

Superintegrable Oscillator and Kepler Systems on Spaces of Nonconstant Curvature via the Stäckel Transform^{*}

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Abstract. The Stäckel transform is applied to the geodesic motion on Euclidean space, through the harmonic oscillator and Kepler–Colomb potentials, in order to obtain maximally superintegrable classical systems on N -dimensional Riemannian spaces of nonconstant curvature. By one hand, the harmonic oscillator potential leads to two families of superintegrable systems which are interpreted as an intrinsic Kepler–Colomb system on a hyperbolic curved space and as the so-called Darboux III oscillator. On the other, the Kepler–Colomb potential gives rise to an oscillator system on a spherical curved space as well as to the Taub–NUT oscillator. Their integrals of motion are explicitly given. The role of the (flat/curved) Fradkin tensor and Laplace–Runge–Lenz N -vector for all of these Hamiltonians is highlighted throughout the paper. The corresponding quantum maximally superintegrable systems are also presented.

Key words: coupling constant metamorphosis; integrable systems; curvature; harmonic oscillator; Kepler–Coulomb; Fradkin tensor; Laplace–Runge–Lenz vector; Taub–NUT; Darboux surfaces

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1 Introduction

The coupling constant metamorphosis or Stäckel transform was formerly introduced in [1, 2] and further developed and applied to several classical and quantum Hamiltonian systems in [3, 4, 5, 6, 7]. This approach has proven to be a useful tool in order to relate different (super)integrable systems together with their associated symmetries and to deduce new integrable Hamiltonian systems starting from known ones.

For our purposes, the classical Stäckel transform can be briefly summarized as follows [3, 4]. Consider the conjugate coordinates and momenta $\mathbf{q}, \mathbf{p} \in \mathbb{R}^N$ with canonical Poisson bracket

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$\{q_i, p_j\} = \delta_{ij}$ and the notation:

$$\mathbf{q}^2 = \sum_{i=1}^N q_i^2, \quad \mathbf{p}^2 = \sum_{i=1}^N p_i^2, \quad |\mathbf{q}| = \sqrt{\mathbf{q}^2}.$$

Let H be an “initial” Hamiltonian, H_U an “intermediate” one and \tilde{H} the “final” system given by

$$H = \frac{\mathbf{p}^2}{\mu(\mathbf{q})} + V(\mathbf{q}), \quad H_U = \frac{\mathbf{p}^2}{\mu(\mathbf{q})} + U(\mathbf{q}), \quad \tilde{H} = \frac{H}{U} = \frac{\mathbf{p}^2}{\tilde{\mu}(\mathbf{q})} + \tilde{V}(\mathbf{q}), \quad (1.1)$$

such that $\tilde{\mu} = \mu U$ and $\tilde{V} = V/U$. Then, each *second-order* integral of motion (symmetry) S of H leads to a new one \tilde{S} corresponding to \tilde{H} through an “intermediate” symmetry S_U of H_U . In particular, if S and S_U are written as

$$S = \sum_{i,j=1}^N a^{ij}(\mathbf{q}) p_i p_j + W(\mathbf{q}) = S_0 + W(\mathbf{q}), \quad S_U = S_0 + W_U(\mathbf{q}), \quad (1.2)$$

then one gets a second-order symmetry of \tilde{H} in the form

$$\tilde{S} = S_0 - W_U \tilde{H}. \quad (1.3)$$

The aim of this paper is to apply the above procedure when the “initial” Hamiltonian H is the N -dimensional (ND) free Euclidean motion, and when H_U is either the isotropic harmonic oscillator or the Kepler–Coulomb (KC) Hamiltonian. It is well known that these three systems are maximally superintegrable (MS), that is, they are endowed with the maximum number of $2N - 1$ functionally independent integrals of motion (in these cases, all of them are quadratic in the momenta). These three systems and their MS property are briefly recalled in the next section, and we will see that the Stäckel transform gives rise to several MS systems \tilde{H} that are defined on Riemannian spaces of *nonconstant curvature*. Moreover, we will show that the new potential \tilde{V} can be interpreted as either an (intrinsic) oscillator or a KC potential on the corresponding curved manifold. In this way, by starting from the Euclidean Fradkin tensor [8], formerly studied by Demkov in [9], and the Laplace–Runge–Lenz (LRL) N -vector, the Stäckel transform provides for each case its curved analogue (see [10, 11, 12] and references therein).

In particular, we show in Section 3 that if H_U is chosen to be the harmonic oscillator we obtain two different final MS Hamiltonians, for which \tilde{H} is endowed with a curved Fradkin tensor; these are a KC system on a hyperbolic space of nonconstant curvature and the so-called Darboux III oscillator [13, 14, 15]. In Section 4 we take H_U as the (flat) KC Hamiltonian and the Stäckel transform leads to other two different MS systems together with their curved LRL N -vector; both of them are interpreted as intrinsic oscillators on curved Riemannian manifolds. Surprisingly enough, one of them is the ND generalization of the Taub–NUT oscillator [16, 17, 18, 19, 20, 21, 22, 23]. We stress that for some systems the dimension $N = 2$ is rather special as the underlying manifold remains flat, meanwhile for $N \geq 3$ such systems are defined on proper curved spaces (see Sections 3.1 and 4.1). This is similar to what happens in the classifications of 2D and 3D integrable systems on spaces of constant curvature (including the flat Euclidean one) [24, 25, 26, 27, 28, 29, 30, 31] which exhibit some differences according to the dimension and, in general, the 3D case is usually the cornerstone for the generalization of a given system to arbitrary dimension.

As a byproduct of this construction, the “growth” of the Fradkin tensor and the LRL vector from their Euclidean “seeds” to their curved counterparts can be highlighted from a global perspective. These results are comprised in Table 1 in the last section. Furthermore we also present in Table 2 the MS quantization for all of the above systems together with their “additional” quantum Fradkin/LRL symmetries.

2 Harmonic oscillator and Kepler potentials on Euclidean space

In order to fix a suitable common framework, we briefly recall the well-known basics of the superintegrability properties of the Hamiltonians describing free motion, harmonic oscillator and KC potentials on the ND Euclidean space \mathbf{E}^N .

As the “initial” Hamiltonian H (1.1) for the Stäckel procedure we consider the one defining the geodesic motion on \mathbf{E}^N plus a *relevant* constant α :

$$H = \frac{1}{2} \mathbf{p}^2 + \alpha. \quad (2.1)$$

Obviously this is a MS system, and there are many possibilities to choose its integrals of motion. We shall make use, throughout the paper, of the following results.

Proposition 1.

(i) *The Hamiltonian (2.1) is endowed with the following constants of motion.*

- $(2N - 3)$ angular momentum integrals ($m = 2, \dots, N$):

$$\begin{aligned} S^{(m)} &= \sum_{1 \leq i < j \leq m} (q_i p_j - q_j p_i)^2, & S_{(m)} &= \sum_{N-m < i < j \leq N} (q_i p_j - q_j p_i)^2, \\ S^{(N)} &= S_{(N)} \equiv \mathbf{L}^2, \end{aligned} \quad (2.2)$$

where \mathbf{L}^2 is the square of the total angular momentum.

- N^2 integrals which are the “seeds” of the Fradkin tensor ($i, j = 1, \dots, N$):

$$S_{ij} = p_i p_j \quad \text{such that} \quad \sum_{i=1}^N S_{ii} = 2(H - \alpha). \quad (2.3)$$

- N integrals which are the “seeds” of the components of the LRL vector ($i = 1, \dots, N$):

$$S_i = \sum_{k=1}^N p_k (q_k p_i - q_i p_k) \quad \text{such that} \quad \sum_{i=1}^N S_i^2 = 2\mathbf{L}^2(H - \alpha). \quad (2.4)$$

- (ii) *Each of the three sets $\{H, S^{(m)}\}$, $\{H, S_{(m)}\}$ ($m = 2, \dots, N$) and $\{S_{ii}\}$ ($i = 1, \dots, N$) is formed by N functionally independent functions in involution.*
- (iii) *Both sets $\{H, S^{(m)}, S_{(m)}, S_{ii}\}$ and $\{H, S^{(m)}, S_{(m)}, S_i\}$ ($m = 2, \dots, N$ and a fixed index i) are constituted by $2N - 1$ functionally independent functions.*

As “intermediate” Hamiltonians H_U (1.1) we consider either the harmonic oscillator or the KC system. Since both of them are central potentials, the angular momentum integrals (2.2) are valid for both cases, that is, $S_U^{(m)} \equiv S^{(m)}$ and $S_{U,(m)} \equiv S_{(m)}$ in (1.2). We recall that, in fact, the spherical symmetry of a central potential on \mathbf{E}^N directly provides such $(2N - 3)$ independent angular momentum integrals, so they characterize a quasi-MS system [32, 33]. However what makes rather special the harmonic oscillator and KC systems is the existence of one more independent integral, which is extracted from a new set of integrals that ensure their MS property and is related to the fact that these two systems are the only ones fulfilling the classical Bertrand’s theorem [34]. In this respect, each of the sets of integrals (2.3) and (2.4) gives rise to one known set of additional constants for the harmonic oscillator and KC system, respectively.

Proposition 2.

(i) *The harmonic oscillator Hamiltonian defined by*

$$H_U = \frac{1}{2}\mathbf{p}^2 + \beta\mathbf{q}^2 + \gamma \quad (2.5)$$

has the $(2N-3)$ angular momentum integrals (2.2) together with N^2 additional ones given by the components of the ND Fradkin tensor $(i, j = 1, \dots, N)$:

$$S_{U,ij} = p_i p_j + 2\beta q_i q_j \quad \text{such that} \quad \sum_{i=1}^N S_{U,ii} = 2(H_U - \gamma).$$

(ii) *Each of the three sets $\{H_U, S^{(m)}\}$, $\{H_U, S_{(m)}\}$ ($m = 2, \dots, N$) and $\{S_{U,ii}\}$ ($i = 1, \dots, N$) is formed by N functionally independent functions in involution.*

(iii) *The set $\{H_U, S^{(m)}, S_{(m)}, S_{U,ii}\}$ ($m = 2, \dots, N$ and a fixed index i) provides $2N-1$ functionally independent functions.*

Proposition 3.

(i) *The KC Hamiltonian given by*

$$H_U = \frac{1}{2}\mathbf{p}^2 + \frac{\delta}{|\mathbf{q}|} + \xi \quad (2.6)$$

has the $(2N-3)$ angular momentum integrals (2.2) together with the N components of the LRL vector $(i = 1, \dots, N)$:

$$S_{U,i} = \sum_{k=1}^N p_k (q_k p_i - q_i p_k) - \frac{\delta q_i}{|\mathbf{q}|} \quad \text{such that} \quad \sum_{i=1}^N S_{U,i}^2 = 2\mathbf{L}^2(H_U - \xi) + \delta^2.$$

(ii) *Each of the two sets $\{H_U, S^{(m)}\}$ and $\{H_U, S_{(m)}\}$ ($m = 2, \dots, N$) is formed by N functionally independent functions in involution.*

(iii) *The set $\{H_U, S^{(m)}, S_{(m)}, S_{U,i}\}$ ($m = 2, \dots, N$ and a fixed index i) is constituted by $2N-1$ functionally independent functions.*

In the two next sections we apply the Stäckel transform to each of these two MS systems. Notice that the proper isotropic harmonic oscillator arises whenever $\beta = \omega^2/2$ with frequency ω and $\gamma = 0$, while the Kepler one corresponds to set $\delta = -K$ and $\xi = 0$. We remark that in this approach the constant α is essential in order to obtain a curved potential while the others β , γ , δ and ξ enter in both the kinetic and the potential term giving rise to MS oscillator/KC potentials on Riemannian spaces of nonconstant curvature, so that they can be regarded as classical “deformation parameters”.

3 Superintegrable systems from harmonic oscillator potential

If we consider as the initial Hamiltonian H the free system (2.1) and as the intermediate one the harmonic oscillator H_U (2.5), then we obtain the final Hamiltonian \tilde{H}

$$\tilde{H} = \frac{\mathbf{p}^2}{2(\gamma + \beta\mathbf{q}^2)} + \frac{\alpha}{\gamma + \beta\mathbf{q}^2}, \quad (3.1)$$

so that the relations (1.1) read as

$$\mu = 2, \quad V = \alpha, \quad U = \gamma + \beta \mathbf{q}^2, \quad \tilde{\mu} = 2(\gamma + \beta \mathbf{q}^2), \quad \tilde{V} = \frac{\alpha}{\gamma + \beta \mathbf{q}^2}.$$

As far as the symmetries $S = S_0 + W$ (1.2) are concerned, we find from Proposition 1 that

$$\begin{aligned} S_0^{(m)} &= S^{(m)}, & W^{(m)} &= 0, & S_{0,(m)} &= S_{(m)}, \\ W_{(m)} &= 0, & S_{0,ij} &= S_{ij}, & W_{ij} &= 0, \end{aligned}$$

while from Proposition 2 we obtain the elements W_U for the decompositions of $S_U = S_0 + W_U$,

$$W_U^{(m)} = 0, \quad W_{U,(m)} = 0, \quad W_{U,ij} = 2\beta q_i q_j,$$

where $m = 2, \dots, N$ and $i, j = 1, \dots, N$.

Consequently, the Hamiltonian \tilde{H} (3.1) is Stäckel equivalent to the free Euclidean motion, through the harmonic oscillator potential, and its integrals of motion \tilde{S} come from (1.3) and turn out to be

$$\tilde{S}^{(m)} = S^{(m)}, \quad \tilde{S}_{(m)} = S_{(m)}, \quad \tilde{S}_{ij} = p_i p_j - 2\beta q_i q_j \tilde{H}(\mathbf{q}, \mathbf{p}). \quad (3.2)$$

Thus we have obtained the $(2N - 3)$ angular momentum integrals $S^{(m)}$ and $S_{(m)}$, together with N^2 ones, \tilde{S}_{ij} , which form a *curved* Fradkin tensor. The quasi-MS property of \tilde{H} is ensured by the preservation of the $(2N - 3)$ angular momentum integrals, that is, each of the two sets $\{\tilde{H}, S^{(m)}\}$, $\{\tilde{H}, S_{(m)}\}$ ($m = 2, \dots, N$) is formed by N functionally independent functions in involution. Hence, from now on, we assume this fact and only pay attention to the additional constants \tilde{S}_{ij} which characterize (3.1) as a MS system.

In order to perform a preliminary geometrical analysis of \tilde{H} we recall that, in general, any Hamiltonian of the form

$$\mathcal{H} = \frac{\mathbf{p}^2}{2f(|\mathbf{q}|)^2} + \mathcal{V}(|\mathbf{q}|)$$

can be interpreted as describing a particle (with unit mass) on an ND spherically symmetric space \mathcal{M} under the action of the central potential $\mathcal{V}(|\mathbf{q}|)$ [14]. The metric and scalar curvature of \mathcal{M} are given by [35]

$$\begin{aligned} ds^2 &= f(|\mathbf{q}|)^2 d\mathbf{q}^2, \\ R &= -(N-1) \left(\frac{(N-4)f'(r)^2 + f(r)(2f''(r) + 2(N-1)r^{-1}f'(r))}{f(r)^4} \right), \end{aligned} \quad (3.3)$$

where we have introduced the radial coordinate $r = |\mathbf{q}|$. For general results on 2D and 3D (super)integrable systems on conformally flat spaces we refer to [36, 37, 38].

Furthermore, the conformal factor $f(|\mathbf{q}|) = f(r)$ is directly related, under the following prescription, with the *intrinsic KC and oscillator potentials* on \mathcal{M} :

$$\mathcal{U}_{\text{KC}}(r) := \int^r \frac{dr'}{r'^2 f(r')}, \quad \mathcal{U}_{\text{O}}(r) := \frac{1}{\mathcal{U}_{\text{KC}}(r)^2}, \quad (3.4)$$

that was introduced in [14] up to additive and multiplicative constants.

With these ideas in mind, we now analyze the specific systems defined by \tilde{H} (3.1) according to the values of the parameters β and γ . Notice that α is the constant which governs the potential, so to setting $\alpha = 0$ leads to geodesic motion on \mathcal{M} , and that β must be always different from zero, since otherwise \tilde{H} is again the initial H . Therefore we are led to consider two different cases with generic α : (i) $\beta \neq 0, \gamma = 0$; and (ii) $\beta \neq 0, \gamma \neq 0$.

3.1 The case with $\beta \neq 0$ and $\gamma = 0$: a curved hyperbolic KC system

If $\gamma = 0$ we scale \tilde{H} to deal with the Hamiltonian

$$\mathcal{H}_{\text{KC}} = \beta \tilde{H} = \frac{\mathbf{p}^2}{2\mathbf{q}^2} + \frac{\alpha}{\mathbf{q}^2}. \quad (3.5)$$

Then, $f(|\mathbf{q}|) = |\mathbf{q}| = r$ so the metric and scalar curvature (3.3) on \mathcal{M} reduces to

$$ds^2 = \mathbf{q}^2 d\mathbf{q}^2, \quad R = -\frac{3(N-1)(N-2)}{r^4}, \quad (3.6)$$

while the intrinsic KC and oscillator potentials (3.4) on this curved space would be

$$\mathcal{U}_{\text{KC}}(r) = -\frac{1}{2r^2}, \quad \mathcal{U}_{\text{O}}(r) = 4r^4.$$

The latter result shows that \mathcal{H}_{KC} (3.5) always defines an intrinsic KC potential on the space \mathcal{M} . Nevertheless the curvature (3.6) vanishes for $N = 2$, while the space is of nonconstant curvature for $N \geq 3$. Therefore, for $N = 2$ the Hamiltonian must correspond to the usual KC system on the Euclidean space. This fact can be explicitly proven by applying to \mathcal{H}_{KC} (3.5) the Kustaanheimo–Stiefel canonical transformation defined by [39, 40]

$$\tilde{q}_1 = \frac{1}{2}(q_1^2 - q_2^2), \quad \tilde{p}_1 = \frac{p_1 q_1 - p_2 q_2}{q_1^2 + q_2^2}, \quad \tilde{q}_2 = q_1 q_2, \quad \tilde{p}_2 = \frac{p_2 q_1 + p_1 q_2}{q_1^2 + q_2^2}, \quad (3.7)$$

so with canonical Poisson bracket $\{\tilde{q}_i, \tilde{p}_j\} = \delta_{ij}$. In this way we recover the 2D KC Hamiltonian

$$\mathcal{H}_{\text{KC}} = \frac{p_1^2 + p_2^2}{2(q_1^2 + q_2^2)} + \frac{\alpha}{q_1^2 + q_2^2} = \frac{1}{2}(\tilde{p}_1^2 + \tilde{p}_2^2) + \frac{\alpha}{2\sqrt{\tilde{q}_1^2 + \tilde{q}_2^2}}$$

and the *five* symmetries (3.2) reduce to *three* integrals of motion, namely

$$\begin{aligned} S^{(2)} = S_{(2)} = \mathbf{L}^2 &= (q_1 p_2 - q_2 p_1)^2 = 4(\tilde{q}_1 \tilde{p}_2 - \tilde{q}_2 \tilde{p}_1)^2, \\ \tilde{S}_{11} = p_1^2 - 2q_1^2 \mathcal{H}_{\text{KC}} &= 2\tilde{p}_2(\tilde{q}_2 \tilde{p}_1 - \tilde{q}_1 \tilde{p}_2) - \frac{\alpha \tilde{q}_1}{\sqrt{\tilde{q}_1^2 + \tilde{q}_2^2}} - \alpha, \quad \tilde{S}_{22} = -\tilde{S}_{11} - 2\alpha, \\ \tilde{S}_{12} = \tilde{S}_{21} = p_1 p_2 - 2q_1 q_2 \mathcal{H}_{\text{KC}} &= 2\tilde{p}_1(\tilde{q}_1 \tilde{p}_2 - \tilde{q}_2 \tilde{p}_1) - \frac{\alpha \tilde{q}_2}{\sqrt{\tilde{q}_1^2 + \tilde{q}_2^2}}. \end{aligned}$$

Hence, by taking into account Proposition 3 for $N = 2$, we find that, under the above canonical transformation, the only angular momentum integral $S^{(2)}$ is kept, while the four constants coming from the 2D Fradkin tensor reduce to the two components of the LRL vector: $(\tilde{S}_{11}, \tilde{S}_{22}) \rightarrow \tilde{S}_1$ and $(\tilde{S}_{12}, \tilde{S}_{21}) \rightarrow \tilde{S}_2$.

Consequently, a proper curved KC system arises whenever $N \geq 3$ and its full integrability properties can be summarized as follows.

Proposition 4.

- (i) For $N \geq 3$, the Hamiltonian \mathcal{H}_{KC} (3.5) determines an intrinsic KC system on the hyperbolic space of nonconstant curvature (3.6).
- (ii) \mathcal{H}_{KC} is endowed with the $(2N - 3)$ angular momentum integrals (2.2) together with N^2 ones which are the components of an ND curved Fradkin tensor $(i, j = 1, \dots, N)$:

$$\tilde{S}_{ij} = p_i p_j - 2q_i q_j \mathcal{H}_{\text{KC}} \quad \text{such that} \quad \sum_{i=1}^N \tilde{S}_{ii} = -2\alpha \quad \text{and} \quad \{\tilde{S}_{ii}, \tilde{S}_{jj}\} = 0.$$

- (iii) The set $\{\mathcal{H}_{\text{KC}}, S^{(m)}, S_{(m)}, \tilde{S}_{ii}\}$ ($m = 2, \dots, N$ and a fixed index i) is formed by $2N - 1$ functionally independent functions.

3.2 The case with $\beta \neq 0$ and $\gamma \neq 0$: the Darboux III oscillator

If both $\beta, \gamma \neq 0$, we can write the Hamiltonian (3.1) in the form

$$\mathcal{H}_\lambda = \gamma \tilde{H} - \alpha = \frac{\mathbf{p}^2}{2(1 + \lambda \mathbf{q}^2)} - \frac{\lambda \alpha \mathbf{q}^2}{1 + \lambda \mathbf{q}^2}, \quad \lambda = \beta/\gamma. \quad (3.8)$$

Then the metric and scalar curvature (3.3) on the corresponding manifold \mathcal{M} are given by

$$ds^2 = (1 + \lambda \mathbf{q}^2) d\mathbf{q}^2, \quad R = -\lambda \frac{(N-1)(2N + 3(N-2)\lambda r^2)}{(1 + \lambda r^2)^3}, \quad (3.9)$$

and the intrinsic potentials (3.4) read

$$\mathcal{U}_{\text{KC}}(r) = -\frac{\sqrt{1 + \lambda r^2}}{r}, \quad \mathcal{U}_{\text{O}}(r) = \frac{r^2}{1 + \lambda r^2}. \quad (3.10)$$

In this way, we recover the ND spherically symmetric generalization of the Darboux surface of type III [41, 42, 43, 44] introduced in [14, 35]. Notice that the domain of $r = |\mathbf{q}|$ and the type of the underlying curved manifold depends on the sign of λ [15]:

$$\begin{aligned} \lambda > 0: \quad R(0) &= -2\lambda N(N-1), \quad r \in [0, \infty); \\ \lambda < 0: \quad R(0) &= 2|\lambda|N(N-1), \quad r \in [0, 1/\sqrt{|\lambda|}), \end{aligned}$$

where we have written the value of the scalar curvature (3.9) at the origin $r = 0$. We stress that $R(0)$ coincides either with the scalar curvature of the ND *hyperbolic space* with negative constant sectional curvature equal to -2λ for $\lambda > 0$, or with that corresponding to the ND *spherical space* with sectional curvature equal to $2|\lambda|$ for $\lambda < 0$.

By taking into account the above geometrical considerations and expressions (3.10), we find that \mathcal{H}_λ comprises both an intrinsic hyperbolic oscillator potential and a spherical one on \mathcal{M} according to the sign of λ . Strictly speaking the curved oscillator potentials arise by introducing the frequency $\omega^2 = -2\lambda\alpha$ and, in that form, the limit $\lambda \rightarrow 0$ gives rise to the harmonic oscillator on \mathbf{E}^N , so λ behaves as a classical deformation parameter governing the curvature and the potential. The MS property of \mathcal{H}_λ is then characterized by [13, 15]:

Proposition 5.

- (i) The Hamiltonian \mathcal{H}_λ (3.8) defines an intrinsic curved hyperbolic Darboux oscillator for $\lambda > 0$ and $r \in [0, \infty)$ and a curved spherical Darboux one for $\lambda < 0$ and $r \in [0, 1/\sqrt{|\lambda|})$.
- (ii) Besides the $(2N-3)$ angular momentum integrals (2.2), \mathcal{H}_λ Poisson-commutes with the N^2 components of the ND curved Fradkin tensor ($i, j = 1, \dots, N$) given by

$$\tilde{S}_{ij} = p_i p_j - 2\lambda q_i q_j (\mathcal{H}_\lambda + \alpha) \quad \text{such that} \quad \sum_{i=1}^N \tilde{S}_{ii} = 2\mathcal{H}_\lambda \quad \text{and} \quad \{\tilde{S}_{ii}, \tilde{S}_{jj}\} = 0.$$

- (iii) The set $\{\mathcal{H}_\lambda, S^{(m)}, S_{(m)}, \tilde{S}_{ii}\}$ ($m = 2, \dots, N$ and a fixed index i) is formed by $2N-1$ functionally independent functions.

4 Superintegrable systems from the Kepler–Coulomb potential

In this case we consider the initial Hamiltonian H (2.1) and the KC system H_U (2.6) for the intermediate one; this provides the final Hamiltonian \tilde{H} (1.1)

$$\tilde{H} = \frac{|\mathbf{q}|\mathbf{p}^2}{2(\delta + \xi|\mathbf{q}|)} + \frac{\alpha|\mathbf{q}|}{\delta + \xi|\mathbf{q}|}, \quad (4.1)$$

where

$$\mu = 2, \quad V = \alpha, \quad U = \frac{\delta + \xi|\mathbf{q}|}{|\mathbf{q}|}, \quad \tilde{\mu} = \frac{2(\delta + \xi|\mathbf{q}|)}{|\mathbf{q}|}, \quad \tilde{V} = \frac{\alpha|\mathbf{q}|}{\delta + \xi|\mathbf{q}|}.$$

From Proposition 1 we obtain the decomposition of the symmetries $S = S_0 + W$ (2.2) and (2.4) ($m = 2, \dots, N$ and $i = 1, \dots, N$):

$$S_0^{(m)} = S^{(m)}, \quad W^{(m)} = 0, \quad S_{0,(m)} = S_{(m)}, \quad W_{(m)} = 0, \quad S_{0,i} = S_i, \quad W_i = 0,$$

while from Proposition 3 we find the one corresponding to $S_U = S_0 + W_U$ (1.2)

$$W_U^{(m)} = 0, \quad W_{U,(m)} = 0, \quad W_{U,i} = -\frac{\delta q_i}{|\mathbf{q}|}.$$

Therefore, the Hamiltonian \tilde{H} (4.1) is Stäckel equivalent to the free Euclidean motion, through the KC potential, and its integrals of motion \tilde{S} (1.3) are given by

$$\tilde{S}^{(m)} = S^{(m)}, \quad \tilde{S}_{(m)} = S_{(m)}, \quad \tilde{S}_i = \sum_{k=1}^N p_k (q_k p_i - q_i p_k) + \frac{\delta q_i}{|\mathbf{q}|} \tilde{H}(\mathbf{q}, \mathbf{p}). \quad (4.2)$$

Hence, \tilde{H} (4.1) is endowed with the $(2N - 3)$ angular momentum integrals (2.2) together with a *curved* LRL N -vector with components \tilde{S}_i .

Notice that the parameter δ cannot vanish in order to avoid the initial system H . Thus, similarly to the previous section, we study two systems covered by \tilde{H} (4.1): (i) $\delta \neq 0$, $\xi = 0$; and (ii) $\delta \neq 0$, $\xi \neq 0$.

4.1 The case with $\delta \neq 0$ and $\xi = 0$: a curved spherical oscillator system

If $\xi = 0$ we have the Hamiltonian system

$$\mathcal{H}_O = \delta \tilde{H} = \frac{1}{2} |\mathbf{q}| \mathbf{p}^2 + \alpha |\mathbf{q}|. \quad (4.3)$$

We stress that for $N = 3$ this system was early considered in [2]. The metric and scalar curvature (3.3) give

$$ds^2 = \frac{1}{|\mathbf{q}|} d\mathbf{q}^2, \quad R = \frac{3(N-1)(N-2)}{4r}, \quad (4.4)$$

and the intrinsic KC and oscillator potentials (3.4) turn out to be

$$\mathcal{U}_{\text{KC}}(r) = -\frac{2}{\sqrt{r}}, \quad \mathcal{U}_O(r) = \frac{r}{4}.$$

Hence \mathcal{H}_O (4.3) determines an intrinsic oscillator potential on \mathcal{M} . However, for $N = 2$ the curvature is equal to zero, so this case should actually be the 2D harmonic oscillator. This can be proven by means of the canonical transformation

$$\begin{aligned} \tilde{q}_1 &= \frac{q_2}{\left(\sqrt{q_1^2 + q_2^2} - q_1\right)^{1/2}}, & \tilde{p}_1 &= \frac{(p_1 q_2 - 2p_2 q_1) \left(\sqrt{q_1^2 + q_2^2} - q_1\right) + p_2 q_2^2}{\left(\sqrt{q_1^2 + q_2^2} - q_1\right)^{3/2}}, \\ \tilde{q}_2 &= \left(\sqrt{q_1^2 + q_2^2} - q_1\right)^{1/2}, & \tilde{p}_2 &= \frac{p_2 q_2 - p_1 \left(\sqrt{q_1^2 + q_2^2} - q_1\right)}{\left(\sqrt{q_1^2 + q_2^2} - q_1\right)^{1/2}}, \end{aligned}$$

which is just the inverse of the canonical transformation (3.7); this yields the expected system

$$\mathcal{H}_O = \frac{1}{2}\sqrt{q_1^2 + q_2^2}(p_1^2 + p_2^2) + \alpha\sqrt{q_1^2 + q_2^2} = \frac{1}{4}(\tilde{p}_1^2 + \tilde{p}_2^2) + \frac{1}{2}\alpha(\tilde{q}_1^2 + \tilde{q}_2^2).$$

The canonical transformation of the *three* symmetries (4.2) gives

$$\begin{aligned} S^{(2)} = S_{(2)} = \mathbf{L}^2 &= (q_1 p_2 - q_2 p_1)^2 = \frac{1}{4}(\tilde{q}_1 \tilde{p}_2 - \tilde{q}_2 \tilde{p}_1)^2, \\ \tilde{S}_1 &= p_2(q_2 p_1 - q_1 p_2) + \frac{q_1}{\sqrt{q_1^2 + q_2^2}} \mathcal{H}_O = \frac{1}{4}(\tilde{p}_1^2 - \tilde{p}_2^2) + \frac{1}{2}\alpha(\tilde{q}_1^2 - \tilde{q}_2^2), \\ \tilde{S}_2 &= p_1(q_1 p_2 - q_2 p_1) + \frac{q_2}{\sqrt{q_1^2 + q_2^2}} \mathcal{H}_O = \frac{1}{2}\tilde{p}_1 \tilde{p}_2 + \alpha \tilde{q}_1 \tilde{q}_2. \end{aligned}$$

Then the four components of the 2D Euclidean Fradkin tensor \tilde{S}_{ij} are recovered, in the new canonical variables, from the set of constants $(\mathcal{H}_O, \tilde{S}_1, \tilde{S}_2)$ by setting

$$\begin{aligned} \tilde{S}_{11} &= 2(\mathcal{H}_O + \tilde{S}_1) = \tilde{p}_1^2 + 2\alpha\tilde{q}_1^2, & \tilde{S}_{22} &= 2(\mathcal{H}_O - \tilde{S}_1) = \tilde{p}_2^2 + 2\alpha\tilde{q}_2^2, \\ \tilde{S}_{12} &= \tilde{S}_{21} = 2\tilde{S}_2 = \tilde{p}_1 \tilde{p}_2 + 2\alpha\tilde{q}_1 \tilde{q}_2. \end{aligned}$$

Therefore the proper curved system arises whenever $N \geq 3$, which yields the following

Proposition 6.

- (i) For $N \geq 3$, the Hamiltonian \mathcal{H}_O (4.3) defines an intrinsic oscillator potential on the spherical space of nonconstant curvature (4.4).
- (ii) \mathcal{H}_O Poisson-commutes with the $(2N - 3)$ angular momentum integrals (2.2) and with the components of the LRL N -vector given by $(i = 1, \dots, N)$:

$$\tilde{S}_i = \sum_{k=1}^N p_k (q_k p_i - q_i p_k) + \frac{q_i}{|\mathbf{q}|} \mathcal{H}_O \quad \text{such that} \quad \sum_{i=1}^N \tilde{S}_i^2 = \mathcal{H}_O^2 - 2\alpha\mathbf{L}^2.$$

- (iii) The set $\{\mathcal{H}_O, S^{(m)}, S_{(m)}, \tilde{S}_i\}$ ($m = 2, \dots, N$ and a fixed index i) is formed by $2N - 1$ functionally independent functions.

4.2 The case with $\delta \neq 0$ and $\xi \neq 0$: the Taub-NUT oscillator

We scale the Hamiltonian (4.1) as

$$\mathcal{H}_\eta = \xi \tilde{H} = \frac{|\mathbf{q}|\mathbf{p}^2}{2(\eta + |\mathbf{q}|)} + \frac{\alpha|\mathbf{q}|}{\eta + |\mathbf{q}|}, \quad \eta = \delta/\xi. \quad (4.5)$$

Notice that the limit $\eta \rightarrow 0$ reduces to the free Hamiltonian in Euclidean space. The metric and scalar curvature (3.3) on the corresponding manifold \mathcal{M} turn out to be

$$ds^2 = \frac{\eta + |\mathbf{q}|}{|\mathbf{q}|} d\mathbf{q}^2, \quad R = \eta(N - 1) \frac{(4(N - 3)r + 3(N - 2)\eta)}{4r(\eta + r)^3}, \quad (4.6)$$

so that the domain of $r = |\mathbf{q}|$ in \mathcal{M} depends on the sign of η :

$$\eta > 0: \quad r \in (0, \infty); \quad \eta < 0: \quad r \in [|\eta|, \infty). \quad (4.7)$$

The intrinsic potentials (3.4) are given by

$$\mathcal{U}_{\text{KC}}(r) = -\frac{2}{\eta} \sqrt{\frac{\eta+r}{r}}, \quad \mathcal{U}_{\text{O}}(r) = \frac{\eta^2 r}{4(\eta+r)}. \quad (4.8)$$

Consequently, \mathcal{H}_η defines two intrinsic oscillators, which are different systems according to (4.7).

It is worth comparing (4.5) with the Taub-NUT system [16, 17, 18, 19, 20, 21, 22, 23] which can be written as [14]:

$$\begin{aligned} \mathcal{H}_{\text{Taub-NUT}} &= \frac{\mathbf{p}^2}{2(1+4m/|\mathbf{q}|)} + \frac{\mu^2}{2(4m)^2} \left(1 + \frac{4m}{|\mathbf{q}|}\right) \\ &= \frac{|\mathbf{q}|\mathbf{p}^2}{2(4m+|\mathbf{q}|)} + \frac{\mu^2|\mathbf{q}|/(4m)^2}{2(4m+|\mathbf{q}|)} + \frac{\mu^2}{2|\mathbf{q}|(4m+|\mathbf{q}|)} + \frac{\mu^2/(4m)}{4m+|\mathbf{q}|}. \end{aligned} \quad (4.9)$$

The relationship with \mathcal{H}_η is established by setting

$$\eta = 4m, \quad \alpha = -\frac{\mu^2}{2(4m)^2},$$

which gives

$$\mathcal{H}_{\eta=4m} + \frac{\mu^2}{(4m)^2} = \frac{|\mathbf{q}|\mathbf{p}^2}{2(4m+|\mathbf{q}|)} + \frac{\mu^2|\mathbf{q}|/(4m)^2}{2(4m+|\mathbf{q}|)} + \frac{\mu^2/(4m)}{4m+|\mathbf{q}|},$$

so that we recover three terms in the ‘‘expanded’’ expression for $\mathcal{H}_{\text{Taub-NUT}}$ (4.9); namely, the kinetic term defining the geodesic motion on the Taub-NUT space (4.6), the intrinsic oscillator potential (4.8) and the one which comes out by adding a constant to the oscillator potential. There is one missing term, the third one in (4.9), which corresponds to the Dirac monopole. However we notice that this can be derived from the angular momentum by introducing hyper-spherical coordinates in the form [14]

$$\mathbf{p}^2 = p_r^2 + r^{-2}\mathbf{L}^2 \quad \text{and next} \quad \mathbf{L}^2 \rightarrow \mathbf{L}^2 + \mu^2.$$

From this viewpoint, \mathcal{H}_η can be regarded as an ND MS generalization of the Taub-NUT system which is recovered for $\eta > 0$, being the case with $\eta < 0$ a different physical oscillator potential.

The symmetry properties for \mathcal{H}_η are summarized in

Proposition 7.

- (i) The Hamiltonian \mathcal{H}_η (4.5) characterizes two intrinsic oscillator potentials on the corresponding Riemannian space of nonconstant curvature (4.6) according to (4.7).
- (ii) \mathcal{H}_η is endowed with the $(2N-3)$ angular momentum integrals (2.2) together with the components of the curved LRL N -vector given by $(i=1, \dots, N)$:

$$\tilde{S}_i = \sum_{k=1}^N p_k (q_k p_i - q_i p_k) + \eta \frac{q_i}{|\mathbf{q}|} \mathcal{H}_\eta \quad \text{such that} \quad \sum_{i=1}^N \tilde{S}_i^2 = 2\mathbf{L}^2(\mathcal{H}_\eta - \alpha) + \eta^2 \mathcal{H}_\eta^2.$$

- (iii) The set $\{\mathcal{H}_\eta, S^{(m)}, S_{(m)}, \tilde{S}_i\}$ ($m=2, \dots, N$ and a fixed index i) is formed by $2N-1$ functionally independent functions.

Table 1. Maximally superintegrable classical oscillator and KC Hamiltonians in N dimensions.

<ul style="list-style-type: none"> • Geodesic motion on Euclidean space 	
$H = \frac{1}{2} \mathbf{p}^2 + \alpha$	
* Common $(2N - 3)$ angular momentum integrals of motion	
$S^{(m)} = \sum_{1 \leq i < j \leq m} (q_i p_j - q_j p_i)^2$,	$S_{(m)} = \sum_{N-m < i < j \leq N} (q_i p_j - q_j p_i)^2$,
$S^{(N)} = S_{(N)} \equiv \mathbf{L}^2$	
* “Seeds” of the ND Fradkin tensor	* “Seeds” of the LRL N -vector
$S_{ij} = p_i p_j$	$S_i = \sum_{k=1}^N p_k (q_k p_i - q_i p_k)$
• Harmonic oscillator	• Euclidean KC
$H_U = \frac{1}{2} \mathbf{p}^2 + \beta \mathbf{q}^2 + \gamma$	$H_U = \frac{1}{2} \mathbf{p}^2 + \frac{\delta}{ \mathbf{q} } + \xi$
* Flat ND Fradkin tensor	* Flat LRL N -vector
$S_{U,ij} = p_i p_j + 2\beta q_i q_j$	$S_{U,i} = \sum_{k=1}^N p_k (q_k p_i - q_i p_k) - \delta \frac{q_i}{ \mathbf{q} }$
• Curved hyperbolic KC ($N \geq 3$)	• Curved spherical oscillator ($N \geq 3$)
$\mathcal{H}_{\text{KC}} = \frac{\mathbf{p}^2}{2\mathbf{q}^2} + \frac{\alpha}{\mathbf{q}^2}$	$\mathcal{H}_{\text{O}} = \frac{1}{2} \mathbf{q} \mathbf{p}^2 + \alpha \mathbf{q} $
* Curved ND Fradkin tensor	* Curved LRL N -vector
$\tilde{S}_{ij} = p_i p_j - 2q_i q_j \mathcal{H}_{\text{KC}}$	$\tilde{S}_i = \sum_{k=1}^N p_k (q_k p_i - q_i p_k) + \frac{q_i}{ \mathbf{q} } \mathcal{H}_{\text{O}}$
• Darboux III oscillator	• Taub-NUT oscillator
$\mathcal{H}_\lambda = \frac{\mathbf{p}^2}{2(1 + \lambda \mathbf{q}^2)} - \frac{\lambda \alpha \mathbf{q}^2}{1 + \lambda \mathbf{q}^2}$	$\mathcal{H}_\eta = \frac{ \mathbf{q} \mathbf{p}^2}{2(\eta + \mathbf{q})} + \frac{\alpha \mathbf{q} }{\eta + \mathbf{q} }$
* Curved ND Fradkin tensor	* Curved LRL N -vector
$\tilde{S}_{ij} = p_i p_j - 2\lambda q_i q_j (\mathcal{H}_\lambda + \alpha)$	$\tilde{S}_i = \sum_{k=1}^N p_k (q_k p_i - q_i p_k) + \eta \frac{q_i}{ \mathbf{q} } \mathcal{H}_\eta$

5 Outlook and superintegrable quantization

So far we have obtained and interpreted four MS classical Hamiltonian systems on Riemannian spaces of nonconstant curvature by starting from free motion on \mathbf{E}^N and applying the Stäckel transform through the harmonic oscillator and KC potentials. The main results here obtained are displayed in Table 1 where the transition from the “seeds” of the Fradkin tensor and the LRL vector up to their curved analogues is laid bare by reading the table through its two columns. Recall, however, that the Darboux III and the Taub-NUT oscillators give rise, each of them, to two different physical systems according to the sign of the parameters λ and η , respectively.

Some related comments are in order. All the Hamiltonians shown in Table 1 are constructed on spherically symmetric spaces so that they are endowed with an $\mathfrak{so}(N)$ Lie–Poisson symmetry. In particular, let us consider the generators of rotations $J_{ij} = q_i p_j - q_j p_i$ with $i < j$ and $i, j = 1, \dots, N$ which span the $\mathfrak{so}(N)$ Lie–Poisson algebra

$$\{J_{ij}, J_{ik}\} = J_{jk}, \quad \{J_{ij}, J_{jk}\} = -J_{ik}, \quad \{J_{ik}, J_{jk}\} = J_{ij}, \quad i < j < k.$$

Then the “common” $(2N - 3)$ angular momentum integrals $S^{(m)}$ and $S_{(m)}$ (2.2) can be written

Table 2. Maximally superintegrable quantum oscillator and KC Hamiltonians in N dimensions.

* Common $(2N - 3)$ quantum angular momentum operators	
$\hat{S}^{(m)} = \sum_{1 \leq i < j \leq m} (\hat{q}_i \hat{p}_j - \hat{q}_j \hat{p}_i)^2,$	$\hat{S}_{(m)} = \sum_{N-m < i < j \leq N} (\hat{q}_i \hat{p}_j - \hat{q}_j \hat{p}_i)^2,$
$\hat{S}^{(N)} = \hat{S}_{(N)} \equiv \hat{\mathbf{L}}^2$	
• Quantum hyperbolic KC ($N \geq 3$)	• Quantum spherical oscillator ($N \geq 3$)
$\hat{\mathcal{H}}_{\text{KC}} = \frac{1}{2\hat{\mathbf{q}}^2} \hat{\mathbf{p}}^2 + \frac{\alpha}{\hat{\mathbf{q}}^2}$	$\hat{\mathcal{H}}_{\text{O}} = \frac{1}{2} \hat{\mathbf{q}} \hat{\mathbf{p}}^2 + \alpha \hat{\mathbf{q}} $
* Quantum ND Fradkin tensor	* Quantum LRL N -vector
$\hat{S}_{ij} = \hat{p}_i \hat{p}_j - 2\hat{q}_i \hat{q}_j \hat{\mathcal{H}}_{\text{KC}}$	$\hat{S}_i = \frac{1}{2} \sum_{k=1}^N \hat{p}_k (\hat{q}_k \hat{p}_i - \hat{q}_i \hat{p}_k) + \frac{1}{2} \sum_{k=1}^N (\hat{q}_k \hat{p}_i - \hat{q}_i \hat{p}_k) \hat{p}_k + \frac{\hat{q}_i}{ \hat{\mathbf{q}} } \hat{\mathcal{H}}_{\text{O}}$
$\sum_{i=1}^N \hat{S}_{ii} = -2\alpha$	$\sum_{i=1}^N \hat{S}_i^2 = \hat{\mathcal{H}}_{\text{O}}^2 - 2\alpha \hat{\mathbf{L}}^2 - \frac{1}{2} (N-1)^2 \hbar^2 \alpha$
• Quantum Darboux III oscillator	• Quantum Taub-NUT oscillator
$\hat{\mathcal{H}}_{\lambda} = \frac{1}{2(1 + \lambda \hat{\mathbf{q}}^2)} \hat{\mathbf{p}}^2 - \frac{\lambda \alpha \hat{\mathbf{q}}^2}{1 + \lambda \hat{\mathbf{q}}^2}$	$\hat{\mathcal{H}}_{\eta} = \frac{ \hat{\mathbf{q}} }{2(\eta + \hat{\mathbf{q}})} \hat{\mathbf{p}}^2 + \frac{\alpha \hat{\mathbf{q}} }{\eta + \hat{\mathbf{q}} }$
* Quantum ND Fradkin tensor	* Quantum LRL N -vector
$\hat{S}_{ij} = \hat{p}_i \hat{p}_j - 2\lambda \hat{q}_i \hat{q}_j (\hat{\mathcal{H}}_{\lambda} + \alpha)$	$\hat{S}_i = \frac{1}{2} \sum_{k=1}^N \hat{p}_k (\hat{q}_k \hat{p}_i - \hat{q}_i \hat{p}_k) + \frac{1}{2} \sum_{k=1}^N (\hat{q}_k \hat{p}_i - \hat{q}_i \hat{p}_k) \hat{p}_k + \eta \frac{\hat{q}_i}{ \hat{\mathbf{q}} } \hat{\mathcal{H}}_{\eta}$
$\sum_{i=1}^N \hat{S}_{ii} = 2\hat{\mathcal{H}}_{\lambda}$	$\sum_{i=1}^N \hat{S}_i^2 = 2\hat{\mathbf{L}}^2 (\hat{\mathcal{H}}_{\eta} - \alpha) + \eta^2 \hat{\mathcal{H}}_{\eta}^2 + \frac{1}{2} (N-1)^2 \hbar^2 (\hat{\mathcal{H}}_{\eta} - \alpha)$

as the quadratic Casimirs of some rotation subalgebras $\mathfrak{so}(m) \subset \mathfrak{so}(N)$:

$$S^{(m)} = \sum_{1 \leq i < j \leq m} J_{ij}^2, \quad S_{(m)} = \sum_{N-m < i < j \leq N} J_{ij}^2,$$

with $S^{(N)} = S_{(N)} = \mathbf{L}^2$ being the quadratic Casimir of $\mathfrak{so}(N)$. In this respect, we also notice that all the LRL constants of motion ($S_i, S_{U,i}, \tilde{S}_i$) given in Table 1 are transformed as N -vectors under the action of the generators of $\mathfrak{so}(N)$ (as it should be):

$$\{J_{ij}, \tilde{S}_k\} = \delta_{ik} \tilde{S}_j - \delta_{jk} \tilde{S}_i.$$

Furthermore, all of these systems possess an $\mathfrak{sl}(2, \mathbb{R})$ coalgebra symmetry as well [14]. If we denote $J_- = \mathbf{q}^2$, $J_+ = \mathbf{p}^2$ and $J_3 = \mathbf{q} \cdot \mathbf{p}$ we have that

$$\{J_3, J_+\} = 2J_+, \quad \{J_3, J_-\} = -2J_-, \quad \{J_-, J_+\} = 4J_3,$$

and the common integrals (2.2) are just the m th (left and right) coproducts of the Casimir of $\mathfrak{sl}(2, \mathbb{R})$. This set of $(2N - 3)$ integrals is “universal” for any Hamiltonian function defined by $\mathcal{H} = \mathcal{H}(\mathbf{q}^2, \mathbf{p}^2, \mathbf{q} \cdot \mathbf{p})$ so that this always provides, at least, a quasi-MS system [32, 33]. Therefore the Hamiltonians shown in Table 1 are distinguished systems since they have “additional” symmetries.

To end with, we shall present the MS quantization of the four curved classical systems. For this purpose, we remark that the MS quantization of the Darboux III oscillator has been recently obtained in [45], and the corresponding quantum dynamics has been fully solved for $\lambda > 0$ (the case with $\lambda < 0$ is still an open problem). This quantization has been obtained by applying the so called “Schrödinger quantization” procedure [46], and its relationship with the Laplace–Beltrami and position-dependent-mass quantizations has been established in [47] by means of similarity transformations (see also [48]).

Therefore, let us consider the quantum position and momenta operators, $\hat{\mathbf{q}}, \hat{\mathbf{p}}$, with canonical Lie bracket $[\hat{q}_i, \hat{p}_j] = i\hbar\delta_{ij}$. The resulting MS quantum Hamiltonians are summarized in the following final statement.

Proposition 8. *Let \mathcal{H} be one of the classical Hamiltonians given in Propositions 4–7.*

- (i) *The Schrödinger quantization of \mathcal{H} and its quantum symmetries are given in Table 2.*
- (ii) *The quantum Hamiltonian $\hat{\mathcal{H}}$ is endowed with $(2N - 3)$ quantum angular momentum operators $\hat{S}^{(m)}$ and $\hat{S}_{(m)}$, such that $\{\hat{\mathcal{H}}, \hat{S}^{(m)}\}$ or $\{\hat{\mathcal{H}}, \hat{S}_{(m)}\}$ ($m = 2, \dots, N$) is a set of N algebraically independent commuting observables.*
- (iii) *If $\hat{\mathcal{H}} \in (\hat{\mathcal{H}}_{\text{KC}}, \hat{\mathcal{H}}_{\lambda})$ then it commutes with the N^2 components, \hat{S}_{ij} , of a quantum Fradkin tensor ($i, j = 1, \dots, N$). The set $\{\hat{\mathcal{H}}, \hat{S}^{(m)}, \hat{S}_{(m)}, \hat{S}_{ii}\}$ ($m = 2, \dots, N$ and a fixed index i) is formed by $2N - 1$ algebraically independent commuting observables.*
- (iv) *When $\hat{\mathcal{H}} \in (\hat{\mathcal{H}}_{\text{O}}, \hat{\mathcal{H}}_{\eta})$ this commutes with the N components, \hat{S}_i , of a quantum LRL vector ($i = 1, \dots, N$). The set $\{\hat{\mathcal{H}}, \hat{S}^{(m)}, \hat{S}_{(m)}, \hat{S}_i\}$ ($m = 2, \dots, N$ and a fixed index i) is constituted by $2N - 1$ algebraically independent commuting observables.*

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