Solvable Structures Associated to the Nonsolvable Symmetry Algebra $\mathfrak{sl}(2,\mathbb{R})$

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Abstract. Third-order ordinary differential equations with Lie symmetry algebras isomorphic to the nonsolvable algebra $\mathfrak{sl}(2,\mathbb{R})$ admit solvable structures. These solvable structures can be constructed by using the basis elements of these algebras. Once the solvable structures are known, the given equation can be integrated by quadratures as in the case of solvable symmetry algebras.

Key words: first integral; solvable structure; $\mathcal{C}^\infty$-symmetry; nonsolvable algebra

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1 Introduction

It is well known that an nth-order ordinary differential equation (ODE) possessing an n-dimensional solvable Lie symmetry algebra can be integrated by quadratures [14, 15, 23, 24, 31]. This is a sufficient condition, but not necessary [8] and extensions of the classical approach have been amply investigated in the recent literature (hidden symmetries [1, 2], nonlocal symmetries [3, 10, 11], $\mathcal{C}^\infty$-symmetries [19], etc.).

In this paper we focus on a generalization of solvable algebras called solvable structures [4, 5, 13, 30] and their applications to integrate ODEs which admit nonsolvable symmetry algebras. The concept of solvable structure [5] refers to systems of independent vector fields that are in involution; for a scalar ODE, this (trivially involutive) system is formed by just one element, the vector field **A** associated to the given ODE. In this case, a solvable structure involves an ordered set of generalized vector fields that, in general, are not symmetries of the ODE and do not form a solvable algebra. Nevertheless, if a solvable structure for the ODE is known, then the equation can be (locally) solved by quadratures [5, 13]. Thus, it is important to have methods that allow the determination of solvable structures for ODEs in order to solve them by quadratures.

We investigate in this paper if a solvable structure can be constructed from a nonsolvable Lie symmetry algebra. In this case, the integrability by quadratures of the ODEs admitting nonsolvable symmetry algebras would be warranted, as in the case of solvable symmetry algebras. This study addresses the case of third-order ODEs which admit a Lie symmetry algebra isomorphic to the nonsolvable algebra $\mathfrak{sl}(2,\mathbb{R})$. This class of equations is well determined and known in the literature: in fact, symmetry analysis of third-order ODEs and the classification according to their symmetries have been extensively studied [7, 12, 18, 28].

Previous studies in the literature [6, 16] show that any third-order ODE admitting $\mathrm{SL}(2,\mathbb{C})$ as symmetry group can be solved via a pair of quadratures and the solution to a Riccati equation, or, equivalently, a second-order linear ODE. The analysis performed in [6] proves this result by using that the three inequivalent actions of the unimodular Lie group $\mathrm{SL}(2,\mathbb{C})$ on the complex plane [17] are directly connected via the standard prolongation process. A different approach is presented in [16]: a two-dimensional subalgebra $\mathcal{L}_2$ is used to reduce the given third-order ODE

to a first-order equation, which cannot be integrated by quadratures, but can be transformed into a Riccati equation by using a nonlocal symmetry. In [20] it is shown that this nonlocal symmetry comes from one of the symmetries of the original third-order ODE and can be recovered as a $\mathcal{C}^\infty$-symmetry for the first-order equation. The general solution of the Riccati equation becomes a second-order ODE in the original variables, which can be integrated by quadratures by using $\mathcal{L}_2$. Both procedures require a previous transformation to map the basis elements of the Lie symmetry algebra into one of the canonical realizations of $\mathfrak{sl}(2,\mathbb{C})$ [9, 17].

The starting point of the approach presented in this paper is a basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ of the Lie invariance algebra for the given third-order ODE satisfying the following commutation relations:

$$[\mathbf{v}_1, \mathbf{v}_3] = \mathbf{v}_1, \qquad [\mathbf{v}_1, \mathbf{v}_2] = 2\mathbf{v}_3, \qquad [\mathbf{v}_3, \mathbf{v}_2] = \mathbf{v}_2.$$

Although this symmetry Lie algebra is isomorphic to the nonsolvable symmetry algebra $\mathfrak{sl}(2,\mathbb{R})$, it is shown that solvable structures can be explicitly constructed from the second-order prolongations of the basis elements (denoted by $\mathbf{v}_i^{(2)}$ for $i = 1, 2, 3$). In fact, the main result in this paper shows that there exist two functions F_1 and F_2 such that the vector fields $F_1 \mathbf{v}_1^{(2)}$, $F_2 \mathbf{v}_2^{(2)}$, and $\mathbf{v}_3^{(2)}$ can be used to construct two solvable structures with respect to the vector field $\mathbf{A}$ associated to the given ODE (Theorem 4.3). As a conclusion, the original ODE can be solved (at least locally) by quadratures, provided the functions F_1 and F_2 are known.

This result is based on the compatibility of two systems of partial differential equations (PDEs) (see systems (4.14)). We present a constructive proof of the existence of solutions for systems (4.14), which provides a method to construct the solvable structure in practice. For this purpose, instead of reducing the original ODE to a first-order ODE in the usual way (by using the solvable subalgebra generated by $\mathbf{v}_1$ and $\mathbf{v}_2$, as in [6, 16]), we use $\mathbf{v}_3$ to reduce the original equation to a second-order ODE. The basis elements $\mathbf{v}_1$ and $\mathbf{v}_2$ are lost as Lie point symmetries for the reduced equation, but can be recovered as $\mathcal{C}^\infty$-symmetries [20]. These $\mathcal{C}^\infty$-symmetries can be used to calculate two functionally independent first integrals I_1 and I_2 of the reduced equation. A procedure to compute them by quadratures [22] is explained in Section 3, provided that particular solutions of systems (3.9) are known. Particular solutions F_1 and F_2 of systems (4.14) can be directly found by using I_1 and I_2 (see (4.13)) or by using solutions of systems (3.9) (see (5.5)).

In Section 5 the solvable structures are used to give three different strategies that can be followed to integrate by quadratures any third-order ODE admitting a Lie symmetry algebra that is isomorphic to $\mathfrak{sl}(2,\mathbb{R})$. Remarkably, no further integration is necessary when the mentioned functions F_1 and F_2 come from first integrals I_1 and I_2 of the reduced equation or from particular solutions of systems (3.9).

The method is illustrated in Sections 6 and 7 with two examples of third-order ODEs which admit the nonsolvable Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ as symmetry algebra. For each example, the procedure provides a complete set of first integrals for the third-order equation in terms of two independent solutions of a second-order linear equation. As in the previous methods in the literature, the complete solution of the original ODE involves the solutions of a second-order linear ODE, but the use of the procedure we present in this paper (see Method 2 in Section 5) does not require additional quadratures.

2   Symmetries and solvable structures

In this section we recall the notion of solvable structure and some of its properties [5, 30]. Functions, vector fields, and differential forms are assumed to be smooth and well defined on an open and simply connected subset D of either $\mathbb{R}^n$ or an n-dimensional manifold $\mathcal{M}_n$.

Given a system $\mathcal{S}$ of vector fields on D, span$(\mathcal{S})$ stands for the space of the linear combinations of the elements of $\mathcal{S}$ over the ring of the smooth functions on D. In what follows, two systems of vector fields, $\mathcal{S}$ and $\widetilde{\mathcal{S}}$, are called *equivalent* if span$(\mathcal{S}) =$ span$(\widetilde{\mathcal{S}})$ [13].

The following concepts refer to an involutive system of vector fields, i.e., a set of vector fields $\mathcal{A} = \{\mathbf{A}_1, \ldots, \mathbf{A}_r\}$ such that $[\mathbf{A}_i, \mathbf{A}_j] \in$ span$(\mathcal{A})$ for $i, j \in \{1, \ldots, r\}$. In this paper angle brackets are used to indicate that the order of the elements of the set must be taken into account.

Definition 2.1. Let $\mathcal{A} = \{\mathbf{A}_1, \ldots, \mathbf{A}_r\}$ be a system of $r < n$ independent vector fields on D which are in involution.

1. A smooth vector field $\mathbf{X}$ on D is called a *symmetry* of $\mathcal{A}$ if the following conditions hold:

    (a) $\mathbf{A}_1, \ldots, \mathbf{A}_r$, and $\mathbf{X}$ are independent;
    (b) $[\mathbf{X}, \mathbf{A}_i] \in$ span$(\mathcal{A})$ for $1 \le i \le r$.

2. Let $\mathcal{S} = \langle \mathbf{X}_1, \ldots, \mathbf{X}_{n-r} \rangle$ be an ordered set of independent vector fields on D. The ordered system $\mathcal{A} \cup \mathcal{S} = \langle \mathbf{A}_1, \ldots, \mathbf{A}_r, \mathbf{X}_1, \ldots, \mathbf{X}_{n-r} \rangle$ is a *solvable structure* with respect to $\mathcal{A}$ if

    (a) $\mathcal{S}_j = \{\mathbf{A}_1, \ldots, \mathbf{A}_r, \mathbf{X}_1, \ldots, \mathbf{X}_j\}$ is in involution for $j = 1, \ldots, n-r$;
    (b) $\mathbf{X}_1$ is a symmetry of $\mathcal{A}$;
    (c) $\mathbf{X}_{j+1}$ is a symmetry of $\mathcal{S}_j$ for $j = 1, \ldots, n-r-1$.

2.1 Symmetries and solvable structures in the context of ODEs

The notion of symmetry given in Definition 2.1 represents a generalization of the concept of Lie point symmetry. Given a scalar nth-order ODE

$$u_n = \phi(x, u, u_1, \ldots, u_{n-1}), \tag{2.1}$$

where x denotes the independent variable, u is the dependent variable, and $u_j = \frac{d^j u}{dx^j}$ for $1 \le j \le n$, let

$$\mathbf{A} = \partial_x + u_1 \partial_u + \cdots + \phi(x, u, u_1, \ldots, u_{n-1}) \partial_{u_{n-1}} \tag{2.2}$$

denote the vector field associated to equation (2.1). Equation (2.1) is defined for points of the corresponding nth-order jet space whose projections to the $(n-1)$th-order jet belong to the domain of ϕ. Let $M \subset \mathbb{R}^2$ be an open set of the projection of this domain to the zero-order jet space.

A smooth vector field $\mathbf{v} = \xi(x, u) \partial_x + \eta(x, u) \partial_u$ on M is a Lie point symmetry of equation (2.1) if and only if

$$\left[\mathbf{v}^{(n-1)}, \mathbf{A}\right] = -\mathbf{A}(\xi) \cdot \mathbf{A},$$

where $\mathbf{v}^{(n-1)}$ stands for the $(n-1)$th-order prolongation of $\mathbf{v}$ [15, 23, 24, 31]. Therefore, $\mathbf{v}^{(n-1)}$ is a symmetry of the (trivially involutive) system $\mathcal{A} = \{\mathbf{A}\}$ in the sense of Definition 2.1. The same result holds for generalized symmetries for which the infinitesimals ξ and η can depend on derivatives of u with respect to x [23, 24].

The notion of solvable structure given in Definition 2.1 generalizes the concept of solvable symmetry algebra. If equation (2.1) admits a solvable symmetry algebra $\mathcal{G}$ of dimension n, then there exists an ordered basis $\langle \mathbf{v}_1, \ldots, \mathbf{v}_n \rangle$ of $\mathcal{G}$ such that $[\mathbf{v}_i, \mathbf{v}_j] = \sum_{k=1}^{j-1} c_{ij}^k \mathbf{v}_k$ for $1 \le i < j \le n$ and where $c_{ij}^k \in \mathbb{R}$. Therefore, $\langle \mathbf{A}, \mathbf{v}_1^{(n-1)}, \ldots, \mathbf{v}_n^{(n-1)} \rangle$ is a solvable structure with respect to $\{\mathbf{A}\}$.

The integrability by quadratures of an nth-order ODE which admits a solvable symmetry algebra $\mathcal{G}$ of dimension n is well known. In fact, the integrability by quadratures can be characterized through solvable structures:

Proposition 2.2 ([5, Proposition 6])**.** *An involutive system $\mathcal{A}$ is locally integrable by quadratures if and only if there exists a solvable structure with respect to $\mathcal{A}$.*

Now we recall the method [5, 13] to construct n independent first integrals for equation (2.1) when a solvable structure $\langle \mathbf{A}, \mathbf{X}_1, \ldots, \mathbf{X}_n \rangle$ with respect to $\{\mathbf{A}\}$ is known. The solvable structure is used to define the differential 1-forms given by

$$\boldsymbol{\omega}_i = \frac{\mathbf{X}_n \lrcorner \cdots \lrcorner \widehat{\mathbf{X}_i} \lrcorner \cdots \lrcorner \mathbf{X}_1 \lrcorner \mathbf{A} \lrcorner \boldsymbol{\Omega}}{\mathbf{X}_n \lrcorner \cdots \lrcorner \mathbf{X}_1 \lrcorner \mathbf{A} \lrcorner \boldsymbol{\Omega}}, \qquad i = 1, \ldots, n, \tag{2.3}$$

where $\widehat{\mathbf{X}_i}$ indicates omission of $\mathbf{X}_i$, $\lrcorner$ denotes the interior product, and $\boldsymbol{\Omega} = dx \wedge du \wedge \cdots \wedge du_{n-1}$. The system $\{\boldsymbol{\omega}_1, \ldots, \boldsymbol{\omega}_n\}$ has distinguishing closure properties [13]: $d\boldsymbol{\omega}_n = 0$ and for $1 \leq i < n$, $d\boldsymbol{\omega}_i \in \mathcal{I}\{\boldsymbol{\omega}_{i+1}, \ldots, \boldsymbol{\omega}_n\}$, where $\mathcal{I}\{\boldsymbol{\omega}_{i+1}, \ldots, \boldsymbol{\omega}_n\}$ denotes the ideal generated by $\boldsymbol{\omega}_{i+1}, \ldots, \boldsymbol{\omega}_n$ under taking exterior products.

These properties permit the integration by quadratures (at least locally) of the 1-forms (2.3) by proceeding as follows: $\boldsymbol{\omega}_n$ is locally exact and any of its primitives I_n is a first integral of $\mathbf{A}$. The restriction of $\boldsymbol{\omega}_{n-1}$ to each submanifold defined by $I_n = c_n$, $c_n \in \mathbb{R}$, is closed, and a primitive I_{n-1} can be found by a quadrature. We can continue in this fashion by further restricting the submanifolds at each stage until we have fully integrated the system of 1-forms. By the definition of the 1-forms (2.3), the functions $\{I_1, \ldots, I_n\}$ are functionally independent first integrals of $\mathbf{A}$. These results provide the following theorem [5, 13]:

Theorem 2.3. *Let* (2.2) *be the vector field associated to equation* (2.1) *and assume that* $\langle \mathbf{A}, \mathbf{X}_1, \ldots, \mathbf{X}_n \rangle$ *is a solvable structure with respect to* $\{\mathbf{A}\}$. *Then the given ODE* (2.1) *can be* (*at least locally*) *solved by quadratures alone.*

3   Symmetries and $\mathcal{C}^\infty$-symmetries for second-order ODEs

In this section we establish some relationships between symmetries and $\mathcal{C}^\infty$-symmetries (also called λ-symmetries) for the integrability by quadratures of second-order ODEs that will be used later.

We consider a second-order equation

$$w_2 = \widetilde{\phi}(y, w, w_1), \tag{3.1}$$

where y is the independent variable, w is the dependent variable, and $w_i = \frac{d^i w}{dy^i}$ for $i \in \mathbb{N}$. Let $M_1 \subset \mathbb{R}^2$ be an open set of the projection of the domain of $\widetilde{\phi}$ to the corresponding zero-order jet space. Throughout this section $\mathbf{A}$ denotes the vector field associated to (3.1) and $\mathbf{D}_y$ is the total derivative operator with respect to y, i.e., $\mathbf{D}_y = \partial_y + w_1 \partial_w + \cdots + w_k \partial_{w_{k+1}} + \cdots$.

We recall [19] that a $\mathcal{C}^\infty$-symmetry of (3.1) is a pair $(\mathbf{v}, \lambda)$, where $\mathbf{v} = \xi(y, w)\partial_y + \eta(y, w)\partial_w$ is a vector field on M_1 and $\lambda = \lambda(y, w, w_1)$ is a smooth function, such that

$$\left[\mathbf{v}^{[\lambda,(1)]}, \mathbf{A}\right] = \lambda \mathbf{v}^{[\lambda,(1)]} - (\mathbf{A} + \lambda)(\xi)\mathbf{A}, \tag{3.2}$$

where $\mathbf{v}^{[\lambda,(1)]}$ stands for the first-order λ-prolongation of $\mathbf{v}$

$$\mathbf{v}^{[\lambda,(1)]} = \xi \partial_y + \eta \partial_w + ((\mathbf{D}_y + \lambda)(\eta) - (\mathbf{D}_y + \lambda)(\xi)w_1) \, \partial_{w_1}. \tag{3.3}$$

Two $\mathcal{C}^\infty$-symmetries $(\overline{\mathbf{v}}_1, \lambda_1)$ and $(\overline{\mathbf{v}}_2, \lambda_2)$ of equation (3.1) are called $\mathbf{A}$-equivalent [21] (or simply equivalent) if the systems $\left\{\mathbf{A}, \overline{\mathbf{v}}_1^{[\lambda_1,(1)]}\right\}$ and $\left\{\mathbf{A}, \overline{\mathbf{v}}_2^{[\lambda_2,(1)]}\right\}$ are equivalent, i.e.,

$$\mathrm{span}\big(\big\{\mathbf{A}, \overline{\mathbf{v}}_1^{[\lambda_1,(1)]}\big\}\big) = \mathrm{span}\big(\big\{\mathbf{A}, \overline{\mathbf{v}}_2^{[\lambda_2,(1)]}\big\}\big).$$

We assume that $(\overline{\mathbf{v}}_1, \lambda_1)$ and $(\overline{\mathbf{v}}_2, \lambda_2)$ are two inequivalent $\mathcal{C}^\infty$-symmetries of the equation (3.1) and denote $\mathbf{Y}_i = \overline{\mathbf{v}}_i^{[\lambda_i,(1)]}$ for $i = 1, 2$. The corresponding expressions (3.2) become

$$[\mathbf{Y}_i, \mathbf{A}] = \lambda_i \mathbf{Y}_i + \mu_i \mathbf{A}, \tag{3.4}$$

where $\mu_i = -(\mathbf{A} + \lambda_i)(\mathbf{Y}_i(y))$ for $i = 1, 2$. Assuming that $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$, (3.4) shows that $\mathbf{Y}_1$ and $\mathbf{Y}_2$ are not symmetries of $\{\mathbf{A}\}$ in the sense of Definition 2.1. Relations (3.4) imply that the systems $\{\mathbf{A}, \mathbf{Y}_1\}$ and $\{\mathbf{A}, \mathbf{Y}_2\}$ are in involution; by Frobenius theorem [23], there exist two functionally independent first integrals $I_1 = I_1(y, w, w_1)$ and $I_2 = I_2(y, w, w_1)$ of $\mathbf{A}$ such that $\mathbf{Y}_1(I_1) = \mathbf{Y}_2(I_2) = 0$. Such nonconstant function I_1 (resp. I_2) will be called a first integral of $\mathbf{A}$ associated to the $\mathcal{C}^\infty$-symmetry $(\overline{\mathbf{v}}_1, \lambda_1)$ (resp. $(\overline{\mathbf{v}}_2, \lambda_2)$).

A procedure to calculate by quadratures two first integrals associated to two inequivalent $\mathcal{C}^\infty$-symmetries of a second-order ODE is described in the following subsection.

3.1 $\mathcal{C}^\infty$-symmetries and integrability by quadratures for second-order ODEs

Any given $\mathcal{C}^\infty$-symmetry $(\mathbf{v}, \lambda)$ of equation (3.1) is equivalent to the $\mathcal{C}^\infty$-symmetry

$$(\partial_w, \lambda_Q), \qquad \text{where} \quad \lambda_Q = \lambda + \frac{\mathbf{A}(Q)}{Q}, \tag{3.5}$$

and Q denotes the characteristic $Q = \eta - \xi \cdot w_1$ of $\mathbf{v}$. This is a consequence of the relation

$$\mathbf{v}^{[\lambda,(1)]} = Q(\partial_w)^{[\lambda_Q,(1)]} + \xi\mathbf{A}, \tag{3.6}$$

which follows from (3.3). This pair (∂_w, λ_Q) is called the *canonical representative* of $(\mathbf{v}, \lambda)$.

If $(\overline{\mathbf{v}}_1, \lambda_1)$ and $(\overline{\mathbf{v}}_2, \lambda_2)$ are two inequivalent $\mathcal{C}^\infty$-symmetries of equation (3.1) and Q_1 and Q_2 are the respective characteristics, then the corresponding expressions (3.6) can be written as

$$\mathbf{Y}_1 = Q_1\mathbf{X}_1 + \xi_1\mathbf{A}, \qquad \mathbf{Y}_2 = Q_2\mathbf{X}_2 + \xi_2\mathbf{A}, \tag{3.7}$$

where $\mathbf{X}_i = (\partial_w)^{[\lambda_{Q_i},(1)]}$ and $\mathbf{Y}_i = \overline{\mathbf{v}}_i^{[\lambda_i,(1)]}$ for $i = 1, 2$.

Although, in general, $\mathbf{Y}_1$ and $\mathbf{Y}_2$ are not in involution, the vector fields $\mathbf{X}_1$ and $\mathbf{X}_2$ form a two-dimensional algebra. In fact, it can be checked that

$$[\mathbf{X}_1, \mathbf{A}] = \lambda_{Q_1}\mathbf{X}_1, \qquad [\mathbf{X}_2, \mathbf{A}] = \lambda_{Q_2}\mathbf{X}_2, \qquad [\mathbf{X}_1, \mathbf{X}_2] = \rho(\mathbf{X}_1 - \mathbf{X}_2), \tag{3.8}$$

where

$$\rho = \frac{\mathbf{X}_1(\lambda_{Q_2}) - \mathbf{X}_2(\lambda_{Q_1})}{\lambda_{Q_1} - \lambda_{Q_2}}.$$

By using (3.8) and the properties of the Lie bracket, it can be proved that if $\overline{h}_1, \overline{h}_2 \in \mathcal{C}^\infty(M_1^{(1)})$ satisfy the following systems

$$\mathbf{A}(\overline{h}_1) = \lambda_{Q_1}\overline{h}_1, \qquad \mathbf{X}_2(\overline{h}_1) = \rho\overline{h}_1;$$
$$\mathbf{A}(\overline{h}_2) = \lambda_{Q_2}\overline{h}_2, \qquad \mathbf{X}_1(\overline{h}_2) = \rho\overline{h}_2, \tag{3.9}$$

then $\{\mathbf{A}, \overline{h}_1\mathbf{X}_1, \overline{h}_2\mathbf{X}_2\}$ is an abelian algebra; hence the set $\{\overline{h}_1\mathbf{X}_1, \overline{h}_2\mathbf{X}_2\}$ is a system of commuting symmetries of $\{\mathbf{A}\}$. The compatibility of systems (3.9) has been proved in [22].

If $\overline{h}_1$ and $\overline{h}_2$ are some known particular solutions of the respective system in (3.9) then equation (3.1) can be integrated by quadratures: by using Theorem 2.3 it can be checked that the differential 1-forms

$$\boldsymbol{\beta}_1 = \mu_1(\mathbf{X}_1 \lrcorner \mathbf{A} \lrcorner \mathbf{\Omega}), \qquad \boldsymbol{\beta}_2 = \mu_2(\mathbf{X}_2 \lrcorner \mathbf{A} \lrcorner \mathbf{\Omega}), \tag{3.10}$$

where

$$\mu_1 = \frac{1}{\overline{h}_2 \left(\mathbf{X}_2 \lrcorner \mathbf{X}_1 \lrcorner \mathbf{A} \lrcorner \mathbf{\Omega} \right)} = \frac{1}{\overline{h}_2(\lambda_{Q_1} - \lambda_{Q_2})},$$

$$\mu_2 = \frac{1}{\overline{h}_1 \left(\mathbf{X}_1 \lrcorner \mathbf{X}_2 \lrcorner \mathbf{A} \lrcorner \mathbf{\Omega} \right)} = \frac{1}{\overline{h}_1(\lambda_{Q_2} - \lambda_{Q_1})}, \tag{3.11}$$

are (locally) exact. Let $I_i = I_i(y, w, w_1)$ be a function such that $\boldsymbol{\beta}_i = dI_i$ for $i = 1, 2$. By (3.10), such functions I_1 and I_2 are first integrals of $\mathbf{A}$ and satisfy $\mathbf{X}_1(I_1) = \mathbf{X}_2(I_2) = 0$. According to (3.7), I_1 (resp. I_2) is a common first integral for the involutive system $\{\mathbf{A}, \mathbf{Y}_1\}$ (resp. $\{\mathbf{A}, \mathbf{Y}_2\}$).

Previous discussion shows that two functions $\overline{h}_1$ and $\overline{h}_2$ satisfying the corresponding system in (3.9) can be used to construct, by quadratures, two first integrals, I_1 and I_2, of $\mathbf{A}$, associated to two inequivalent $\mathcal{C}^\infty$-symmetries of the equation. We want to point out that $\overline{h}_1$ and $\overline{h}_2$ are determined by the constructed functions I_1 and I_2, because by (3.10), (3.11), and (3.7), we can write

$$\overline{h}_1 = \frac{1}{\boldsymbol{\beta}_2(\mathbf{X}_1)} = \frac{1}{\mathbf{X}_1(I_2)} = \frac{Q_1}{\mathbf{Y}_1(I_2)}, \qquad \overline{h}_2 = \frac{1}{\boldsymbol{\beta}_1(\mathbf{X}_2)} = \frac{1}{\mathbf{X}_2(I_1)} = \frac{Q_2}{\mathbf{Y}_2(I_1)}. \tag{3.12}$$

In fact, let I_1, I_2 be two arbitrary first integrals of $\mathbf{A}$ associated to $(\overline{\mathbf{v}}_1, \lambda_1)$ and $(\overline{\mathbf{v}}_2, \lambda_2)$, respectively; since $\mathbf{A}$, $\mathbf{X}_1$, and $\mathbf{X}_2$ are independent, then $\mathbf{X}_1(I_2) \neq 0$ and $\mathbf{X}_2(I_1) \neq 0$ and the functions

$$\overline{h}_1 = \frac{1}{\mathbf{X}_1(I_2)}, \qquad \overline{h}_2 = \frac{1}{\mathbf{X}_2(I_1)} \tag{3.13}$$

are well defined. From (3.8) it follows that $\mathbf{A}(\mathbf{X}_1(I_2)) = -[\mathbf{X}_1, \mathbf{A}](I_2) = -\lambda_{Q_1}\mathbf{X}_1(I_2)$, because $\mathbf{A}(I_2) = 0$. Therefore

$$\mathbf{A}(\overline{h}_1) = \mathbf{A}\left(\frac{1}{\mathbf{X}_1(I_2)} \right) = -\frac{\mathbf{A}\left(\mathbf{X}_1(I_2) \right)}{\left(\mathbf{X}_1(I_2) \right)^2} = \frac{\lambda_{Q_1}}{\mathbf{X}_1(I_2)} = \lambda_{Q_1}\overline{h}_1. \tag{3.14}$$

Equality $\mathbf{A}(\overline{h}_2) = \lambda_{Q_2}\overline{h}_2$ can be proved in a similar way. The third relation in (3.8) provides $\mathbf{X}_1(\mathbf{X}_2(I_1)) = [\mathbf{X}_1, \mathbf{X}_2](I_1) = -\rho\mathbf{X}_2(I_1)$ and $\mathbf{X}_2(\mathbf{X}_1(I_2)) = -[\mathbf{X}_1, \mathbf{X}_2](I_2) = -\rho\mathbf{X}_1(I_2)$, because $\mathbf{X}_1(I_1) = \mathbf{X}_2(I_2) = 0$. By proceeding as in (3.14), it follows that $\mathbf{X}_1(\overline{h}_2) = \rho\overline{h}_2$ and $\mathbf{X}_2(\overline{h}_1) = \rho\overline{h}_1$. This proves that functions $\overline{h}_1$ and $\overline{h}_2$ in (3.13) are solutions of the respective systems in (3.9).

These results are collected in the following theorem for further reference.

Theorem 3.1. *Let $(\overline{\mathbf{v}}_1, \lambda_1)$ and $(\overline{\mathbf{v}}_2, \lambda_2)$ be two inequivalent $\mathcal{C}^\infty$-symmetries of equation (3.1) and consider their respective canonical representatives, $(\partial_w, \lambda_{Q_1})$ and $(\partial_w, \lambda_{Q_2})$. Denote $\mathbf{X}_i = (\partial_w)^{[\lambda_{Q_i},(1)]}$ and $\mathbf{Y}_i = \overline{\mathbf{v}}_i^{[\lambda_i,(1)]}$ for $i = 1, 2$.*

1. *If $\overline{h}_1$ and $\overline{h}_2$ are particular solutions of the respective systems in (3.9), then two functionally independent first integrals of $\mathbf{A}$ associated to the given $\mathcal{C}^\infty$-symmetries can be found by quadratures as primitives of the 1-forms defined in (3.10).*

2. *Conversely, for $i = 1, 2$, let I_i be a first integral of $\mathbf{A}$ associated to $(\overline{\mathbf{v}}_i, \lambda_i)$. Then the functions $\overline{h}_1$ and $\overline{h}_2$ given by (3.13) satisfy the corresponding systems in (3.9).*

4   Solvable structures from $\mathfrak{sl}(2, \mathbb{R})$ for third-order ODEs

Let us consider a third-order ODE

$$u_3 = \phi(x, u, u_1, u_2), \tag{4.1}$$

that admits a Lie symmetry algebra that is isomorphic to $\mathfrak{sl}(2,\mathbb{R})$. In this section we investigate how a solvable structure for (4.1) can be explicitly constructed by using the basis elements of the symmetry algebra. Once this is achieved, the equation can be integrated by quadratures, as in the case of solvable symmetry algebras, although $\mathfrak{sl}(2,\mathbb{R})$ is not solvable.

A basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ of the Lie symmetry algebra of equation (4.1) verifying

$$[\mathbf{v}_1, \mathbf{v}_3] = \mathbf{v}_1, \qquad [\mathbf{v}_1, \mathbf{v}_2] = 2\mathbf{v}_3, \qquad [\mathbf{v}_3, \mathbf{v}_2] = \mathbf{v}_2 \tag{4.2}$$

can always be chosen [9]. Most of the approaches to integrate equations of the form (4.1) that admit a Lie symmetry algebra isomorphic to $\mathfrak{sl}(2,\mathbb{R})$ [6, 14, 16] use $\mathbf{v}_1$ or $\mathbf{v}_2$ to reduce (4.1) because any of them determines a two-dimensional algebra with $\mathbf{v}_3$. The use of $\mathbf{v}_3$ seems to be the worst choice to reduce the order of (4.1), because both $\mathbf{v}_1$ and $\mathbf{v}_2$ are lost as Lie point symmetries for the reduced equation, i.e., they are type I hidden symmetries [1, 2]. Nevertheless, these basis elements can be recovered as $\mathcal{C}^\infty$-symmetries [20], which, as it is shown in this section, will play an important role in the construction of the solvable structure.

If we choose the Lie point symmetry $\mathbf{v}_3$ to reduce the order of equation (4.1), then we can introduce canonical coordinates (y, α) for $\mathbf{v}_3$, i.e., a local change of variables on an open set M of the variables (x, u) of equation (4.1),

$$\varphi(x, u) = (y(x, u), \alpha(x, u)),$$

in which $\mathbf{v}_3$ becomes ∂_α. Let $\alpha_1 = \frac{d\alpha}{dy}$ be denoted by w and let $w_i = \alpha_{i+1}$ for $1 \leq i \leq 2$. Locally, equation (4.1) can be written in terms of the invariants $\{y, w, w_1, w_2\}$ of $\mathbf{v}_3^{(2)}$ as a reduced equation

$$w_2 = \widetilde{\phi}(y, w, w_1), \tag{4.3}$$

defined for $(y, w) \in M_1$ for some open set M_1. In this section $\mathbf{A}_{(x,u)}$ will denote the vector field associated to equation (4.1), $\mathbf{A}_{(y,\alpha)}$ will be the vector field associated to equation $\alpha_3 = \widetilde{\phi}(y, \alpha_1, \alpha_2)$, and $\mathbf{A}_{(y,w)}$ will be the vector field associated to equation (4.3).

The basis elements $\mathbf{v}_1$ and $\mathbf{v}_2$ are lost as Lie point symmetries for equation (4.3), because $\mathbf{v}_1^{(1)}$ and $\mathbf{v}_2^{(1)}$ are not well-defined vector fields in the (y, w)-coordinates (they are exponential vector fields [23]). However, they can be recovered as $\mathcal{C}^\infty$-symmetries for equation (4.3) [20]. For that purpose we consider two nonzero functions $\varsigma_1, \varsigma_2 \in \mathcal{C}^\infty(M)$ such that

$$\mathbf{v}_3(\varsigma_1) = \varsigma_1, \qquad \mathbf{v}_3(\varsigma_2) = -\varsigma_2. \tag{4.4}$$

Observe that ς_2 can be constructed from ς_1 as $\varsigma_2 = 1/\varsigma_1$ and vice versa. With this choice we get

$$\left[\mathbf{v}_3^{(1)}, \varsigma_1\mathbf{v}_1^{(1)}\right] = \left[\mathbf{v}_3^{(1)}, \varsigma_2\mathbf{v}_2^{(1)}\right] = 0, \tag{4.5}$$

which can be checked by using (4.2) and the properties of the Lie bracket. The vector fields $\varsigma_1\mathbf{v}_1^{(1)}$ and $\varsigma_2\mathbf{v}_2^{(1)}$ are projectable [27] with respect to the projection

$$\begin{aligned} \pi_{\mathbf{v}_3}^{(1)}: \quad \varphi^{(1)}\big(M^{(1)}\big) &\rightarrow M_1, \\ (y, \alpha, w) &\mapsto (y, w), \end{aligned}$$

because (4.5) holds. Let

$$\overline{\mathbf{v}}_1 = \big(\pi_{\mathbf{v}_3}^{(1)}\big)_*\big(\varsigma_1\mathbf{v}_1^{(1)}\big), \qquad \overline{\mathbf{v}}_2 = \big(\pi_{\mathbf{v}_3}^{(1)}\big)_*\big(\varsigma_2\mathbf{v}_2^{(1)}\big) \tag{4.6}$$

denote the corresponding projected vector fields. By Theorem 3 in [20] the pairs $(\overline{\mathbf{v}}_1, \lambda_1)$ and $(\overline{\mathbf{v}}_2, \lambda_2)$ are $\mathcal{C}^\infty$-symmetries of the equation (4.3) for the functions

$$\lambda_1 = -\frac{\mathbf{A}_{(y,\alpha)}(\varsigma_1)}{\varsigma_1}, \qquad \lambda_2 = -\frac{\mathbf{A}_{(y,\alpha)}(\varsigma_2)}{\varsigma_2}, \tag{4.7}$$

respectively. In what follows we denote

$$\mathbf{Y}_1 = \overline{\mathbf{v}}_1^{[\lambda_1,(1)]}, \qquad \mathbf{Y}_2 = \overline{\mathbf{v}}_2^{[\lambda_2,(1)]}. \tag{4.8}$$

For $i = 1, 2$, let $I_i = I_i(x, w, w_1)$ be a nonconstant first integral of $\mathbf{A}_{(y,w)}$ associated to the $\mathcal{C}^\infty$-symmetry $(\overline{\mathbf{v}}_i, \lambda_i)$, i.e., $\mathbf{Y}_i(I_i) = \mathbf{A}_{(y,w)}(I_i) = 0$. The existence of such functions is warranted by Frobenius theorem, as it was discussed in Section 3. A moment of reflection reveals that these two first integrals, written in terms of the original variables (x, u, u_1, u_2), are also first integrals of the original third-order equation (4.1) [25, 26]. In fact,

$$\mathbf{A}_{(x,u)}(I_1) = \mathbf{v}_1^{(2)}(I_1) = 0, \qquad \mathbf{A}_{(x,u)}(I_2) = \mathbf{v}_2^{(2)}(I_2) = 0. \tag{4.9}$$

Since I_1 and I_2 can be written in terms of the differential invariants of $\mathbf{v}_3$ then

$$\mathbf{v}_3^{(2)}(I_1) = \mathbf{v}_3^{(2)}(I_2) = 0. \tag{4.10}$$

Previous discussion provides the following result:

Theorem 4.1. *Let $I_1 = I_1(y, w, w_1)$ and $I_2 = I_2(y, w, w_1)$ be two first integrals of $\mathbf{A}_{(y,w)}$ associated to the $\mathcal{C}^\infty$-symmetries $(\overline{\mathbf{v}}_1, \lambda_1)$ and $(\overline{\mathbf{v}}_2, \lambda_2)$ defined by (4.6) and (4.7). Then the functions*

$$I_i = I_i(y(x, u), w(x, u, u_1), w_1(x, u, u_1, u_2)), \qquad i = 1, 2, \tag{4.11}$$

are two functionally independent first integrals of $\mathbf{A}_{(x,u)}$ such that

$$\mathbf{v}_1^{(2)}(I_1) = \mathbf{v}_3^{(2)}(I_1) = 0, \qquad \mathbf{v}_2^{(2)}(I_2) = \mathbf{v}_3^{(2)}(I_2) = 0. \tag{4.12}$$

Relations (4.9) and (4.10) imply that $\mathbf{v}_2^{(2)}(I_1) \neq 0$ and $\mathbf{v}_1^{(2)}(I_2) \neq 0$, because $\mathbf{A}_{(x,u)}$, $\mathbf{v}_1^{(2)}$, $\mathbf{v}_2^{(2)}$, and $\mathbf{v}_3^{(2)}$ are independent vector fields on the four-dimensional space of variables (x, u, u_1, u_2); hence the functions

$$F_1 = \frac{1}{\mathbf{v}_1^{(2)}(I_2)}, \qquad F_2 = \frac{1}{\mathbf{v}_2^{(2)}(I_1)} \tag{4.13}$$

are well defined. Our aim is to prove that these functions can be used to construct a solvable structure with respect to $\{\mathbf{A}_{(x,u)}\}$ by using the basis elements of the Lie symmetry algebra. Before that, we need to establish some properties satisfied by the functions F_1 and F_2 defined in (4.13).

Lemma 4.2. *The functions F_1 and F_2 defined by (4.13) satisfy*

$$\begin{aligned} \mathbf{v}_3^{(2)}(F_1) &= F_1, & \mathbf{A}_{(x,u)}(F_1) &= 0, & \mathbf{v}_2^{(2)}(F_1) &= 0; \\ \mathbf{v}_3^{(2)}(F_2) &= -F_2, & \mathbf{A}_{(x,u)}(F_2) &= 0, & \mathbf{v}_1^{(2)}(F_2) &= 0. \end{aligned} \tag{4.14}$$

Consequently, both systems in (4.14) are compatible.

Proof. The equality $\left[\mathbf{v}_1^{(2)}, \mathbf{v}_3^{(2)}\right](I_2) = \mathbf{v}_1^{(2)}(I_2)$, which comes from (4.2), yields

$$\mathbf{v}_3^{(2)}\big(\mathbf{v}_1^{(2)}(I_2)\big) = -\mathbf{v}_1^{(2)}(I_2),$$

because of (4.12). Thus,

$$\mathbf{v}_3^{(2)}(F_1) = \mathbf{v}_3^{(2)}\left(\frac{1}{\mathbf{v}_1^{(2)}(I_2)}\right) = -\frac{\mathbf{v}_3^{(2)}\big(\mathbf{v}_1^{(2)}(I_2)\big)}{\big(\mathbf{v}_1^{(2)}(I_2)\big)^2} = \frac{\mathbf{v}_1^{(2)}(I_2)}{\big(\mathbf{v}_1^{(2)}(I_2)\big)^2} = F_1.$$

The relation $\mathbf{v}_3^{(2)}(F_2) = -F_2$ can be deduced in a similar way, from $\left[\mathbf{v}_3^{(2)}, \mathbf{v}_2^{(2)}\right](I_1) = \mathbf{v}_2^{(2)}(I_1)$. Therefore

$$\mathbf{v}_3^{(2)}(F_1) = F_1, \qquad \mathbf{v}_3^{(2)}(F_2) = -F_2. \tag{4.15}$$

Since $\mathbf{v}_i$ is a Lie point symmetry of (4.1), then $\left[\mathbf{v}_i^{(2)}, \mathbf{A}_{(x,u)}\right] = \rho_i \mathbf{A}_{(x,u)}$, where $\rho_i = -\mathbf{A}_{(x,u)}(\mathbf{v}_i(x))$ for $i = 1, 2$. Therefore

$$\mathbf{A}_{(x,u)}\big(\mathbf{v}_1^{(2)}(I_2)\big) = -\left[\mathbf{v}_1^{(2)}, \mathbf{A}_{(x,u)}\right](I_2) = -\rho_1 \mathbf{A}_{(x,u)}(I_2) = 0,$$

because $\mathbf{A}_{(x,u)}(I_2) = 0$. Similarly, $\mathbf{A}_{(x,u)}\big(\mathbf{v}_2^{(2)}(I_1)\big) = 0$. Consequently,

$$\mathbf{A}_{(x,u)}(F_1) = \mathbf{A}_{(x,u)}\left(\frac{1}{\mathbf{v}_1^{(2)}(I_2)}\right) = 0, \qquad \mathbf{A}_{(x,u)}(F_2) = \mathbf{A}_{(x,u)}\left(\frac{1}{\mathbf{v}_2^{(2)}(I_1)}\right) = 0. \tag{4.16}$$

By using (4.2) we can write $\left[\mathbf{v}_1^{(2)}, \mathbf{v}_2^{(2)}\right](I_1) = 2\mathbf{v}_3^{(2)}(I_1)$, which yields $\mathbf{v}_1^{(2)}\big(\mathbf{v}_2^{(2)}(I_1)\big) = 0$, because of (4.12). By taking I_2 instead of I_1, the relation $\mathbf{v}_2^{(2)}\big(\mathbf{v}_1^{(2)}(I_2)\big) = 0$ also holds. Therefore

$$\mathbf{v}_1^{(2)}(F_1) = \mathbf{v}_2^{(2)}\left(\frac{1}{\mathbf{v}_1^{(2)}(I_2)}\right) = 0, \qquad \mathbf{v}_2^{(2)}(F_2) = \mathbf{v}_1^{(2)}\left(\frac{1}{\mathbf{v}_2^{(2)}(I_1)}\right) = 0. \tag{4.17}$$

Relations (4.15), (4.16) and (4.17) prove that the functions (4.13) satisfy (4.14). ∎

The existence of nontrivial solutions for systems (4.14) is the key to construct a solvable structure from the basis elements of the Lie symmetry algebra, as it is shown in the following theorem.

Theorem 4.3. *Let F_1 and F_2 be two functions satisfying (4.14). Then the ordered sets $\left\langle \mathbf{A}_{(x,u)}, \mathbf{v}_3^{(2)}, F_1\mathbf{v}_1^{(2)}, F_2\mathbf{v}_2^{(2)} \right\rangle$ and $\left\langle \mathbf{A}_{(x,u)}, \mathbf{v}_3^{(2)}, F_2\mathbf{v}_2^{(2)}, F_1\mathbf{v}_1^{(2)} \right\rangle$ are solvable structures with respect to $\{\mathbf{A}_{(x,u)}\}$.*

Proof. Since for $i = 1, 2, 3$, the vector field $\mathbf{v}_i = \xi_i(x, u)\partial_x + \eta_i(x, u)\partial_u$ is a Lie point symmetry of (4.1) then

$$\left[\mathbf{v}_i^{(2)}, \mathbf{A}_{(x,u)}\right] = \rho_i \mathbf{A}_{(x,u)}, \tag{4.18}$$

where $\rho_i = -\mathbf{A}_{(x,u)}(\xi_i)$. Obviously $\mathbf{v}_3^{(2)}$ is a symmetry of $\{\mathbf{A}_{(x,u)}\}$, in the sense of Definition 2.1. By using (4.18) and that F_1 and F_2 satisfy (4.14), the following commutation relations can be checked:

$$\begin{aligned}\left[F_1\mathbf{v}_1^{(2)}, \mathbf{A}_{(x,u)}\right] &= F_1\rho_1\mathbf{A}_{(x,u)}, \qquad \left[F_2\mathbf{v}_2^{(2)}, \mathbf{A}_{(x,u)}\right] = F_2\rho_2\mathbf{A}_{(x,u)}, \\ \left[\mathbf{v}_3^{(2)}, F_1\mathbf{v}_1^{(2)}\right] &= \left[\mathbf{v}_3^{(2)}, F_2\mathbf{v}_2^{(2)}\right] = 0, \qquad \left[F_1\mathbf{v}_1^{(2)}, F_2\mathbf{v}_2^{(2)}\right] = 2F_1F_2\mathbf{v}_3^{(2)}.\end{aligned} \tag{4.19}$$

According to Definition 2.1, these relations prove that:

1. $F_1\mathbf{v}_1^{(2)}$ and $F_2\mathbf{v}_2^{(2)}$ are symmetries of $\left\{\mathbf{A}_{(x,u)}, \mathbf{v}_3^{(2)}\right\}$.

2. $F_2\mathbf{v}_2^{(2)}$ (resp. $F_1\mathbf{v}_1^{(2)}$) is a symmetry of $\left\{\mathbf{A}_{(x,u)}, \mathbf{v}_3^{(2)}, F_1\mathbf{v}_1^{(2)}\right\}$ (resp. $\left\{\mathbf{A}_{(x,u)}, \mathbf{v}_3^{(2)}, F_2\mathbf{v}_2^{(2)}\right\}$).

The result follows from Definition 2.1. ∎

5 Strategies for obtaining a complete system of first integrals

The previous discussion shows that any pair of particular solutions F_1, F_2 of the respective system in (4.14) permits the construction of two solvable structures for a third-order equation with Lie symmetry algebra isomorphic to $\mathfrak{sl}(2,\mathbb{R})$. By Theorem 4.3, such functions F_1 and F_2 provide the solvable structures $\left\langle \mathbf{A}_{(x,u)}, \mathbf{v}_3^{(2)}, F_1\mathbf{v}_1^{(2)}, F_2\mathbf{v}_2^{(2)} \right\rangle$ and $\left\langle \mathbf{A}_{(x,u)}, \mathbf{v}_3^{(2)}, F_2\mathbf{v}_2^{(2)}, F_1\mathbf{v}_1^{(2)} \right\rangle$ with respect to $\{\mathbf{A}_{(x,u)}\}$. Therefore, the integrability by quadratures of the given ODE is warranted by Theorem 2.3. In this section we analyze three different strategies that can be followed to integrate completely the given equation.

Method 1: Once two particular solutions F_1 and F_2 of (4.14) have been found, the method based on solvable structures [5] (see also [4, 13, 30]) can be applied to find by quadratures three independent first integrals of $\mathbf{A}_{(x,u)}$. Denote $\mathbf{\Omega} = dx \wedge du \wedge du_1 \wedge du_2$ and consider the corresponding differential 1-forms (2.3) associated to the solvable structure $\left\langle \mathbf{A}, \mathbf{v}_3^{(2)}, F_1\mathbf{v}_1^{(2)}, F_2\mathbf{v}_2^{(2)} \right\rangle$:

$$
\boldsymbol{\omega}_3 = \frac{1}{F_2} \cdot \frac{\mathbf{v}_1^{(2)} \lrcorner\, \mathbf{v}_3^{(2)} \lrcorner\, \mathbf{A}_{(x,u)} \lrcorner\, \mathbf{\Omega}}{\mathbf{v}_2^{(2)} \lrcorner\, \mathbf{v}_1^{(2)} \lrcorner\, \mathbf{v}_3^{(2)} \lrcorner\, \mathbf{A}_{(x,u)} \lrcorner\, \mathbf{\Omega}},
$$

$$
\boldsymbol{\omega}_2 = \frac{1}{F_1} \cdot \frac{\mathbf{v}_2^{(2)} \lrcorner\, \mathbf{v}_3^{(2)} \lrcorner\, \mathbf{A}_{(x,u)} \lrcorner\, \mathbf{\Omega}}{\mathbf{v}_2^{(2)} \lrcorner\, \mathbf{v}_1^{(2)} \lrcorner\, \mathbf{v}_3^{(2)} \lrcorner\, \mathbf{A}_{(x,u)} \lrcorner\, \mathbf{\Omega}},
$$

$$
\boldsymbol{\omega}_1 = \frac{\mathbf{v}_2^{(2)} \lrcorner\, \mathbf{v}_1^{(2)} \lrcorner\, \mathbf{A}_{(x,u)} \lrcorner\, \mathbf{\Omega}}{\mathbf{v}_2^{(2)} \lrcorner\, \mathbf{v}_1^{(2)} \lrcorner\, \mathbf{v}_3^{(2)} \lrcorner\, \mathbf{A}_{(x,u)} \lrcorner\, \mathbf{\Omega}}. \tag{5.1}
$$

The 1-form $\boldsymbol{\omega}_3$ is (locally) exact and a function Θ_1 such that

$$
d\Theta_1 = \boldsymbol{\omega}_3 \tag{5.2}
$$

is a common first integral to the system $\left\{ \mathbf{A}_{(x,u)}, \mathbf{v}_3^{(2)}, F_1\mathbf{v}_1^{(2)} \right\}$.

Since $\left\langle \mathbf{A}, \mathbf{v}_3^{(2)}, F_2\mathbf{v}_2^{(2)}, F_1\mathbf{v}_2^{(2)} \right\rangle$ is also a solvable structure with respect to $\{\mathbf{A}_{(x,u)}\}$, the roles of $F_1\mathbf{v}_1^{(2)}$ and $F_2\mathbf{v}_2^{(2)}$ can be interchanged and thus $\boldsymbol{\omega}_2$ is also (locally) exact. A function Θ_2 such that

$$
d\Theta_2 = \boldsymbol{\omega}_2 \tag{5.3}
$$

is a common first integral to the system $\left\{ \mathbf{A}_{(x,u)}, \mathbf{v}_3^{(2)}, F_2\mathbf{v}_2^{(2)} \right\}$.

Finally, $\boldsymbol{\omega}_1$ is exact modulo $\boldsymbol{\omega}_2$ and $\boldsymbol{\omega}_3$, i.e., $d\boldsymbol{\omega}_1 \in \mathcal{I}\{\boldsymbol{\omega}_2, \boldsymbol{\omega}_3\}$, where $\mathcal{I}\{\boldsymbol{\omega}_2, \boldsymbol{\omega}_3\}$ stands for the ideal generated by $\boldsymbol{\omega}_2$ and $\boldsymbol{\omega}_3$ under taking exterior products. A function Θ_3 such that

$$
d\Theta_3 = \boldsymbol{\omega}_1 \qquad \mathrm{mod}\ \{\boldsymbol{\omega}_2, \boldsymbol{\omega}_3\} \tag{5.4}
$$

completes the set $\{\Theta_1, \Theta_2, \Theta_3\}$ of independent first integrals of the vector field $\mathbf{A}_{(x,u)}$.

Method 2: We recall that the compatibility of systems (4.14) has been proved by constructing the particular solutions given in (4.13):

$$
F_1 = \frac{1}{\mathbf{v}_1^{(2)}(I_2)}, \qquad F_2 = \frac{1}{\mathbf{v}_2^{(2)}(I_1)}.
$$

By Theorem 4.1 the functions I_1 and I_2 can be found through (4.11) from two known first integrals of the reduced equation (4.3). Although these solutions F_1 and F_2 could be used to follow Method 1, the construction and integration of (5.1) is not necessary: the functions I_1, I_2 given in (4.11) and F_1, F_2 are themselves first integrals of $\mathbf{A}_{(x,u)}$ because of (4.9) and Lemma 4.2, respectively.

Let us prove that $\{I_1, F_1, F_2\}$ are functionally independent. If $F_1 = \psi(I_1, F_2)$, for some function ψ, then $\mathbf{v}_1^{(2)}(F_1) = 0$, because $\mathbf{v}_1^{(2)}(F_2) = \mathbf{v}_1^{(2)}(I_1) = 0$. By (4.14), $\mathbf{v}_2^{(2)}(F_1) = 0$, and hence $[\mathbf{v}_1^{(2)}, \mathbf{v}_2^{(2)}](F_1) = 0$. Therefore (4.2) implies that $\mathbf{v}_3^{(2)}(F_1) = 0$, which cannot happen by (4.14). A similar reasoning proves that $\{I_2, F_1, F_2\}$ is also a complete set of first integrals of $\mathbf{A}_{(x,u)}$.

Method 3: By Theorem 4.1 the function I_1 given in (4.11) is a common first integral to the set $\{\mathbf{A}_{(x,u)}, \mathbf{v}_3^{(2)}, \mathbf{v}_1^{(2)}\}$; by (5.1) and (5.2), Θ_1 is a common first integral to the set $\{\mathbf{A}_{(x,u)}, \mathbf{v}_3^{(2)}, F_1\mathbf{v}_1^{(2)}\}$. Since both sets of vector fields are equivalent, Θ_1 and I_1 are functionally dependent. Similarly, Θ_2 and I_2 must be functionally dependent. Therefore, if Θ_3 satisfies (5.4), then the set $\{I_1, I_2, \Theta_3\}$ is a complete system of first integrals of $\mathbf{A}_{(x,u)}$.

In consequence, provided the first integrals (4.11) given in Theorem 4.1 are known, the complete solution of (4.1) arises by finding by quadrature a primitive Θ_3 of $\boldsymbol{\omega}_1$, restricted to $I_1 = C_1$, $I_2 = C_2$, where $C_1, C_2 \in \mathbb{R}$. We point out that $\boldsymbol{\omega}_1$ can be directly computed from the basis elements $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

The second and the third of the described methods use the functions (4.11), obtained from two first integrals of the reduced equation (4.3), to perform the complete integration of equation (4.1). The procedure described in Section 3.1 can be applied to integrate by quadratures the reduced equation (4.3), because this equation admits the pairs $(\overline{\mathbf{v}}_1, \lambda_1)$, $(\overline{\mathbf{v}}_2, \lambda_2)$ defined by (4.6) and (4.7) as $\mathcal{C}^\infty$-symmetries. That procedure works provided that two particular solutions $\overline{h}_1$ and $\overline{h}_2$ of the corresponding system in (3.9) are known (Theorem 3.1).

A remarkable fact is that the particular solutions F_1, F_2 of (4.14) given in (4.13) can be directly expressed in terms of $\overline{h}_1$ and $\overline{h}_2$, without the need to compute I_1 and I_2. In fact, functions (4.13) can be written in the form

$$F_1 = \frac{1}{\mathbf{v}_1^{(2)}(I_2)} = \frac{\varsigma_1}{\mathbf{Y}_1(I_2)} = \frac{\varsigma_1\overline{h}_1}{Q_1}, \qquad F_2 = \frac{1}{\mathbf{v}_2^{(2)}(I_1)} = \frac{\varsigma_2}{\mathbf{Y}_2(I_1)} = \frac{\varsigma_2\overline{h}_2}{Q_2}, \tag{5.5}$$

where ς_1 and ς_2 are given by (4.4) and Q_1, Q_2 are the respective characteristics of the vector fields $\overline{\mathbf{v}}_1$ and $\overline{\mathbf{v}}_2$ given in (4.6). Expressions (5.5) are easy to check by taking (4.6), (4.10), and (3.12) into account. All the functions involved in (5.5) are assumed to be written in terms of the original variables (x, u, u_1, u_2).

The described methods to integrate by quadratures equation (4.1) can be significantly simplified when two particular solutions $\overline{h}_1$ and $\overline{h}_2$ of the corresponding system in (3.9) are known. The next alternatives can be followed:

Option 1: Use the function $\overline{h}_2$ to construct by quadrature a primitive I_1 of the 1-form $\boldsymbol{\beta}_1$ defined in (3.10). Once written in variables (x, u, u_1, u_2), the function I_1, and the functions F_1, F_2 given in (5.5) are three functionally independent first integrals associated to the original third-order equation. Alternatively, the function $\overline{h}_1$ can be used to compute by quadrature a first integral I_2 from a primitive of $\boldsymbol{\beta}_2$ defined in (3.10) and to construct the complete set of first integrals $\{I_2, F_1, F_2\}$. For this option, only one quadrature (to compute either I_1 or I_2) is required.

Option 2: Use the functions $\overline{h}_1$ and $\overline{h}_2$ to construct by means of two quadratures both first integrals I_1 and I_2 as primitives of (3.10). Finally, find by quadrature a primitive Θ_3 of $\boldsymbol{\omega}_1$, restricted to $I_1 = C_1$, $I_2 = C_2$, where $C_1, C_2 \in \mathbb{R}$.

Option 3: Use the functions $\overline{h}_1$ and $\overline{h}_2$ to construct the functions F_1 and F_2 given in (5.5) and follow the Method 1. For this option, three successive quadratures to find the primitives Θ_1, Θ_2 and Θ_3 satisfying (5.2), (5.3), and (5.4), are necessary.

In the following sections this procedure is applied to integrate by quadratures two third-order ODEs admitting Lie symmetry algebras that are isomorphic to $\mathfrak{sl}(2,\mathbb{R})$. The corresponding first integrals and general solutions can be expressed in terms of two independent solutions of second-order *linear* equations.

6   Example I

The third-order ordinary differential equation

$$2u_1 u_3 - 3u_2^2 - 2uu_1^4 = 0. \tag{6.1}$$

admits the following Lie point symmetries

$$\mathbf{v}_1 = \partial_x, \qquad \mathbf{v}_2 = x^2 \partial_x, \qquad \mathbf{v}_3 = x\partial_x, \tag{6.2}$$

which satisfy relations (4.2) and generate a Lie symmetry algebra isomorphic to $\mathfrak{sl}(2,\mathbb{R})$. Our aim is to find solutions F_1 and F_2 of (4.14) to construct the solvable structures warranted by Theorem 4.3. Once this is achieved, equation (6.1) can be integrated by quadratures by following any of the strategies described in Section 5.

The direct search of particular solutions F_1, F_2 of (4.14) seems not to be an easy task. We use the results given in Section 3.1 to construct F_1, F_2 as in (5.5) by using solutions of the corresponding systems (3.9).

Solutions of systems (3.9). Systems (3.9) refer to the second-order equation obtained from (6.1) by using the Lie point symmetry $\mathbf{v}_3$. Such reduced equation becomes

$$w_2 = \frac{3w_1^2}{2w} + \frac{1}{2}w^3 - yw, \tag{6.3}$$

by using the invariants $y = u$ and $w = \frac{1}{xu_1}$ of $\mathbf{v}_3^{(1)}$.

The Lie point symmetries $\mathbf{v}_1$ and $\mathbf{v}_2$ can be recovered as $\mathcal{C}^\infty$-symmetries of (6.3) by using two functions ς_1, ς_2 satisfying (4.4), which can be easily calculated

$$\varsigma_1(x,u) = x, \qquad \varsigma_2(x,u) = \frac{1}{x}. \tag{6.4}$$

According to (4.6) and (4.7), these inherited $\mathcal{C}^\infty$-symmetries are defined by the pairs $(-w\partial_w, w)$ and $(w\partial_w, w)$, respectively. By using the characteristics

$$Q_1(y,w,w_1) = -w, \qquad Q_2(y,w,w_1) = w, \tag{6.5}$$

it can be checked that the first-order λ-prolongations of their respective canonical representatives (3.5) become

$$\mathbf{X}_1 = \partial_w + \left(-w + \frac{w_1}{w}\right)\partial_{w_1}, \qquad \mathbf{X}_2 = \partial_w + \left(w + \frac{w_1}{w}\right)\partial_{w_1}.$$

Systems (3.9) can be constructed with these vector fields and the vector field $\mathbf{A}_{(y,w)}$ associated to equation (6.3). Two of their particular solutions $\overline{h}_1 = \overline{h}_1(y,w,w_1)$ and $\overline{h}_2 = \overline{h}_2(y,w,w_1)$ are given by

$$\overline{h}_1 = -\frac{\left((w^2 - w_1)\Psi_2(y) - 2w\Psi_2'(y)\right)^2}{4w^2 W(\Psi_1, \Psi_2)(y)}, \qquad \overline{h}_2 = \frac{\left((w^2 + w_1)\Psi_2(y) + 2w\Psi_2'(y)\right)^2}{4w^2 W(\Psi_1, \Psi_2)(y)}, \tag{6.6}$$

where $\Psi_1 = \Psi_1(y)$ and $\Psi_2 = \Psi_2(y)$ are two independent solutions of the Airy equation

$$\Psi_{yy} = \frac{1}{2}y\Psi, \tag{6.7}$$

and $W(\Psi_1, \Psi_2)(y) = \Psi_1(y)\Psi_2'(y) - \Psi_1'(y)\Psi_2(y)$ denotes the corresponding Wronskian.

Solutions of systems (4.14) and solvable structures for equation (6.1). Once the functions (6.6) have been determined, the two particular solutions (5.5) for systems (4.14) can

be determined without any additional integration. Expressions (5.5) use the functions (6.4), (6.5), and (6.6), written in the original variables (x, u, u_1, u_2):

$$F_1(x, u, u_1, u_2) = \frac{\left((xu_2 + 2u_1)\,\Psi_1(u) - 2xu_1^2\Psi_1'(u)\right)^2}{4u_1^3 W(\Psi_1, \Psi_2)(u)},$$

$$F_2(x, u, u_1, u_2) = \frac{\left(u_2\Psi_1(u) - 2u_1^2\Psi_1'(u)\right)^2}{4u_1^3 W(\Psi_1, \Psi_2)(u)}. \tag{6.8}$$

By Theorem 4.3, functions (6.8) permit the construction of the solvable structures $\left\langle \mathbf{A}_{(x,u)}, \mathbf{v}_3^{(2)}, F_1\mathbf{v}_1^{(2)}, F_2\mathbf{v}_2^{(2)} \right\rangle$ and $\left\langle \mathbf{A}_{(x,u)}, \mathbf{v}_3^{(2)}, F_1\mathbf{v}_1^{(2)}, F_2\mathbf{v}_2^{(2)} \right\rangle$ with respect to the vector field $\mathbf{A}_{(x,u)}$ associated to equation (6.1).

Complete sets of first integrals of equation (6.1) obtained by quadratures and general solution. Functions (6.6) are all what we need to complete the solution by quadratures by following any of the alternatives enumerated in Section 5.

Option 1: Once the functions (6.6) have been determined, two independent first integrals for equation (6.3) can be calculated by quadratures as primitives of the 1-forms (3.10). Such functions written in variables (x, u, u_1, u_2) provide two functionally independent first integrals of equation (6.1), which can be expressed in terms of the independent solutions Ψ_1 and Ψ_2 of the Airy equation (6.7):

$$I_1(x, u, u_1, u_2) = \frac{u_2\Psi_1(u) - 2u_1^2\Psi_1'(u)}{u_2\Psi_2(u) - 2u_1^2\Psi_2'(u)},$$

$$I_2(x, u, u_1, u_2) = \frac{(xu_2 + 2u_1)\Psi_1(u) - 2xu_1^2\Psi_1'(u)}{(xu_2 + 2u_1)\Psi_2(u) - 2xu_1^2\Psi_2'(u)}. \tag{6.9}$$

Any of the sets $\{I_1, F_1, F_2\}$ or $\{I_2, F_1, F_2\}$ is a complete system of functionally independent first integrals for the equation (6.1).

Option 2: In this case the corresponding 1-form $\boldsymbol{\omega}_1$ in (5.1) becomes

$$\boldsymbol{\omega}_1 = \frac{2u_1u_2 + xu_2^2 - 2xuu_1^4}{2u_1^3}du - \frac{u_1 + 2u_2x}{u_1^2}du_1 + \frac{x}{u_1}du_2.$$

The restriction of $\boldsymbol{\omega}_1$ to the submanifold defined by $I_1 = c_1$ and $I_2 = c_2$, $(c_1, c_2 \in \mathbb{R})$, is exact and

$$\Theta_3(x, u, u_1, u_2) = \ln(x) + \ln\left(\frac{\Psi_1(u) - I_1\Psi_2(u)}{\Psi_1(u) - I_2\Psi_2(u)}\right) \tag{6.10}$$

is a primitive. As it was discussed in Section 5, the set $\{I_1, I_2, \Theta_3\}$ defined by the functions (6.9) and (6.10) is a complete system of first integrals for equation (6.1).

Option 3: The functions F_1 and F_2 given in (6.8) can be used to construct the 1-forms (5.1), which can be successively integrated by quadratures. We omit the expressions for the first integrals $\{\Theta_1, \Theta_2\}$ corresponding to (5.2) and (5.3), because they are functionally dependent of the functions I_1, I_2 given in (6.9). The computation of a remaining first integral Θ_3 could be achieved as in Option 2.

The general solution of equation (6.1) can be obtained, for instance, by setting $I_1 = c_1$, $I_2 = c_2$, $\Theta_3 = \ln(c_3)$ and becomes

$$x = c_3\frac{\Psi_1(u) - c_2\Psi_2(u)}{\Psi_1(u) - c_1\Psi_2(u)}, \tag{6.11}$$

where $c_i \in \mathbb{R}$ for $i = 1, 2, 3$, $c_1 \neq c_2$, and Ψ_1 and Ψ_2 are two independent solutions of the Airy equation (6.7).

Remark 6.1. Equation (6.1) has been chosen for purposes of illustration of the procedures presented in this paper; it could have been solved by using other methods that appear in the literature.

1. By following [16], equation (6.1) can be reduced to a first-order ODE by using the differential invariants

$$y = u, \qquad m = \frac{u_2}{u_1^2}$$

of the two-dimensional subalgebra $\mathcal{L}_2$ generated by $\mathbf{v}_1$ and $\mathbf{v}_3$. The reduced first-order equation is the Riccati equation

$$m_1 = -\frac{1}{2}m^2 + y, \tag{6.12}$$

which becomes the Airy equation (6.7) by means of the standard transformation $m(y) = \frac{\Psi'(y)}{2\Psi(y)}$. In consequence, the general solution of (6.12) can be expressed in terms of two independent solutions, $\Psi_1 = \Psi_1(y)$ and $\Psi_2 = \Psi_2(y)$, of equation (6.7). Such general solution, written in terms of the original variables (x, u, u_1, u_2), yields the second-order equation

$$\frac{u_2}{u_1^2} = \frac{C\Psi_1'(u) + \Psi_2'(u)}{2(C\Psi_1(u) + \Psi_2(u))}, \tag{6.13}$$

where $C \in \mathbb{R}$. Although (6.13) admits $\mathcal{L}_2$, and may therefore be integrated by quadratures, the expression obtained for its general solution

$$x = C_1 + C_2 \int \frac{du}{(C\Psi_1(u) + \Psi_2(u))^2},$$

requires a primitive which apparently cannot be easily evaluated. The procedure presented in this paper overcomes this difficulty, because provides expression (6.11) for the general solution of (6.1). This general solution could be also reached by following the procedure described in [6], see also [14].

2. Apart from (6.2), equation (6.1) admits three additional Lie point symmetries, whose infinitesimals can be expressed in terms of solutions of the Airy equation (6.7) as follows

$$\Psi_1(u)^2 \partial_u, \qquad \Psi_2(u)^2 \partial_u, \qquad \Psi_1(u)\Psi_2(u)\partial_u. \tag{6.14}$$

In (6.14) $\Psi_1 = \Psi_1(u)$ and $\Psi_2 = \Psi_2(u)$ denote two linearly independent solutions of the equation $\Psi''(u) = \frac{1}{2}u\Psi(u)$.

Indeed, it can be checked that equation (6.1) satisfies the conditions that appear in [29, p. 235], although the corresponding solution algorithm could present some difficulties because a rational solution of equation (6.12) seems to be required.

Alternatively, equation (6.1) can be transformed by a point transformation into the representative third-order equation with six-dimensional stabilizer (see [7, Section IV, Case B.4]). The transformation can be found by using basis elements of the form (6.14) such that $\Psi_1 = \Psi_1(u)$ and $\Psi_2 = \Psi_2(u)$ verify $W(\Psi_1, \Psi_2)(u) = 1$. It can be checked that by introducing the new dependent variable $z = \frac{\Psi_2(u)}{\Psi_1(u)}$ equation (6.1) becomes

$$z_3 = \frac{3z_2^2}{2z_1}.$$

This equation can be easily integrated by quadratures and its general solution $z(x) = \frac{k_1}{k_2+x} + k_3$ provides an alternative expression for the general solution of equation (6.1) obtained in (6.11).

7   Example II

For the third-order equation

$$u_3\big(u_1^2 - 2uu_2\big)u^2 + 1 = 0, \tag{7.1}$$

the determining equations for the infinitesimals $\xi = \xi(x,u)$ and $\eta = \eta(x,u)$ of a Lie point symmetry reduce to

$$\eta = u\xi_u, \qquad \xi_{xxx} = 0, \qquad \xi_u = 0. \tag{7.2}$$

If follows that the Lie invariance algebra of equation (7.1) is three-dimensional and it is generated by

$$\mathbf{v}_1 = \partial_x, \qquad \mathbf{v}_2 = x^2\partial_x + 2xu\partial_u, \qquad \mathbf{v}_3 = x\partial_x + u\partial_u. \tag{7.3}$$

By (7.2) any Lie point symmetry of (7.1) must be a linear combination of (7.3). Vector fields (7.3) satisfy the commutation relations (4.2) and generate the nonsolvable Lie algebra $\mathfrak{sl}(2,\mathbb{R})$.

     If we follow, for instance, the Option 2 in Section 5 to integrate equation (7.1), the procedure provides three independent first integrals that can be expressed in terms of two independent solutions, $\psi_1 = \psi_1(z)$, $\psi_2 = \psi_2(z)$, of the linear equation

$$z\psi''(z) - \psi'(z) - 4z^4\psi(z) = 0. \tag{7.4}$$

     The following functions correspond to the first integrals (4.11) given in Theorem 4.1:

$$I_1(x,u,u_1,u_2) = \frac{2(-2u + u_1x)(u_1^2 - 2uu_2)\psi_1(u_1^2 - 2uu_2) + x\psi_1'(u_1^2 - 2uu_2)}{2(-2u + u_1x)(u_1^2 - 2uu_2)\psi_2(u_1^2 - 2uu_2) + x\psi_2'(u_1^2 - 2uu_2)},$$

$$I_2(x,u,u_1,u_2) = \frac{2u_1(u_1^2 - 2uu_2)\psi_1(u_1^2 - 2uu_2) + \psi_1'(u_1^2 - 2uu_2)}{2u_1(u_1^2 - 2uu_2)\psi_2(u_1^2 - 2uu_2) + \psi_2'(u_1^2 - 2uu_2)}. \tag{7.5}$$

     It can be checked that a primitive of the corresponding 1-form $\boldsymbol{\omega_1}$ defined in (5.1), restricted to the submanifold defined by $I_1 = C_1$ and $I_2 = C_2$, is given by

$$\Theta_3(x,u,u_1,u_2) = \ln\left(\frac{C_1\psi_2(u_1^2 - 2uu_2) - \psi_1(u_1^2 - 2uu_2)}{x\big(C_2\psi_2(u_1^2 - 2uu_2) - \psi_1(u_1^2 - 2uu_2)\big)}\right).$$

After replacing C_1 and C_2 by the functions I_1 and I_2 given in (7.5), the function $I_3 = \exp(\Theta_3)$ becomes

$$I_3(x,u,u_1,u_2) = \frac{2u_1(u_1^2 - 2uu_2)\psi_2(u_1^2 - 2uu_2) + \psi_2'(u_1^2 - 2uu_2)}{2(u_1^2 - 2uu_2)(u_1x - 2u)\psi_2(u_1^2 - 2uu_2) + x\psi_2'(u_1^2 - 2uu_2)}. \tag{7.6}$$

The functions given in (7.5) and (7.6) are three independent first integrals for equation (7.1); they provide its implicit solution

$$I_1(x,u,u_1,u_2) = C_1, \qquad I_2(x,u,u_1,u_2) = C_2, \qquad I_3(x,u,u_1,u_2) = C_3, \tag{7.7}$$

where $C_i \in \mathbb{R}$ for $i = 1,2,3$. The elimination of u_1 and u_2 from (7.7) to obtain an explicit solution of (4.1) seems practically impossible: the functions ψ_1 and ψ_2 and their derivatives in (7.7) are evaluated on $u_1^2 - 2uu_2$. Our aim is to obtain a parametric solution for equation (7.1). For that purpose, we consider any solution $\psi = \psi(z)$ of (7.4) and introduce a parameter $t > 0$ such

that $z(t) = \sqrt{2t}$. With this choice, the function $\phi(t) = \psi(z(t))$ is a solution of the Schrödinger equation

$$\phi''(t) - 4\sqrt{2t}\phi(t) = 0. \tag{7.8}$$

Conversely, if $\phi = \phi(t)$ is a solution of (7.8), then $\psi(z) = \phi\left(\frac{z^2}{2}\right)$ satisfies $\psi'(z) = z\phi'\left(\frac{z^2}{2}\right)$ and $\psi = \psi(z)$ is a solution of (7.4). The implicit solution (7.7) expressed in terms of the solutions ϕ_1 and ϕ_2 of (7.8) associated to ψ_1 and ψ_2, respectively, becomes

$$\frac{2(-2u + u_1 x)\phi_1(s) + x\phi_1'(s)}{2(-2u + u_1 x)\phi_2(s) + x\phi_2'(s)} = C_1, \qquad \frac{2u_1\phi_1(s) + \phi_1'(s)}{2u_1\phi_2(s) + \phi_2'(s)} = C_2,$$

$$\frac{2u_1\phi_2(s) + \phi_2'(s)}{2(u_1 x - 2u)\phi_2(s) + x\phi_2'(s)} = C_3, \tag{7.9}$$

where $s = \frac{(u_1^2 - 2uu_2)^2}{2}$. From (7.9), we obtain the parametric solution for equation (7.1)

$$x(s) = \frac{\phi_1(s) - C_1\phi_2(s)}{C_3(\phi_1(s) - C_2\phi_2(s))}, \qquad u(s) = \frac{(C_2 - C_1)W(\phi_1, \phi_2)(s)}{4C_3(\phi_1(s) - C_2\phi_2(s))^2}, \tag{7.10}$$

where ϕ_1 and ϕ_2 are two independent solutions of the Schrödinger equation (7.8) and $W(\phi_1, \phi_2)$ stands for the corresponding Wronskian.

Remark 7.1. The presence of a Schrödinger equation in the solution of a third-order equation with $\mathfrak{sl}(2, \mathbb{R})$ is not new in the literature: it appears in [6] (see also [14]) by using the relations by prolongation of the three possible realizations of $\mathfrak{sl}(2, \mathbb{C})$ on the (x, u) plane. Such relations could have been also used to complete the solution of equation (7.1). It should be remarked that in our method no previous transformation is needed to map the basis elements into one of the canonical realizations of $\mathfrak{sl}(2, \mathbb{C})$.

Reductions of equation (7.1) to Riccati equations or to the associated second-order linear equations can be also obtained by the original Lie theory or by following the procedure in [16]. For instance, it can be checked that equation (7.1) reduces to the Riccati equation

$$m_1 = 2y^2 m^2 - y, \tag{7.11}$$

where $m = \frac{1}{u_1}$ and $y = \frac{1}{2}u_1^2 - uu_2$. Equation (7.11) becomes the linear equation $y\varphi''(y) - 2\varphi'(y) - 2y^4\varphi(y) = 0$ by means of the standard transformation $m(y) = -\frac{\varphi'(y)}{2y^2\varphi(y)}$. By using two independent solutions $\varphi_1 = \varphi_1(y)$ and $\varphi_2 = \varphi_2(y)$ of that linear equation, the general solution of (7.11), written in terms of the original variables (x, u, u_1, u_2), yields the second-order equation

$$u_1 = -2\left(\frac{1}{2}u_1^2 - uu_2\right)^2 \frac{C\varphi_2\left(\frac{1}{2}u_1^2 - uu_2\right) - \varphi_1\left(\frac{1}{2}u_1^2 - uu_2\right)}{C\varphi_2'\left(\frac{1}{2}u_1^2 - uu_2\right) - \varphi_1'\left(\frac{1}{2}u_1^2 - uu_2\right)}. \tag{7.12}$$

Equation (7.12) needs to be integrated in order to recover the general solution of (7.1). This can be avoided by following the method presented in this paper, which provides the parametric solution (7.10) in terms of a slight different second-order linear ODE (equation (7.8)).

8   Concluding remarks and further extensions

The well-known method to integrate by quadratures an ODE with a solvable symmetry algebra is not available for third-order ODEs admitting a Lie symmetry algebra that is isomorphic to the nonsolvable symmetry algebra $\mathfrak{sl}(2, \mathbb{R})$. Basis elements $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ of the Lie symmetry algebra satisfying the relations (4.2) are used to construct explicitly solvable structures with respect to

the vector field $\mathbf{A}$ associated to the ODE. In consequence, the given ODE can be integrated by quadratures.

Such solvable structures are of the form $\left\langle \mathbf{A}, \mathbf{v}_3^{(2)}, F_1\mathbf{v}_1^{(2)}, F_2\mathbf{v}_2^{(2)} \right\rangle$ and $\left\langle \mathbf{A}, \mathbf{v}_3^{(2)}, F_2\mathbf{v}_2^{(2)}, F_1\mathbf{v}_1^{(2)} \right\rangle$, where F_1 and F_2 are solutions of systems (4.14). The existence of such solutions can be proved by a constructive procedure, which involves a second-order ODE obtained by reducing the order of the original ODE with $\mathbf{v}_3$. Such reduced equation admits two $\mathcal{C}^\infty$-symmetries inherited from $\mathbf{v}_1$ and $\mathbf{v}_2$. Two known first integrals of the reduced equation associated to the $\mathcal{C}^\infty$-symmetries can be used to construct the functions F_1 and F_2 (see (4.13)). It is noteworthy that such first integrals can be determined by quadratures when two particular solutions $\overline{h}_1$, $\overline{h}_2$ of systems (3.9) are known (Theorem 3.1). In fact, these solutions $\overline{h}_1$, $\overline{h}_2$ lead directly to the construction of the functions F_1 and F_2 given in (5.5), because the functions ς_1, ς_2, Q_1, and Q_2 are known from the basis elements of the Lie symmetry algebra. Once these functions F_1 and F_2 are known, the given ODE can be integrated by quadratures (as in the case of solvable symmetry algebras) by using any of the three different strategies described in Section 5.

In the presented procedure it is not necessary to map the basis elements of the Lie symmetry algebra into one of the four inequivalent canonical realizations of $\mathfrak{sl}(2,\mathbb{R})$ on the plane. For each realization of $\mathfrak{sl}(2,\mathbb{R})$ the corresponding class of third-order $\mathfrak{sl}(2,\mathbb{R})$-invariant ODEs involves arbitrary elements/functions of single arguments. The method has been illustrated with two particular examples that correspond to the first and second realizations of $\mathfrak{sl}(2,\mathbb{R})$, respectively. In both cases the general solution can be expressed in terms of two linearly independent solutions of some second-order linear equations, as in previous methods in the literature [6, 16] (see also [14]). Further applications of the method to arbitrary third-order $\mathfrak{sl}(2,\mathbb{R})$-invariant equations, including the study of the two remaining inequivalent realizations of $\mathfrak{sl}(2,\mathbb{R})$ on the plane not considered in this paper, are currently being investigated. In particular, it remains the question whether the functions F_1 and F_2 can be always determined in terms of solutions of some second-order linear ODEs, as occurs in the particular examples presented in this paper. These extensions will be addressed in a separate work.

It is expected that the methods developed in this work can be adapted to integrate by quadratures equations with other nonsolvable symmetry algebras. A work in this line is also currently in progress.

Acknowledgements

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