Matter in Loop Quantum Gravity*

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Abstract. Loop quantum gravity, a non-perturbative and manifestly background free, quantum theory of gravity implies that at the kinematical level the spatial geometry is discrete in a specific sense. The spirit of background independence also requires a non-standard quantum representation of matter. While loop quantization of standard model fields has been proposed, detail study of its implications is not yet available. This review aims to survey the various efforts and results.

Key words: loop quantization; loop quantum gravity; matter in loop quantum gravity

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1 Introduction

For most physical situations, we have the comfort of a background space-time geometry – be it the space and time of the non-relativistic regime or the Minkowski space-time of special relativistic regimes or even more exotic black hole space-times. Even to have a well defined, causally well behaved and deterministic classical theory of fields, the background space-times are limited to the class of *globally hyperbolic* geometries. Quantum field theories can be defined in all such space-times [31]. However, the familiar Fock representation or the particle interpretation can be defined *uniquely* only for a sub-class of space-times namely those that are *stationary* or at least asymptotically (in future and past) stationary. This is because in relativistic theories, the Fock representation depends crucially on having a notion of positive (negative) energy solutions, i.e. a notion of preferred time [63].

However, in regimes of early universe or evaporating black holes, we loose this comfort of having a fixed background space-time. In these cases, not only do we have to face 'quantum gravity' but also face the issue of quantization of other matter quantum fields *in the absence of a background space-time*.

One strategy to construct quantum theories is to use the Gelfand–Naimark–Segal theory [23]. Given a commutative C^* algebra, for every positive linear functional on it, there is a cyclic, unitary representation. The strategy is to look for a suitable sub-algebra of the Poisson bracket algebra of classical observables, identify its commutative sub-algebra and construct various representation. Choose among these, those on which the full sub-algebra is suitably represented. Under favourable conditions, a unique representation is picked out. A well known example of this is the unique, weakly continuous representation of the Weyl–Heisenberg C^* algebra (non-commutative) [4, 19]. The other example is the holonomy-flux representation of Loop Quantum Gravity (LQG) [24, 45].

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A natural commutative C^* algebra is constructed from suitable *configuration space variables*. For a linear configuration space such as $\mathbb{R}^N$, the coordinates themselves form suitable variables. For a more general non-linear configuration space, coordinates have only local meaning and are not suitable variables. Instead, one constructs a set of *functions* on the configuration space which are invariant under change of local coordinates. These will be more in number than the dimension of the configuration space (manifold) and may possibly have some relations among them. For example, for the configuration space S^2, we can choose 3 functions on it – namely $X^i(\theta,\phi)$, $i = 1,2,3$ satisfying $\sum_i (X^i)^2 = 1$ (say). The chosen class of functions is required to constitute a *separating set of functions* so that these functions serve to distinguish different points of the configuration space. A set of functions $\{f_i\}$ on a space Q forms a separating set for Q if, for every pair of distinct points, there are at-least two functions in the set which have distinct values at these points.

For field theories, there is a further feature. The configuration space is heuristically described in terms of (tensor etc) fields on a manifold. The 'delta function' in the canonical brackets, means that the canonical coordinates are to be understood as suitably smeared fields. For example, for scalar field ϕ, $F_f(\phi) := \int_\Sigma d^3x f(x)\phi(x)$, for all scalar densities of weight 1, f's, defines functions on the configuration space. For a gauge field $T_i A_a^i dx^a$, the holonomies $h_e(A) := \mathrm{Pexp}(\int_e A)$, is another example of such functions.

It turns out that suitable representation of the Poisson bracket algebra can also be constructed using a special class of functions on the configuration space, the so called *cylindrical functions*, in conjunction with projective techniques [6]. When applied to formulation of gravity in terms of the real SU(2) connection, this constructs the *kinematical Hilbert space* of LQG[1]. On this Hilbert space, geometrical operators have discrete spectra and their eigenstates are associated with appropriate class of graphs. This is paraphrased by saying that quantum geometry lives on *graphs*. The quantum excitations of matter are thus also expected to be associated with *vertices and edges* of graphs.

Construction of such a representation constitutes only a first step in the construction of a quantum theory of gravity. Due to the diffeomorphism invariance, the system has *gauge invariances* and the (Dirac) quantization procedure is split into *three* stages: construction of the *kinematical Hilbert space* followed by (distributional) solutions of the constraints followed by specification of physical Hilbert space.

In this review we will discuss the construction of kinematical Hilbert space. Our discussion is confined to *four* space-time dimensions wherein the matter fields can be *scalars, spinors and 1-form gauge fields*. We will not discuss the gravitational sector (discussed in [6] for instance) except to use the real SU(2) variables – connection and densitized triad – when needed in the matter sector. Our discussion is within a canonical framework leaving out discussion of matter in a spin foam framework. Each type of matter is discussed in a separate section. Section 2 discusses loop quantization of scalars and some of its implications. Both cases of scalar fields ranging over a compact set (Higgs scalars in the adjoint representation of a gauge group [58]) and those ranging over the full $\mathbb{R}$ [7] are discussed. Section 3 describes fermions, their Hamiltonian formulation and loop quantization while Maxwell and Yang–Mills fields are discussed in Section 4. Propagation of matter waves on quantum geometry is briefly discussed in Section 5. Finally, in Section 6 we make a few remarks.

[1]The cylindrical functions in this case are finite linear combinations of *spin network functions*. To define these, take an oriented, piece-wise analytic, closed graph γ, embedded in the 'spatial manifold', Σ_3. Associate to each of its E edges an $SU(2)$ group element in a representation j. These are the holonomies of a (generalized) connection. Associate to each of its V vertices, an intertwiner c and finally take the trace (contract all indices). These functions of connections are naturally labelled by $(\gamma, j_1, \ldots, j_E, c_1, \ldots, c_V)$. Similar functions are defined for matter fields as well.

2 Scalars

In loop quantum gravity approach, the scalar fields, like other matter fields, remain comparatively under-studied. Often the scalar field has been used to play a secondary role for describing dynamics of gravitational degrees of freedom. In particular, it has been used as a 'clock' variable in loop quantum cosmology. Nevertheless, there have been some notable efforts for studying quantum scalar fields themselves. In this section, we survey these different efforts involving quantum scalar fields.

2.1 Scalar field as a 'clock'

We begin the review for scalar fields by considering the situations where scalar fields have been used as 'clock' variables. In these studies, mostly for cosmological scenario, scalar fields have been quantized using standard quantization. Of course, one should use loop quantization also for scalar fields, as done for gravitational degrees of freedom.

In a general background the equation of motion of a *massless, free* scalar field ϕ is

$$\partial_\mu\big(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi\big) = 0, \tag{2.1}$$

where g is the determinant of the metric $g_{\mu\nu}$. For cosmological backgrounds, i.e. in a *spatially homogeneous* space-time, the equation of motion (2.1) reduces to

$$\partial_t\big(\sqrt{q(t)}N^{-1}\partial_t\phi\big) = 0. \tag{2.2}$$

Now if one makes the choice of lapse function N to be $\sqrt{q(t)}$ where q is the determinant of the spatial metric then the equation (2.2) implies scalar field $\phi \propto t$. This suggests that one may choose a massless scalar field as an 'internal clock' to describe the dynamics of the remaining degrees of freedom. Use of 'internal time' to describe the dynamics, instead of the coordinate time t, has the advantage of avoiding the so-called 'problem of time' in quantum gravity.

By choosing such a lapse function, the total Hamiltonian density for a cosmological system with a massless free scalar field can be written as ($N = \sqrt{q}$ chosen)

$$H = H_g + \frac{1}{2}\pi_\phi^2,$$

where H_g represents gravitational Hamiltonian. Now imposition of Hamiltonian constraint on the Hilbert space, i.e. $\hat{H}\Psi = 0$ implies

$$\partial_\phi^2\Psi = 2\hat{H}_g\Psi,$$

where we have used $\hat{\pi}_\phi = -i\partial_\phi$. Thus, the positive and negative frequency solutions can be viewed as satisfying a set of first order evolution equations of the form

$$\pm i\partial_\phi\Psi = \sqrt{-2\hat{H}_g}\Psi =: \hat{\mathbb{H}}\Psi.$$

By identifying the field ϕ to play the role of time and $\hat{\mathbb{H}}$ that of a proper Hamiltonian, one may note that the positive or negative frequency solutions satisfy appropriate Schrödinger equation with respect to the internal time ϕ. This approach of considering scalar field as an internal time has been used to describe the quantum dynamics of isotropic cosmological models such as Friedmann–Robertson–Walker space-time [8] as well as anisotropic cosmology such as Bianchi I space-time [11]. It has been shown for isotropic models that universe undergoes a *bounce* [9] when quantum dynamics is described with respect to an internal time.

2.2 Compact scalars

The Higgs scalars plays a very important role in standard model of particle physics. It provides a mechanism for generation of masses for different elementary particles. Naturally, in any approach to quantum gravity, one would expect to have an appropriate representation of the Higgs scalars in the given quantum theory.

In the standard model of particle physics, while constructing the quantum framework one specifically uses the fact that the background space-time is a Minkowski space-time. A natural kinematical measure available in constructing a suitable Hilbert space in a such space-time is Gaussian measure. Use of this measure leads to the standard Fock Hilbert space. However, one cannot use Gaussian measure in a background independent construction of the Hilbert space such as those used in loop quantum gravity. This is because Gaussian measure is not diffeomorphism invariant.

In order to avoid this problem, Thiemann proposes [58, 59, 60, 61] to use a set of bounded variables for quantizing Higgs scalars. In particular, bounded variables of the form $U(x) = e^{\phi_I(x)\tau^I}$ are used as the configuration space variables rather than the scalars ϕ_I themselves. Here, scalars ϕ_I have been made dimensionless with the help of a constant scale of $\sqrt{8\pi G}$ where G is the Newton's constant of gravitation. These variables are the analogues of the holonomies of connection variables. Unlike the gauge holonomies which are labelled by curves, these are labelled by *points* (no smearing) and are referred to as 'point-holonomies'. These are valued in $\mathcal{G}$ since the $\phi_I(x)\tau^I$ are valued in its Lie algebra. Note that $U(x)$ is a matrix representing a *group element* in a certain representation determined by the generators τ^I. Matrix elements of $U(x)$ therefore determine functions of the Higgs scalars. If one considers a single real scalar field, valued in a compact interval $[0, 2\pi]$ (say), then the basic variables can be chosen to be $U(x) = e^{i\phi(x)}$ which are valued in $U(1)$. For constructing the kinematical Hilbert space, one considers products of finitely many such matrix elements labelled by a finite set of points of the spatial manifold together with unitary representations of $\mathcal{G}$ associated with them. Finite linear combinations of these functions generate the (quantum) configuration space $\overline{\mathcal{U}}$ and the kinematical Hilbert space is then obtained as the space of square integrable functions $L_2(\overline{\mathcal{U}}, d\mu_H)$ where $d\mu_H$ is naturally available Haar measure on the group $\mathcal{G}$. On this, the point holonomies, but not the scalar field, are well defined (multiplicative) operators. The momentum is a density weight 1 object and its integral over three-dimensional regions are analogues of the flux variables of the gravity sector. Their action on the polynomials needs careful definition for which we refer to [58]. All other operators of interest are to be constructed using these basic operators.

We may recall that the classical dynamics of a real scalar field is governed by the Hamiltonian

$$H_\phi = \int d^3x N \left[\frac{\pi_\phi^2}{2\sqrt{q}} + \frac{1}{2}\sqrt{q} q^{ab} \nabla_a \phi \nabla_b \phi + \sqrt{q} V(\phi) \right]. \tag{2.3}$$

Formally, one can construct the quantum Hamiltonian operator corresponding to the classical Hamiltonian (2.3). Regularization of this quantum Hamiltonian operator is performed by using the triangulation of manifold which is analogous to the construction of gravity sector operators. While formal construction of the scalar Hamiltonian has been performed by Thiemann, understanding it's physical consequences remains an open issue.

2.3 Non-compact scalars

In Thiemann's construction [58], the configuration space variables are essentially *periodic* functions of the scalar field. This is sufficient to describe the scalar fields which range over a compact interval. However, such treatment of scalars is inadequate to describe the Klein–Gordon type real-valued scalar field which takes values in $\mathbb{R}^1$ rather than in $U(1)$. To circumvent this problem, Ashtekar, Lewandowski and Sahlmann propose the use of *almost periodic* functions as the

configuration space variables [7]. The formal construction of the kinematical Hilbert space for a single non-compact scalar is carried out as follows: First one considers finite sets of points in $\mathbb{R}^3$ referred to as the vertex sets. For a given vertex set $V = \{x_1, \ldots, x_n\}$, the corresponding vector space is generated by linear combinations of the functions of the form

$$\psi_{V,\vec{\lambda}}(\phi) = e^{i\sum_j \lambda_j \phi(x_j)},$$

where λ_j's are real numbers. After considering all possible sets of vertices, the union of corresponding vector spaces and together with its completion, one construct the quantum configuration space required for loop representation. If one denotes this space as $\overline{\mathcal{A}}$ then the corresponding Hilbert space is given by $L_2(\overline{\mathcal{A}}, d\mu)$. The measure $d\mu$ is defined as follows

$$\int_{\bar{\mathcal{A}}} d\mu \psi_{V,\vec{\lambda}} = \begin{cases} 1, & \text{if } \lambda_j = 0 \ \forall\, j, \\ 0, & \text{otherwise.} \end{cases} \tag{2.4}$$

The elementary configuration operators are analogous of the holonomy operator for connection variables and they act by multiplications as

$$\hat{h}(x,\lambda)\Psi = e^{i\lambda\phi(x)}\Psi.$$

Use of the measure (2.4), makes the elementary holonomy operator $h(x,\lambda) = e^{i\lambda\phi(x)}$ *not* to be weakly continuous in the parameter λ. This implies that one cannot construct an operator corresponding to the scalar field $\phi(x)$ itself. The conjugate variable to the holonomy which is promoted as an operator in the quantum theory, is taken to be the smeared field momentum

$$P(f) = \int dx^3 \pi_\phi(x) f(x),$$

where π_ϕ is conjugate momentum field and $f(x)$ is a test function. Their Poisson bracket is given by

$$\{h(x,\lambda), P(f)\} = i\lambda f(x) h(x,\lambda). \tag{2.5}$$

Subsequently, one looks for a representation of the Poisson bracket (2.5) in the quantum theory as a commutator of the elementary operators. The non-trivial commutator bracket of the elementary operators is

$$[\hat{h}(x,\lambda), \hat{P}(f)] = i\lambda f(x)\hat{h}(x,\lambda),$$

as holonomy operators and smeared momentum operators commute with themselves. After having the kinematical framework ready, one may construct the scalar Hamiltonian operator to understand its quantum dynamics.

2.4 Parametrized scalar fields

Within loop approach, scalar fields have been studied in the context of parametrized field theory (PFT). In the PFT approach, scalar field theory on a flat space-time is recast in the form as if the fields reside in a generally covariant background. In the references [43, 44], Laddha and Varadarajan have studied parametrized field theory for a free scalar field ϕ in 2-dimensional flat space-time. The action for such a scalar field with respect to an inertial frame is

$$S[\phi] = \int d^2X \eta^{AB} \partial_A \phi \partial_B \phi, \tag{2.6}$$

where metric $\eta_{AB} = \text{diag}(-1, 1)$ and $A = 0, 1$. Now one may choose to use an arbitrary coordinate system involving x^a with $a = 0, 1$ such that X^A can be viewed as 'parametrized' by x^a as $X^A = X^A(x^a)$. Then the action (2.6) can be rewritten as

$$S[\phi] = \int d^2x \sqrt{g} g^{ab} \partial_a \phi \partial_a \phi,$$

where $g_{ab} = \eta_{ab} \partial_a X^A \partial_b X^B$ and g is the absolute value of the determinant of the metric g_{ab}.

In the parametrized field theory, the action $S[\phi]$ is treated as not only a functional of ϕ, but also a functional of X^A's. In other words the action for parametrized field theory is taken to be

$$S[\phi, X^A] = \int d^2x \sqrt{g(X)} g^{ab}(X) \partial_a \phi \partial_a \phi. \tag{2.7}$$

The variation of the action (2.7) with respect to the field ϕ leads to $\partial_a(\sqrt{g} g^{ab} \phi \partial_a \phi) = 0$. This equation is basically the same as the flat space equation of motion for ϕ, i.e. $\eta^{AB} \partial_A \partial_B \phi = 0$ but written in x^a coordinates. The variation with respect to X^A leads to the equations which are automatically satisfied as long as $\eta^{AB} \partial_A \partial_B \phi = 0$. In other words, choice of the new coordinates remains undetermined as expected.

For canonical formulation of this theory, it is convenient to choose the light-cone coordinates defined as $X^{\pm} \equiv X^0 \pm X^1$. One may denote the corresponding conjugate momenta by $\Pi_{\pm}$. In terms of these coordinates the action (2.7) can be expressed as

$$S = \int dt \int dx \big[\pi_\phi \dot{\phi} + \Pi_+ \dot{X}^+ + \Pi_- \dot{X}^- - N^+ H_+ - N^- H_- \big],$$

where $N^{\pm}$ are the Lagrange multipliers for the respective constraints $H_{\pm}$. As constructed, the non-trivial Poisson brackets are

$$\{\phi(x), \pi_\phi(x')\} = \delta(x, x'), \{X^{\pm}(x), \Pi_{\pm}(x')\} = \delta(x, x'). \tag{2.8}$$

In the standard Fock quantization, one seeks an appropriate representation of the Poisson brackets (2.8) as commutator brackets of the operators in the quantum theory.

To proceed to polymer quantization, one notes that in 1-space dimension any tensor can be viewed as a density of weight equal to its covariant rank minus its contravariant rank. Conversely, a density weight 1 object can be viewed as a 1-form. Hence $\Pi_{\pm}, Y_{\pm} := \pi_f \pm \partial_x \phi$ are all 1-forms which are integrated along intervals. Exponentials of the integrals are the holonomies (*not point holonomies*). The polymer quantization of the system consists of choosing holonomies of $\Pi_{\pm}$ as multiplicative operators for the $X^{\pm}$, $\Pi_{\pm}$ sector and choosing holonomies of both $Y_{\pm}$ in the matter sector as the basic variables (non-commuting). Details should be seen in the references cited above.

Using this quantization it is shown that one can construct a suitable state in the polymer Hilbert space which reproduces the Fock space two points function in the large wavelength limit. However, understanding the behaviour of two-point function in the strong polymer regime remains an open issue.

2.5 Polymer quantized scalars in Fourier space

A different study of scalar fields using loop quantum gravity techniques has done by Hossain, Husain and Seahra [34]. In their approach, polymer quantization is performed for the Fourier modes of the free scalar field in flat space-time. This has the advantage of dealing with a known Hamiltonian which is a sum of decoupled harmonic oscillators. One then computes 2-point correlation function directly in the energy eigenfunction basis.

The phase space variables for the free scalar field are the canonical pair $(\phi(\mathbf{x},t),\pi(\mathbf{x},t))$ satisfying the Poisson bracket

$$\{\phi(t,\mathbf{x}),\pi(t,\mathbf{y})\}=\delta^{(3)}(\mathbf{x}-\mathbf{y}).$$

The Hamiltonian is

$$H_\phi=\int d^3\mathbf{x}\left[\frac{\pi^2}{2}+\frac{1}{2}\eta^{ab}\partial_a\phi\partial_b\phi\right], \tag{2.9}$$

where the space-time metric is $ds^2=-dt^2+\eta_{ab}dx^a dx^b$. It is convenient though not essential to put the system in a box by restricting to a finite region of flat 3-space so that the volume $V=\int d^3x\sqrt{\eta}$ is finite. One can then perform Fourier expansion of the field into the 3-momentum space as

$$\phi(t,\mathbf{x})=\frac{1}{\sqrt{V}}\sum_{\mathbf{k}}\phi_{\mathbf{k}}(t)e^{i\mathbf{k}\cdot\mathbf{x}},\qquad \phi_{\mathbf{k}}(t)=\frac{1}{\sqrt{V}}\int d^3x\; e^{-i\mathbf{k}\cdot x}\phi(\mathbf{x},t),$$

along with a similar expansion for $\pi(\mathbf{x},t)$. After a suitable redefinition of the Fourier modes, one can express scalar field Hamiltonian (2.9) in terms of *real-valued* $\phi_{\mathbf{k}}$ and $\pi_{\mathbf{k}}$ as

$$H_\phi=\sum_{\mathbf{k}}H_{\mathbf{k}}=\sum_{\mathbf{k}}\left[\frac{\pi_{\mathbf{k}}^2}{2}+\frac{1}{2}k^2\phi_{\mathbf{k}}^2\right],$$

with the Poisson bracket being $\{\phi_{\mathbf{k}},\pi_{\mathbf{k}'}\}=\delta_{\mathbf{k}\mathbf{k}'}$. Clearly, the Hamiltonian is a sum of *decoupled* harmonic oscillators. The polymer quantization of each mode then follows that of a quantum oscillator. In particular, the variables used in polymer quantization are $\phi_{\mathbf{k}}$ and $U_{\lambda\mathbf{k}}=e^{i\lambda\pi_k}$ which satisfy the Poisson bracket $\{\phi_{\mathbf{k}},U_{\lambda\mathbf{k}}\}=i\lambda U_{\lambda\mathbf{k}}$. The parameter λ has dimensions of $(\text{length})^{1/2}$.

As in the case of polymer quantization, π_k itself cannot be a well-defined operator in the quantum theory as the action of $U_{\lambda\mathbf{k}}$ is not *weakly* continuous with respect to λ. To represent momentum operator, instead one uses $\pi_k^\star=(U_{\lambda_\star\mathbf{k}}-U^\dagger_{\lambda_\star\mathbf{k}})/2i\lambda_\star$. It may be emphasized that the polymer quantization method introduces a new length scale (which is $\lambda_\star$ here) in addition to Planck's constant into the quantum theory. One can define a dimensionless parameter as

$$g=\lambda_\star^2|\mathbf{k}|\equiv\frac{|\mathbf{k}|}{M_\star}\propto\frac{\text{polymer length scale}}{\text{spatial wavelength}},$$

where $M_\star^{-1}$ is the fundamental length scale associated with the polymer quantization of ϕ. Clearly, $g=0$ should recover the results that one get from Fock quantization. The standard 2-point correlation function is defined as

$$\langle 0|\hat{\phi}(\mathbf{x},t)\hat{\phi}(\mathbf{x}',t')|0\rangle\equiv\frac{1}{V}\sum_{\mathbf{k}}e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}D_{\mathbf{k}}(t-t'),$$

where $|0\rangle=\Pi_{\mathbf{k}}\otimes|0_{\mathbf{k}}\rangle$ is the vacuum state and the matrix element is

$$D_{\mathbf{k}}(t-t')=\langle 0_{\mathbf{k}}|e^{i\hat{H}_{\mathbf{k}}t}\hat{\phi}_{\mathbf{k}}e^{-i\hat{H}_{\mathbf{k}}t}e^{i\hat{H}_{\mathbf{k}}t'}\hat{\phi}_{\mathbf{k}}e^{-i\hat{H}_{\mathbf{k}}t'}|0_{\mathbf{k}}\rangle, \tag{2.10}$$

where $\hat{H}_{\mathbf{k}}$ is the Hamiltonian operator. The matrix element (2.10) can be computed using the polymer oscillator spectrum $\hat{H}_{\mathbf{k}}|n_{\mathbf{k}}\rangle=E_n^{(\mathbf{k})}|n_{\mathbf{k}}\rangle$, and the expansion of the state $\hat{\phi}_{\mathbf{k}}|0_{\mathbf{k}}\rangle$ in the energy eigenstates as $\hat{\phi}_{\mathbf{k}}|0_{\mathbf{k}}\rangle=\sum_n c_n|n_{\mathbf{k}}\rangle$. By defining 4-momentum $p\equiv(\omega,\mathbf{k})$, one can write the momentum space propagator as

$$D_p=\sum_n\frac{2i\Delta E_n|c_n|^2}{p^2+\Delta E_n^2-|\mathbf{k}|^2-i\epsilon}, \tag{2.11}$$

where $p^2 = -\omega^2 + |\mathbf{k}|^2$ and choice of the sign of $i\epsilon$ corresponds to the Feynman propagator. One can recover the expected Fock space result $D_p = i/(p^2 - i\epsilon)$ by using properties of the Schrödinger oscillator, i.e. $c_n = \delta_{1,n}/\sqrt{2|\mathbf{k}|}$ and $\Delta E_n = n|\mathbf{k}|$. The propagator (2.11) with polymer corrections in the *infrared limit* ($g \ll 1$) can be written as

$$D_p = \frac{i(1-2g)}{p^2 - g|\mathbf{k}|^2 - i\epsilon} + \mathcal{O}(g^2).$$

The pole of the propagator implies an effective dispersion relation

$$\omega^2 = |\mathbf{k}|^2(1 - |\mathbf{k}|/M_\star),$$

which violates Lorentz symmetry. On the other hand in the *ultraviolet limit* ($g \gg 1$), the propagator with leading order corrections is

$$D_p = \frac{i/8g^2}{p^2 + 4g^2|\mathbf{k}|^2 - i\epsilon} + \mathcal{O}\left(\frac{1}{g^6}\right).$$

The corresponding dispersion relation

$$\omega^2 = 4|\mathbf{k}|^4/M_\star^2$$

also violates Lorentz invariance. One may also note that the propagation amplitude at high momentum is suppressed by a factor $1/g^2$.

Polymer quantization introduces a length scale $\lambda^2 = M_*^{-1}$. This is unavoidable. Such a scale could be provided by an underlying quantum geometry. However at the present level of understanding, it is to be treated as an arbitrary parameter. Current observational limits on Lorentz violation may be then used to constrain its value.

2.6 Other studies of scalar fields

Ultraviolet behaviour of polymer quantized scalar fields has been studied by Husain and Kreienbuehl by constructing Fock-like states in flat space-time [35]. Instead of Fourier space, as discussed in section (2.5), here one applies polymer quantization in real space. It is shown that the vacuum expectation values of the commutator and anti-commutator of the creation and annihilation operators become energy dependent, and tends to show some sort of fermionic behavior at high energy. Furthermore, the modified dispersion relation that arises leads to violation of Lorentz invariance.

In the context of spherically symmetry, Gambini, Pullin, and Rastgoo have studied polymer quantization of scalar field [27] coupled to gravity. Their study is performed in the midi-superspace context where even after imposition of the given symmetry, field theoretical nature of scalar fields, i.e. with infinitely many degrees of freedom, is retained. After choosing a quantum geometry state corresponding to locally flat geometry, quantization of the spherically symmetric scalar field is carried out using a discretization followed by a 'polymerization' done in two different ways (the field and the momentum). Their notable conclusion is that the propagator obtained is not Lorentz invariant.

Polymer quantized free massless scalar field in a homogeneous and isotropic cosmological space-time has been studied by Hossain, Husain and Seahra [33]. Use of semi-classical Friedman equation yields a non-singular and non-bouncing universe, without quantum gravity. The system exhibits an early de Sitter-like inflationary phase with sufficient expansion to resolve the horizon and entropy problems, and a built in mechanism for a graceful exit from inflation.

The propagation of polymer quantized scalar field in flat space-times has been studied in [32]. Polymer quantization is performed in real space there unlike in Fourier space as discussed in

section (2.5). Using semi-classical states, it has been shown that effective wave equation is both nonlinear and Lorentz invariance violating. Furthermore, it is demonstrated that polymer effects tend to accumulate with time for plane-symmetric waveforms. Possibility of measuring deviations from the Klein–Gordon equation in particle accelerators or astrophysical observations is also discussed.

3 Fermions

Fermions coupled to gravity has been discussed in the literature [12, 29, 39, 49, 50, 51, 64], at both classical and quantum level. In the formulation of general relativity in terms of real SU(2) connection, Thiemann discussed loop quantization of standard model fields [58, 59, 60, 61]. The fermions were treated in the second order form, i.e. fermions couple to gravity through the spin connection (torsion free Lorentz connection). Perez and Rovelli returned to fermions in presence of the Holst term and found that the Barbero–Immirzi parameter, γ, inverse of the coefficient of the Holst term, becomes classically observable [53]. Mercuri [47, 48] discovered that with a further addition of suitable non-minimal fermionic couplings, γ can be made classically unobservable. He also noted that the added terms (Holst plus non-minimal) can be expressed as the Nieh–Yan topological term after using the connection equation of motion. This strategy of adding non-minimal couplings to keep γ classically unobservable was followed for $N = 1, 2$ and 4 supergravities also [41]. Canonical analysis and loop quantization of fermions with non-minimal couplings was discussed by Bojowald and Das [14, 15].

It was subsequently realised that γ will automatically be classically unobservable provided it is the (inverse) of the coefficient of the Nieh–Yan term (a total divergence) in the Lagrangian density. Thus, instead of the Holst terms alone, if the Nieh–Yan term (Holst + (torsion)^2 piece) is used in conjunction with the Hilbert–Palatini, then for *arbitrary* matter and their couplings, the γ will drop out of the classical equations of motion[2]. But now, since the action is modified, it was not obvious that the real SU(2) formulation will result from the new action. It turned out that it is possible to systematically derive the real SU(2) Hamiltonian formulation from such an action [21, 57]. Since it is in presence of fermions that non-trivial torsion results from the equation of motion of the Lorentz connection, fermions were also included in the canonical analysis and real SU(2) formulation was seen to emerge. The canonical analysis leading to real SU(2) formulation has since been extended to include the other two topological terms namely the Pontryagin and the Euler classes [42, 54].

It is straight forward to derive the real SU(2) formulation from the Hilbert–Palatini action with γ^{-1} times the Nieh–Yan term [20, 21]. Its main points may be summarised as follows. In the *time gauge parametrization*[3], to begin with there are 13 components of the co-tetrad (N, N^a, V^i_a) and 24 components of the Lorentz connection $(\omega^{IJ}_t, \omega^{IJ}_a)$. Of these, 10 variables – N, N^a, ω^{IJ}_t – occur as Lagrange multipliers and 18 variables – $E^a_i \sim V^a_i$ (densitized triad) and 9 combinations A^i_a of ω^{IJ}_a – explicitly appear in the form $p\dot{q}$. The coefficients of the Lagrange multipliers form the primary constraints $\mathcal{H}_a$, $\mathcal{H}$, $\mathcal{G}^{0i}$ and $\mathcal{G}^{ij}$. The remaining 9 combinations of the Lorentz connections, have no velocities and lead to additional 9 primary constraints, say $\pi_i \approx 0 \approx \pi^{ij}$ (symmetric in i, j). Of these, the 3 primary constraints, $\pi_i \approx 0$ form a second class system with the primary constraints $\mathcal{G}^{0i}$ and are eliminated easily using Dirac brackets. Preservation of the remaining 6 primary constraints $\pi^{ij} \approx 0$ lead to further 6 secondary constraints $S_{ij} \approx 0$, with which they form a second class system. These are again eliminated using Dirac brackets. The left over system has the 18 phase space variables, (A^i_a, E^a_i), and 7 first class constraints and is

[2] A necessary condition for a topological origin of γ is thus satisfied.

[3] $I, J = 0, 1, 2, 3$ are the Lorentz indices, $a = 1, 2, 3$ denote the spacial indices. The co-tetrad components are $e^0_t := N$, $e^i_t := N^a V^i_a$, $e^i_a := V^i_a$, $q_{ab} := V^i_a V^j_b \delta_{ij}$, $q := \det(q_{ab}) \neq 0$. The Lorentz connection components are denoted as $\omega^{0i}_a =: K^i_a$, $\omega^{ij}_a =: \mathcal{E}^{ij}{}_k \Gamma^k_a$ [20].

the real SU(2) connection formulation. This counting and the steps in the constraint analysis remain the same when matter is included.

This derivation also introduces factors of $\mathrm{sgn}(e) := \mathrm{sign}(\det(e^I_\mu))$ $(= N\,\mathrm{sign}(\det(V^i_a))$ in the time gauge parametrization) in appropriate places. For instance, we are naturally lead to the definitions: $E^a_i := \mathrm{sgn}(e)\sqrt{q}V^a_i$ and $A^i_a := \gamma^{-1}\mathrm{sgn}(e)\,K^i_a - \Gamma^i_a$ Under an *improper* orthogonal transformation $\Lambda^i{}_j \in \mathrm{O}(3)$ acting on the index i, the triad changes its handedness and the $\mathrm{sgn}(e)$ factor changes sign leaving the handedness of E^a_i unchanged. This is as it should be since the index i on E represents adjoint representation of SO(3) while on V it represents the defining representation. For SO(3), both are equivalent but *not for* O(3). Under an *inversion*, $\Lambda^i{}_j = -\delta^i{}_j$, quantities in the defining representation change sign while those in the adjoint don't. The same reasoning applies to the definition of the connection. Now the connection is also even under inversion. The $\mathrm{sgn}(e)$ factor also change the behaviour of E^a_i and A^i_a under the action of *orientation reversing diffeomorphisms*. These factors are relevant for discussion of 'parity' properties of the canonical formulation (discussed later). This is independent of matter couplings.

When a Dirac fermion is included, the solution of the secondary, second class constraint $S_{ij} \approx 0$ changes since fermions couple to Lorentz connection apart from coupling to the triad. This solution leads to non-trivial Dirac brackets between the SU(2) connection and the fermions. One can however make natural shifts in the definition of the connection to recover the canonical brackets. This also simplifies the constraints. Four Fermi interaction terms however survive in the Hamiltonian and are signatures of first order formulation. In the second order formulation where fermions couple to the torsion free connection, there are no terms quartic in the fermions.

Fermions are also tied with possible parity violations [14, 15, 25]. There are two distinct notions of 'parity': one related to orientation of the space-time manifold (*parity*) and one related to the improper Lorentz transformation (*Lorentz parity*). Depending upon the definitions of the basic canonical variables (with or without the $\mathrm{sgn}(e)$ factors in this work (Section 3.3)), the canonical framework and the action are (non-)invariant under *one* of the notions of parity. These possibilities are distinguished and discussed in [20]. Bojowald and Das discuss the non-invariance under *Lorentz parity* [14, 15] in the context of non-minimally coupled fermion.

We now turn to the classical, canonical form of a Dirac fermion minimally coupled to gravity in the first order formulation and discuss the loop quantization of fermions in the following sub-section.

3.1 The Hamiltonian formulation

The starting point is a choice of (tensor/spinor) fields and a corresponding generally covariant, local action on 4-dimensional space-time $M \simeq \mathbb{R} \times \Sigma_3$. The next step is to carry out a 3+1 decomposition to identify the Lagrangian which is a function of (tensor) fields on Σ_3 together with their velocities with respect to the chosen time coordinate. The fields whose velocities appear in the Lagrangian are *potentially* the configuration space variables while those without velocities appearing in the Lagrangian are *Lagrange multipliers* whose coefficients will be *primary constraints*. This Lagrangian leads to the *kinematical phase space*. Now a constraint analysis a la Dirac is performed. If there are second class constraints, one may hope to simplify the analysis by solving the second class constraints. However, now one must use the Dirac brackets. These may not have the canonical form for the remaining variables (i.e. may not be Darboux coordinates) and a new choice of variables may be necessary. This is particularly relevant for Lagrangians which are *linear* in the velocities eg the Hilbert–Palatini–Nieh–Yan and the Dirac Lagrangians which typically do have second class constraints. The classical Hamiltonian formulation is completed when the action is expressed in the Hamiltonian form together with first class constraints. We also have fields coordinatizing the kinematical phase space

(after the second class constraints are eliminated) with the configuration space coordinates identified.

We begin with the Lagrangian 4-forms built from the basic fields the co-tetrad $e^I_\mu dx^\mu$ and the Lorentz connection $\omega_\mu^{IJ} dx^\mu$ ($\kappa := 8\pi G$)

$$\begin{aligned}
&\mathcal{L}_{\rm HP}(e,\omega) = \frac{1}{2\kappa}\left[\mathrm{sgn}(e)\frac{1}{2}\mathcal{E}_{IJKL}R^{IJ}(\omega)\wedge e^K\wedge e^L\right],\\
&\mathcal{L}_{\rm NY}(e,\omega) = \left[T^I(e,\omega)\wedge T_I(e,\omega) - R_{IJ}(\omega)\wedge e^I\wedge e^J\right],\\
&\mathcal{L}_{\rm grav} := \mathcal{L}_{\rm HP} + \frac{\eta}{2\kappa}\mathcal{L}_{\rm NY},\\
&\mathcal{L}_{\rm Dirac} = -\frac{i}{2}|e|\left[\bar\lambda e^\mu_I\gamma^I D_\mu(\omega,A,\ldots)\lambda - \overline{D_\mu(\omega,A,\ldots)\lambda}e^\mu_I\gamma^I\lambda\right],\\
&D_\mu(\omega)\lambda := \partial_\mu\lambda + \frac{1}{2}\omega_\mu^{\ IJ}\sigma_{IJ}\lambda + ie'A_\mu\lambda + \cdots + \lambda,\\
&\overline{D_\mu(\omega)\lambda} := \left\{\partial_\mu\lambda^\dagger + \frac{1}{2}\omega_\mu^{\ IJ}\lambda^\dagger\sigma^\dagger_{IJ} - ie'A_\mu\lambda^\dagger + \cdots + \lambda^\dagger\right\}\gamma^0.
\end{aligned}$$

The ... refer to possible couplings of the Dirac fermion to other gauge fields, e.g. the Maxwell field. These are suppressed in the following[4].

Here $R := d\omega + \omega\wedge\omega$ and $T := de + \omega\wedge e$ are the usual curvature and torsion 2-forms. The factor of $\mathrm{sgn}(e)$ is present because only then the Hilbert–Palatini Lagrangian matches with the $\sqrt{|g|}R(g)$. This arises from noting that determinant of the co-tetrad is given by, $e = \mathrm{sgn}(e)\sqrt{|g|}$.

A 3+1 decomposition is carried out as usual by choosing a foliation defined by *a time function*, $\mathcal{T} : M \to \mathbb{R}$ and a vector field $t^\mu\partial_\mu$, transversal to its leaves. The vector field is normalised by $t\cdot\partial\mathcal{T} = 1$ so that the parameters of its integral curves, serve as the time coordinate. Given such a decomposition, we choose a parametrization of the co-tetrad which leads to the usual ADM parametrization in terms of the metric, $g_{\mu\nu} := e^I_\mu e^J_\nu\eta_{IJ}$. Both the co-tetrad and the corresponding tetrad in this parametrization are displayed below

$$\begin{aligned}
&e^I_t = Nn^I + N^aV^I_a, \qquad e^I_a = V^I_a, \qquad n^In_I = -1, \qquad n^IV^J_a\eta_{IJ} = 0, &(3.1)\\
&e^t_I = -N^{-1}n_I, \qquad e^a_I = N^{-1}n_IN^a + V^a_I, \qquad n^IV^a_I = 0,\\
&\text{with}\quad V^a_IV^J_a = \delta^J_I + n_In^J, \qquad V^a_IV^I_b = \delta^a_b. &(3.2)
\end{aligned}$$

We will now restrict to configurations such that $n_i = 0$, $n_0 = -1$. This also implies that $V^0_a = 0 = V^a_0$ and that V^a_i are invertible with V^i_a as the inverse[5]. We also define the 3-metric $q_{ab} := V^i_aV^j_b\delta_{ij}$ (which is positive definite in classical theory) and denote $q := \det(q_{ab})$.

For future convenience we introduce $\Psi := q^{1/4}\lambda$, $\Psi^\dagger := q^{1/4}\lambda^\dagger$. This absorbs away the $\sqrt{q}$ factors in the Lagrangian as well as in the constraints. Note that the terms involving the derivatives of $\sqrt{q}$ cancel out. The λ fermionic variables being of density weight zero, the Ψ fermionic variables are of density weight 1/2. From now on we will use the half density variables.

Substituting the 3+1 parametrization of the tetrad and using the time-gauge, the Lagrangian can be written as

$$\mathcal{L}_{\rm Dirac} = \frac{i}{2}(\Psi^\dagger\partial_t\Psi - \partial_t(\Psi^\dagger)\Psi) - \omega_{t0i}\mathcal{G}^{0i}_F - \frac{1}{2}\omega_{tij}\mathcal{E}^{ijk}\mathcal{G}^F_k - N^{a'}\mathcal{H}^F_{a'} - N\mathcal{H}_F,$$

[4]Our conventions are: Greek letters denote space-time indices, lower case Latin letters denote space indices, upper case Latin ones denote Lorentz indices. Our signature is $(-+++)$. The (metric independent) Levi-Civita symbols are $\mathcal{E}^{txyz} = +1 = \mathcal{E}_{0123}$. The determinant is defined by $e\ \mathcal{E}^{IJKL} = -\mathcal{E}^{\mu\nu\alpha\beta}e^I_\mu e^J_\nu e^K_\alpha e^L_\beta$. For the Dirac matrices: $2\eta^{IJ}\mathbb{1} = \gamma^I\gamma^J + \gamma^J\gamma^I$, $\sigma^{IJ} := \frac{1}{4}\left[\gamma^I,\gamma^J\right]$, $\gamma_5 := i\gamma^0\gamma^1\gamma^2\gamma^3$, $\bar\lambda := \lambda^\dagger\gamma^0$.

[5]This would correspond to the choice of the so-called *time gauge* if we started without restricting the configurations a priori.

where

$$\mathcal{G}_F^{0i}=0,\qquad \mathcal{G}_i^F=-\frac{i}{2}\mathcal{E}_{ijk}\Psi^\dagger\sigma^{jk}\Psi=\Psi^\dagger\gamma_5\sigma_{0i}\Psi,$$
$$\mathcal{H}_{a'}^F=-\frac{i}{2}\left(\bar\Psi\gamma^0 D_{a'}\Psi-\overline{D_{a'}\Psi}\gamma^0\Psi\right),\qquad \mathcal{H}_F=\frac{i}{2}V_i^a\left(\bar\Psi\gamma^i D_a\Psi-\overline{D_a\Psi}\gamma^i\Psi\right).$$

For the fermions, the action being linear in velocities, we have primary constraints, $\pi_\lambda\sim\lambda^\dagger$, $\pi_{\lambda^\dagger}\sim\lambda$, which are second class. Also these variables fail to be Darboux coordinates – do not have vanishing Poisson brackets[6] with the gravitational variables due to the $\sqrt{q}$ factor. The shift to Ψ, $\Psi^\dagger$ variables makes the matter and gravitational variables Poisson-commute. Defining Dirac brackets relative to these primary, second class constraints allows us to use Ψ, $\Psi^\dagger$ as basic variables with Dirac brackets given by

$$\left\{\Psi^\alpha(x),\Psi^\dagger_\beta(y)\right\}=-i\delta^\alpha_\beta\delta^3(x,y).$$

From the details given in [20], we have the final expressions:

$$P_i^a:=(\kappa\gamma)^{-1}E_i^a=(\kappa\gamma)^{-1}\mathrm{sgn}(e)V_i^a\sqrt{q},\qquad A_a^i=\gamma\,\mathrm{sgn}(e)K_a^i-\Gamma_a^i(V),$$
$$\Gamma_a^i(V):=\frac{\mathcal{E}^{ijk}}{2}V_k^b\left\{\partial_bV_{aj}-\partial_aV_{bj}+V_j^cV_a^l\partial_bV_{cl}\right\},$$
$$\{A_a^i(x),P_j^b(y)\}=\delta_a^b\delta_j^i\delta^3(x,y),\qquad \{\Psi^\alpha(x),\Psi^\dagger_\beta(y)\}_+=-i\delta^\alpha_\beta\delta^3(x,y),$$
$$\mathcal{G}_i=\partial_aP_i^a+\mathcal{E}_{ij}{}^kA_a^jP_k^a-\frac{i}{2}\mathcal{E}_{ijk}\Psi^\dagger\sigma_{jk}\Psi,\qquad \sigma_{jk}:=\frac{1}{4}[\gamma_j,\gamma_k],$$
$$\mathcal{H}_a=F_{ab}^iP_i^b-\frac{i}{2}\left(\bar\Psi\gamma^0\mathcal{D}_a\Psi-\overline{\mathcal{D}_a\Psi}\gamma^0\Psi\right),\qquad \mathcal{D}_a\Psi:=\left(\partial_a-iA_a^i\gamma_5\sigma_{0i}\right)\Psi,$$
$$\mathcal{H}:=\kappa\gamma^2\frac{1}{2}\frac{P_j^bP_k^c}{\sqrt{q}}\left\{\mathcal{E}^{jk}{}_lF_{bc}^l(A)-(1+\gamma^2)\left(K_b^jK_c^k-K_c^jK_b^k\right)\right\}+\gamma\partial_a\left(\mathrm{sgn}(e)V_i^a\mathcal{G}^i_{\mathrm{vac}}\right)$$
$$+\frac{i}{2}\frac{\kappa\gamma\,\mathrm{sgn}(e)P_i^a}{\sqrt{q}}\left\{\bar\Psi\gamma^i\mathcal{D}_a\Psi-\overline{\mathcal{D}_a\Psi}\gamma^i\Psi\right\}(A)+\left[\left(\frac{\kappa\gamma^2}{2\sqrt{q}}K_a^iP_i^a\right)\left(\Psi^\dagger\gamma_5\Psi\right)\right]$$
$$-\left[\frac{3}{16}\kappa\frac{\left(\Psi^\dagger\gamma_5\Psi\right)^2}{\sqrt{q}}-\kappa\frac{(\Psi^\dagger\gamma_5\sigma^{0i}\Psi)(\Psi^\dagger\gamma_5\sigma^0{}_i\Psi)}{\sqrt{q}}\right].\tag{3.3}$$

In the above, $\mathcal{G}_i^{\mathrm{vac}}=\partial_aP_i^a+\mathcal{E}_{ij}{}^kA_a^jP_k^a:=\mathcal{G}^i-\mathcal{G}_F^i$.

Remark 1. *Dimensionally*, $\kappa\sim L^2$, $(A,K,\partial)\sim L^{-1}$, $E\sim L^0$, $P\sim L^{-2}$, $(\Psi,\bar\Psi)\sim L^{-3/2}$, $\mathcal{G}\sim L^{-3}$, $(\mathcal{H}_a,\mathcal{H})\sim L^{-4}$.

The *density weights are:* $(P,\mathcal{G},\mathcal{H}_a,\mathcal{H})=+1$, $(\Psi,\bar\Psi)=1/2$ and $(A,K,V,\Gamma)=0$.

Under[7] *Lorentz parity:* $(V,K,\mathrm{sgn}(e))$ are odd, $(\Gamma,A,P,\Psi,\bar\Psi)$ and the constraints are all even.

Under *parity combined with* $\gamma\to-\gamma$*:* $(V,K,\Gamma,A,P,\gamma\,\mathrm{sgn}(e))$ and the constraints are all even.

The A and K above correspond to the vacuum case for which the Thiemann identities hold. The inverse square root of q and K_a^i appearing above are manipulated exactly as in the vacuum case.

Explicitly, the identities are

$$\mathrm{sgn}(e)\mathcal{E}^{bca}V_a^i=\mathcal{E}^{ijk}\frac{E_j^bE_k^c}{\sqrt{\det(E_i^a)}},\qquad q:=(\det(V_a^i))^2=\det(E_i^a),$$
$$\kappa\gamma\frac{\mathrm{sgn}(e)}{2}V_a^i(x)=\left\{A_a^i(x),\int d^3y\sqrt{q}\right\}\quad\Rightarrow$$

[6] Strictly *Generalized Poisson brackets* [30], due to the Grassmann nature of the fermions.

[7] Discussed in Subsection 3.3.

$$\mathcal{E}^{ijk}\frac{E^b_j E^c_k}{\sqrt{\det(E^a_i)}} = \frac{2}{\kappa\gamma}\mathcal{E}^{bca}\left\{A^i_a(x), \int d^3y\sqrt{q}\right\},$$
$$H_E(1) := \frac{\kappa\gamma^2}{2}\int \frac{P^b_j P^c_k}{\sqrt{q}}\mathcal{E}^{jk}_{\ \ l}F^l_{bc}, \qquad \overline{K} := \int d^3y\,\mathrm{sgn}(e)K^i_a P^a_i \quad \Rightarrow$$
$$\overline{K} = (\kappa\gamma^3)^{-1}\left\{H_E(1), \int d^3y\sqrt{q}\right\}, \qquad \mathrm{sgn}(e)K^i_a(x) = \{A^i_a(x), \overline{K}\}.$$

These identities suffice to derive a quantization the Hamiltonian constraint from that of the 'Euclidean Hamiltonian constraint' (the first term in the Hamiltonian constraint) and of the volume operator. This completes the classical canonical formulations as it follows from the action.

3.2 Constraint algebra

It is easy to see that the gauge constraint generates correct gauge transformation of the basic fields. Specifically, with $\mathcal{G}(\Lambda) := \int_{\Sigma_3} d^3x\Lambda^i\mathcal{G}_i$,

$$\{A^i_a(x), \mathcal{G}(\Lambda)\} = -\mathcal{D}_a\Lambda^i = -\partial_a\Lambda^i - \mathcal{E}^i_{\ jk}A^j_a\Lambda^k,$$
$$\{P^a_i(x), \mathcal{G}(\Lambda)\} = +\mathcal{E}_{ij}^{\ \ k}\Lambda^j P^a_k,$$
$$\{\Psi^\alpha(x), \mathcal{G}(\Lambda)\} = -i\Lambda^i(\gamma_5\sigma_{0i}\Psi)^\alpha,$$
$$\{\Psi^\dagger_\alpha(x), \mathcal{G}(\Lambda)\} = +i\Lambda^i(\Psi^\dagger\gamma_5\sigma_{0i})_\alpha.$$

If we compute the infinitesimal action of the $\mathcal{H}_a$ constraint on the basic variables, we see that it equals the Lie derivatives of the basic variables only up to an SU(2) gauge transformation. We are however free to modify the constraints by adding suitable combinations of themselves. So we *define* the diffeomorphism constraint as

$$\mathcal{C}(\vec{N}) := \int_{\Sigma_3} d^3x N^a\mathcal{C}_a \quad \text{with}$$
$$\mathcal{C}_a := \mathcal{H}_a - A^i_a\mathcal{G}_i = P^b_i\partial_a A^i_b - \partial_b(A^i_a P^b_i) + \frac{i}{2}\big(\Psi^\dagger\partial_a\Psi - \partial_a\Psi^\dagger\cdot\Psi\big),$$

which leads to the infinitesimal transformations,

$$\{A^i_a(x), \mathcal{C}(\vec{N})\} = \mathcal{L}_{\vec{N}}A^i_a = \partial_a(N^b A^i_b) + N^b(\partial_b A^i_a - \partial_a A^i_b),$$
$$\{P^a_i(x), \mathcal{C}(\vec{N})\} = \mathcal{L}_{\vec{N}}P^a_i = N^b\partial_b P^a_i - P^b_i\partial_b N^a + 1\cdot(\partial_b N^b)P^a_i,$$
$$\{\Psi^\alpha(x), \mathcal{C}(\vec{N})\} = \mathcal{L}_{\vec{N}}\Psi^\alpha = N^b\partial_b\Psi^\alpha + \frac{1}{2}\cdot(\partial_b N^b)\Psi^\alpha,$$
$$\{\Psi^\dagger_\alpha(x), \mathcal{C}(\vec{N})\} = \mathcal{L}_{\vec{N}}\Psi^\dagger_\alpha = N^b\partial_b\Psi^\dagger_\alpha + \frac{1}{2}\cdot(\partial_b N^b)\Psi^\dagger_\alpha.$$

This implies that $\{\mathrm{var}, \int N^a C_a\} = \mathcal{L}_{N^a}(\mathrm{var})$ for all variables. The Gauge constraint already generates the correct gauge transformation of the basic variables. By inspection, it follows that the gauge constraints (weakly) commute with the diffeomorphism and the Hamiltonian constraint, the gauge constraint and the diffeomorphism constraints form sub-algebras and the diffeomorphism constraint transforms the Hamiltonian constraint by the Lie derivative. The non-trivial bracket is the bracket of two Hamiltonian constraints.

Before turning to loop quantization, we briefly draw attention to the invariance (or lack of it) under two distinct 'parity' operations.

3.3 Parity and Lorentz parity

Recall that we begin with the (co-)tetrad field e^I_μ, the Lorentz connection ω^{IJ}_μ and the fermion fields λ, $\bar{\lambda}$ (or Ψ, $\bar{\Psi}$) defined over a manifold $M \sim \mathbb{R} \times \Sigma_3$ which is assumed to be orientable. With the topology specified, M can be taken to be *time-orientable* with respect to all the metric tensors constructed by the parametrization (3.1), (3.2). Obviously, Σ_3 is orientable as well.

There are two distinct sets of 'parity' transformations: orientation reversing diffeomorphism of M and a $\mathrm{O}(1,3)$ transformation with determinant $= -1$. Note that these distinctions do not exist in flat space-time with Minkowski metric where the allowed diffeomorphisms are the isometries of the Minkowski metric which are the same as $\mathrm{O}(1,3)$ transformations. We will keep the time orientation fixed. Orientation reversing diffeomorphisms of M will then be reversing the orientation of Σ_3. We will refer to these as *parity* transformations. The improper Lorentz transformations Λ^I_J, will also be taken so that $\det \Lambda = -1$ and $\Lambda^0{}_0 = 1$ and will be referred to as *Lorentz parity transformations*.

After going to the canonical framework in the 'time gauge', we have the fields A^i_a, P^a_i, Ψ, $\bar{\Psi}$ defined on Σ_3. A parity transformation is now an orientation reversing diffeomorphism of Σ_3 while the improper $\mathrm{O}(3)$ transformation, *inversion*, will be taken as the Lorentz parity transformation.

In the Lagrangian framework, the Hilbert–Palatini action is invariant under both sets of transformations while the Nieh–Yan (and the Pontryagin) actions *change signs under parity* but are *invariant under Lorentz parity.* Hence the combined action is invariant under Lorentz parity and *non*-invariant under parity. The $\mathrm{sgn}(e)$ factor in the Hilbert–Palatini action is crucial for this.

The variables of the canonical framework are defined in terms of those of the Lagrangian framework. These definitions of the $\mathrm{SU}(2)$ connection in terms of K and Γ and the conjugate momentum in terms of the triad *are* consistent with the $\mathrm{SO}(3)$ gauge transformation extended to include the Lorentz parity. Thus the triad which transforms by the defining representation changes sign under Lorentz parity. The 'densitised triad' (or the conjugate momentum) transforms by the adjoint representation and should be invariant under Lorentz parity. The $\mathrm{sgn}(e)$ factor in their definitions precisely takes care of this. The same can be seen in the definition of the connection. It is easy to see that the symplectic structure and the constraints (vacuum) are *all invariant under Lorentz parity.*

When fermions are included, these are *scalars* under orientation preserving diffeomorphisms[8], and transform as $\Psi \to \gamma^0\Psi$ under Lorentz parity. All the constraints including fermions are invariant under Lorentz parity. This is true in both the Lagrangian and the canonical frameworks.

With regards to parity the situation is different. The action is not invariant under parity, due to the Nieh–Yan term. In the canonical framework, the connection is not simply even/odd under parity since the K term changes sign while the Γ does not. The 'densitised triad' also acquires an extra minus sign under parity (behaves as a 'pseudo-vector of weight 1'). The symplectic structure thus is *not* invariant. The constraints also are not invariant under parity and this is consistent with the non-invariance of the action.

The action *is* invariant under parity *combined with* $\gamma \to -\gamma$. Our definitions have the appropriate factors of γ to restore the simple (even) behaviour of the basic canonical variable resulting also in the invariance of the Poisson brackets and constraints.

In short, *Lorentz parity is an invariance of the action as well as the canonical framework while parity is not. However, parity combined with* $\gamma \to -\gamma$ *is an invariance of both action and the canonical framework.*

[8]Definition of spinors depends on the orientation. Thus the action of parity on spinors needs a much more careful statement. We will assume for simplicity that the under orientation reversing diffeomorphism, the fermionic action remains invariant.

Remark 2. We could consider a canonical formulation *ab initio*, say by making a canonical transformation from the ADM variables, without any reference to an action. We could then *define* of the basic canonical variables A_a^i, E_i^a, without the factors of $\mathrm{sgn}(e)$. This will restore the 'densitized triad' to its usual density weight 1 vector density status and the connection to its 1-form status. The canonical framework is then *invariant* under parity (without changing sign of γ). Under Lorentz parity, however, the connection does not have simple behaviour and the conjugate momentum is odd. The reason of course is that the internal index 'i' on the triad and on the extrinsic curvature refers to the defining representation while that on the connection and its conjugate refer to the adjoint representation and these are distinguished by the improper orthogonal transformations. This results in non-invariance of the canonical framework under Lorentz parity. If the sign of γ is changed along with the Lorentz parity transformation, then the basic variables are even, the symplectic structure is invariant and so are the constraints *and the action*. Thus, the definitions without the $\mathrm{sgn}(e)$ factors, interchanges the role of Lorentz parity and parity appropriately combined with $\gamma \to -\gamma$.

Which of these notions is 'appropriate'? If we were to consider formulation in terms of the metric tensor, then the notion of Lorentz parity is not even definable as there is no internal Lorentz transformation. On the other hand, with fermionic matter we *have to* consider oriented manifolds and the tetrad formulation, introducing the possibility of Lorentz parity. If the orientation of the manifold is regarded as a fixed background structure, the parity transformations are excluded by definition and Lorentz parity alone remains. Which of these notions is *relevant* for experimental observation, is unclear and so is the issue of 'gravitational parity violation/invariance' in LQG.

In the next subsection, we briefly summarise the loop quantization of fermions [14, 15, 58, 59, 60, 61].

3.4 Loop quantization of fermions

The loop quantization of the gravitational fields as well as of the fermions has been already given by Thiemann and those arguments remain valid. We have already gone from λ fermions to the half density Ψ fermions. In our derivation from the action, this is simply seen as the choice of Darboux coordinates. We also do not work with the 'conjugate' variables π_Ψ, $\pi_{\bar\Psi}$. Instead we solve the primary second class constraints and use Dirac brackets relative to these. Thus we take, for a single Dirac fermion, Ψ^A, $\bar\Psi_A$ to be complex, Grassmann valued, half densities: $\bar\Psi_A := (\Psi^A)^*$, $A = 1, \ldots, 4$. This can easily be generalized to many Dirac fermions or Weyl fermions[9].

As mentioned in the introduction, the first step is to construct functions on the fermionic phase space. Here we discuss Thiemann's proposal presented in a slightly different form. It is based on a particular, naturally available, class of smearing densities.

Consider a decomposition of Σ_3 into cells B_n, closed subsets of Σ_3. Let $\epsilon^3 := \mu(B_n)$ be the *Lebesgue measure* ('coordinate volume') of the cell B_n and let v_n be the *centre* of B_n (a marked point in the interior of B_n). Define the *indicator* or (*characteristic*) function:

$$\chi_\epsilon(v_n, x) := \chi_n(x) := \begin{cases} 1, & \text{if } x \in \mathrm{Int}(B_n), \\ 0, & \text{otherwise.} \end{cases}$$

Then the quantity, $\chi_n(x)/\epsilon_n^{3w}$ has density weight w. This is because, the Lebesgue measure of a cell, transforms as a scalar density of weight -1 under a coordinate transformation and the indicator function is of course invariant. Note that eventually we are interested in the

[9] The overbar on a Ψ with an index A, denotes the complex conjugate while one without an index denotes the Dirac conjugate: $\bar\Psi = \Psi^\dagger\gamma^0$.

limit of infinite refinement of the cell decomposition, $\epsilon_n \to 0$, and the stated behaviour of the Lebesgue measure is valid in this limit. Using these quantities, labelled by the cells of a cell decomposition, we can smear any scalar density of weight $(1-w)$ and construct a function on the configuration/phase space.

For a Grassmann variable Ψ (the spinorial index is suppressed) of weight 1/2, we define the Grassmann valued functions

$$\theta_n(\Psi) := \int_{\Sigma_3} d^3x\, \frac{\chi_n(x)}{\epsilon_n^{3/2}}\Psi(x) = \int_{B_n} d^3x\, \frac{\chi_n(x)}{\epsilon_n^{3/2}}\Psi(x).$$

Under a diffeomorphism: $B_n \to B'_n$, its Lebesgue measure $\epsilon_n^3 \to \epsilon_n'^3 = |\frac{\partial x'}{\partial x}|\epsilon_n^3$, $d^3x' = |\frac{\partial x'}{\partial x}|d^3x$ while $\Psi'(x') = \sqrt{|\frac{\partial x}{\partial x'}|}\Psi(x)$. This implies that

$$\begin{aligned}\theta'_n &:= \int_{B'_n} d^3x' \frac{\chi_n(x')\Psi'(x')}{\epsilon_n'^{3/2}} = \int_{B_n} d^3x \frac{\chi_n(x)\Psi(x)|\frac{\partial x'}{\partial x}||\frac{\partial x}{\partial x'}|^{1/2}}{\epsilon_n^{3/2}|\frac{\partial x'}{\partial x}|^{1/2}} \\ &= \int_{B_n} d^3x \frac{\chi_n(x)\Psi(x)}{\epsilon_n^{3/2}} = \theta_n.\end{aligned}$$

Thus, the θ variables are indeed *invariant* under diffeomorphism[10]. It is immediate that $\bar\theta_n = (\theta_n)^*$.

In the limit of finer cell decomposition, $\epsilon_n \to 0$, we get the following expressions:

$$\begin{aligned}&\lim_{\epsilon_n\to 0} \chi_{\epsilon_n}(v_n, x) = \delta_{v_n,x} \qquad \text{(Kroneker delta)}, \\ &\lim_{\epsilon_n\to 0} \frac{\chi_{\epsilon_n}(v_n,x)}{\epsilon_n^3} := \delta^3(v_n,x) \qquad (\delta\text{-distribution}).\end{aligned}$$

From these it follows that

$$\begin{aligned}\{\theta_m, \bar\theta_n\} &= \int_{B_m} d^3x \int_{B_n} d^3y \frac{\chi_m(x)}{\epsilon_m^{3/2}}\frac{\chi_n(y)}{\epsilon_n^{3/2}}\{\Psi(x), \bar\Psi(y)\} \\ &= -i\int_{B_m} d^3x \int_{B_n} d^3y \frac{\chi_m(x)}{\epsilon_m^{3/2}}\frac{\chi_n(y)}{\epsilon_n^{3/2}}\delta^3(x,y) \\ &= -i\delta_{m,n}\int_{B_n} d^3x \frac{\chi_n(x)}{\epsilon_n^3} = -i\delta_{m,n}, \qquad \because \chi_m(x)\chi_n(x) = \delta_{mn}\chi_m(x).\end{aligned}$$

With these, we obtain the Poisson brackets among the θ variables (functions on the phase space). The identity in the third line above holds because the $\chi_n(x) = 0$ unless x is in the *interior* of the cell B_n.

At finite but small ϵ_n, we can write, $\theta_n \approx \epsilon_n^{3/2}\Psi(v_n)$. Notice that the θ_n variables are effectively associated with the points v_n. This is particularly useful in expressing the constraint expressions in terms of the θ's as illustrated below.

Recall from equation (3.3) that the smeared Hamiltonian constraint has fermion bilinears with and without derivative as well as terms quartic in fermions. Using a cell decomposition, we can express the integral as a sum by restricting the integrals over the cells. In the limit of small ϵ_n, these integrals can be approximated by using the mean value theorem. The factors

[10] In the units where $\hbar = 1 = c$, these variables are also *dimensionless*. This is because the action is dimensionless which implies $\Psi \sim L^{-3/2}$ and $\epsilon \sim L$. This is similar to the holonomy variables constructed from connections. For the similar variables corresponding to scalar fields, this is not so.

of ϵ_n's available can be distributed with the fermions to go from $\Psi(v_n)$ to θ_n variables. The correct density weight of $+1$ ensures that no factors of ϵ_n's remain unabsorbed. In equations

$$\begin{aligned}
\int_{\Sigma_3} d^3x \bar{\Psi}_A(x) M^A_{\ B}(x) \Psi^B(x) &\approx \sum_n \epsilon_n^3 \bar{\Psi}_A(v_n) M^A_{\ B}(v_n) \Psi^B(v_n) \\
&= \sum_n \left(\epsilon_n^{3/2} \bar{\Psi}_A(v_n)\right) M^A_{\ B}(v_n) \left(\epsilon_n^{3/2} \Psi^B(v_n)\right) \approx \sum_n \bar{\theta}_A(v_n) M^A_{\ B}(v_n) \theta^B(v_n), \\
\int_{\Sigma_3} d^3x \bar{\Psi}_A(x) (\sigma^i f_i^a(x))^A_{\ B} \frac{\partial \Psi^B(x)}{\partial x^a} &\approx \sum_n \epsilon_n^3 \bar{\Psi}_A(v_n) (\sigma^i f_i^a(v_n))^A_{\ B} \frac{\partial \Psi^B(v_n)}{\partial v_n^a} \\
&\approx \sum_n \bar{\theta}_A(v_n) (\sigma^i f_i^a(v_n))^A_{\ B} \frac{\partial \theta^B(v_n)}{\partial v_n^a}.
\end{aligned}$$

This takes care of the fermionic bilinears. The quartic terms have a factor of $\sqrt{q}(x)$ in the denominator. Following [6, 14, 15], we first express it as a Poisson bracket with only positive powers of the volume of the cell containing the point x. From the definition of the determinant and our conventions given in the footnote 4, it follows that

$$\frac{\mathrm{sgn}(e)}{\sqrt{q}} = \frac{1}{6} \mathcal{E}^{abc} \mathcal{E}_{ijk} \frac{V_a^i V_b^j V_c^k}{q}.$$

Let $\mathcal{V}_n := \int_{B_n} d^3x \sqrt{q}(x)$. Then

$$\begin{aligned}
\{A_a^i(x), \mathcal{V}_n^{1/3}\} &= \frac{1}{3} \mathcal{V}_n^{-2/3} \{A_a^i(x), \mathcal{V}_n\} = \begin{cases} \frac{\kappa\gamma}{6} V_a^i \mathcal{V}_n^{-2/3}, & \text{if } x \in B_n, \\ 0, & \text{otherwise}, \end{cases} \\
\therefore \frac{\mathrm{sgn}(e)}{\sqrt{q}(x)} &= \frac{36}{(\kappa\gamma)^3} \{A_a^i, \mathcal{V}_n^{1/3}\} \{A_b^j, \mathcal{V}_n^{1/3}\} \{A_c^k, \mathcal{V}_n^{1/3}\} \mathcal{E}^{abc} \mathcal{E}_{ijk} \frac{\mathcal{V}_n^2}{q(x)}.
\end{aligned}$$

For small ϵ_n, and for non-zero answer ($x \in B_n$), $\mathcal{V}_n \approx \epsilon_n^3 \sqrt{q}(x)$ and therefore $\mathcal{V}_n^2/q(x) \approx \epsilon_n^6$. These factors of ϵ_n are neatly combined with the 4 Ψ, $\bar{\Psi}$ variables to go over to the θ variables. The ϵ_n^3 from the integration measure combines with the three A_a^i to produce the combinations $\mathrm{Tr}(\mathcal{E}^{IJK} h_I^{-1}\{h_I, \dots\} h_J^{-1}\{h_J, \dots\} h_K^{-1}\{h_K, \dots\})$ with the $\mathcal{E}$ ensuring that the three edges are non-coplanar and the Tr ensuring the SU(2) invariance.

Thus all integral expression in the constraints are expressible in terms of the θ variables. Since the transition from the Ψ variables to the θ variables is a linear operation, the Grassmann properties of the θ variables are the same as those of the Ψ variables. We have thus the classical phase space for Grassmann valued fields.

To construct a quantum theory, we first construct the space of complex valued (and Grassmann valued) functions and define an inner product on it. The elementary θ variables, are to represented as multiplicative operators while the their complex conjugates ("momenta") are to be represented by derivative operators. The procedure is standard [13, 22, 58, 59, 60, 61]. The Grassmann nature restricts the functions to be polynomials eg for a single pair of complex Grassmann variables, θ, $\bar{\theta}$, the most general function is: $f(\bar{\theta}, \theta) = a + b\theta + c\bar{\theta} + d\bar{\theta}\theta$ where a, b, c, d are complex numbers. A *holomorphic* function, has no $\bar{\theta}$ dependence. For k number of complex pairs, the vector space of holomorphic functions is 2^k-dimensional. On this space one has the usual *Berezin* measure, $\int d\bar{\theta} d\theta f(\bar{\theta}, \theta) = d$, however as pointed out by Thiemann, this is not positive definite and thus unsuitable for constructing a Hilbert space. The modification proposed by Thiemann is: $\mu(\bar{\theta}, \theta) := e^{\bar{\theta}\theta} d\bar{\theta} d\theta$. The inner product defined by $\langle f, g\rangle := \int \mu(\bar{\theta}, \theta) f^*(\bar{\theta}) g(\theta)$, turns the function space into a Hilbert space.

On this, define the operators

$$[\hat{\theta} f](\theta) := \theta f(\theta), \qquad [\hat{\bar{\theta}} f](\theta) := \hbar \frac{df(\theta)}{d\theta}.$$

These operators satisfy: $[\hat{\theta}, \hat{\bar{\theta}}]_+ = \hbar$ as desired by the general quantization rule, $\{A, B\} \to (-i/\hbar)[\hat{A}, \hat{B}]$, valid for the generalized Poisson bracket. It follows that $\hat{\bar{\theta}} = (\hat{\theta})^\dagger$ also holds reflecting the classical relation $\bar{\theta} = \theta^*$ and $\mu(\bar{\theta}, \theta)$ is the unique normalised measure selected by the above adjointness property [58, 59, 60, 61]. Generalization to finitely many Grassmann variables is immediate. For fermionic fields (infinitely many variables) one proceeds via projective limit.

For d-Weyl fermion fields, we will have $2d$ θ variables, $\theta^1, \theta^2, \ldots, \theta^{2d}$ at each point of Σ_3. The holomorphic functions are polynomials of maximal degree 2^{2d} in the θ variables at *each* point. Denote a *monomial* of degree k at a point v by

$$F_{v,\{i\}_{k(v)}} := \theta^{i_1}(v)\theta^{i_2}(v)\cdots\theta^{i_{k(v)}}(v), \qquad i_1, \ldots, i_k \in [1, 2d], \qquad k(v) \in [0, 2^{2d}].$$

For every finite set of points[11], $\{\vec{v}\} = \{v_1, v_2, \ldots, v_n\}$ and a corresponding vector of labels, $\vec{I}_{\vec{k}} := \{\{i\}_{k(v_1)}, \{i\}_{k(v_2)}, \ldots, \{i\}_{k(v_n)}\}$, define the *elementary functions of θ's*,

$$\mathcal{F}_{\vec{I}_{\vec{k}}} := \prod_{j=1,n} F_{v_j,\{i\}_{k(v_j)}}.$$

Finite linear combinations of these elementary functions are called (fermionic) cylindrical functions which are orthonormal with respect to the inner product defined point-wise. Closure of the set of these cylindrical functions defines the *Hilbert space of the loop quantization of fermions*. Quantization of more general observables proceeds by first expressing these observables in terms of the θ variables and promoting them to the corresponding operators.

3.5 Implications for loop quantum cosmology

Implications of fermionic matter has also been discussed in the context of homogeneous models by Bojowald and Das in [14, 15]. The symmetric fermionic fields are only restricted to be constants on the spatial manifold by homogeneity. Isotropy however requires these constants to be zero. One way to see this is to note that from the presumed non-zero Ψ's we can construct spatial vectors as $V_i^a \bar{\Psi}\gamma^i\Psi$ and by isotropy there cannot be any non-zero constant vector. For homogeneous, anisotropic models however non-zero fermionic constants are allowed[12].

This has a major implication for diagonal models. Recall that the homogeneous connection and triad are of the form: $A_a^i(x) = \Phi_I^i \omega_a^I(x)$, $E_i^a(x) = \sqrt{g_0} P_I^a X_I^a(x)$ where, ω_a^I are the Maurer–Cartan forms and X_I^a their dual vector fields. The sub-class of *diagonal* models is defined by the restriction: $\Phi_I^i := c_I \Lambda_I^i$, $P_i^I := p^I \Lambda_i^I$ (no sum over I), where $\Lambda_i^I \Lambda_J^i = \delta_J^I$, $\Lambda_I^i \Lambda_J^j \Lambda_K^k \mathcal{E}_{ijk} = \mathcal{E}_{IJK}$. With this choice, the homogeneous, gravitational part of the Gauss Law constraint vanishes identically. With the possibility of non-zero, homogeneous fermionic degrees of freedom, the homogeneous $\mathcal{G}_F^i \neq 0$ and therefore the gravitational part *cannot* be zero. Consequently, the connection and the triad variables cannot be diagonalised simultaneously.

A way out, suggested in [14, 15] is to use different Λ's for connection and triad. The corresponding dynamical system is still complicated and a specific choice is made to get a simplified

[11] When matter and gravity are considered together, it is natural (sufficient) to take the set of points to be the *vertices* of graph labelling a spin-network function in the gravity sector (and matter gauge sector if present).

[12] Only fermionic matter is sensitive to the action of the gravitational gauge group (subgroup of local Lorentz transformations). All other matter fields are *scalars* with respect to this action and thus do not contribute to the Gauss constraint.

model, wherein the constant fermionic vector is taken to be along, 1-axis. This permits a identification of canonical coordinates, 4 in number. To explore parity invariance more explicitly, the model is further specialized to Bianchi I LRS class. The main conclusion drawn is that in this specific case, parity invariance at the level of underlying difference equation is violated only if there are parity violating interactions in the *matter sector*. However, the difference equation remains deterministic even when parity is violated [14, 15].

4 Gauge fields

The construction of the kinematical Hilbert space for gauge fields proceeds exactly as in the case of the gravitational, SU(2) gauge fields. For identical logic, in a background independent context, the gauge fields are quantized through the use of the holonomies and fluxes of the electric fields. The new feature is the structure of the corresponding contributions to the Hamiltonian constraint and of course the new Gauss constraints corresponding to the additional gauge invariances. As remarked in the footnote 12, there is no contribution to the rotation constraint. The contribution to the diffeomorphism constraint is of the same form as in the gravitational sector. The couplings of these matter fields to the gravitational ones are through the triad variables for minimally coupled matter and there are also non-gravitational couplings among the matter fields. The underlying background independence which forces the use of holonomy and fluxes impacts the quantization of these interaction terms as well. Loop quantization of gauge fields has been discussed by Thiemann [58, 59, 60, 61] and Ashtekar–Lewandowski [6] (see also [1]). The salient points are noted below.

4.1 Maxwell field theory

The gauge group in the Maxwell case case is Abelian which simplifies some of the technical details. In particular, since the holonomies are simple numbers (not matrices), the Poisson bracket of holonomy with 3-dimensionally smeared electric field is proportional to the holonomy and the 3-smeared electric field maps cylindrical functions to cylindrical functions [6]. We will however use 2-dimensional smearing and the fluxes Poisson commute in both cases.

The classical analysis beginning with the Maxwell action followed by canonical formulation using the time gauge parametrization is summarised below

$$\begin{aligned}
&\mathcal{L}_{\text{Maxwell}} := \frac{1}{16\pi}\sqrt{|g|}g^{\mu\alpha}g^{\nu\beta}\mathbb{F}_{\mu\nu}\mathbb{F}_{\alpha\beta},\\
&H(N,N^a,\Lambda) = \int \frac{N}{8\pi}\frac{q_{ab}}{\sqrt{q}}\big(\mathbb{P}^a\mathbb{P}^b + \mathbb{B}^a\mathbb{B}^b\big) + \frac{N^a}{8\pi}\mathbb{F}_{ab}\mathbb{P}^b + \frac{\Lambda}{8\pi}\partial_a\mathbb{P}^a,\\
&\mathbb{B}^a := \frac{1}{2}\mathcal{E}^{abc}\mathbb{F}_{bc}, \qquad \mathbb{E}^a := \mathbb{P}^a = -\sqrt{q}\; N\; \mathbb{F}^{ta}
\end{aligned}$$

with $\mathbb{A}_a$, $\mathbb{P}^b$ being the canonical coordinates. Note that $\mathbb{P}^a$, $\mathbb{B}^a$ are both of density weight 1. As usual, before quantization, we need to express the constraints (and other observables of interest), in terms of the holonomies and fluxes.

We have the common factor of $q_{ab}/\sqrt{q}$. As before, introduce a cell decomposition with cells B_n, containing points v_n and with Lebesgue measure ϵ_n^3 and define $\mathcal{V}_n := \int_{B_n} d^3x\sqrt{q(x)}$. Noting that

$$\{A_a^i(x), \mathcal{V}_n^l\} = l\mathcal{V}_n^{l-1}\frac{\kappa\gamma}{2}\text{sgn}(e)V_a^i, \quad l \in (0,1), \qquad \mathcal{V}_n \approx \epsilon_n^3\sqrt{q}(v_n) \quad \text{for small} \;\; \epsilon_n,$$

we write

$$\frac{q_{ab}}{\sqrt{q}}(x) \approx \frac{16}{(\kappa\gamma)^2}\epsilon_n^3\delta_{ij}\{A_a^i(x), \sqrt{\mathcal{V}_n}\}\{A_b^j(x), \sqrt{\mathcal{V}_n}\}.$$

Next, observe that for 2-smeared fluxes of $\mathbb{P}^a$, we have $\Phi_S := \int_S \mathbb{P}^a \mathcal{E}_{abc} dS^{bc} \approx \mathbb{P}^a(v) n_a(S) \epsilon_S^2$ where S is a (small) surface, v is a point in it and ϵ_S^2 (small) is its Lebesgue measure. The 'normal' to S is given by, $n_a(S) := \mathcal{E}_{abc} \frac{\partial x^b}{\partial \xi_\alpha} \frac{\partial x^c}{\partial \xi_\beta} \mathcal{E}^{\alpha\beta}$ where ξ^α, $\alpha = 1, 2$ denote the local coordinates *on* S while $x^a(\xi)$ denote the embedding of the 2-surface in Σ_3. For a small triangular surface, we can also replace $n_a(S)\epsilon_S^2 \approx \frac{1}{2}\mathcal{E}_{abc} \delta S_I^b \delta S_J^c$ where δS_I^b, δS_J^c denote coordinate lengths of a pair of edges of the triangle. Likewise, for an infinitesimal curve e, $h_e := \mathrm{Pexp}(\int_e A) = \exp(\int_e A)$ implies $h_e^{-1}\{h_e, \dots\} \approx \{\int_e A, \dots\} \approx \{\delta t \dot{e}^a A_a^i \tau_i, \dots\}$.

To regularise the Hamiltonian, we introduce a cell decomposition and approximate the integral by a sum-over-cells. This provides a ϵ^3 and the $q_{ab}/\sqrt{q}$ expression provides another factor of ϵ^3 for each term. The 6 factors of ϵ can be distributed as 2 each for the two $\mathbb{P}$'s and 1 each for the two gravitational connections in the Poisson brackets. To express the variables in terms of fluxes and holonomies, we need to choose two surfaces and two curves paying attention to the *contraction* of a, b indices which is part of specification of gravitational coupling.

We recall from the construction done in the gravitational sector. We choose a triangulation of Σ_3 by elementary tetrahedra Δ_n [58, 59, 60, 61]. In anticipation of the quantization step, we choose the edges of the tetrahedra to be analytic. *One* vertex of each tetrahedron is distinguished and at the most finitely many tetrahedra meet at such a vertex. Edges meeting at a vertex are taken to be out-going. The three edges of each tetrahedron, provide three linearly independent tangent vectors at its vertex v_n, $\dot{e}^a_I$, $I = 1, 2, 3$, such that the a^{th} coordinate interval of the I^{th} edge $= \delta t_I \dot{e}^a_I$, no sum over I and I is *not* to be confused with the Lorentz index which is no longer relevant now. δt_I refers parametrization of the I^{th} edge. Each tetrahedron, also provides three non-coplanar surfaces S_{IJ}, bounded by the edges e_I, e_J and the 'opposite' edge. We have thus a natural choice of surfaces and edges. For these surfaces and edges, we have, $\Phi_{IJ} \approx \frac{1}{2}\mathbb{P}^a \mathcal{E}_{abc} (\dot{e}^b_I \delta t_I)(\dot{e}^c_J \delta t_J)$ and $h_K^{-1}\{h_K, \sqrt{\mathcal{V}_n}\} \approx \{\delta t_K \dot{e}^a_K A^i_a \tau_i, \sqrt{\mathcal{V}_n}\}$. Clearly

$$\frac{1}{6}\mathcal{E}^{IJK}\Phi_{IJ} h_K^{-1}\{h_K, \sqrt{\mathcal{V}_n}\} \approx \frac{1}{12}\mathcal{E}^{IJK}\mathbb{P}^a \mathcal{E}_{abc} (\delta t_I \dot{e}^b_I)(\delta t_J \dot{e}^c_J)(\delta t_K \dot{e}^d_K)\{A^i_d \tau_i, \sqrt{\mathcal{V}_n}\}.$$

Now

$$\mathcal{E}^{IJK} \dot{e}^b_I \dot{e}^c_J \dot{e}^d_K = \mathcal{E}^{bcd} \det(\dot{e}^a_I), \qquad \det(\dot{e}^a_I)\delta t_1 \delta t_2 \delta t_3 =: \epsilon^3, \qquad \mathcal{E}_{abc}\mathcal{E}^{bcd} = 2\delta^d_a.$$

Using these, it follows

$$\begin{aligned}
\frac{1}{8\pi}\int_{\Sigma_3} N \frac{q_{ab}}{\sqrt{q}} \mathbb{P}^a \mathbb{P}^b &\approx \frac{2}{\kappa^2\gamma^2\pi} \sum_n N(v_n) \sum_{\Delta_n} \delta_{ij} \big(\epsilon_n^3 \mathbb{P}^c \delta^a_c \{A^i_a, \sqrt{\mathcal{V}_\Delta}\}\big)\big(\epsilon_n^3 \mathbb{P}^d\ \delta^b_d\ \{A^j_b, \sqrt{\mathcal{V}_\Delta}\}\big) \\
&= \left(-\frac{4}{\kappa^2\gamma^2\pi}\right) \sum_n N(v_n) \sum_{\Delta_n} \frac{\mathcal{E}^{IJK}}{6} \frac{\mathcal{E}^{I'J'K'}}{6} \Phi_{IJ}\Phi_{I'J'} \\
&\quad \times \mathrm{Tr}\left[h_K^{-1}\{h_K, \sqrt{\mathcal{V}_\Delta}\} h_{K'}^{-1}\{h_{K'}, \sqrt{\mathcal{V}_\Delta}\}\right].
\end{aligned}$$

In the last equation, we have included $\mathrm{Tr}\,\tau_i\tau_j = -\frac{1}{2}\delta_{ij}$. All the factors of ϵ's have been neatly absorbed, again thanks to density weight 1 of the Hamiltonian. The sum over n is a sum over vertices of the triangulation. The sum over Δ_n includes those tetrahedra whose distinguished vertex is v_n and this is a finite sum. Note that some vertices of the triangulation may not be distinguished vertices of *any* tetrahedron. Such vertices do not contribute to the sum. Therefore, the sums over v_n and Δ_n together, denote sum over the cells of the decomposition. Notice that surfaces over which the *electric flux* is taken is correlated with the edge along which the *gravitational holonomy* is to be taken thanks to the $\mathcal{E}^{IJK}$ factors.

In the passage to quantum theory, the triangulation is adapted to the underlying graph of a cylindrical function[13] by aligning the edges of the elementary tetrahedra to be segments of corresponding edge of the graph and the distinguished vertex to be the vertex of the graph [58, 59, 60, 61]. Upon quantization, only those tetrahedra will contribute which have an edge overlapping with the edge of the holonomy. In each contributing cell (tetrahedron), the action of the electric flux operator, Φ_{IJ} will be non-zero *only* on the Maxwell holonomy along the edge e_K which is transversal to the surface S_{IJ} and will give $\hbar q_K$. This also selects the edge K in the gravitational holonomy, h_K. Here q_K, an integer, is the charge (in units of electric charge) in the Maxwell holonomy. On any cylindrical function, the action of the regulated quantum operator, results in a finite sum of terms since only vertices of the underlying graph contribute to the sum, regardless of refinement of the triangulation.

We have deviated somewhat from the procedure given in [6, 58, 59, 60, 61]. We have not done the point splitting and explicitly made passage to the flux operators absorbing the factors of ϵ's (in the limit of small ϵ). The correlation between the edge in the gravitational holonomy and the surface in the Maxwell flux is preserved through the use of triplet of edges and corresponding unique transversal surfaces provided by an elementary tetrahedron.

For the magnetic term, we can proceed identically by defining a flux $\Psi_S := \int_S \mathbb{B}^a \mathcal{E}_{abc} dS^{bc}$ which will lead to an equation same as the one above with Φ_S replaced by Ψ_S. Unlike the electric flux however, this magnetic flux is *not* an elementary variable. For

$$\begin{aligned}\Psi_S &:= \int_S \frac{1}{2}\mathcal{E}_{abc}\mathbb{B}^a dx^b \wedge dx^c = \int_S \mathcal{E}_{abc}\frac{\mathcal{E}^{aef}}{4}\mathbb{F}_{ef} dx^b \wedge dx^c \\ &= \int_S (\delta^e_b\delta^f_c - \delta^e_c\delta^f_c)\frac{1}{4}\mathbb{F}_{ef} dx^b \wedge dx^c = \int_S \mathbb{F} \approx \frac{1}{2}\left(\mathbb{h}(e_S) - \mathbb{h}^{-1}(e_S)\right),\end{aligned}$$

where e_S is a small close curve bounding the surface S. When the triangulation is adapted to the underlying graph of a cylindrical function, the curve e_S may be taken to be one of the loops α_{IJ} of Thiemann [58, 59, 60, 61], which bounds the face S_{IJ}. Because of the presence of the Maxwell holonomies (with unit charge), the underlying graph of a cylindrical state is changed by adding these loops. But there are no gravitational holonomies along these loops and hence the spin labels of these extra loops are zero.

We note in passing that if we have a spin and charge network state, i.e. a graph with both the spin and the charge labels for its edges some of which could be zero, then the action of the electric part of the Hamiltonian will be *non-zero only on those edges which have both the spin and the charge labels to be non-zero* – the volume operator will kill edges with no spin while the electric flux operator will kill the edge with no charge. This feature enables the operator to be well defined on the kinematical Hilbert space even in the limit of infinite refinement. By contrast, the magnetic part will *not* kill an edge with zero charge but will add loops with unit charge. This prevents the operator to be well defined on the kinematical Hilbert space in the limit of infinite refinement. It could however be defined on a subspace of the algebraic dual of the space of the cylindrical functions [6].

4.2 Yang–Mills field theory

Apart from the non-Abelian nature of the gauge field, in form, the expressions proceed in the same manner as for the Maxwell case. The details are given in [58, 59, 60, 61].

[13] The cylindrical functions in this case are analogous to the gravitational spin network functions except for the replacement of the spin label by a *integer charge* label with charge conservation at each vertex and are called *charge network* functions [62]. These have been called *flux networks* in [6].

5 Propagation of matter waves on quantum geometry

While there are good arguments to justify search for a quantum gravity, its observable signatures have been quite difficult to come by. Various different approaches to quantum gravity have contained a suggestion that space-time of quantum gravity is different from the continuum classical geometry be it discreteness and/or non-commutativity etc. Coupled with a heuristic idea that existence of a fundamental length scale may indicate a conflict with Lorentz invariance, a possible violation of Lorentz invariance has been considered as a possible signature of quantum gravity. The violation is thought to be manifested as a deviation from the Lorentz invariant *dispersion relation* for particle propagation on a 'quantum space-time'. Its cumulative effect over cosmological distances are estimated to be detectable. From LQG Hamiltonian constraint, such deviation have been extracted [1, 2, 3, 26] using the following set of assumptions[14]. The Lorentz violations in the Polymer quantized scalar in Minkowski background has already been discussed in Section 2.5.

One begins by assuming a quantum state to be of a product form with a suitable state for geometry as well as for matter. For example, the early works, e.g. [26], the geometry state was a *weave* state [10, 17, 28, 36, 37, 38, 65] which provides a length scale $L_{\rm weave} \gg \ell_{\rm P}$ such that for larger scales the 3-geometry may be approximated by a continuum while the discrete structure manifests near the Planck scale. The scales relevant for observations of interest are assumed to be much larger than $L_{\rm weave}$. The matter state is chosen to be a coherent state so that the expectation values of the matter operators can be approximated by classical fields (fields obeying classical equations to leading order). These fields are assumed to be varying slowly over a length scale of $\lambda \gg L_{\rm weave}$. Taking expectation value of the Hamiltonian constraint, the geometry operators are replaced by the metric used in the construction of the weave plus corrections of the order of $\ell_{\rm P}/L_{\rm weave}$. The expectation values of the matter part of the Hamiltonian constraint is taken as defining an *effective Hamiltonian*. The *space-time metric* is taken to be a static metric determined by the 3-metric and a lapse N. The matter equations of motion following from the effective Hamiltonian contain the modifications implied by quantum gravity. In computing the effective Hamiltonian, one expands the classical fields at the vertices in a region of size $L_{\rm weave}$ around a central point of the region and averages over these vertices. The symmetries of the 3-geometry (eg rotational invariance of the flat geometry) is used to restrict the form of the averages. In turns out that there are corrections in the quadratic order in the Maxwell fields which imply polarization dependent speed of propagation suggesting birefringence. The computations are varied by different choice of states for expectation values eg more general semi-classical states rather than weaves or even using only qualitative properties of 'would be semi-classical states' [1]. These computations also give additional corrections. Subsequently, the method is also applied to propagation of neutrinos [2, 3]. These early computations show how Lorentz invariance violations – as manifested by dispersion relations – could potentially arise in LQG. But do they *imply* that LQG violates Lorentz invariance?

This is not easy to answer. The above computations are essentially at a kinematical level. This however need not be regarded as a strong drawback, because the very characterization of Lorentz violation pre-supposes a background space-time in which one can find locally inertial observers. Introducing a background is akin to a gauge fixing and heuristically at least the use of kinematical states could be justified. There are still not satisfactory semi-classical states of the geometry and matter system which will describe 'test fields' on a given space-time background. The proper framework for this would be quantum field theory on curved space-times. To obtain it from the full quantum theory of gravity and matter, is a challenging task. First steps in this

[14] The subject of Lorentz invariance violation in all its aspects is rather large. We have included it here only to the extent of indicating how a computation of potentially observable signatures involving matter could be done. For review, we refer the reader to [46].

direction are discussed by Sahlmann and Thiemann [55, 56]. A cautionary note on a different aspect, has also been sounded in [16], pointing out that staying within Hamiltonian framework for computation including *approximation* leading to higher order derivatives, could be misleading because the corrections could be due to additional degrees of freedom with a Lorentz invariant dynamics. There is yet another aspect. Dispersion relation, relativistic or otherwise, need a background space-time. The presumed space-times in the first approaches have been static space-times with spatial geometry being described by a semi-classical state. Thus the 'time' part of the geometry does not have anything to do with the quantised spatial geometry. One has not used a quantum space-time, only quantum space. An extension to quantum space-time has been considered in the context of FLRW cosmology [5]. Such extensions are probably needed to discuss matter wave propagations on quantum space-time and its implications.

6 Concluding remarks

We began by noting that non-Fock quantization of matter fields is necessary once 'space-time' is dynamical. However, the illusion of a fixed background space-time with Fock quantized fields on it, is also extremely persuasive. In these circumstances, gravitational interactions of matter fields are negligible compared to intra-matter interactions. Consequently, the observed semi-classicality of geometry itself would conceivably be sensitive to matter content as well as their interactions. Matter interactions must play a role in getting our 'late universe' with its essentially classical space-time emerge from the quantum gravity regime of the early universe. Study of (loop) quantized matter has the potential to provide selection criteria from the stability of semi-classical states of geometry. We are still quite far away from this goal.

There are other 'applications' of matter fields. They can serve as probes for understanding properties of background independent quantization. For instance, in a fixed background space-time such as the Minkowski space-time, the short distance behaviour of quantum fields play a role in admitting the chiral (and the trace) anomalies. One of these, namely the axial anomaly, has played a historic role in understanding the $\pi^0 \to 2\gamma$ process. Cancellation of gauge anomalies has also provided constraints on model building. In a background independent quantization, we loose the handle of short distance behaviour. How then are the anomalies to be understood? This is an old issue more recently mentioned by Nicolai et al. [52], still awaiting an understanding.

The focus of analysis of matter in quantum gravity has been to study its propagation in the regime of semi-classical gravity. These have indicated that one could expect dispersion relations not compliant with Lorentz invariance. This has been seen in the propagation of massless fermions (neutrinos), light as well as massless scalar fields. For fermions and light, quantum effects of spatial geometry are incorporated while space-time is constructed classically. The case of the scalar field discussed in Section 2.5, is qualitatively different from these. Here the relativistic (massless) scalar field is classically expressed as a collection of classical 'harmonic oscillators' with frequencies $\omega_{\mathbf{k}} \sim |\mathbf{k}|$ and there is no Lorentz violation at this stage. For the usual Fock quantization, the energy spectrum, in units of $\hbar\omega_{\mathbf{k}}$ is linearly spaced and the Lorentz invariant dispersion results from the propagator. With polymer quantization, the spectrum is not linearly spaced and leads to Lorentz violating dispersion relation. While one can construct Lorentz generators from the $\pi_{\mathbf{k}}$, $\phi_{\mathbf{k}}$ basic variables, one cannot do so with the $U_{\lambda\mathbf{k}} - U^{\dagger}_{\lambda\mathbf{k}}$ variables for $\lambda \neq 0$. Clearly, it is the polymer quantization which requires the use of $U_{\lambda\mathbf{k}}$ with non-zero λ which is the source of this Lorentz violation at the level of 'free field'. Importing the Lorentz violating dispersion relation into the usual computational scheme of perturbative quantum field theory based on *Fock* quantization with its UV divergences (especially the 'quadratic divergences'), will lead to the 'fine tuning' problem discussed by Collins et al. [18] (see also [40]). To check the viability of a polymer quantized field theory on a Minkowski background, one will need to develop a computational scheme for a polymer quantized, interacting theory and then

test if at 'low energies' Lorentz violating effects are suppressed or not. Currently this lies only in the realm of possibilities.

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References

[1] Alfaro J., Morales-Técotl H.A., Urrutia L.F., Loop quantum gravity and light propagation, *Phys. Rev. D* **65** (2002), 103509, 18 pages, hep-th/0108061.

[2] Alfaro J., Morales-Técotl H.A., Urrutia L.F., Quantum gravity corrections to neutrino propagation, *Phys. Rev. Lett.* **84** (2000), 2318–2321, gr-qc/9909079.

[3] Alfaro J., Morales-Técotl H.A., Urrutia L.F., Quantum gravity and spin-1/2 particle effective dynamics, *Phys. Rev. D* **66** (2002), 124006, 19 pages, hep-th/0208192.

[4] Ashtekar A., Fairhurst S., Willis J.L., Quantum gravity, shadow states and quantum mechanics, *Classical Quantum Gravity* **20** (2003), 1031–1061, gr-qc/0207106.

[5] Ashtekar A., Kaminski W., Lewandowski J., Quantum field theory on a cosmological, quantum space-time, *Phys. Rev. D* **79** (2009), 064030, 12 pages, arXiv:0901.0933.

[6] Ashtekar A., Lewandowski J., Background independent quantum gravity: a status report, *Classical Quantum Gravity* **21** (2004), R53–R152, gr-qc/0404018.

[7] Ashtekar A., Lewandowski J., Sahlmann H., Polymer and Fock representations for a scalar field, *Classical Quantum Gravity* **20** (2003), L11–L21, gr-qc/0211012.

[8] Ashtekar A., Pawlowski T., Singh P., Quantum nature of the big bang: an analytical and numerical investigation, *Phys. Rev. D* **73** (2006), 124038, 33 pages, gr-qc/0604013.

[9] Ashtekar A., Pawlowski T., Singh P., Quantum nature of the big bang: improved dynamics, *Phys. Rev. D* **74** (2006), 084003, 23 pages, gr-qc/0607039.

[10] Ashtekar A., Rovelli C., Smolin L., Weaving a classical metric with quantum threads, *Phys. Rev. Lett.* **69** (1992), 237–240, hep-th/9203079.

[11] Ashtekar A., Wilson-Ewing E., Loop quantum cosmology of Bianchi type I models, *Phys. Rev. D* **79** (2009), 083535, 21 pages, arXiv:0903.3397.

[12] Baez J.C., Krasnov K.V., Quantization of diffeomorphism-invariant theories with fermions, *J. Math. Phys.* **39** (1998), 1251–1271, hep-th/9703112.

[13] Berezin F.A., The method of second quantization, *Pure and Applied Physics*, Vol. 24, Academic Press, New York, 1966.

[14] Bojowald M., Das R., Canonical gravity with fermions, *Phys. Rev. D* **78** (2008), 064009, 16 pages, arXiv:0710.5722.

[15] Bojowald M., Das R., Fermions in loop quantum cosmology and the role of parity, *Classical Quantum Gravity* **25** (2008), 195006, 23 pages, arXiv:0806.2821.

[16] Bojowald M., Morales-Técotl H.A., Sahlmann H., Loop quantum gravity phenomenology and the issue of Lorentz invariance, *Phys. Rev. D* **71** (2005), 084012, 7 pages, gr-qc/0411101.

[17] Borissov R., Weave states for plane gravitational waves, *Phys. Rev. D* **49** (1994), 923–929.

[18] Collins J., Perez A., Sudarsky D., Urrutia L., Vucetich H., Lorentz invariance and quantum gravity: an additional fine-tuning problem?, *Phys. Rev. Lett.* **93** (2004), 191301, 4 pages, gr-qc/0403053.

[19] Date G., Lectures on LQG/LQC, arXiv:1004.2952.

[20] Date G., Revisiting canonical gravity with fermion, arXiv:1110.3416.

[21] Date G., Kaul R.K., Sengupta S., Topological interpretation of Barbero–Immirzi parameter, *Phys. Rev. D* **79** (2009), 044008, 7 pages, arXiv:0811.4496.

[22] DeWitt B., Supermanifolds, Cambridge Monographs on Mathematical Physics, Cambridge University Press, Cambridge, 1984.

[23] Fell J.M.G., Doran R.S., Representations of $*$-algebras, locally compact groups, and Banach $*$-algebraic bundles, Academic Press, Boston, 1988.

[24] Fleischhack C., Representations of the Weyl algebra in quantum geometry, *Comm. Math. Phys.* **285** (2009), 67–140, math-ph/0407006.

[25] Freidel L., Minic D., Takeuchi T., Quantum gravity, torsion, parity violation, and all that, *Phys. Rev. D* **72** (2005), 104002, 6 pages, hep-th/0507253.

[26] Gambini R., Pullin J., Nonstandard optics from quantum space-time, *Phys. Rev. D* **59** (1999), 124021, 4 pages, gr-qc/9809038.

[27] Gambini R., Pullin J., Rastgoo S., Quantum scalar field in quantum gravity: the propagator and Lorentz invariance in the spherically symmetric case, *Gen. Relativity Gravitation* **43** (2011), 3569–3592, arXiv:1105.0667.

[28] Grot N., Rovelli C., Weave states in loop quantum gravity, *Gen. Relativity Gravitation* **29** (1997), 1039–1048.

[29] Hehl F.W., von der Heyde P., Kerlick G.D., Nester J.M., General relativity with spin and torsion: foundations and prospect, *Rev. Modern Phys.* **48** (1976), 393–416.

[30] Henneaux M., Teitelboim C., Quantization of gauge systems, Princeton University Press, Princeton, NJ, 1992.

[31] Hollands S., Wald R.M., Axiomatic quantum field theory in curved spacetime, *Comm. Math. Phys.* **293** (2010), 85–125, arXiv:0803.2003.

[32] Hossain G.M., Husain V., Seahra S.S., Background independent quantization and wave propagation, *Phys. Rev. D* **80** (2009), 044018, 12 pages, arXiv:0906.4046.

[33] Hossain G.M., Husain V., Seahra S.S., Nonsingular inflationary universe from polymer matter, *Phys. Rev. D* **81** (2010), 024005, 5 pages, arXiv:0906.2798.

[34] Hossain G.M., Husain V., Seahra S.S., Propagator in polymer quantum field theory, *Phys. Rev. D* **82** (2010), 124032, 5 pages, arXiv:1007.5500.

[35] Husain V., Kreienbuehl A., Ultraviolet behavior in background independent quantum field theory, *Phys. Rev. D* **81** (2010), 084043, 7 pages, arXiv:1002.0138.

[36] Iwasaki J., Basis states for gravitons in non-perturbative loop representation space, gr-qc/9807013.

[37] Iwasaki J., Rovelli C., Gravitons as embroidery on the weave, *Internat. J. Modern Phys. D* **1** (1992), 533–557.

[38] Iwasaki J., Rovelli C., Gravitons from loops: non-perturbative loop-space quantum gravity contains the graviton-physics approximation, *Classical Quantum Gravity* **11** (1994), 1653–1676.

[39] Jacobson T., Fermions in canonical gravity, *Classical Quantum Gravity* **5** (1988), L143–L148.

[40] Jain P., Ralston J.P., Supersymmetry and the Lorentz fine tuning problem, *Phys. Lett. B* **621** (2005), 213–218, hep-ph/0502106.

[41] Kaul R.K., Holst actions for supergravity theories, *Phys. Rev. D* **77** (2008), 045030, 8 pages, arXiv:0711.4674.

[42] Kaul R.K., Sengupta S., Topological parameters in gravity, *Phys. Rev. D* **85** (2012), 024026, 15 pages, arXiv:1106.3027.

[43] Laddha A., Varadarajan M., Polymer parametrized field theory, *Phys. Rev. D* **78** (2008), 044008, 22 pages, arXiv:0805.0208.

[44] Laddha A., Varadarajan M., Polymer quantization of the free scalar field and its classical limit, *Classical Quantum Gravity* **27** (2010), 175010, 45 pages, arXiv:1001.3505.

[45] Lewandowski J., Okołów A., Sahlmann H., Thiemann T., Uniqueness of diffeomorphism invariant states on holonomy-flux algebras, *Comm. Math. Phys.* **267** (2006), 703–733, gr-qc/0504147.

[46] Mattingly D., Modern tests of Lorentz invariance, *Living Rev. Relativity* **8** (2005), 5, 84 pages, gr-qc/0502097.

[47] Mercuri S., Fermions in the Ashtekar–Barbero connection formalism for arbitrary values of the Immirzi parameter, *Phys. Rev. D* **73** (2006), 084016, 14 pages, gr-qc/0601013.

[48] Mercuri S., Nieh–Yan invariant and Fermions in Ashtekar–Barbero–Immirzi formalism, gr-qc/0610026.

[49] Morales-Técotl H.A., Esposito G., Self-dual action for fermionic fields and gravitation, *Nuovo Cimento B* **109** (1994), 973–982, gr-qc/9506073.

[50] Morales-Técotl H.A., Rovelli C., Fermions in quantum gravity, *Phys. Rev. Lett.* **72** (1994), 3642–3645, gr-qc/9401011.

[51] Morales-Técotl H.A., Rovelli C., Loop space representation of quantum fermions and gravity, *Nuclear Phys. B* **451** (1995), 325–361.

[52] Nicolai H., Peeters K., Zamaklar M., Loop quantum gravity: an outside view, *Classical Quantum Gravity* **22** (2005), R193–R247, hep-th/0501114.

[53] Perez A., Rovelli C., Physical effects of the Immirzi parameter in loop quantum gravity, *Phys. Rev. D* **73** (2006), 044013, 3 pages, gr-qc/0505081.

[54] Rezende D.J., Perez A., Four-dimensional Lorentzian Holst action with topological terms, *Phys. Rev. D* **79** (2009), 064026, 11 pages, arXiv:0902.3416.

[55] Sahlmann H., Thiemann T., Towards the QFT on curved spacetime limit of QGR. I. A general scheme, *Classical Quantum Gravity* **23** (2006), 867–908, gr-qc/0207030.

[56] Sahlmann H., Thiemann T., Towards the QFT on curved spacetime limit of QGR. II. A concrete implementation, *Classical Quantum Gravity* **23** (2006), 909–954, gr-qc/0207031.

[57] Sengupta S., Kaul R.K., Canonical supergravity with Barbero–Immirzi parameter, *Phys. Rev. D* **81** (2010), 024024, 7 pages, arXiv:0909.4850.

[58] Thiemann T., Kinematical Hilbert spaces for fermionic and Higgs quantum field theories, *Classical Quantum Gravity* **15** (1998), 1487–1512, gr-qc/9705021.

[59] Thiemann T., Quantum spin dynamics (QSD), *Classical Quantum Gravity* **15** (1998), 839–873, gr-qc/9606089.

[60] Thiemann T., Quantum spin dynamics (QSD). II. The kernel of the Wheeler–DeWitt constraint operator, *Classical Quantum Gravity* **15** (1998), 875–905, gr-qc/9606090.

[61] Thiemann T., Quantum spin dynamics (QSD). V. Quantum gravity as the natural regulator of the Hamiltonian constraint of matter quantum field theories, *Classical Quantum Gravity* **15** (1998), 1281–1314, gr-qc/9705019.

[62] Varadarajan M., Photons from quantized electric flux representations, *Phys. Rev. D* **64** (2001), 104003, 9 pages, gr-qc/0104051.

[63] Wald R.M., Quantum field theory on curved space-times and black hole thermodynamics, The University of Chicago Press, Chicago, 1994.

[64] Weyl H., A remark on the coupling of gravitation and electron, *Phys. Rev.* **77** (1950), 699–701.

[65] Zegwaard J., The weaving of curved geometries, *Phys. Lett. B* **300** (1993), 217–222, hep-th/9210033.