersurface curved orbit M_4

On the complement $M - M_5$, $\sigma = 0$ (and so μ is invariant); this simplifies many formulae there. First, $\varepsilon = -H_\Phi(\mathbb{S}, \mathbb{S}) = -\mu_a \mu^a$. Substituting in (3.10), (3.13), (3.14), (3.15) and using that expression for ε yields (on M_4)

$$\xi^a = \mu_b \phi^{ba}, \tag{5.9}$$

$$I_{ab} = 3\mu^c \mu_{[c} \phi_{ab]} = -\varepsilon \phi_{ab} - 2\mu_{[a} \xi_{b]}, \tag{5.10}$$

$$J_{ab} = 3\mu^c \phi_{[ca} \theta_{b]} = \mu^c (*\phi)_{cab}, \tag{5.11}$$

$$K_{ab} = -2\mu^c \mu_{[a} \phi_{b]c} = 2\mu_{[a} \xi_{b]}. \tag{5.12}$$

Denote by $M_{\mathbf{S}}$ the set on which the line field $\mathbf{S}$ is defined (recall from Section 5.3 that $\mathbf{S} := \langle \mu \rangle$ on the space where $\sigma = 0$ and $\mu \neq 0$). By the proof of Proposition 5.5, this is M_4 in the Ricci-negative case, $M_4 \cup M_2^+ \cup M_2^-$ in the Ricci-positive case, and $M_4 \cup M_2$ in the Ricci-flat case.

Proposition 5.22. *On the submanifold $M_{\mathbf{S}}$, $\mathbf{S}^\perp = TM_{\mathbf{S}} \subset TM|_{M_{\mathbf{S}}}$.*
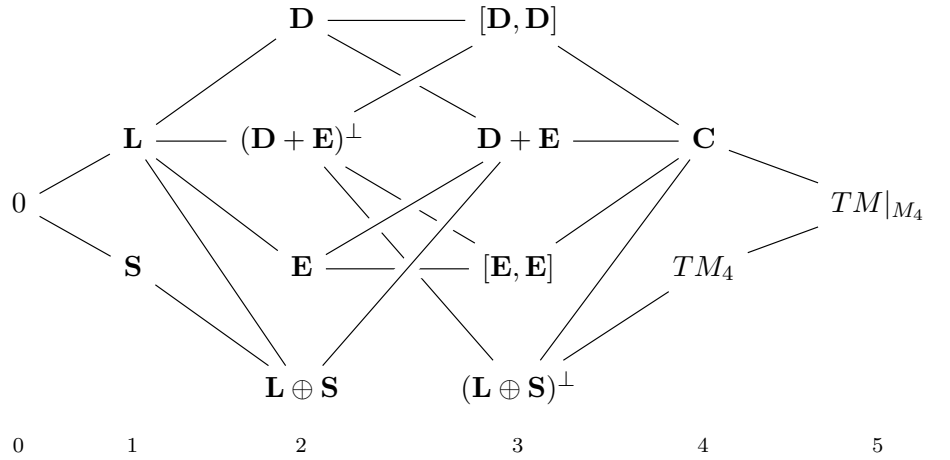
Proof. The set $M_{\mathbf{S}}$ is precisely where $\sigma = 0$ and $\mu^a = \sigma^{,a} \neq 0$, so σ is a defining function for $M_{\mathbf{S}}$ and hence $TM_{\mathbf{S}} = \ker \mu$ there. ∎

5.5.1   The canonical lattices of hypersurface distributions

Recall that if $\mathbb{S}$ is nonisotropic (if σ is not Ricci-flat) it determines a direct sum decomposition $\mathcal{V} = \mathcal{W} \oplus \langle \mathbb{S} \rangle$, where $\mathcal{W} := \langle \mathbb{S} \rangle^\perp$, and if $\mathbb{S}$ is isotropic (if σ is Ricci-flat), it determines a filtration associated to (3.3): $\mathcal{V} \supset \mathcal{W} \supset \operatorname{im} \mathbb{K} \supset \ker K \supset \langle \mathbb{S} \rangle \supset \{0\}$. On M_4, $\mathbb{S} \in \langle X \rangle^\perp - \operatorname{im}(X \times \cdot)$, so $X \times \mathbb{S} \in \ker(X \times \cdot) - \langle X \rangle$. In particular, $X \times \mathbb{S}$ is isotropic but nonzero, and hence it determines an analogous filtration of $\mathcal{V}$.

Forming the intersections and spans of the components of the filtrations determined by $\mathbb{S}$ and $X \times \mathbb{S}$ gives a lattice of vector subbundles of $\mathcal{V}$ under the operations of span and intersection (in fact, it is a graded lattice graded by rank). It has 22 elements in the non-Ricci-flat case and 26 in the Ricci-flat case, so for space reasons we do not reproduce these here. However, since $\mathcal{W}/\langle X \rangle \cong TM[-1]$, the sublattice of vector bundles $\mathcal{N}$ satisfying $\langle X \rangle \preceq \mathcal{N} \preceq \mathcal{W}$ descends to a natural lattice of subbundles of $TM|_{M_4}$. We record these lattices (they are different in the Ricci-flat and non-Ricci-flat cases), which efficiently encode the incidence relationships among the subbundles, in the following proposition. (We omit the proof, which is tedious but straightforward, and which can be achieved by working in an adapted frame.)
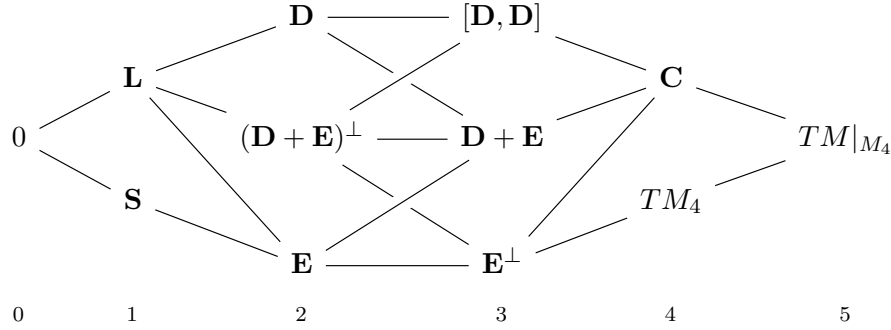
Proposition 5.23. *The bundle $TM|_{M_4}$ admits a natural lattice of vector subbundles under the operations of span and intersection: If σ is non-Ricci-flat, the lattice is*
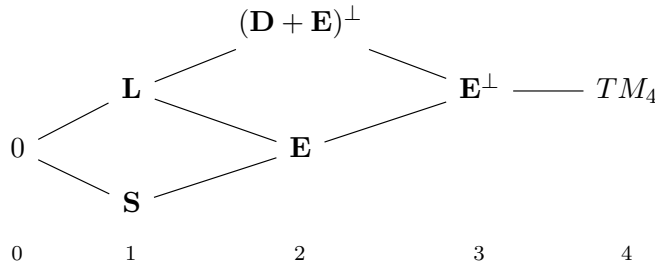
$$\mathbf{D} \text{———} [\mathbf{D},\mathbf{D}]$$

$$\mathbf{L} \text{——} (\mathbf{D}+\mathbf{E})^\perp \qquad \mathbf{D}+\mathbf{E} \text{——} \mathbf{C}$$

$$0 \qquad\qquad TM|_{M_4}$$

$$\mathbf{S} \qquad \mathbf{E} \text{——} [\mathbf{E},\mathbf{E}] \qquad TM_4$$

$$\mathbf{L}\oplus\mathbf{S} \qquad (\mathbf{L}\oplus\mathbf{S})^\perp$$

0	1	2	3	4	5

In particular, this contains a full flag field

$$0 \subset \mathbf{L} \subset (\mathbf{D}+\mathbf{E})^\perp \subset (\mathbf{L}\oplus\mathbf{S})^\perp \subset TM_4$$

on TM_4. The subbundles in the lattice that depend only on $\mathcal{D}[\mathbf{D};\sigma]$ and not $\mathbf{D}$ are 0, $\mathbf{L}$, $\mathbf{S}$, $\mathbf{L}\oplus\mathbf{S}$, $(\mathbf{L}\oplus\mathbf{S})^\perp$, $\mathbf{C}$, TM_4, $TM|_{M_4}$. If σ is Ricci-flat, the lattice is

$$\mathbf{D} \text{———} [\mathbf{D},\mathbf{D}]$$

$$\mathbf{L} \qquad\qquad \mathbf{C}$$

$$0 \qquad (\mathbf{D}+\mathbf{E})^\perp \text{——} \mathbf{D}+\mathbf{E} \qquad TM|_{M_4}$$

$$\mathbf{S} \qquad\qquad TM_4$$

$$\mathbf{E} \text{———} \mathbf{E}^\perp$$

0	1	2	3	4	5

This determines a natural sublattice

$$(\mathbf{D}+\mathbf{E})^\perp$$

$$\mathbf{L} \qquad\qquad \mathbf{E}^\perp \text{——} TM_4$$

$$0 \qquad \mathbf{E}$$

$$\mathbf{S}$$

0	1	2	3	4

of vector subbundles of TM_4. The subbundles in the first lattice that depend only on the 1-parameter family $\mathcal{D}[\mathbf{D};\sigma]$ and not $\mathbf{D}$ are 0, $\mathbf{L}$, $\mathbf{S}$, $\mathbf{E}$, $\mathbf{E}^\perp$, $\mathbf{C}$, TM_4, $TM|_{M_4}$.

    In both cases, the restriction $\mathbf{L}|_{M_4}$ (which depends only on $\mathcal{D}[\mathbf{D};\sigma]$) is the intersection of the distribution $\mathbf{D}$ and the tangent space TM_4 of the hypersurface.

    In the lattices, all bundles are implicitly restricted to M_4, the numbers indicate the ranks of the bundles in their respective columns, and (in the case of the two large lattices) the diagram is arranged so that each bundle is positioned horizontally opposite its $\mathbf{g}$-orthogonal bundle.

5.5.2   The hypersurface leaf space L_3

Recall that $L_3 := \pi_L(M_4)$. Since $\mathbf{S}$ is spanned by the invariant component μ of $\mathbb{S}$ and $\mathcal{L}_\xi \mathbb{S} = 0$, $\mathbf{S}$ descends to a line field $\hat{\mathbf{S}} \subset TL|_{L_3}$. This line field is contained in TL_3 iff $\mathbf{S}$ is contained in $TM_4 = \mathbf{S}^\perp$, that is (by Proposition 5.23) iff σ is Ricci-flat.

Similarly, since the flow of ξ preserves $\mathbf{g}$, it also preserves $\mathbf{C} \cap TM_4 = (\mathbf{L} \oplus \mathbf{S})^{\perp}$. Then, because $\mathbf{L} \subset \mathbf{C} \cap TM_4 \subset \mathbf{S}^{\perp} = TM_4$, $\mathbf{C} \cap TM_4$ descends to a 2-plane distribution $\mathbf{H} \subset TL_3$.

Proposition 5.24. *The* 2*-plane distribution* $\mathbf{H} \subset TL_3$ *defined as above is contact.*

Proof. Since $\mathbf{C} \cap TM_4 = (\ker \xi^{\flat}) \cap TM_4$, we can write this bundle as $\ker \iota_{M_4}^{*} \xi^{\flat}$, where $\iota_{M_4} \colon M_4 \hookrightarrow M$ denotes inclusion. By construction, $\iota_{M_4}^{*} \xi^{\flat}$ (where we have trivialized $\xi^{\flat}$ with respect to an arbitrary scale τ) is also the pullback $\pi_L^{*} \beta$ of a defining 1-form $\beta \in \Gamma(TL_3)$ for $\mathbf{H}$. Thus, $\pi_L^{*}(\beta \wedge d\beta) = \pi_L^{*} \beta \wedge d(\pi_L^{*} \beta) = \iota_{M_4}^{*} \xi^{\flat} \wedge d(\iota_{M_4}^{*} \xi^{\flat}) = \iota_{M_4}^{*}(\xi^{\flat} \wedge d\xi^{\flat})$, but computing in an adapted frame shows that $\xi^{\flat} \wedge d\xi^{\flat}$ vanishes nowhere, and hence the same holds for $\beta \wedge d\beta$; equivalently, $\mathbf{H}$ is contact. ∎

Now, consider the component $\zeta^{a}{}_{b} := -\sigma \psi^{a}{}_{b} - \mu_c \chi^{ca}{}_{b} - \rho \phi^{a}{}_{b} \in \Gamma(\mathrm{End}_{\circ}(TM))$ of $\mathbb{K}^{A}{}_{B}$ in the splitting (3.9) with respect to a scale τ. Let $\mathbf{J} \colon \mathbf{H} \to TL|_{L_3}$ be the map that lifts a vector $\hat{\eta} \in \mathbf{H}_{\hat{x}}$ to any $\eta \in T_x M_4$ for arbitrary $x \in \pi_L^{-1}(\hat{x})$, applies ζ, and then pushes forward back to $T_{\hat{x}} L$ by π_L.

We show that this map is well-defined, that it is independent of the choice of τ, and that we may regard it as an endomorphism of $\mathbf{H}$: By Lemma 3.7, $\mathcal{L}_{\xi} \mathbb{K} = -[L_0^{\mathcal{A}}(\xi), \mathbb{K}] = -[\mathbb{K}, \mathbb{K}] = 0$, so $\mathbb{K}$ and hence ζ is itself invariant under the flow of ξ, and hence that ζ is independent of choice of basepoint x of the lift. Now, any two lifts $\eta, \eta' \in T_x M_4$ differ by an element of $\ker T_x \pi_L = \langle \xi_x \rangle$; on the other hand, expanding (3.2) in terms of the splitting determined by τ, taking a particular component equation, and evaluating at $\sigma = 0$ gives the identity $\zeta^{b}{}_{a} \xi^{a} = \alpha \xi^{b}$ for some smooth function α, so $T_x \pi_L \cdot \zeta(\xi) = 0$, and hence $\mathbf{J}$ is well-defined. Finally, under a change of scale, ζ is transformed to $\zeta^{a}{}_{b} \mapsto \zeta^{a}{}_{b} + \Upsilon^{a} \xi_b - \xi^{a} \Upsilon_b$ for some form $\Upsilon_a \in \Gamma(T^{*}M)$ [4]. A lift η^{b} of $\hat{\eta} \in \mathbf{H}$ is an element of $\mathbf{C} \cap TM_4 \subset \mathbf{C} = \ker \xi^{\flat}$, $\Upsilon^{a} \xi_b \eta^{b} = 0$. The term $\xi^{a} \Upsilon_b \eta^{b}$ is again in $\ker T_x \pi_L$, and we conclude that $\mathbf{J}$ is independent of the scale τ.

Now, in the notation of the previous paragraph, we have $\mu_b \zeta^{b}{}_{c} \eta^{c} = \mu_b(-\mu_d \chi^{db}{}_{c} - \rho \phi^{b}{}_{c}) \eta^{c} = -\rho \nu_b \phi^{b}{}_{c} \eta^{c}$. This is $-\rho \xi_c \eta^{c}$, and we saw above that $\xi_c \eta^{c} = 0$, so $\mu_b \zeta^{b}{}_{c} \eta^{c} = 0$, that is, $\zeta^{b}{}_{c} \eta^{c} \in \ker \mu = \mathbf{S}^{\perp}$. Using the $\mathbf{g}$-skewness of ζ gives $\xi_b \zeta^{b}{}_{c} \eta^{c} = -\eta_b \zeta^{b}{}_{c} \xi^{c} = -\eta_b(\alpha \xi^{b}) = -\alpha \eta_b \xi^{b}$, but again $\eta_b \xi^{b} = 0$, so we also have $\zeta^{b}{}_{c} \eta^{c} \in \ker \xi^{\flat} = \mathbf{L}^{\perp}$. Thus, $\zeta(\eta) \in \mathbf{L}^{\perp} \cap \mathbf{S}^{\perp} = \mathbf{C} \cap TM_4$, and pushing forward by π_L gives $\mathbf{J}(\hat{\eta}) \in \mathbf{H}$, so we may view $\mathbf{J}$ as an endomorphism of $\mathbf{H}$.

Proposition 5.25. *The endomorphism* $\mathbf{J} \in \Gamma(\mathrm{End}(\mathbf{H}))$ *defined as above is an ε-complex structure.*

Proof. In the above notation, unwinding (twice) the definition of $\mathbf{J}$ gives that $\mathbf{J}^2(\hat{\eta}) = T_x \pi_L \cdot \zeta^2(\eta)$. Now, another component equation of (3.2) is $\zeta^{a}{}_{c} \zeta^{c}{}_{b} - \xi^{a} \nu_b - \nu^{a} \xi_b = \varepsilon \delta^{a}{}_{b} + \mu^{a} \mu_b$ for some $\nu \in \Gamma(T^{*}M)$. The above observations about the terms $\Upsilon^{a} \xi_b$ and $\xi^{a} \Upsilon_b$ apply just as well to $\xi^{a} \nu_b$ and $\nu^{a} \xi_b$, and since $\eta \in \mathbf{C} \cap TM_4 \subset \mathbf{S}^{\perp} = \ker \mu$, we have $\mu^{a} \mu_b \eta^{b} = 0$, so the above component equation implies $\mathbf{J}^2 = \varepsilon \, \mathrm{id}_{\mathbf{H}}$. In the case $\varepsilon = +1$ one can verify that the (± 1)-eigenspaces of $\mathbf{J}$ are both 1-dimensional, that is, $\mathbf{J}$ is an almost ε-complex structure on $\mathbf{H}$. ∎

In the Ricci-negative case, this shows precisely that $(L_3, \mathbf{H}, \mathbf{J})$ is an almost CR structure (in fact, it turns out to be integrable, see the next subsubsection), and one might call the resulting structure in the general case an almost ε-CR structure. The three signs of ε (equivalently, the three signs of the Einstein constant) give three qualitatively distinct structures, so we treat them separately.

5.5.3   Ricci-negative case: The classical Fefferman conformal structure

If σ is Ricci-negative, then by Example 5.2, $M - M_5 = M_4$ inherits a conformal structure $\mathbf{c_S}$ of signature $(1, 3)$. We can identify the standard tractor bundle $\mathcal{V}_{\mathbf{S}}$ of $\mathbf{c_S}$ with the restriction $\mathcal{W}|_{\Sigma}$ of

the $\nabla^{\mathcal{V}}$ parallel subbundle $\mathcal{W} := \langle \mathbb{S} \rangle^{\perp}$, and under this identification the normal tractor connection on $\mathcal{V}_{\mathbf{S}}$ coincides with the restriction of $\nabla^{\mathcal{V}}$ to $\mathcal{W}|_{\Sigma}$ [33]. In particular, $\mathrm{Hol}(\mathbf{c_S}) \leq \mathrm{SU}(1,2)$, but this containment characterizes (locally) the 4-dimensional conformal structures that arise from the classical Fefferman conformal construction [16, 51], which canonically associates to any nondegenerate partially integrable almost CR structure of hypersurface type on a manifold a conformal structure on a natural $\mathbb{S}^1$-bundle over that manifold [30], [18, Example 3.1.7, Section 4.2.4].

Proposition 5.26. *Let $\mathcal{D}$ denote a 1-parameter family of conformally isometric oriented $(2,3,5)$ distributions related by a Ricci-negative almost Einstein scale σ.*

1. *The conformal structure $\mathbf{c_S}$ of signature $(1,3)$ determined on the hypersurface curved orbit M_4 is a Fefferman conformal structure.*

2. *The infinitesimal generator of the (local) $\mathbb{S}^1$-action is $\xi|_{M_4} = \iota_7(\sigma)|_{M_4}$, so the line field it spans is $\mathbf{L}|_{M_4} = \mathbf{D} \cap TM_4$ for every $\mathbf{D} \in \mathcal{D}$.*

3. *The 3-dimensional CR-structure underlying the Fefferman conformal structure $(M_4, \mathbf{c_S})$ is $(L_3, \mathbf{H}, \mathbf{J})$.*

Proof. The first claim is deduced in the paragraph before the proposition. The latter claims follow from (1), unwinding definitions, and the proof of [16, Corollary 2.3]. ∎

5.5.4 Ricci-positive case: A paracomplex analogue of the Fefferman conformal structure

This case is similar to the Ricci-negative case, but differs qualitatively in two ways.

First, the endomorphism $\mathbf{J} \in \mathrm{End}(\mathbf{H})$ is a paracomplex structure rather than a complex one; let $\mathbf{H}_{\pm}$ denote its (± 1)-eigendistributions, which are both line fields. A contact distribution on a 3-manifold equipped with a direct sum decomposition into line fields is the 3-dimensional specialization of a *Legendrean contact structure*, the paracomplex analogue of a partially integrable almost CR structure of hypersurface type [18, Section 4.2.3]. These correspond to regular, normal parabolic geometries of type $(\mathrm{SL}(3,\mathbb{R}), P_{12})$ where $P_{12} < \mathrm{SL}(3,\mathbb{R})$ is a Borel subgroup. The analog of the classical Fefferman conformal structure associates to any Legendrean contact structure on a manifold N a neutral conformal structure on a natural $\mathrm{SO}(1,1)$-bundle over the manifold [43, 53]. By analogy with the construction discussed in Section 5.5.3, we call a conformal structure that locally arises this way a *para-Fefferman conformal structure*.

Second, the conformal structure $\mathbf{c_S}$ is defined on the union $M_4 \cup M_2^+ \cup M_2^-$, but only its restriction to M_4 is induced by the analogue of the classical Fefferman construction (indeed, recall from Section 5.5 that the vector field ξ whose integral curves comprise the leaf space L vanishes on $M_2^{\pm}$).

The paracomplex analogue of Proposition 5.26 is the following:

Proposition 5.27. *Let $\mathcal{D}$ denote a 1-parameter family of conformally isometric $(2,3,5)$ distributions related by a Ricci-positive almost Einstein scale σ.*

1. *The conformal structure $\mathbf{c_S}|_{M_4}$ of signature $(2,2)$ determined on the hypersurface curved orbit M_4 is a para-Fefferman conformal structure.*

2. *The infinitesimal generator of the (local) $\mathrm{SO}(1,1)$-action is $\xi|_{M_4} = \iota_7(\sigma)|_{M_4}$, so the line field it spans is $\mathbf{L}|_{M_4} = \mathbf{D} \cap TM_4$ for every $\mathbf{D} \in \mathcal{D}$.*

3. *The 3-dimensional Legendrean contact structure underlying $(M_4, \mathbf{c_S}|_{M_4})$ is $(L_3, \mathbf{H}_+ \oplus \mathbf{H}_-)$, where $\mathbf{H}_{\pm}$ are the (± 1)-eigenspaces of the paracomplex structure $\mathbf{J}$ on $\mathbf{H}$.*

The geometry of 3-dimensional Legendrean contact structures admits another concrete, and indeed classical (local) interpretation, namely as that of second-order ordinary differential equations (ODEs) modulo point transformations: We can regard a second-order ODE $\ddot{y} = F(x, y, \dot{y})$ as a function $F(x, y, p)$ on the jet space $J^1 := J^1(\mathbb{R}, \mathbb{R})$, and the vector fields $D_x := \partial_x + p\partial_y + F(x, y, p)\partial_p$ and ∂_p span a contact distribution (namely the kernel of $dy - p\,dx \in \Gamma(T^*J^1)$), so $\langle D_x \rangle \oplus \langle \partial_p \rangle$ is a Legendrean contact structure on J^1. Point transformations of the ODE, namely those given by prolonging to J^1 (local) coordinate transformations of $\mathbb{R}^2_{xy}$, are precisely those that preserve the Legendrean contact structure (up to diffeomorphism) [25].

5.5.5 Ricci-flat case: A fibration over a special conformal structure

In this case, Example 5.2 gives that the hypersurface curved orbit M_4 locally fibers over the space $\widetilde{L}$ of integral curves of $\mathbf{S}$ (nota bene the fibrations $\pi_L|_{M_4}\colon M_4 \to L_3$ in the non-Ricci-flat cases above are instead along the integral curves of $\mathbf{L}$), and that $\widetilde{L}$ inherits a conformal structure $\mathbf{c}_{\widetilde{L}}$ of signature $(1, 2)$. Considering the sublattice of the last lattice in Proposition 5.23 of the distributions containing $\mathbf{S}$ and forming the quotient bundles modulo $\mathbf{S}$ yields a complete flag field of $T\widetilde{L}$ that we write as $0 \subset \mathbf{E}/\mathbf{S} \subset \mathbf{E}^\perp/\mathbf{S} \subset T\widetilde{L}$ it depends only on $\mathcal{D}$. Since $\mathbf{E}$ is totally $\mathbf{c}$-isotropic, the line field $\mathbf{E}/\mathbf{S}$ is $\mathbf{c}_{\widetilde{L}}$-isotropic, and by construction it is orthogonal to $\mathbf{E}^\perp/\mathbf{S}$ with respect to $\mathbf{c}_{\widetilde{L}}$. Thus, we may regard the induced structure on $\widetilde{L}$ as a Lorentzian conformal structure equipped with an isotropic line field.

Similarly, the fibration along the integral curves of $\mathbf{L}$ determines a complete flag field that we denote $0 \subset \mathbf{E}/\mathbf{L} \subset \mathbf{H} \subset TL_3$. Computing in a local frame gives $\mathbf{E}/\mathbf{L} = \ker \mathbf{J} = \operatorname{im} \mathbf{J}$, and this line the kernel of the (degenerate) conformal (negative semidefinite) bilinear form $\mathbf{c}$ determines on $\mathbf{H}$.

5.6 The high-codimension curved orbits $M_2^\pm$, M_2, $M_0^\pm$

Recall that on these orbits, $\xi = 0$ and $K = 0$, and hence $\mathbf{E}$ is not defined. Recall also that if σ is Ricci-negative, all three of these curved orbits are empty. If σ is Ricci-flat, only M_2 and $M_0^\pm$ occur, and if σ is Ricci-positive, only $M_2^\pm$ occur. Since the curved orbits $M_0^\pm$ are 0-dimensional, they inherit no structure.

5.6.1 The curved orbits $M_2^\pm$: Projective surfaces

By Theorem 5.1 and the orbit decomposition of the flat model in Section 5.2, the holonomy reduction determines parabolic geometries of type $(\mathrm{SL}(3, \mathbb{R}), P_1)$ and $(\mathrm{SL}(3, \mathbb{R}), P_2)$ on $M_2^\pm$. Torsion-freeness of the normal conformal Cartan connection immediately implies that these parabolic geometries are torsion-free and hence determine underlying torsion-free projective structures (that is, equivalence classes of torsion-free affine connections having the same unparametrized geodesics).

Again, the formulae for various objects simplify on this orbit: By the proof of Proposition 5.5 we have $\sigma = 0$, $\xi = 0$, and $\mu^c \theta_c = \pm 1$ here, and substituting in (3.13), (3.14), (3.15) gives

$$I = -\phi, \qquad J = \pm\phi, \qquad K = 0.$$

These specializations immediately give the Ricci-positive analog of Corollary 5.7:

Proposition 5.28. *Let $(M, \mathbf{D})$ be an oriented $(2, 3, 5)$ distribution and σ a Ricci-positive scale for $\mathbf{c_D}$. Then, the limiting normal conformal Killing forms $\phi_{\mp\infty} := \pm I + J$ respectively vanish precisely on $M_2^\pm$, so the distributions $\mathbf{D}_{\mp\infty}$ they respectively determine are respectively defined precisely on $M_5 \cup M_4 \cup M_2^\mp$.*

Proposition 5.29. *For all $x \in M_2^\pm$, $T_x M_2^\pm = \mathbf{D}_x$ (for every $\mathbf{D} \in \mathcal{D}$).*

Proof. By Proposition 5.5, $M_2^\pm = \{x \in M \colon \xi_x = 0\}$ and so $TM_2^\pm \subseteq \ker \nabla \xi$. On the other hand, as in the proof of that proposition we have $\xi^a{}_{,b} = -\zeta^a{}_b - \theta_c \mu^c \delta^a{}_b$, and computing in an adapated frame shows that on $M_2^\pm$, $\xi^a{}_{,b}$ has rank 3. Equivalently, the kernel has dimension $2 = \dim TM_2^\pm = 2$, so $\ker \nabla \xi = T_x M_2^\pm$.

Writing $\nabla_c^{\mathcal{V}} \mathbb{K}^A{}_B = 0$ in components gives $\xi^b{}_{,c} = -\zeta^b{}_{,c} - \mu^d \theta_d \delta^b{}_c$, and as in the proof of Proposition 5.5, $-\zeta^b{}_c = \sigma \psi^b{}_c + \mu_d \chi^{db}{}_c + \rho \phi^b{}_c$. Since $x \in M_2^\pm$, $\mu^d \phi_d = \pm 1$ and $\sigma = 0$. For $\eta \in \mathbf{D}_x$, Proposition 2.10(2) gives that $\phi^b{}_c \eta^c = 0$, and computing in an adapted frame gives that $\mu_d \chi^{db}{}_c$ restricts to $\mathrm{id}_{\mathbf{D}}$ on $M_2^\pm$. Substituting then gives $\xi^b{}_{,c} \eta^c = 0$, so by dimension count $T_x M_2^\pm = \mathbf{D}_x$. $\hfill\blacksquare$

5.6.2 The curved orbit M_2

As for the hypersurface curved orbits, forming the intersections and spans of the components of the filtrations determined by $\mathbb{S}$ and $X \times \mathbb{S}$ in this case yields a lattice of (14) vector subbundles of $\mathcal{V}$, and determining the lattice of (10) vector subbundles of $TM|_{M_2}$ this induces shows in particular that one has a distinguished line field $\mathbf{S} = \mathbf{D} \cap TM_2$ on M_2. Specializing the formulae for I, J, K as in the previous cases gives that on M_2, $I = J = K = 0$.

6 Examples

In this section, we give three conformally nonflat examples, one for each sign of the Einstein constant; each is produced using a different method. To the knowledge of the authors, before the present work there were no examples in the literature of nonflat $(2, 3, 5)$ conformal structures known to admit a non-Ricci-flat almost Einstein scale.[7] In particular, these examples show that none of the holonomy reductions considered in this article force local flatness of the underlying conformal structure.

Example 6.1 (a distinguished rolling distribution). We construct a homogeneous Sasaki–Einstein metric of signature $(3, 2)$ whose negative determines a Ricci-negative conformal structure. Each $(2, 3, 5)$ distribution in the corresponding family is diffeomorphic to a particular special so-called *rolling distribution*.

Let $(\mathbb{S}^2, h_+, \mathbb{J}_+)$ and $(\mathbb{H}^2, h_-, \mathbb{J}_-)$ respectively denote the round sphere and hyperbolic plane with their usual Kähler structures, rescaled so that their respective scalar curvatures are ± 12. In the usual respective polar coordinates (r, φ) and (s, ψ),

$$h_+ := \frac{2}{3} \cdot \frac{1}{(r^2 + 1)^2} (dr^2 + r^2 d\varphi^2), \qquad \mathbb{J}_+ := r \partial_r \otimes d\varphi - \tfrac{1}{r} \partial_\varphi \otimes dr,$$

$$h_- := \frac{2}{3} \cdot \frac{1}{(s^2 - 1)^2} (ds^2 + s^2 d\psi^2), \qquad \mathbb{J}_- := s \partial_s \otimes d\psi - \tfrac{1}{s} \partial_\psi \otimes ds.$$

Then, the triple $(\mathbb{S}^2 \times \mathbb{H}^2, \hat{g}, \hat{K})$, where $\hat{g} := h_+ \oplus -h_-$ and $\hat{K} := \mathbb{J}_+ \oplus \mathbb{J}_-$, is a Kähler structure satisfying $\hat{R}_{ab} = 6 \hat{g}_{ab}$. The Kähler form $\hat{g}_{ac} \hat{K}^c{}_b$ is equal to $(d\alpha)_{ab}$, where

$$\alpha := \frac{2}{3} \left(-\frac{\varphi r \, dr}{(r^2 + 1)^2} + \frac{\psi s \, ds}{(s^2 - 1)^2} \right).$$

The infinitesimal symmetries of the Kähler structure are spanned by the lifts of the infintesimal symmetries of $(\mathbb{S}^2, h_+, \mathbb{J}_+)$ and $(\mathbb{H}^2, h_-, \mathbb{J}_-)$, and so the infinitesimal symmetry algebra is $\mathfrak{aut}(\hat{g}, \hat{K}) \cong \mathfrak{so}(3, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R})$.

---

[7] We recently learned from Bor and Nurowski that they have, in work in progress, also constructed examples [8].

On the canonical $\mathbb{S}^1$-bundle $\pi\colon M \to \mathbb{S}^2 \times \mathbb{H}^2$ defined in Section 5.4.6, with standard fiber coordinate λ, define $\beta := d\lambda - 2\pi^*\alpha$. Then, the associated Sasaki–Einstein structure is (M, g, ∂_λ), where $g := \pi^*\hat{g} + \beta^2$. The normalizations of the scalar curvatures of the sphere and hyperbolic plane were chosen so that $R_{ab} = 4g_{ab}$. The 1-parameter family $\{\mathbf{D}_v\}$ of corresponding oriented $(2, 3, 5)$ distributions, which in particular induce the conformal class $[-g]$, is

$$\mathbf{D}_v = \Big\langle 3\big(r^2 + 1\big)s\partial_r + 3\big(s^2 - 1\big)s\cos\gamma\partial_s + 3\big(s^2 - 1\big)\sin\gamma\partial_\psi + 4s\left(\frac{s\psi\cos\gamma}{s^2 - 1} - \frac{r\varphi}{r^2 + 1}\right)\partial_\lambda,$$
$$3\big(r^2 + 1\big)s\partial_\varphi + 3\big(s^2 - 1\big)s\sin\gamma\partial_s - 3r\big(s^2 - 1\big)\cos\gamma\partial_\psi + \frac{4s}{s^2 - 1}r\psi\sin\gamma\partial_\lambda\Big\rangle,$$

where

$$\gamma := \frac{r^2 - 1}{r^2 + 1}\varphi + \frac{s^2 + 1}{s^2 - 1}\psi - 3\lambda + v.$$

One can compute the tractor connection explicitly (the explicit expression is unwieldy, so we do not reproduce it here) and use it to compute that the conformal holonomy $\mathrm{Hol}([-g])$ is the full group $\mathrm{SU}(1, 2)$. In particular, this shows that in the Ricci-negative case the holonomy reduction considered in this case does not automatically entail a holonomy reduction to a smaller group. Since almost Einstein scales are in bijective correspondence with parallel standard tractors, the space of Einstein scales is 1-dimensional (as an independent parallel standard tractor would further reduce the holonomy). One can compute that $\mathfrak{aut}(\mathbf{D}_v) \cong \mathfrak{aut}(\hat{g}, \hat{K}) \cong \mathfrak{so}(3, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R})$ and $\mathfrak{aut}([-g]) \cong \mathfrak{aut}(g) \cong \mathfrak{aut}(\hat{g}, \hat{K}) \oplus \langle\xi\rangle \cong \mathfrak{so}(3, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R}) \oplus \mathbb{R}$.

One can show that every distribution $\mathbf{D}_v$ is equivalent to the so-called rolling distribution for the Riemannian surfaces $(\mathbb{S}^2, g_+)$ and $(\mathbb{H}^2, g_-)$. The underlying space of this distribution, which we can informally regard as the space of relative configurations of $\mathbb{S}^2$ and $\mathbb{H}^2$ in which the surfaces are tangent at a single point, is the twistor bundle [2] over $\mathbb{S}^2 \times \mathbb{H}^2$ whose fiber over (x_+, x_-) is the circle $\mathrm{Iso}(T_{x_+}\mathbb{S}^2, T_{x_-}\mathbb{H}^2) \cong \mathbb{S}^1$ of isometries. The distribution is the one characterized by the so-called no-slip, no-twist conditions on the relative motions of the two surfaces [13, Section 3].

We can produce a para-Sasaki analogue of this example, which in particular has full holonomy group $\mathrm{SL}(3, \mathbb{R})$ and hence shows the holonomy reduction to that group again does not automatically entail a reduction to a smaller group. Let $(\mathbb{L}^2, h, \mathbb{J})$ denote the para-Kähler Lorenztian surface with

$$h := \frac{2}{3\big(r^2 + 1\big)^2}\big(-dr^2 + r^2 d\varphi^2\big), \qquad \mathbb{J} := r\partial_r \otimes d\varphi + \tfrac{1}{r}\partial_\varphi \otimes dr.$$

Then, the triple $(\mathbb{L}^2 \times \mathbb{L}^2, h \oplus h, \mathbb{J} \oplus \mathbb{J})$, is a suitably normalized para-Kähler structure and we can proceed as before. Every $(2, 3, 5)$ distribution in the determined family is diffeomorphic to the Lorentzian analogue of the rolling distribution for the surfaces $(\mathbb{L}^2, h)$ and $(\mathbb{L}^2, -h)$.[8]

Example 6.2 (a cohomogeneity 1 distribution from a homogeneous projective surface)**.** We construct an example of a Ricci-positive almost Einstein $(2, 3, 5)$ conformal structure by specifying a para-Fefferman conformal structure $\mathbf{c}_N$ on a 4-manifold N and solving a natural geometric Dirichlet problem: We produce a conformal structure $\mathbf{c}$ on $N \times \mathbb{R}$ equipped with a holonomy reduction to $\mathrm{SL}(3, \mathbb{R})$ for which the hypersurface curved orbit is N and the induced structure there is $\mathbf{c}_N$. In particular, this yields an example of an almost Einstein $(2, 3, 5)$ distribution for which the zero locus of the almost Einstein scale is nonempty, and hence for which the curved orbit decomposition has more than one nonempty curved orbit.

---

[8]This para-Kähler–Einstein structure is isometric to [22, equation (4.21)], which is attributed there to Nurowski.

Consider the projective structure $[\nabla]$ on $\mathbb{R}^2_{xy}$ containing the torsion-free connection ∇ characterized by

$$\nabla_{\partial_x}\partial_x = 3xy^2\partial_x + x^3\partial_y, \qquad \nabla_{\partial_x}\partial_y = \nabla_{\partial_y}\partial_x = 0, \qquad \nabla_{\partial_y}\partial_y = x^3\partial_x - 3x^2y\partial_y.$$

Eliminating the parameter in the geodesic equations for ∇ yields the ODE $\ddot{y} = (x\dot{y} - y)^3$, which corresponds (recall Section 5.5.4) to the function $F(x,y,p) = (xp - y)^3$. The point symmetry algebra of the ODE (that is, the symmetry algebra of the Legendrean contact structure on $J := \{xp - y > 0\} \subset J^1(\mathbb{R}, \mathbb{R})$) is $\mathfrak{sl}(2,\mathbb{R})$ and acts infinitesimally transitively. Hence, we may identify J with an open subset of (some cover of) $\mathrm{SL}(2,\mathbb{R})$. With respect to the left-invariant local frame

$$E_X := -(xp - y)^2\partial_p, \qquad E_H := x\partial_x + y\partial_y, \qquad E_Y := \frac{1}{xp - y}(\partial_x + p\partial_y).$$

of J, the line fields spanning the contact distribution are $\langle\partial_p\rangle = \langle E_X\rangle$, and $\langle D_x\rangle = \langle E_Y - 3E_X\rangle$. The Fefferman conformal structure $(N, \mathbf{c}_N)$ is again homogeneous: Its (5-dimensional) symmetry algebra $\mathfrak{aut}(\mathbf{c}_N)$ contains an infinitesimally transitive subalgebra isomorphic to $\mathfrak{gl}(2,\mathbb{R})$. A (local) left-invariant frame of N realizing this subalgebra is given by

$$\hat{E}_X = E_X + x(xp - y)\partial_a, \qquad \hat{E}_H = E_H - \partial_a, \qquad \hat{E}_Y = E_Y, \qquad \partial_a,$$

where a is the standard coordinate on the fiber of $N \to J$ and our notation uses the natural (local) decomposition $N \cong J \times \mathbb{R}_a$. In the dual left-invariant coframe $\{\chi, \eta, \upsilon, \alpha\}$, the conformal structure $\mathbf{c}_N$ has left-invariant representative $g_N := -\chi\upsilon - \eta^2 + \eta\alpha - \upsilon^2$.

The scale $\sigma_N := e^{a/2}\sqrt{xp - y}$ (given here with respect to the scale corresponding to g_N) is an almost Einstein scale, and hence $g_E := \sigma_N^{-2}g_N$ is Einstein (in fact, Ricci-flat). The conformal class $\mathbf{c}$ on $M := N \times \mathbb{R}_r$ containing $g' := g_E - dr^2$ admits the almost Einstein scale r (here given with respect to g'): $g := r^{-2}g'|_{\{\pm r > 0\}} \in \mathbf{c}|_{\{\pm r > 0\}}$ is a *Poincaré–Einstein metric* for $\mathbf{c}_N$, and in particular is Ricci-positive, and $\mathbf{c}_N$ is a *conformal infinity* for g; see [29, Section 4]. (We suppress the notation for the pullback by the canonical projection $M = N \times \mathbb{R} \to N$.)

So, the curved orbits are $M_5^\pm = \{(p, r) \in N \times \mathbb{R} : \pm r > 0\}$, $M_4 = N \times \{0\} \leftrightarrow N$, and $M_2^\pm = \varnothing$. On M_5, $g := r^{-2}g'$ is Ricci-positive, and $(N, \mathbf{c}_N)$ is a conformal infinity for either of $(M_5^\pm, g|_{M_5^\pm})$. The infinitesimal symmetry algebra $\mathfrak{aut}(\mathbf{c})$ of $\mathbf{c}$ has dimension 6, and is spanned by $\mathcal{X} := y\partial_x - p^2\partial_p + p\partial_a$, $\mathcal{H} := -x\partial_x + y\partial_y + 2p\partial_p - \partial_a$, $\mathcal{Y} := x\partial_y + \partial_p$, $\mathcal{Z} := e^{-a}[(xp - y)\partial_p - x\partial_a]$, $\mathcal{A} := -2\partial_a + r\partial_r$, ∂_r. Now, $\mathcal{X} \wedge \mathcal{H} \wedge \mathcal{Y} \wedge \mathcal{A} \wedge \partial_r = -2(xp - y)^2\partial_x \wedge \partial_y \wedge \partial_p \wedge \partial_a \wedge \partial_r$, which vanishes nowhere on M, so $(M, \mathbf{c})$ is homogeneous.

Computing the compatible parallel tractor 3-forms, and in particular using (4.6), gives that one 1-parameter family of conformally isometric oriented $(2,3,5)$ distributions $\mathbf{D}_t^\mp$ that induce $\mathbf{c}$ and are related by the Einstein scale r is given on M_5 as

$$\mathbf{D}_t^\mp := \left\langle \pm\frac{re^{-a\mp t}}{xp - y}\hat{E}_X + 2\partial_a, \mp\left[2e^{2a\pm t}(xp - y)^2 + \frac{1}{2}e^{\mp t}r^2\right]\hat{E}_X \right.$$
$$\left. + e^a r(xp - y)\hat{E}_H \pm 2(xp - y)^2 e^{2a\pm t}\hat{E}_Y + \frac{1}{r}\partial_a + \partial_r \right\rangle.$$

Computing the wedge product of the two spanning fields shows that this span extends smoothly across M_4 to a $(2,3,5)$ distribution on all of M. By definition this family is $\mathcal{D}(\mathbf{D}_0^-; r)$, and the corresponding conformal Killing field is $\iota_7(r) = \mathcal{A}$. The infinitesimal symmetry algebra of $\mathbf{D}_t^\pm$ is $\mathfrak{aut}(\mathbf{D}_t^\pm) = \langle \mathcal{X}, \mathcal{H}, \mathcal{Y}, \pm e^{\pm t}\mathcal{Z} - 2\partial_r \rangle \cong \mathfrak{gl}(2,\mathbb{R})$. In particular, this furnishes an example of an inhomogeneous $(2,3,5)$ distribution that induces a homogeneous conformal structures.

The metric g' is itself Ricci-flat, so the conformal structure $\mathbf{c}$ admits two linearly independent almost Einstein scales. In the scale of $\mathbf{c}$ determined by g', $\mathrm{aEs}(\mathbf{c}) = \langle 1, r\rangle$, and the corresponding

conformal Killing fields are spanned by $\iota_7(1) = \mathcal{Z} + \partial_r$ and $\iota_7(r) = \mathcal{A}$. These scales correspond to two linearly independent parallel tractors, which reduces the conformal holonomy $\mathrm{Hol}(\mathbf{c})$ to a proper subgroup of $\mathrm{SL}(3,\mathbb{R})$; computing gives $\mathrm{Hol}(\mathbf{c}) \cong \mathrm{SL}(2,\mathbb{R}) \ltimes \mathbb{R}^2$.

Example 6.3 (submaximally symmetric $(2,3,5)$ distributions). In Cartan's analysis [21] of the equivalence problem for $(2,3,5)$ distributions, he showed that if the dimension of the infinitesimal symmetry algebra of a $(2,3,5)$ distribution $\mathbf{D}$ has infinitesimal symmetry algebra of dimension < 14 (equivalently, if it is not locally flat) and satisfies a natural uniformity condition, then $\dim \mathfrak{aut}(\mathbf{D}) \leq 7$. (It was shown much more recently, in [48], that the uniformity condition is unnecessary.) Moreover, equality holds iff the distribution is locally equivalent, up to a suitable notion of complexification, to the distribution

$$\mathbf{D}_I := \big\langle \partial_q, \partial_x + p\partial_y + q\partial_p - \tfrac{1}{2}\big[q^2 + \tfrac{10}{3}Ip^2 + \big(1 + I^2\big)y^2\big]\partial_z \big\rangle \tag{6.1}$$

on $\mathbb{R}^5_{xypqz}$ for some constant I.[9]

The almost Einstein geometry of the distributions $\mathbf{D}_I$ is discussed in detail in [64]: The induced conformal structure $\mathbf{c}_I := \mathbf{c}_{\mathbf{D}_I}$ contains the representative metric

$$\begin{aligned} g_I := {}&\big[-\tfrac{3}{2}\big(I^2 + 1\big)y^2 + 2Ip^2 - \tfrac{1}{2}q^2\big]dx^2 - 4Ip\,dx\,dy + q\,dx\,dp \\ &- 3p\,dx\,dq - 3\,dx\,dz - 3I\,dy^2 + 3\,dy\,dq - 2\,dp^2. \end{aligned}$$

The trivializations by g_I of the almost Einstein scales of $\mathbf{c}_I$ are the pullbacks by the projection $\mathbb{R}^5_{xypqz} \to \mathbb{R}_x$ of the solutions of the homogeneous ODE $\sigma'' - \tfrac{1}{3}I\sigma = 0$ in x, and all of these turn out to be Ricci-flat. In particular the vector space of almost Einstein scales of $\mathbf{c}_I$ is 2-dimensional, so by Theorem 3.5 $\dim \mathfrak{aut}(\mathbf{c}_I) = \dim \mathfrak{aut}(\mathbf{D}_I) + \dim \mathrm{aEs}(\mathbf{c}_I) = 9$. For all I, $\mathrm{Hol}(\mathbf{c}_I)$ is isomorphic to the 5-dimensional Heisenberg group. Unlike for the non-Ricci-flat cases, the authors are aware of no example of a $(2,3,5)$ distribution $\mathbf{D}$ for which $\mathbf{c}_{\mathbf{D}}$ is equal to the full (8-dimensional) stabilizer $\mathrm{SL}(2,\mathbb{R}) \ltimes Q_+$ in G_2 of an isotropic vector in the standard representation.

These distributions are contained in the first class of examples of $(2,3,5)$ distributions whose induced conformal structures locally admit Einstein representatives [52, Example 6].[10]

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[9]The coefficient $\tfrac{10}{3}$ corrects an arithmetic error in [21, Section 9, equation (6)]. Also, note that we have specialized the formula given there to constant I.

[10]In that reference, these distributions were given in a form not immediately recognizable as diffeomorphic to those in (6.1). For $I \neq \pm\tfrac{3}{4}$, the distribution $\mathbf{D}_I$ is diffeomorphic to the distribution defined via [52, equation (55)] by the function $F(q) = q^m$, where $k = 2m - 1$ is any value that satisfies

$$I^2 = \frac{(k^2 + 1)^2}{(k^2 - 9)(\tfrac{1}{9} - k^2)};$$

when $I = \pm\tfrac{3}{4}$, one may take $F(q) = \log q$ [26].

A Tabular summary of the curved orbit decomposition.

M_a	ε −1	0	+1	Section	S	$S \cap P$	structure	A	leaf space structure	X	$\mathbf{L}$	$\mathbf{S}$
$M_5^\pm$	•				$SU(1,2)$	$SU(1,1)$	Sasaki–Einstein	$U(1,1)$	Kähler–Einstein			
		•		Section 5.4	$SL(2,\mathbb{R}) \ltimes Q_+$	$SL(2,\mathbb{R})$	null-Sasaki–Einstein	$GL(2,\mathbb{R})$	null-Kähler–Einstein	$\pm H_\Phi(X,\mathbb{S}) > 0$	$\mathbf{L} \subset [\mathbf{D},\mathbf{D}]$ $\mathbf{L} \pitchfork \mathbf{D}$	–
			•		$SL(3,\mathbb{R})$	$SL(2,\mathbb{R})$	para-Sasaki–Einstein	$GL(2,\mathbb{R})$	para-Kähler–Einstein			
M_4	•			Section 5.5.3	$SU(1,2)$	P_-	Fefferman conformal (sig. $(1,3)$)	$P_{SU(1,2)}$	3-dim. CR structure			
		•		Section 5.5.5	$SL(2,\mathbb{R}) \ltimes Q_+$	$\mathbb{R} \ltimes \mathbb{R}^3$	fibration over conformal (sig. $(1,2)$) + isotropic line field	$*^{\dagger}$	complete flag field‡	$H(X,\mathbb{S}) = 0$ $(X \times \mathbb{S}) \wedge X \neq 0$	$\mathbf{L} \subset \mathbf{D}$	$\mathbf{S} \pitchfork [\mathbf{D},\mathbf{D}]$
			•	Section 5.5.4	$SL(3,\mathbb{R})$	P_+	para-Fefferman conformal (sig. $(2,2)$)	P_{12}	second-order ODE			
$M_2^\pm$			•	Section 5.6.1	$SL(3,\mathbb{R})$	P_1 or P_2	2-dim. projective	–	–	$X \times \mathbb{S} = \pm X$	–	$\mathbf{S} \subset [\mathbf{D},\mathbf{D}]$ $\mathbf{S} \pitchfork \mathbf{D}$
M_2		•		Section 5.6.2	$SL(2,\mathbb{R}) \ltimes Q_+$	$\mathbb{R} \ltimes (\mathbb{R}^4 \ltimes \mathbb{R})$	line field	–	–	$X \times \mathbb{S} = 0$ $X \wedge \mathbb{S} \neq 0$	–	$\mathbf{S} \subset \mathbf{D}$
$M_0^\pm$		•			–	$SL(2,\mathbb{R}) \ltimes Q_+$	$SL(2,\mathbb{R}) \ltimes Q_+$	trivial	–	–	$X \wedge \mathbb{S} = 0$	–	–

† This is the solvable 5-dimensional group $\mathbb{R} \ltimes ((\mathbb{R}^2 \ltimes \mathbb{R}) \oplus \mathbb{R})$.
‡ The leaf space also inherits a degenerate conformal bilinear form with respect to which the line field is isotropic and the plane distribution is isotropic but not totally isotropic.

M_a	ξ^a	I_{ab}	J_{ab}	K_{ab}
$M_5^\pm$	θ^a	$\frac{1}{2}(-\varepsilon\phi_{ab} + \bar\phi_{ab})$	$-\theta^c \chi_{cab}$	$\frac{1}{2}(-\varepsilon\phi_{ab} - \bar\phi_{ab})$
M_4	$\mu_b \phi^{ba}$	$-\varepsilon\phi_{ab} - 2\mu_{[a}\xi_{b]}$	$3\mu^c \phi_{[ca}\theta_{b]}$	$2\mu_{[a}\xi_{b]}$
$M_2^\pm$	0	$-\phi_{ab}$	ϕ_{ab}	0
M_2	0	0	0	0
$M_0^\pm$	0	0	0	0

The formulae for the open orbits $M_5^\pm$ are given in the scale σ.

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