Quantum Curve and the First Painlevé Equation

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Abstract. We show that the topological recursion for the (semi-classical) spectral curve of the first Painlevé equation $P_{\rm I}$ gives a WKB solution for the isomonodromy problem for $P_{\rm I}$. In other words, the isomonodromy system is a quantum curve in the sense of [Dumitrescu O., Mulase M., *Lett. Math. Phys.* **104** (2014), 635–671, arXiv:1310.6022] and [Dumitrescu O., Mulase M., arXiv:1411.1023].

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Dedicated to Professor Takahiro Kawai on his seventieth birthday

1 Introduction

Painlevé transcendents are remarkable special functions which appear in many areas of mathematics and physics (e.g., [17]). These are solutions of certain nonlinear ordinary differential equations known as *Painlevé equations*. These equations were discovered by Painlevé and Gambier more than 100 years ago [34], and solutions have the so-called *Painlevé property*; i.e., any movable singularity must be a pole. One particular property of the Painlevé equations is existence of the *Lax pair*; that is, each Painlevé equation describes an *isomonodromic deformation* of a certain meromorphic linear ordinary differential equation [20, 21]. The monodromy data of the linear ODEs gives a conserved quantity of the Painlevé transcendents. The *Riemann–Hilbert method*, as well as *exact WKB analysis* are applied to analyze the properties of Painlevé transcendents [4, 17, 24, 25, 36].

On the other hand, *quantum curves* attract both mathematicians and physicists since they are expected to encode the information of many quantum topological invariants, such as Gromov–Witten invariants, quantum knot invariants etc. These are concieved in physics literature including [1, 2, 10, 18]. A quantum curve is an ordinary differential (or difference) equation containing a formal parameter $\hbar$ (which plays the role of the Planck constant), like a Schrödinger equation. The quantum invariants appear in the coefficients of the *WKB* (*Wentzel–Kramers–Brillouin*) *solution* of the quantum curve.

The Eynard–Orantin's *topological recursion* introduced in [16] is closely related to both of the quantum curves and Painlevé equations (and many other topics). Topological recursion is a recursive algorithm to compute the $1/N$-expansion of the correlation functions and the partition function of matrix models from its *spectral curve*, and it is generalized to any algebraic curve which may not come from a matrix model. In this context, quantum curves were first discussed in [6] for the Airy spectral curve, and generalized to spectral curves with various backgrounds (see [11, 12, 13, 18, 29] and the survey article [31]). The spectral curves are recovered as the *semi-classical limit* $\hbar \to 0$ of the quantum curves. Moreover, the topological

recursion is also closely related to integrability [5, 7, 19] as is the relationship between matrix models and integrable systems [9, 28].

The aim of this paper is to relate quantum curves and the *first Painlevé equation* with a formal parameter $\hbar$

$$P_{\mathrm{I}}: \quad \hbar^2 \frac{d^2 q}{dt^2} = 6q^2 + t.$$

The (semi-classical) spectral curve for the isomonodormy system associated with P_{I} is given by

$$y^2 = 4(x - q_0)^2 (x + 2q_0), \tag{1.1}$$

where $q_0 = q_0(t)$ is an explicit function of t. This is a family of algebraic curves in (x, y)-space parametrized by t. (The curve (1.1) appeared in [16, Section 10.6] as the spectral curve of (3,2)-minimal model.) Our main result claims that, starting from the spectral curve (1.1), its *quantization* through the Eynard–Orantin's topological recursion (in the sense of [11, 12]) recovers the *whole* isomonodoromy system for P_{I}.

The precise statement of our main theorem is as follows. Let $W_{g,n}(z_1, \ldots, z_n)$ be the *Eynard–Orantin differential* of type (g, n) defined from the spectral curve (1.1) (see Section 3.1). These are meromorphic multi-differential forms, and z_i's are copies of a coordinate on the spectral curve (1.1). $W_{g,n}$'s also depend on t since the spectral curve depends on t. Then, our main result states the following.

Theorem 1.1 (Theorem 3.3). *The following WKB-type formal series $\psi(x, t, \hbar)$ defined by*

$$\psi(x(z), t, \hbar) := \exp\left(\sum_{g \geq 0, n \geq 1} \hbar^{2g-2+n} \frac{1}{n!} \frac{1}{2^n} \int_{\bar{z}}^{z} \cdots \int_{\bar{z}}^{z} W_{g,n}(z_1, \ldots, z_n)\right) \tag{1.2}$$

satisfies the isomonodromy system associated with P_{I}. Here $x(z)$ is an explicit rational function of z which appears in the parametrization of the spectral curve (1.1), and $\bar{z} = -z$.

The above theorem tells us that the isomonodoromy system associated with P_{I} is a quantum curve, and its particular WKB solution is constructed by the topological recursion as (1.2). The main differences between our theorem and previous results on quantum curves are the following:

- Our quantum curve is a restriction of a certain *partial differential equation* (a holonomic system).

- There are *infinitely many $\hbar$-correction terms* in the quantum curve, and these correction terms are essentially given by the *asymptotic expansion* of the solution of P_{I} for $\hbar \to 0$.

This paper is organized as follows. In Section 2, we briefly review some known facts about P_{I} together with an important result on the WKB analysis of isomonodromic systems developed by Kawai–Takei [25, 26]. Our main theorem will be formulated in Section 3 after recalling the notion of topological recursion. We will give a proof of the main results in Section 4.

Remark 1.2. After writing the draft version of this paper, the authors were informed that B. Eynard also has the same result which has not been published yet, but presented in [14]. See also [15, Chapter 5].

2 The first Painlevé equation and isomonodromy system

Let us consider the *first Painlevé equation* with a formal parameter $\hbar$:

$$P_{\mathrm{I}}: \quad \hbar^2 \frac{d^2 q}{dt^2} = 6q^2 + t.$$

The equation P_{I} is obtained from

$$\frac{d^2 \tilde{q}}{d\tilde{t}^2} = 6\tilde{q}^2 + \tilde{t}$$

via the rescaling $\tilde{t} = \hbar^{-4/5} t$, $\tilde{q} = \hbar^{-2/5} q$. We will regard $\hbar$ as a small parameter (i.e., Planck's constant), and investigate a particular formal solution of P_{I} which has an $\hbar$-expansion.

2.1 Formal solution of P_{I}

P_{I} has the following formal power series solution:

$$q(t, \hbar) = \sum_{n=0}^{\infty} \hbar^{2n} q_{2n}(t) = q_0(t) + \hbar^2 q_2(t) + \hbar^4 q_4(t) + \cdots. \tag{2.1}$$

It contains only even order terms of $\hbar$ since P_{I} is invariant under $\hbar \mapsto -\hbar$. The leading term $q_0 = q_0(t)$ satisfies

$$6q_0^2 + t = 0, \qquad \text{hence } q_0(t) = \sqrt{-t/6}, \tag{2.2}$$

and the subleading terms are recursively determined by

$$q_{2(k+1)}(t) = \frac{1}{12 q_0(t)} \left(\frac{d^2 q_{2k}}{dt^2}(t) - 6 \sum_{k_1 + k_2 = k+1,\, k_i > 0} q_{2k_1}(t) q_{2k_2}(t) \right), \qquad k \geq 1. \tag{2.3}$$

As we will see, the coefficients of the formal series appearing in this paper are multivalued functions of t and are defined on the Riemann surface of q_0. Thus, in what follows, we may use q_0 instead of t when we express coefficients.

The relation (2.3) implies

$$q_{2k} = c_{2k} q_0^{1-5k}, \qquad c_{2k} \in \mathbb{C}.$$

It is obvious that the coefficients $q_{2k}(t)$ have a singularity at $q_0 = 0$ (i.e., $t = 0$). This special point is called a *turning point* of P_{I} [25, Definition 2.1] (see also [26, Section 4]). Throughout the paper, we assume the following:

Assumption 2.1. The independent variable t of P_{I} lies on a domain that doesn't contain the origin.

Remark 2.2. The formal solution (2.1) is called a 0-*parameter solution* of P_{I} in [26] since it doesn't contain free parameters. More general formal solutions having one or two free parameters (called 1- or 2-parameter solutions) are constructed in [4] for all Painlevé equations of second order. See also [3] for a construction of general formal solutions of *higher* order Painlevé equations.

Remark 2.3. The formal solution (2.1) is in fact a *divergent* series. However, [23, Theorem 1.1] proved that the formal solution is *Borel summable* when q_0 satisfies $q_0 \neq 0$ and $\arg q_0 \notin \{\frac{2\ell}{5}\pi \mid \ell \in \mathbb{Z}\}$. The exceptional set is called the *Stokes curve* of P_{I}. (See [25, Definition 2.1] for the notion of Stokes curves of Painlevé equations with a small parameter $\hbar$.) That is, there exists a function which is analytic in $\hbar$ on a sectorial domain with the center at the origin (which is also analytic in t) such that (2.1) is the asymptotic expansion of the function for $\hbar \to 0$ in the sector. The analytic function is called the *Borel sum* of the formal series (2.1), and it gives an analytic solution of P_{I} (see [8] for Borel summation method). This particular asymptotic solution obtained by the Borel summation method is called the *tri-tronquée solution* of P_{I} (see [22]), and the *non-linear Stokes phenomena* on Stokes curves are analyzed by [17, 24, 36].

2.2 Isomonodromy system and the τ-function

It is known that P_{I} describes the compatibility condition for the following system of linear PDEs (cf. [21, Appendix C]):

$$\hbar \frac{\partial \Psi}{\partial x} = A\Psi, \qquad \hbar \frac{\partial \Psi}{\partial t} = B\Psi, \tag{2.4}$$

where

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} := \begin{pmatrix} p & 4(x-q) \\ x^2 + qx + q^2 + \dfrac{t}{2} & -p \end{pmatrix},$$

$$B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} := \begin{pmatrix} 0 & 2 \\ \dfrac{x}{2} + q & 0 \end{pmatrix}.$$

The compatibility condition

$$\hbar \frac{\partial A}{\partial t} - \hbar \frac{\partial B}{\partial x} + [A, B] = 0$$

is equivalent to the following Hamiltonian system

$$\hbar \frac{dq}{dt} = \frac{\partial H}{\partial p}, \qquad \hbar \frac{dp}{dt} = -\frac{\partial H}{\partial q}, \tag{2.5}$$

where the (time-dependent) Hamiltonian is given by

$$H = H(q, p, t) := \frac{1}{2}p^2 - 2q^3 - tq.$$

We can easily check that (2.5) and P_{I} are equivalent. The above system of linear ODEs is called the *isomonodromy system* associated with P_{I} (see [20, 21]).

Let $(q, p) = (q(t, \hbar), p(t, \hbar))$ be a formal power series solution of the Hamiltonian system (2.5); that is, $q(t, \hbar)$ is the formal solution (2.1) of P_{I}, and

$$p(t, \hbar) = \hbar \frac{dq(t, \hbar)}{dt} = \sum_{n=0}^{\infty} \hbar^{2n+1} p_{2n+1}(t).$$

The corresponding Hamiltonian function is denoted by

$$\sigma(t, \hbar) := H\big(q(t, \hbar), p(t, \hbar), t\big). \tag{2.6}$$

We can check that (2.6) is invariant under $\hbar \mapsto -\hbar$, and hence it has the following expansion:

$$\sigma(t, \hbar) = \sum_{n=0}^{\infty} \hbar^{2n} \sigma_{2n}(t). \tag{2.7}$$

Definition 2.4 ([20, 32])**.** The τ-*function* (corresponding to the formal solution (2.1)) of P_I is defined by

$$\hbar^2 \frac{d}{dt} \log \tau(t, \hbar) = \sigma(t, \hbar) \tag{2.8}$$

up to constant.

The τ-function can also be defined in terms of a solution of (2.4) [20] (see also Appendix A). The expansion (2.7) implies that the τ-function (2.8) has an expansion of the form

$$\log \tau(t, \hbar) = \sum_{g=0}^{\infty} \hbar^{2g-2} \tau_{2g}(t).$$

2.3 Spectral curve

In what follows, we assume that the formal solution $(q(t, \hbar), p(t, \hbar))$ of (2.5) constructed above is substituted into the coefficients of the isomonodromy system (2.4). Then, the coefficients of the isomonodromy system has the following $\hbar$-expansions:

$$A = A_0(x, t) + \hbar A_1(x, t) + \hbar^2 A_2(x, t) + \cdots,$$
$$B = B_0(x, t) + \hbar B_1(x, t) + \hbar^2 B_2(x, t) + \cdots,$$

whose top terms are given by

$$A_0(x, t) = \begin{pmatrix} 0 & 4(x - q_0) \\ x^2 + q_0 x + q_0^2 + \dfrac{t}{2} & 0 \end{pmatrix}, \qquad B_0(x, t) = \begin{pmatrix} 0 & 2 \\ \dfrac{x}{2} + q_0 & 0 \end{pmatrix}.$$

Observe that, since q_0 satisfies (2.2), the algebraic curve defined by

$$\det(y - A_0(x, t)) = y^2 - 4(x - q_0)^2(x + 2q_0) = 0 \tag{2.9}$$

has *genus* 0. Actually, this gives a family of algebraic curves in $\mathbb{C}^2_{(x,y)}$ parametrized by t. Since we have assumed that $t \neq 0$, $x = q_0$ and $x = -2q_0$ are distinct.

Definition 2.5. We call the algebraic curve (2.9) *the semi-classical spectral curve*, or *the spectral curve* of (the first equation of) the isomonodromy system (2.4).

Remark 2.6. It is shown in [25, Proposition 1.3] that, for all (second order) Painlevé equations with a formal parameter $\hbar$, the semi-classical spectral curves corresponding to the same type of formal power series solution as (2.1) have *genus* 0.

Remark 2.7. Since we are taking the semi-classical limit (i.e., top term in $\hbar$-expansion), our spectral curve (2.9) is different from usual spectral curves for isomonodromic deformation equations discussed, e.g., in [33, 35]. The spectral curves in the above papers have *higher genus*. Recently, Nakamura [30] investigates the geometry of genus 2 spectral curves which appear in an *autonomous limit* of the *4th order Painlevé equations*, and use them to classify the Painlevé equations. See [27] for the list of 4th order Painlevé equations.

2.4 WKB analysis of isomonodromy system in scalar form

Denote the unknown vector function of (2.4) by $\Psi = {}^t(\psi_1, \psi_2)$. Then, $\psi = \psi_1$ satisfies the following scalar version of isomonodromy system

$$\left(\left(\hbar \frac{\partial}{\partial x} \right)^2 + f \left(\hbar \frac{\partial}{\partial x} \right) + g \right) \psi = 0,$$
$$\hbar \frac{\partial \psi}{\partial t} = \frac{1}{2(x-q)} \left(\hbar \frac{\partial \psi}{\partial x} - p\psi \right), \tag{2.10}$$

where

$$f = f(x, t, \hbar) := -\operatorname{tr} A - \hbar \frac{\partial}{\partial x} \log A_{12} = -\hbar \frac{1}{x-q},$$
$$g = g(x, t, \hbar) := \det A - \hbar \frac{\partial A_{11}}{\partial x} + \hbar A_{11} \frac{\partial}{\partial x} \log A_{12}$$
$$= -\left(4x^3 + 2tx + p^2 - 4q^3 - 2tq \right) + \hbar \frac{p}{x-q}.$$

The coefficients of f and g have an $\hbar$-expansion since q and p are contained in them

$$f = -\hbar \frac{1}{x-q_0} + \hbar^3 \frac{1}{1728q_0^4(x-q_0)^2} + \hbar^5 \frac{49x - 51q_0}{5971968q_0^9(x-q_0)^3} + \cdots, \tag{2.11}$$

$$g = -4(x-q_0)^2(x+2q_0) - \hbar^2 \frac{x+11q_0}{144q_0^2(x-q_0)} - \hbar^4 \frac{7x^2 + 34q_0x - 53q_0^2}{248832q_0^7(x-q_0)^2} + \cdots. \tag{2.12}$$

The top term of g appears in the defining equation of the spectral curve (2.9), and its zeros are called *turning points* of the first equation of (2.10) in the WKB analysis. In particular, under the assumption $t \neq 0$, there is

- a *simple* turning point at $x = -2q_0$ which is a branch point of the spectral curve (2.9), and

- a *double* turning point at $x = q_0$ which is a singular point of the spectral curve (2.9).

Consider the *Riccati equation*

$$\hbar^2 \left(P^2 + \frac{\partial P}{\partial x} \right) + f\hbar P + g = 0. \tag{2.13}$$

This is equivalent to the first equation in (2.10) by

$$\psi = \exp \left(\int^x P dx \right), \qquad \text{i.e.,} \quad P = \frac{1}{\psi} \frac{\partial \psi}{\partial x}.$$

Let

$$P^{(\pm)}(x, t, \hbar) = \sum_{m=0}^{\infty} \hbar^{m-1} P_m^{(\pm)}(x, t)$$

be the formal solutions of (2.13) with the top term

$$P_0^{(\pm)}(x, t) = \pm 2(x-q_0)\sqrt{x + 2q_0}.$$

The coefficients $P_m^{(\pm)}(x,t)$ are recursively determined by

$$2P_0^{(\pm)}P_{m+1}^{(\pm)} + \sum_{\substack{a+b=m+1 \\ a,b\geq 1}} P_a^{(\pm)}\big(P_b^{(\pm)} + f_b\big) + \frac{\partial P_m^{(\pm)}}{\partial x} + g_{m+1} = 0 \qquad \text{for} \quad m \geq 0, \qquad (2.14)$$

where f_a and g_a are the coefficient of $\hbar^a$ in f and g, respectively. Explicit forms of the first few terms are given by

$$P_1^{(\pm)} = -\frac{1}{4(x+2q_0)}, \qquad P_2^{(\pm)} = \pm\frac{x+17q_0}{576q_0^2(x+2q_0)^{5/2}}, \qquad P_3^{(\pm)} = -\frac{2x^2 + 20q_0 x + 77q_0^2}{6912q_0^4(x+2q_0)^4},$$

$$P_4^{(\pm)} = \pm\frac{28x^4 + 500q_0 x^3 + 3684q_0^2 x^2 + 14273q_0^3 x + 27307q_0^4}{3981312q_0^7(x+2q_0)^{11/2}}.$$

It is obvious from (2.14) that $P_m^{(\pm)}(x,t)$ are holomorphic except at the turning points and $x=\infty$ (and multivalued for even m). It also follows from the recursion relation (2.14) that

$$P^{(\pm)}(x,t,\hbar) = \pm\left(\frac{2}{\hbar}x^{3/2} + \frac{t}{2\hbar}x^{-1/2} \mp \frac{1}{4}x^{-1} + \frac{\sigma(t,\hbar)}{2\hbar}x^{-3/2} + O\big(x^{-2}\big)\right) \qquad (2.15)$$

holds when $x \to \infty$.

Remark 2.8. We can check that $P_m^{(\pm)}(x,t)$'s have the following asymptotic expansion for $x \to \infty$

$$P_0^{(\pm)}(x,t) = \pm\left(2x^{3/2} + \frac{t}{2}x^{-1/2} + O\big(x^{-3/2}\big)\right), \qquad P_1^{(\pm)}(x,t) = -\frac{1}{4}x^{-1} + O\big(x^{-3/2}\big),$$

$$P_m^{(\pm)}(x,t) = O\big(x^{-3/2}\big) \qquad \text{for} \quad m \geq 2,$$

and we have (2.15) after summing up $\hbar^{m-1}P_m^{(\pm)}(x,t)$. Once you know that $P^{(\pm)}(x,t,\hbar)$ has an asymptotic expansion in this sense, subleading terms in (2.15) can be computed from the Riccati equation (2.13).

Define

$$P_{\text{odd}}(x,t,\hbar) := \frac{1}{2}\big(P^{(+)}(x,t,\hbar) - P^{(-)}(x,t,\hbar)\big),$$

$$P_{\text{even}}(x,t,\hbar) := \frac{1}{2}\big(P^{(+)}(x,t,\hbar) + P^{(-)}(x,t,\hbar)\big).$$

It is easy to check that (cf. [26, Section 2])

$$P_{\text{even}}(x,t,\hbar) = -\frac{1}{2}\frac{\partial}{\partial x}\log\frac{\hbar P_{\text{odd}}(x,t,\hbar)}{2(x-q(t,\hbar))} \qquad (2.16)$$
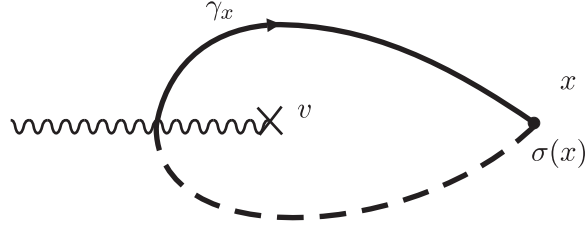
and

$$P_{\text{odd}}(\sigma(x),t,\hbar) = -P_{\text{odd}}(x,t,\hbar) \qquad (2.17)$$

hold. Here x is regarded as a coordinate on the spectral curve, and σ is the covering involution for the spectral curve: $P^{(+)}(\sigma(x),t) = -P^{(-)}(x,t)$.

Since

$$\frac{\hbar P_{\text{odd}}(x,t,\hbar)}{2(x-q(t,\hbar))} = \sqrt{x+2q_0}\big(1 + O(\hbar)\big),$$

Figure 1. For a given x, the path γ_x starts from the point $\sigma(x)$ and ends at x. The wiggly lines designate a branch cut, and the solid (resp. dotted) part represents a part of path on the first (resp. the second) sheet of the spectral curve.

the right hand-side of (2.16) is the derivative of the formal power series

$$-\frac{1}{2}\log\frac{\hbar P_{\mathrm{odd}}(x,t,\hbar)}{2(x-q(t,\hbar))} = -\frac{1}{4}\log(x+2q_0) + O(\hbar).$$

Thus the ambiguity of the branch of the logarithm only appears in the top term, but we care about the ambiguity since it doesn't matter in our computation.

The following theorem was applied in the *transformation theory of Painlevé equations* in [25]. We will use the fact in the proof of our main theorem.

Theorem 2.9 (cf. [25, Proposition 1.2 and Theorem 1.1]).

(i) *The formal series $P^{(\pm)}(x,t,\hbar)$ satisfies*

$$\hbar\frac{\partial}{\partial t}P^{(\pm)}(x,t,\hbar) = \frac{\partial}{\partial x}\left(\frac{\hbar P^{(\pm)}(x,t,\hbar) - p(t,\hbar)}{2(x-q(t,\hbar))}\right). \tag{2.18}$$

In particular, $P_{\mathrm{odd}}(x,t,\hbar)$ satisfies

$$\frac{\partial}{\partial t}P_{\mathrm{odd}}(x,t,\hbar) = \frac{\partial}{\partial x}\left(\frac{P_{\mathrm{odd}}(x,t,\hbar)}{2(x-q(t,\hbar))}\right). \tag{2.19}$$

(ii) *All coefficients of $P^{(\pm)}(x,t,\hbar)$ are holomorphic except at the simple turning point $x = -2q_0$ and $x = \infty$. In particular, they are holomorphic at the double turning point $x = q_0$.*

(iii) *The formal series*

$$\begin{aligned}\psi_{\pm}(x,t,\hbar) &:= \exp\left(\pm\int_v^x P_{\mathrm{odd}}(x',t,\hbar)dx' - \frac{1}{2}\log\frac{\hbar P_{\mathrm{odd}}(x,t,\hbar)}{2(x-q(t,\hbar))}\right) \\ &= \left(\frac{2(x-q(t,\hbar))}{\hbar P_{\mathrm{odd}}(x,t,\hbar)}\right)^{1/2}\exp\left(\pm\int_v^x P_{\mathrm{odd}}(x',t,\hbar)dx'\right)\end{aligned} \tag{2.20}$$

satisfies the isomonodoromy system (2.10). *Here v is the simple turning point $-2q_0$. The integral from v is defined by*

$$\int_v^x P_{\mathrm{odd}}(x',t,\hbar)dx' = \frac{1}{2}\int_{\gamma_x} P_{\mathrm{odd}}(x',t,\hbar)dx', \tag{2.21}$$

where the path γ_x is depicted in Fig. 1 (cf. [26, Section 2]).

Proof. Although the scalar version of isomonodromy system (2.10) is different from that used in [25], they are related by a gauge transformation $\psi \mapsto (x-q)^{1/2}\psi$. Therefore, the equalities (2.18) and (2.19) in (i) together with the holomorphicity of each coefficient of $P_{\mathrm{odd}}(x,t,\hbar)$ at $x = q_0$ follows from [25, Proposition 1.2 and Theorem 1.1]. Then, it turns out that the coefficients of $P_{\mathrm{odd}}(x,t,\hbar)/(x - q(t,\hbar))$ are also holomorphic due to (2.19). Then, (2.16) implies that each coefficient of $P_{\mathrm{even}}(x,t,\hbar)$ is also holomorphic at $x = q_0$. Thus we have proved (ii).

The claim (iii) follows from a straightforward computations

$$
\frac{1}{\psi_\pm}\frac{\partial \psi_\pm}{\partial x} = \pm P_{\mathrm{odd}} - \frac{1}{2}\frac{\partial}{\partial x}\log\left(\frac{\hbar P_{\mathrm{odd}}}{2(x-q)}\right) = \pm P_{\mathrm{odd}} + P_{\mathrm{even}} = P^{(\pm)},
$$

$$
\hbar\frac{1}{\psi_\pm}\frac{\partial \psi_\pm}{\partial t} = \frac{1}{2}\left(\frac{-\hbar(dq/dt)}{x-q} - \frac{\hbar}{P_{\mathrm{odd}}}\frac{\partial P_{\mathrm{odd}}}{\partial t}\right) \pm \int_v^x \hbar\frac{\partial P_{\mathrm{odd}}}{\partial t}dx
$$

$$
= -\frac{p}{2(x-q)} - \frac{\hbar}{P_{\mathrm{odd}}}\left(\frac{1}{2(x-q)}\frac{\partial P_{\mathrm{odd}}}{\partial x} - \frac{P_{\mathrm{odd}}}{2(x-q)^2}\right) \pm \hbar\frac{P_{\mathrm{odd}}}{2(x-q)}
$$

$$
= -\frac{p}{2(x-q)} + \frac{\hbar P^{(\pm)}}{2(x-q)} = \frac{1}{2(x-q)}\left(\frac{\hbar}{\psi_\pm}\frac{\partial \psi_\pm}{\partial x} - p\right). \qquad \blacksquare
$$

As we will see below, an *isomonodromic WKB solution* such as (2.20) is constructed from just a family of algebraic curves (2.9) by *the topological recursion* ([16]). In particular, the first equation in (2.10) gives a *quantization* of the spectral curve (2.9) in the sense of [11, 12].

Remark 2.10. In the above computation the normalization (2.20) is essential. Since P_{odd} is anti-invariant under the covering involution σ as (2.17) and the integral in (2.20) is defined as a contour integral (2.21), we don't need to take care of the branch point v in the computation

$$
\int_v^x \frac{\partial P_{\mathrm{odd}}}{\partial t}dx = \frac{1}{2}\int_{\sigma(x)}^x \frac{\partial}{\partial x}\left(\frac{P_{\mathrm{odd}}}{2(x-q)}\right)dx = \frac{P_{\mathrm{odd}}}{2(x-q)}.
$$

Remark 2.11. We can also construct a WKB-type formal solution of matrix isomonodromy system (2.4). Define

$$
\tilde{\psi}_\pm(x,t,\hbar) = \frac{\hbar\frac{d\psi_\pm}{dx}(x,t,\hbar) - A_{11}(x,t,\hbar)\psi_\pm(x,t,\hbar)}{A_{12}(x,t,\hbar)}
$$

$$
= \frac{\hbar P^{(\pm)}(x,t,\hbar) - A_{11}(x,t,\hbar)}{A_{12}(x,t,\hbar)}\psi_\pm(x,t,\hbar).
$$

Then, the matrix valued formal series

$$
\Psi(x,t,\hbar) = \begin{pmatrix} \psi_+(x,t,\hbar) & \psi_-(x,t,\hbar) \\ \tilde{\psi}_+(x,t,\hbar) & \tilde{\psi}_-(x,t,\hbar) \end{pmatrix} \tag{2.22}
$$

gives a fundamental formal solution of the isomonodoromy system (2.4).

3 Topological recursion and quantum curve theorem

In this section we review the Eynard–Orantin's topological recursion [16] for our spectral curve (2.9), and formulate our main theorem.

3.1 Topological recursion

The topological recursion is an algorithm associating some differential forms $W_{g,n}$ and numbers F_g given the following source data:

- A plane curve $(\mathcal{C}, x, y)$: $\mathcal{C}$ is a compact Riemann surface, $x, y \colon \mathcal{C} \to \mathbb{P}^1$ are meromorphic functions.

- The Bergman kernel B: It is a symmetric differential form on $\mathcal{C} \times \mathcal{C}$ with poles of order 2 along the diagonal, and satisfying some normalization conditions.

In our case, $\mathcal{C} = \mathbb{P}^1$ and x, y are rational functions which parametrize the spectral curve (2.9)

$$x(z) = z^2 - 2q_0, \qquad y(z) = 2z(z^2 - 3q_0). \tag{3.1}$$

Here z is a coordinate on $\mathbb{P}^1$. The Bergman kernel is given by

$$B(z_1, z_2) = \frac{dz_1 dz_2}{(z_1 - z_2)^2},$$

since the spectral curve is of genus 0. Zeros of dx are called *ramification points* of the spectral curve (3.1). Our spectral curve has only one ramification point at $z = 0$.

The topological recursion for our spectral curve (3.1) is formulated as follows (see [16] for general case):

Definition 3.1 ([16, Definition 4.2] (see also [11, Section 3])). The *Eynard–Orantin differential* $W_{g,n}(z_1, \ldots, z_n)$ of type (g, n) is a meromorphic n-differential on the n-times product of the spectral curve (3.1) defined by the following *topological recursion relation*:

- for $2g - 2 + n \leq 0$:

$$W_{0,1}(z_1) := y(z_1) dx(z_1) = 4z_1^2(z_1^2 - 3q_0) dz_1,$$
$$W_{0,2}(z_1, z_2) := B(z_1, z_2) = \frac{dz_1 dz_2}{(z_1 - z_2)^2},$$

- for $2g - 2 + n = 1$:

$$W_{0,3}(z_1, z_2, z_3) := \frac{1}{2\pi i} \oint_{\gamma_0} K(z, z_1) \big[W_{0,2}(z, z_2) W_{0,2}(\bar{z}, z_3) + W_{0,2}(z, z_3) W_{0,2}(\bar{z}, z_2) \big],$$
$$W_{1,1}(z_1) := \frac{1}{2\pi i} \oint_{\gamma_0} K(z, z_1) W_{0,2}(z, \bar{z}),$$

- for $2g - 2 + n \geq 2$:

$$W_{g,n}(z_1, \ldots, z_n) := \frac{1}{2\pi i} \oint_{\gamma_0} K(z, z_1)$$
$$\times \Bigg[\sum_{j=2}^{n} \big(W_{0,2}(z, z_j) W_{g,n-1}(\bar{z}, z_{[\hat{1},\hat{j}]}) + W_{0,2}(\bar{z}, z_j) W_{g,n-1}(z, z_{[\hat{1},\hat{j}]}) \big)$$
$$+ W_{g-1,n+1}(z, \bar{z}, z_{[\hat{1}]}) + \sum_{\substack{g_1+g_2=g \\ I \sqcup J = [\hat{1}]}}^{\text{stable}} W_{g_1,|I|+1}(z, z_I) W_{g_2,|J|+1}(\bar{z}, z_J) \Bigg]. \tag{3.2}$$

Here γ_0 is a small cycle (in z-plane) which encircles the ramification point $z = 0$ in the counter-clockwise direction, $\bar{z} = -z$ is the conjugate of z near the ramification point, and the *recursion kernel* $K(z, z_1)$ is given by

$$K(z, z_1) = -\frac{\omega^{\bar{z}-z}(z_1)}{2(y(z) - y(\bar{z}))dx(z)}, \qquad \omega^{\bar{z}-z}(z_1) = \int_z^{\bar{z}} W_{0,2}(\cdot, z_1).$$

Also, we use the index convention $[\hat{j}] = \{1, \ldots, n\} \setminus \{j\}$ and so on. Lastly, the sum in the third line of (3.2) is taken for indices in the stable range (i.e., only $W_{g,n}$'s with $2g - 2 + n \geq 1$ appear).

The explicit form of some of Eynard–Orantin differentials are given as follows

$$W_{0,3} = \frac{1}{12q_0 z_1^2 z_2^2 z_3^2} dz_1 dz_2 dz_3,$$

$$W_{0,4} = \frac{z_1^2 z_2^2 z_3^2 z_4^2 + 3q_0(z_1^2 z_2^2 z_3^2 + z_2^2 z_3^2 z_4^2 + z_3^2 z_4^2 z_1^2 + z_4^2 z_1^2 z_2^2)}{144 q_0^3 z_1^4 z_2^4 z_3^4 z_4^4} dz_1 dz_2 dz_3 dz_4,$$

$$W_{1,1} = \frac{z_1^2 + 3q_0}{288 q_0^2 z_1^4} dz_1,$$

$$W_{1,2} = \frac{2z_1^4 z_2^4 + 6q_0(z_1^4 z_2^2 + z_1^2 z_2^4) + 3q_0^2(5z_1^4 + 3z_1^2 z_2^2 + 5z_2^4)}{3456 q_0^4 z_1^6 z_2^6} dz_1 dz_2,$$

$$W_{2,1} = \frac{28 z_1^8 + 84 q_0 z_1^6 + 252 q_0^2 z_1^4 + 609 q_0^3 z_1^2 + 945 q_0^4}{1990656 q_0^7 z_1^{10}} dz_1.$$

Eynard–Orantin differentials have the following properties (see [16]):

- As a differential form on each variable z_i, $W_{g,n}$, for $2g - 2 + n \geq 1$, is *holomorphic* except for the ramification point 0 and may have a pole at 0.

- $W_{g,n}$ is *symmetric*; that is, they are invariant under any permutation of variables.

- For $2g - 2 + n \geq 1$, $W_{g,n}$ is *anti-invariant* under the involution $z_i \mapsto \bar{z}_i$ for each variable:

$$W_{g,n}(z_1, \ldots, \bar{z}_j, \ldots, z_n) = -W_{g,n}(z_1, \ldots, z_j, \ldots, z_n) \qquad \text{for} \quad j = 1, \ldots, n.$$

- $W_{g,n}$ is also *holomorphic* in t except for $t = 0$ (i.e., $q_0 = 0$). There is a formula for the derivative of $W_{g,n}$ with respect to t; see Section 3.5.

3.2 Quantum curve theorem

In this section we describe our main result which claims that the scalar isomonodromy system (2.10) gives a *quantum curve*.

Definition 3.2. For $g \geq 0, n \geq 1$ satisfying $2g - 2 + n \geq 1$, define *open free energy* of type (g, n) by

$$F_{g,n}(z_1, \ldots, z_n) := \frac{1}{2^n} \int_{\bar{z}_1}^{z_1} \cdots \int_{\bar{z}_n}^{z_n} W_{g,n}(z_1, \ldots, z_n). \tag{3.3}$$

It follows from the definition that open free energies satisfy

$$d_{z_1} \cdots d_{z_n} F_{g,n}(z_1, \ldots, z_n) = W_{g,n}(z_1, \ldots, z_n),$$

$$F_{g,n}(z_1, \ldots, \bar{z}_j, \ldots, z_n) = -F_{g,n}(z_1, \ldots, z_j, \ldots, z_n) \qquad \text{for} \quad j = 1, \ldots, n.$$

Explicit computation shows that

$$F_{0,3}(z_1, z_2, z_3) = -\frac{1}{12q_0 z_1 z_2 z_3},$$

$$F_{0,4}(z_1, z_2, z_3, z_4) = \frac{z_1^2 z_2^2 z_3^2 z_4^2 + q_0\left(z_1^2 z_2^2 z_3^2 + z_2^2 z_3^2 z_4^2 + z_3^2 z_4^2 z_1^2 + z_4^2 z_1^2 z_2^2\right)}{144q_0^3 z_1^3 z_2^3 z_3^3 z_4^3},$$

$$F_{1,1}(z_1) = -\frac{z_1^2 + q_0}{288q_0^2 z_1^3},$$

$$F_{1,2}(z_1, z_2) = \frac{2z_1^4 z_2^4 + 2q_0\left(z_1^4 z_2^2 + z_1^2 z_2^4\right) + q_0^2\left(3z_1^4 + z_1^2 z_2^2 + 3z_2^4\right)}{3456q_0^4 z_1^5 z_2^5},$$

$$F_{2,1}(z_1) = -\frac{140z_1^8 + 140q_0 z_1^6 + 252q_0^2 z_1^4 + 435q_0^3 z_1^2 + 525q_0^4}{9953280q_0^7 z_1^9}.$$

We also introduce functions $\{S_m(x,t)\}_{m\geq 0}$ by

$$S_0(x,t) := \int_v^x y(z(x'))dx', \qquad S_1(x,t) := -\frac{1}{2}\log\left(\frac{y(z(x))}{2(x-q_0)}\right),$$

and for $m \geq 2$

$$S_m(x,t) := \sum_{\substack{2g-2+n=m-1 \\ g\geq 0, n\geq 1}} \frac{F_{g,n}(z,\ldots,z)}{n!}\Bigg|_{z=z(x)},$$

where $z(x) = \sqrt{x + 2q_0}$ is the inverse function of $x(z)$. After computations we have

$$S_0(x,t) = \frac{4}{5}(x - 3q_0)(x + 2q_0)^{3/2}, \qquad S_1(x,t) = -\frac{1}{4}\log(x + 2q_0),$$

$$S_2(x,t) = -\frac{x + 7q_0}{288q_0^2(x + 2q_0)^{3/2}}, \qquad S_3(x,t) = \frac{2x^2 + 14q_0 x + 35q_0^2}{6912q_0^4(x + 2q_0)^3},$$

$$S_4(x,t) = -\frac{140x^4 + 1580q_0 x^3 + 7476q_0^2 x^2 + 18739q_0^3 x + 23499q_0^4}{9953280q_0^7(x + 2q_0)^{9/2}}.$$

Our main result is the following.

Theorem 3.3. *The formal series $\psi(x,t,\hbar)$ given by*

$$\psi(x,t,\hbar) := \exp(S(x,t,\hbar)t), \tag{3.4}$$

$$S(x,t,\hbar) := \sum_{m=0}^{\infty} \hbar^{m-1} S_m(x,t) \tag{3.5}$$

satisfies both of the differential equations in scalar-version of the isomonodromy system (2.10).
That is, the formal series $S(x,t,\hbar)$ given by (3.5) *satisfies the following differential equations which are equivalent to* (2.10)*:*

$$\hbar^2\left(\left(\frac{\partial S}{\partial x}\right)^2 + \frac{\partial^2 S}{\partial x^2}\right) = \frac{\hbar}{x-q}\left(\hbar\frac{\partial S}{\partial x} - p\right) + \left(4x^3 + 2tx + p^2 - 4q^3 - 2tq\right), \tag{3.6}$$

$$\hbar\frac{\partial S}{\partial t} = \frac{1}{2(x-q)}\left(\hbar\frac{\partial S}{\partial x} - p\right). \tag{3.7}$$

Thus, the *principal specialization* (i.e., setting $z_i = z$ for all $i = 1, \ldots, n$) of the open free energies gives an isomonodromic WKB solution. Theorem 3.3 implies

$$\frac{\partial S}{\partial x}(x, t, \hbar) = P^{(+)}(x, t, \hbar) \tag{3.8}$$

holds (under a suitable choice of the branch of $\sqrt{x + 2q_0}$). The computational results in Section 2.4 show that (3.8) holds up to $\hbar^4$. A full proof of Theorem 3.3 will be given in Section 4 together with that of Theorem 3.7 below.

Remark 3.4. In the topological recursion (3.2), we take residues only at the ramification point $z = 0$. Thus $W_{g,n}$'s defined here are different from those in [12]; in particular, our quantum curve (2.10) has *infinitely many $\hbar$-corrections* as in (2.11) and (2.12) (but recovers the same spectral curve in the semi-classical limit).

Remark 3.5. In Theorem 3.3, the choice of the lower end points of the integral in (3.3) is important. Different choice also give a WKB solution of the first equation in (2.10), but it may not satisfy the second equation in general.

3.3   Closed free energies and the τ-function

The other main result of this paper is giving another proof of the known fact about the relationship between the closed free energies and the τ-function of P_{I} (cf. [9, 16]).

Definition 3.6 ([16, Definition 4.3]). Define the *closed free energy* $F_g = F_g(t)$ for $g \geq 2$ by

$$F_g(t) = \frac{1}{2\pi i(2 - 2g)} \oint_{\gamma_0} \Phi(z) W_{g,1}(z),$$

where

$$\Phi(z) = \int_{z_0}^{z} y(z) dx(z) = \frac{4}{5}z^5 - 4q_0 z^3 + \mathrm{const}$$

and z_0 is a generic point. Free energies F_0 and F_1 for $g = 0, 1$ are also defined but in a different manner (see [16, Sections 4.2.2 and 4.2.3] for the definition).

Note that F_g defined here is different from $F_{g,n}$ defined in the previous subsection. F_g's are also called *symplectic invariants* since they are invariant under symplectic transformations of the spectral curve (see [16]). Explicit computation shows that

$$F_0(t) = -\frac{48q_0^5}{5}, \qquad F_1(t) = -\frac{1}{24}\log(-3q_0),$$

$$F_2(t) = \frac{7}{207360q_0^5}, \qquad F_3(t) = \frac{245}{429981696q_0^{10}}.$$

Theorem 3.7 ([9] and [16, Section 10.6]). *The generating function of the free energy $F_g(t)$ gives a τ-function of P_{I}:*

$$\log \tau(t, \hbar) = \sum_{g=0}^{\infty} \hbar^{2g-2} F_g(t).$$

Namely,

$$\frac{dF_g(t)}{dt} = \sigma_{2g}(t). \tag{3.9}$$

The proof will be given in Section 4. It is worth mentioning that the closed free energies specify one particular τ-function although there is an ambiguity in Definition 2.4.

Proposition 3.8. *For $g \geq 2$, we have*

$$F_g(t) = \int_\infty^t \sigma_{2g}(t')dt'. \tag{3.10}$$

Proof. Let us describe the behavior of the $W_{g,n}$'s when $q_0 \to \infty$ (i.e., $t \to \infty$). When q_0 tends to ∞, no singular point of the integrand in the right hand-side of (3.2) on the z-plane hits the integration cycle γ_0. Thus, we can show that

$$W_{g,n}(z_1, \ldots, z_n) = O\big(q_0^{-(2g-2+n)}\big)$$

for $2g - 2 + n \geq 0$. This implies that

$$F_g(t) = O\big(q_0^{-(2g-2)}\big)$$

holds since $\Phi(z) \sim q_0$ as $q_0 \to \infty$ (but we can verify that F_g for $g \geq 2$ has a stronger decay in the above explicit computations). This completes the proof of (3.10). ∎

3.4   Asymptotics of Eynard–Orantin differnetials

The rest of this section will be devoted to show some important properties of $W_{g,n}$ and $F_{g,n}$. Firstly, we will describe the asymptotic behavior of them near $z_i = \infty$.

Lemma 3.9.

(i) *For $2g - 2 + n \geq 0$, we have*

$$W_{g,n}(z_1, \ldots, z_n) = \left(\frac{c_{g,n}}{z_1^2 \cdots z_n^2} + O\big(z_1^{-4} \cdots z_n^{-4}\big) \right) dz_1 \cdots dz_n \tag{3.11}$$

   as $z_i \to \infty$ for all $i = 1, \ldots, n$. Here $c_{g,n} \in \mathbb{C}$ is a constant.

(ii) *For $2g - 2 + n \geq 0$, we have*

$$F_{g,n}(z_1, \ldots, z_n) = \frac{c'_{g,n}}{z_1 \cdots z_n} + O\big(z_1^{-3} \cdots z_n^{-3}\big), \qquad c'_{g,n} \in \mathbb{C}, \tag{3.12}$$

   as $z_i \to \infty$ for all $i = 1, \ldots, n$.

Proof. The first property (3.11) follows from the analyticity of $W_{g,n}$ at $z_i = \infty$. The second property (3.12) follows from (3.11) immediately because $F_{g,n}(z_1, \ldots, z_n)$ doesn't have a constant term due to the definition (3.3). ∎

As a corollary, the principal specialization of open free energies satisfies

$$F_{g,n}(z, \ldots, z) = O\big(z^{-n}\big) \tag{3.13}$$

when $z \to \infty$.

3.5   Variation of spectral curve

There is a formula (for "variation of spectral curves") that allows us to compute derivatives of $W_{g,n}$ etc. with respect to the parameter t.

Theorem 3.10 (cf. [16, Theorem 5.1])**.**

(i) *For $2g - 2 + n \geq 0$, we have*

$$\frac{\partial}{\partial t} W_{g,n}(z(x_1), \ldots, z(x_n))$$
$$= -2 \operatorname*{Res}_{x_{n+1}=\infty} z(x_{n+1}) W_{g,n+1}\big(z(x_1), \ldots, z(x_n), z(x_{n+1})\big). \tag{3.14}$$

(ii) *For $g \geq 1$, we have*

$$\frac{dF_g}{dt}(t) = -2 \operatorname*{Res}_{x=\infty} z(x) W_{g,1}(z(x)) = - \operatorname*{Res}_{z=\infty} z W_{g,1}(z). \tag{3.15}$$

(iii) *For $2g - 2 + n \geq 1$, we have*

$$\frac{\partial}{\partial t} F_{g,n}(z(x_1), \ldots, z(x_n))$$
$$= -2 \operatorname*{Res}_{x_{n+1}=\infty} z(x_{n+1}) d_{x_{n+1}} F_{g,n+1}(z(x_1), \ldots, z(x_n), z(x_{n+1})),$$

or equivalently,

$$\frac{\partial}{\partial t} F_{g,n}(z(x_1), \ldots, z(x_n))$$
$$= \lim_{z_{n+1} \to \infty} \left(z_{n+1}^2 \frac{\partial}{\partial z_{n+1}} F_{g,n+1}(z_1, \ldots, z_n, z_{n+1}) \right) \Bigg|_{(z_1, \ldots, z_n) = (z(x_1), \ldots, z(x_n))}. \tag{3.16}$$

Proof. Set $\Lambda(z) := z$. Then, we can check $\Lambda(z)$ satisfies the required condition

$$\operatorname*{Res}_{z=\infty} \left(\Lambda(z) W_{0,2}(z, z_1) \right) = -dz_1 = - \left(\frac{\partial y}{\partial t}(z_1) dx(z_1) - \frac{\partial x}{\partial t}(z_1) dy(z_1) \right)$$

to apply [16, Theorem 5.1]. Thus the claim (i) and (ii) are proved. Integrating both hand-sides of (3.14), we have (iii). ∎

3.6   Differential recursion for open free energies

Here we give a key theorem in the proof of our main results. We have the following *differential recursion* which is a modification of the one obtained in [11, 12].

Theorem 3.11. *The open free energies for $2g - 2 + n \geq 2$ satisfy the following equations*

$$\frac{\partial F_{g,n}}{\partial z_1}(z_1, \ldots, z_n) = \sum_{j=2}^{n} \frac{-2z_j}{z_1^2 - z_j^2} \left(\frac{1}{2y(z_1)\frac{dx}{dz}(z_1)} \frac{\partial F_{g,n-1}}{\partial z_1}(z_{[\hat{j}]}) - \frac{1}{2y(z_j)\frac{dx}{dz}(z_j)} \frac{\partial F_{g,n-1}}{\partial z_j}(z_{[\hat{1}]}) \right)$$
$$- \frac{1}{2y(z_1)\frac{dx}{dz}(z_1)} \frac{\partial^2}{\partial u_1 \partial u_2} \bigg(F_{g-1,n+1}(u_1, u_2, z_{[\hat{1}]})$$
$$+ \sum_{\substack{g_1+g_2=g \\ I \sqcup J = [\hat{1}]}}^{\text{stable}} F_{g_1,|I|+1}(u_1, z_I) F_{g_2,|J|+1}(u_2, z_J) \bigg) \Bigg|_{u_1=u_2=z_1}$$

$$+ \frac{s}{\frac{dy}{dz}(s)\frac{dx}{dz}(s)(z_1^2 - s^2)} \left[\sum_{j=2}^{n} \frac{-2z_j}{z_j^2 - s^2} \frac{\partial F_{g,n-1}}{\partial z_1}(s, z_{[\hat{1},\hat{j}]}) \right.$$

$$+ \frac{\partial^2}{\partial u_1 \partial u_2} \left(F_{g-1,n+1}(u_1, u_2, z_{[\hat{1}]}) \right.$$

$$\left. \left. + \sum_{\substack{g_1+g_2=g \\ I \sqcup J = [\hat{1}]}}^{\text{stable}} F_{g_1,|I|+1}(u_1, z_I) F_{g_2,|J|+1}(u_2, z_J) \right) \right|_{u_1=u_2=s} \right]. \tag{3.17}$$

Here $s = (3q_0)^{1/2}$ is a zero of $y(z)$.

Proof. This can be proved by a similar technique used in [11, Theorem 4.7], as follows. Integrating the topological recursion relation (3.2) with respect to $z_2, \ldots, z_n$, we have

$$\frac{\partial}{\partial z_1} F_{g,n}(z_1, \ldots, z_n) = \frac{1}{2^{n-1}} \int_{\bar{z}_2}^{z_2} \cdots \int_{\bar{z}_n}^{z_n} W_{g,n}(z_1, \ldots, z_n)$$

$$= \frac{1}{2\pi i} \frac{1}{2^{n-1}} \oint_{\gamma_0} K(z, z_1) R_{g,n}(z, z_2, \ldots, z_n), \tag{3.18}$$

where

$$R_{g,n}(z, z_2, \ldots, z_n) = \sum_{j=2}^{n} \left[\left(\int_{\bar{z}_j}^{z_j} W_{0,2}(z, z_j) \right) \left(\int_{\bar{z}_{[\hat{1},\hat{j}]}}^{z_{[\hat{1},\hat{j}]}} W_{g,n-1}(\bar{z}, z_{[\hat{1},\hat{j}]}) \right) \right.$$

$$\left. - \left(\int_{\bar{z}_j}^{z_j} W_{0,2}(\bar{z}, z_j) \right) \left(\int_{\bar{z}_{[\hat{1},\hat{j}]}}^{z_{[\hat{1},\hat{j}]}} W_{g,n-1}(z, z_{[\hat{1},\hat{j}]}) \right) \right]$$

$$+ \int_{\bar{z}_{[\hat{1}]}}^{z_{[\hat{1}]}} W_{g-1,n+1}(z, \bar{z}, z_{[\hat{1}]})$$

$$+ \sum_{\substack{g_1+g_2=g \\ I \sqcup J = [\hat{1}]}}^{\text{stable}} \left(\int_{\bar{z}_I}^{z_I} W_{g_1,|I|+1}(z, z_I) \right) \left(\int_{\bar{z}_J}^{z_J} W_{g_2,|J|+1}(\bar{z}, z_J) \right).$$

Here, for a set $L = \{\ell_1, \ldots, \ell_k\} \subset \{1, \ldots, n\}$ of indices, we have used the notation

$$\int_{\bar{z}_L}^{z_L} W_{g,n}(z_1, \ldots, z_n) := \int_{\bar{z}_{\ell_1}}^{z_{\ell_1}} \cdots \int_{\bar{z}_{\ell_k}}^{z_{\ell_k}} W_{g,n}(z_1, \ldots, z_n).$$

On the z-plane, the integrand $K(z, z_1) R_{g,n}(z, z_1, \ldots, z_n)$ in the right hand-side of (3.18) has poles at

- at $z = z_1, \bar{z}_1$ which are poles of $K(z, z_1)$,
- at $z = z_2, \ldots, z_n, \bar{z}_2, \ldots, \bar{z}_n$ which are poles of $W_{0,2}(z, z_j)$ and $W_{0,2}(\bar{z}, z_j)$,
- at $z = s, \bar{s}$ which are poles of $K(z, z_1)$,

and all of them are simple poles. Then, the equalities

$$\int_{\bar{z}_j}^{z_j} W_{0,2}(z, z_j) = \left(\frac{1}{z - z_j} - \frac{1}{z - \bar{z}_j} \right) dz,$$

$$\frac{1}{2^{n-2}} \int_{\bar{z}_{[\hat{1},\hat{j}]}}^{z_{[\hat{1},\hat{j}]}} W_{g,n}(z, z_{[\hat{1},\hat{j}]}) = \frac{\partial F_{g,n-1}}{\partial z_1}(z, z_{[\hat{1},\hat{j}]})$$

and the residue theorem show (3.17). ∎

Remark 3.12. Note that the first two blocks in the right hand-side of (3.17) coincide with that obtained in [11, 12]. Unlike the case of [11, 12], we need more terms arising from $z = s$ corresponding to the singular point $(x, y) = (q_0, 0)$ of the spectral curve (2.9) since it becomes a (simple) pole of the recursion kernel $K(z, z_1)$. It also worth mentioning that the right hand-side of (3.17) doesn't have singularity at $z_j = s$ for $j = 1, \ldots, n$.

Using this differential recursion, we can give an alternative expression of (3.16) as follows.

Theorem 3.13. *For $2g - 2 + n \geq 1$, the following holds:*

$$\frac{\partial}{\partial t} F_{g,n}(z(x_1), \ldots, z(x_n)) = E_{g,n}(z(x_1), \ldots, z(x_n)), \tag{3.19}$$

where

$$
\begin{aligned}
E_{g,n}&(z_1, \ldots, z_n) \\
&:= \sum_{j=1}^{n} \frac{2z_j}{2y(z_j)\frac{dx}{dz}(z_j)} \frac{\partial F_{g,n}}{\partial z_j}(z_1, \ldots, z_n) + \frac{s}{\frac{dy}{dz}(s)\frac{dx}{dz}(s)} \sum_{j=1}^{n} \frac{-2z_j}{z_j^2 - s^2} \frac{\partial F_{g,n}}{\partial u_1}(u_1, z_{[\hat{j}]}) \Bigg|_{u_1 = s} \\
&\quad + \frac{s}{\frac{dy}{dz}(s)\frac{dx}{dz}(s)} \frac{\partial^2}{\partial u_1 \partial u_2} \Bigg(F_{g-1,n+2}(u_1, u_2, z_1, \ldots, z_n) \\
&\quad + \sum_{\substack{g_1 + g_2 = g \\ I \sqcup J = \{1, \ldots, n\}}}^{\text{stable}} F_{g_1, |I|+1}(u_1, z_I) F_{g_2, |J|+1}(u_2, z_J) \Bigg) \Bigg|_{u_1 = u_2 = s}. \tag{3.20}
\end{aligned}
$$

Proof. The equality (3.16) shows that the left hand-side of (3.19) coincides with

$$\lim_{z_{n+1} \to \infty} z_{n+1}^2 \frac{\partial}{\partial z_{n+1}} F_{g,n+1}(z_1, \ldots, z_n, z_{n+1})$$

after the substitution $z_i \mapsto z(x_i)$ for $i = 1, \ldots, n$. Then, the equality follows from the asymptotic behavior (3.12) of $F_{g,n}$'s and the above differential recursion (3.17) for $2g - 2 + (n+1) \geq 2$. ∎

4 Proof of main theorems

4.1 Strategy for the proof

What we will show here is that the formal series $S(x, t, \hbar)$ defined in (3.5) satisfies the system of equations (3.6) and (3.7). In addition, we will also prove the equality (3.9). These equalities will be proved by an induction as follows.

Theorem 4.1. *Let $[\bullet]_{\hbar^m}$ be the coefficient of $\hbar^m$ in a formal series $\bullet$ of $\hbar$. For an even integer $k \geq 2$, assume that*

$$
\begin{aligned}
\frac{\partial S_m}{\partial x}(x, t) &= P_m(x, t) &\quad &\text{for} \quad m = 0, \ldots, k-1, \\
\frac{\partial S_m}{\partial t}(x, t) &= \left[\frac{1}{2(x - q)} \left(\hbar \frac{\partial S}{\partial x} - p \right) \right]_{\hbar^m} &\quad &\text{for} \quad m = 0, \ldots, k-1, \\
\frac{dF_g}{dt}(t) &= \sigma_{2g}(t) &\quad &\text{for} \quad g = k/2 \tag{4.1}
\end{aligned}
$$

holds. Here $P_m(x, t) = P_m^{(+)}(x, t)$ is the coefficient of $\hbar^{m-1}$ in the formal solution $P^{(+)}(x, t, \hbar)$ of the Riccati equation (2.13) constructed in Section 2.4, and σ_{2g} is given in (2.7). Then, we have

(A) *The following equality holds for $m = k$ and $k + 1$:*

$$\left[\hbar^2 \left(\left(\frac{\partial S}{\partial x} \right)^2 + \frac{\partial^2 S}{\partial x^2} \right) \right]_{\hbar^m} = \left[2\hbar^2 \frac{\partial S}{\partial t} + \left(4x^3 + 2tx + p^2 - 4q^3 - 2tq \right) \right]_{\hbar^m}. \quad (4.2)$$

(B) *The following equalities hold:*

$$\frac{\partial S_k}{\partial x}(x,t) = P_k(x,t), \qquad \frac{\partial S_k}{\partial t}(x,t) = \left[\frac{1}{2(x-q)} \left(\hbar \frac{\partial S}{\partial x} - p \right) \right]_{\hbar^k}, \quad (4.3)$$

$$\frac{\partial S_{k+1}}{\partial x}(x,t) = P_{k+1}(x,t), \qquad \frac{\partial S_{k+1}}{\partial t}(x,t) = \left[\frac{1}{2(x-q)} \left(\hbar \frac{\partial S}{\partial x} - p \right) \right]_{\hbar^{k+1}}, \quad (4.4)$$

$$\frac{dF_g}{dt}(t) = \sigma_{2g}(t) \qquad for \quad g = (k+2)/2. \quad (4.5)$$

It is obvious that our main theorems (Theorems 3.3 and 3.7) follow from the statements in (A) and (B). The rest of this section is devoted to give a proof of (A) and (B).

4.2 Proof of (A)

We emphasize that the results shown in Section 4.2.1 below are proved without using the assumption (4.1). We also note that we only use the second equality in assumption (4.1) in Section 4.2.2 to prove (A).

4.2.1 Computation of principal specializations

Define

$$G_{g,n}(z_1, \ldots, z_n) := \frac{\partial F_{g,n}}{\partial z_1}(z_1, \ldots, z_n) - \sum_{j=2}^{n} \frac{-2z_j}{z_1^2 - z_j^2} \left(\frac{1}{2y(z_1)\frac{dx}{dz}(z_1)} \frac{\partial F_{g,n-1}}{\partial z_1}(z_{[\hat{j}]}) \right.$$
$$- \frac{1}{2y(z_j)\frac{dx}{dz}(z_j)} \frac{\partial F_{g,n-1}}{\partial z_j}(z_{[\hat{1}]}) \Big)$$
$$+ \frac{1}{2y(z_1)\frac{dx}{dz}(z_1)} \frac{\partial^2}{\partial u_1 \partial u_2} \left(F_{g-1,n+1}(u_1, u_2, z_{[\hat{1}]}) \right.$$
$$+ \sum_{\substack{g_1+g_2=g \\ I \sqcup J = [\hat{1}]}}^{\text{stable}} F_{g_1,|I|+1}(u_1, z_I) F_{g_2,|J|+1}(u_2, z_J) \Big) \Bigg|_{u_1 = u_2 = z_1}. \quad (4.6)$$

The technique developed in [11, 12] enables us to show the following.

Lemma 4.2 (cf. [11, Theorem 6.5]). *For $m \geq 2$, we have*

$$\left(\frac{2y(z)}{\frac{dx}{dz}(z)} \sum_{\substack{2g-2+n=m \\ g \geq 0, n \geq 1}} \frac{G_{g,n}(z, \ldots, z)}{(n-1)!} \right) \Bigg|_{z=z(x)} = \sum_{\substack{a+b=m+1 \\ a,b \geq 0}} \frac{\partial S_a}{\partial x} \frac{\partial S_b}{\partial x} + \frac{\partial^2 S_m}{\partial x^2} - \frac{1}{x-q_0} \frac{\partial S_m}{\partial x}. \quad (4.7)$$

Proof. As is shown in [11, Theorem 6.5], applying $\sum_{2g-2+n=m} \frac{1}{(n-1)!}$ and the principal specialization to (4.6), we have

$$\sum_{\substack{2g-2+n=m \\ g \geq 0, n \geq 1}} \frac{G_{g,n}(z, \ldots, z)}{(n-1)!} = \frac{1}{2y(z)\frac{dx}{dz}(z)} \left(\sum_{\substack{a+b=m+1 \\ a,b \geq 2}} \frac{\partial S_a(x(z))}{\partial z} \frac{\partial S_b(x(z))}{\partial z} + \frac{\partial^2 S_m(x(z))}{\partial z^2} \right)$$

$$+ \frac{\partial S_{m+1}(x(z))}{\partial z} + \left\{ \frac{\partial}{\partial z} \left(\frac{1}{2y(z)\frac{dx}{dz}(z)} \right) \right\} \frac{\partial S_m(x(z))}{\partial z}.$$

After the coordinate change $z = z(x)$, the right hand-side becomes

$$\frac{\frac{dx}{dz}(z(x))}{2y(z(x))} \left(\sum_{\substack{a+b=m+1 \\ a,b \geq 2}} \frac{\partial S_a}{\partial x} \frac{\partial S_b}{\partial x} + \frac{\partial^2 S_m}{\partial x^2} + 2y(z(x)) \frac{\partial S_{m+1}}{\partial x} - \frac{1}{y(z(x))} \frac{\partial y(z(x))}{\partial x} \frac{\partial S_m}{\partial x} \right).$$

Then, the desired equality (4.7) follows from the above equality and

$$\frac{\partial S_0}{\partial x} = y(z(x)), \qquad \frac{\partial S_1}{\partial x} = -\frac{1}{2y(z(x))} \frac{\partial y(z(x))}{\partial x} + \frac{1}{2(x - q_0)}. \qquad\blacksquare$$

Note that the right hand-side of (4.7) coincides with

$$\left[\hbar^2 \left(\left(\frac{\partial S}{\partial x} \right)^2 + \frac{\partial^2 S}{\partial x^2} \right) \right]_{\hbar^{m+1}} - \frac{1}{x - q_0} \frac{\partial S_m}{\partial x}.$$

Thus, Lemma 4.2 relates the principal specialization of $G_{g,n}$ to the left hand-side of (4.2). Next we also relate them to the right hand-side of (4.2).

Lemma 4.3. *Let $E_{g,n}(z_1, \ldots, z_n)$ be the functions defined by (3.20). Then, the following equality holds for $m \geq 2$*

$$\sum_{\substack{2g-2+n=m \\ g \geq 0, n \geq 2}} \left(\frac{2y(z)}{\frac{dx}{dz}(z)} \frac{G_{g,n}(z, \ldots, z)}{(n-1)!} - \frac{2E_{g,n-1}(z, \ldots, z)}{(n-1)!} \right) \Bigg|_{z=z(x)} = -\frac{1}{x - q_0} \frac{\partial S_m}{\partial x}. \qquad (4.8)$$

Proof. Theorem 3.11 shows that (4.6) can also be written as

$$G_{g,n}(z_1, \ldots, z_n) = \frac{s}{\frac{dy}{dz}(s)\frac{dx}{dz}(s)(z_1^2 - s^2)} \sum_{j=2}^{n} \frac{-2z_j}{z_j^2 - s^2} \frac{\partial F_{g,n-1}}{\partial z_1}(s, z_{[\hat{1},\hat{j}]})$$

$$+ \frac{s}{\frac{dy}{dz}(s)\frac{dx}{dz}(s)(z_1^2 - s^2)} \frac{\partial^2}{\partial u_1 \partial u_2} \Bigg(F_{g-1,n+1}(u_1, u_2, z_{[\hat{1}]})$$

$$+ \sum_{\substack{g_1+g_2=g \\ I \sqcup J = [\hat{1}]}}^{\text{stable}} F_{g_1,|I|+1}(u_1, z_I) F_{g_2,|J|+1}(u_2, z_J) \Bigg) \Bigg|_{u_1=u_2=s}. \qquad (4.9)$$

Taking the principal specialization of (4.9) and (3.20) with $n \mapsto n-1$, we have

$$\frac{2y(z)}{\frac{dx}{dz}(z)} \frac{G_{g,n}(z, \ldots, z)}{(n-1)!} - \frac{2E_{g,n-1}(z, \ldots, z)}{(n-1)!}$$

$$= -\frac{4z}{2y(z)\frac{dx}{dz}(z)} \left(\frac{1}{(n-1)!} \frac{\partial}{\partial z} F_{g,n-1}(z, \ldots, z) \right) \qquad (4.10)$$

for any $g \geq 0$ and $n \geq 2$ satisfying $2g - 2 + n \geq 2$. Then, summing up (4.10) for $g \geq 0$, $n \geq 2$ satisfying $2g - 2 + n = m$, we obtain (4.8) after the coordinate change $z = z(x)$. $\qquad\blacksquare$

On the other hand, Theorem 3.13 implies that

$$\sum_{\substack{2g-2+n=m \\ g \geq 0, n \geq 2}} \frac{2E_{g,n-1}(z, \ldots, z)}{(n-1)!} \Bigg|_{z=z(x)} = 2\frac{\partial}{\partial t} S_m = \left[2\hbar^2 \frac{\partial S}{\partial t} \right]_{\hbar^{m+1}}$$

holds for $m \geq 2$. Therefore we have the following.

Lemma 4.4. *The equality*

$$\left[\hbar^2\left(\left(\frac{\partial S}{\partial x}\right)^2 + \frac{\partial^2 S}{\partial x^2}\right)\right]_{\hbar^{m+1}}$$

$$= \left[2\hbar^2\frac{\partial S}{\partial t}\right]_{\hbar^{m+1}} + \sum_{\substack{2g-2+n=m \\ g\geq 0,\, n\geq 1}} \frac{2y(z)}{\frac{dx}{dz}(z)}\frac{G_{g,n}(z,\ldots,z)}{(n-1)!} - \sum_{\substack{2g-2+n=m \\ g\geq 0,\, n\geq 2}} \frac{2y(z)}{\frac{dx}{dz}(z)}\frac{G_{g,n}(z,\ldots,z)}{(n-1)!}$$

holds for $m \geq 2$.

4.2.2 Completion of the proof of (A)

Lemma 4.4 implies

$$\left[\hbar^2\left(\left(\frac{\partial S}{\partial x}\right)^2 + \frac{\partial^2 S}{\partial x^2}\right)\right]_{\hbar^{m+1}} = \left[2\hbar^2\frac{\partial S}{\partial t}\right]_{\hbar^{m+1}} \qquad \text{if } m \text{ is even,}$$

$$\left[\hbar^2\left(\left(\frac{\partial S}{\partial x}\right)^2 + \frac{\partial^2 S}{\partial x^2}\right)\right]_{\hbar^{m+1}} = \left[2\hbar^2\frac{\partial S}{\partial t}\right]_{\hbar^{m+1}} + \frac{2y}{\frac{dx}{dz}}G_{(m+1)/2,1} \qquad \text{if } m \text{ is odd.} \tag{4.11}$$

On the other hand, it follows from (2.7) that

$$\left[4x^3 + 2tx + p^2 - 4q^3 - 2tq\right]_{\hbar^{m+1}} = \begin{cases} 0 & \text{if } m \text{ is even,} \\ 2\sigma_{m+1} & \text{if } m \text{ is odd.} \end{cases}$$

Therefore, under the assumption (4.1), the desired equality (4.2) follows from (4.11) and Lemma 4.5 below.

Lemma 4.5. *For $g \geq 2$, we have*

$$\frac{2y(z)}{\frac{dx}{dz}(z)}G_{g,1}(z) = 2\frac{dF_g}{dt}(t). \tag{4.12}$$

Proof. Firstly, we note that

$$\frac{2y(z)}{\frac{dx}{dz}(z)}G_{g,1}(z) = \frac{1}{4s^2}\left(\frac{\partial^2 F_{g-1,2}}{\partial z_1 \partial z_2}(s,s) + \sum_{\substack{g_1+g_2=g \\ g_1,g_2\geq 1}} \frac{\partial F_{g_1,1}}{\partial z_1}(s)\frac{\partial F_{g_2,1}}{\partial z_2}(s)\right) \tag{4.13}$$

holds. Using the differnetial recursion (3.17) for $n = 1$, we have

$$\frac{\partial F_{g,1}}{\partial z_1}(z) = -\frac{1}{2y(z)\frac{dx}{dz}(z)}\left(\frac{\partial^2 F_{g-1,2}}{\partial z_1 \partial z_2}(z,z) + \sum_{\substack{g_1+g_2=g \\ g_1,g_2\geq 1}} \frac{\partial F_{g_1,1}}{\partial z_1}(z)\frac{\partial F_{g_2,1}}{\partial z_1}(z)\right)$$

$$+ \frac{s}{\frac{dy}{dz}(s)\frac{dx}{dz}(s)(z^2-s^2)}\left(\frac{\partial^2 F_{g-1,2}}{\partial z_1 \partial z_2}(s,s) + \sum_{\substack{g_1+g_2=g \\ g_1,g_2\geq 1}} \frac{\partial F_{g_1,1}}{\partial z_1}(s)\frac{\partial F_{g_2,1}}{\partial z_1}(s)\right).$$

Then, Lemma 3.9 implies that

$$z\frac{\partial F_{g,1}}{\partial z_1}(z)dz = zW_{g,1}(z) = \frac{1}{8s^2}\left(\frac{\partial^2 F_{g-1,2}}{\partial z_1 \partial z_2}(s,s) + \sum_{\substack{g_1+g_2=g \\ g_1,g_2\geq 1}} \frac{\partial F_{g_1,1}}{\partial z_1}(s)\frac{\partial F_{g_2,1}}{\partial z_2}(s)\right)\frac{dz}{z} + O(1)$$

holds when $z \to \infty$. Then the equality (4.12) follows from (3.15) and (4.13). ∎

4.3 Proof of (B)

One of the desired equality (4.3) is proved as follows.

Lemma 4.6. *Under the assumption* (4.1), *we have*

$$\frac{\partial S_k}{\partial x}(x,t) = P_k(x,t), \tag{4.14}$$

$$S_k(x,t) = \int_\infty^x P_k(x',t)dx', \tag{4.15}$$

$$\frac{\partial S_k}{\partial t}(x,t) = \left[\frac{1}{2(x-q)}\left(\hbar\frac{\partial S}{\partial x}-p\right)\right]_{\hbar^k}. \tag{4.16}$$

Proof. The equality (4.2) for $m = k$ and the second equality in the assumption (4.1) imply

$$\left[\hbar^2\left(\left(\frac{\partial S}{\partial x}\right)^2+\frac{\partial^2 S}{\partial x^2}\right)\right]_{\hbar^k} = \left[\frac{\hbar}{x-q}\left(\hbar\frac{\partial S}{\partial x}-p\right)+\left(4x^3+2tx+p^2-4q^3-2tq\right)\right]_{\hbar^k}.$$

Thus $\partial S_k/\partial x$ and P_k satisfy the same equation (2.14) under our induction hypothesis. Then the uniqueness of the solution of (2.14) implies (4.14).

Since $S_m(x)$ for $m \geq 2$ decay when $x \to \infty$ (cf. (3.13)), the equality (4.15) immediately follows from (4.14). Then, the equality (2.18) shows

$$\frac{\partial}{\partial t}S_k(x,t) = \int_\infty^x\left[\hbar\frac{\partial}{\partial t}P\right]_{\hbar^k}dx = \int_\infty^x\frac{\partial}{\partial x}\left[\frac{\hbar P-p}{2(x-q)}\right]_{\hbar^k}dx = \left[\frac{1}{2(x-q)}\left(\hbar\frac{\partial S}{\partial x}-p\right)\right]_{\hbar^k}.$$

The last equality follows from the assumption (4.1) and the fact that $P_m(x,t)$'s decay when $x \to \infty$ for $m \geq 1$ (see Remark 2.8), and

$$\lim_{x\to\infty}\frac{P_0(x,t)}{(x-q_0)^2} = 0.$$

Thus we have proved (4.16). ∎

Since we have also already proved (4.2) for $m = k + 1$, we can prove (4.4) by the same discussion as the proof of Lemma 4.6 above. Then, finally we obtain

Lemma 4.7. *The equality* (4.5) *is true; namely, we have*

$$\frac{dF_{(k+2)/2}}{dt} = \sigma_{2k+2}. \tag{4.17}$$

Proof. It follows from the equality (4.11) (for the odd number $m = k+1$) and Lemma 4.5 that

$$2\frac{\partial S_0}{\partial x}\frac{\partial S_{k+2}}{\partial x} + \sum_{\substack{a+b=k+2\\a,b\geq 1}}\frac{\partial S_a}{\partial x}\frac{\partial S_b}{\partial x} + \frac{\partial^2 S_{k+1}}{\partial x^2} - 2\frac{\partial S_{k+1}}{\partial t} = 2\frac{dF_{(k+2)/2}}{dt} \tag{4.18}$$

holds. On the other hand, we know that P_{k+2} satisfies

$$2P_0P_{k+2} + \sum_{\substack{a+b=k+2\\a,b\geq 1}}P_aP_b + \frac{\partial P_{k+1}}{\partial x} - \left[\frac{\hbar}{x-q}\left(\hbar P-p\right)\right]_{\hbar^{k+2}} = 2\sigma_{k+2} \tag{4.19}$$

(cf. (2.14)). Under our assumption, comparing (4.18) and (4.19), we have

$$\frac{\partial S_0}{\partial x}\left(\frac{\partial S_{k+2}}{\partial x}-P_{k+2}\right) = \frac{dF_{(k+2)/2}}{dt} - \sigma_{k+2}. \tag{4.20}$$

Note that the right hand-side doesn't depend on x. Then, thanks to the fact

$$\left.\frac{\partial S_0}{\partial x}\right|_{x=q_0} = 0$$

and the holomorphicity of $S_m(x)$ and $P_m(x)$ at the double turning point $x = q_0$ (see Theorem 2.9), we have the desired equality (4.17) by substituting $x = q_0$ into (4.20). ∎

This completes the proof of (B) and Theorem 4.1. Thus we have proved Theorems 3.3 and 3.7.

Remark 4.8. Since the spectral curve (2.9) has only one branch point, we have

$$\int_v^x P_m(x',t)dx' = \int_\infty^x P_m(x',t)dx' \tag{4.21}$$

for all even $m \geq 2$. This implies that the WKB solution (3.4) defined by the topological recursion coincides with the WKB solution (2.20) constructed in Section 2.4. However, the above equality (4.21) may not hold for other Painlevé equations since their spectral curves have more branch points in general.

A   Alternative definition of the τ-function by Jimbo–Miwa–Ueno

There is another definition of τ-function (2.8) in terms of the formal solution (2.22) of the isomonodromy system.

Proposition A.1 ([20, Section 5]; see also [5, Section 4.2] and [7, Section 1.5]). *The τ-function satisfies*

$$\frac{d}{dt}\log\tau(t,\hbar) = -2\operatorname*{Res}_{x=\infty}\left(\frac{1}{\hbar}\frac{\partial T_\infty}{\partial t}(x,t)\mathcal{W}_1(x,t,\hbar)dx\right), \tag{A.1}$$

where

$$T_\infty(x,t) := \frac{4x^{5/2}}{5} + tx^{1/2}$$

(which is the divergent part of $\int^x P_0^{(+)}(x',t)dx'$ as $x \to \infty$), and

$$\mathcal{W}_1(x,t,\hbar) = \frac{\partial\psi_+}{\partial x}(x,t,\hbar)\tilde\psi_-(x,t,\hbar) - \frac{\partial\tilde\psi_+}{\partial x}(x,t,\hbar)\psi_-(x,t,\hbar).$$

Proof. It follows from the definition (2.22) of Ψ that

$$\mathcal{W}_1(x,t,\hbar) = P^{(+)}(x,t,\hbar) + \frac{A_{12}(x,t,\hbar)}{2\hbar P_{\mathrm{odd}}(x,t,\hbar)}\frac{\partial}{\partial x}\left(\frac{\hbar P^{(+)}(x,t,\hbar) - A_{11}(x,t,\hbar)}{A_{12}(x,t,\hbar)}\right).$$

Then, the asymptotics (2.15) of $P^{(\pm)}(x,t,\hbar)$ implies that

$$\mathcal{W}_1(x,t,\hbar) = \frac{2}{\hbar}x^{3/2} + \frac{t}{2\hbar}x^{-1/2} + \frac{\sigma(t,\hbar)}{2\hbar}x^{-3/2} + O(x^{-2})$$

holds when $x \to \infty$, and thus we have (A.1). ∎

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