

Structure Relations and Darboux Contractions for 2D 2nd Order Superintegrable Systems^{*}

Robin HEINONEN[†], Ernest G. KALNINS[‡], Willard MILLER Jr.[†] and Eyal SUBAG[§]

[†] School of Mathematics, University of Minnesota, Minneapolis, Minnesota 55455, USA

E-mail: heino027@umn.edu, miller@ima.umn.edu

URL: <http://www.ima.umn.edu/miller/>

[‡] Department of Mathematics, University of Waikato, Hamilton, New Zealand

E-mail: math0236@waikato.ac.nz

[§] School of Mathematical Science, Tel Aviv University, Tel Aviv 69978, Israel

E-mail: eyalsubag@gmail.com

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Abstract. Two-dimensional quadratic algebras are generalizations of Lie algebras that include the symmetry algebras of 2nd order superintegrable systems in 2 dimensions as special cases. The superintegrable systems are exactly solvable physical systems in classical and quantum mechanics. Distinct superintegrable systems and their quadratic algebras can be related by geometric contractions, induced by Inönü–Wigner type Lie algebra contractions. These geometric contractions have important physical and geometric meanings, such as obtaining classical phenomena as limits of quantum phenomena as $\hbar \rightarrow 0$ and nonrelativistic phenomena from special relativistic as $c \rightarrow \infty$, and the derivation of the Askey scheme for obtaining all hypergeometric orthogonal polynomials as limits of Racah/Wilson polynomials. In this paper we show how to simplify the structure relations for abstract nondegenerate and degenerate quadratic algebras and their contractions. In earlier papers we have classified contractions of 2nd order superintegrable systems on constant curvature spaces and have shown that all results are derivable from free quadratic algebras contained in the enveloping algebras of the Lie algebras $e(2, \mathbb{C})$ in flat space and $o(3, \mathbb{C})$ on nonzero constant curvature spaces. The quadratic algebra contractions are induced by generalizations of Inönü–Wigner contractions of these Lie algebras. As a special case we obtained the Askey scheme for hypergeometric orthogonal polynomials. After constant curvature spaces, the 4 Darboux spaces are the 2D manifolds admitting the most 2nd order Killing tensors. Here we complete this theoretical development for 2D superintegrable systems by showing that the Darboux superintegrable systems are also characterized by free quadratic algebras contained in the symmetry algebras of these spaces and that their contractions are also induced by Inönü–Wigner contractions. We present tables of the contraction results.

Key words: contractions; quadratic algebras; superintegrable systems; Darboux spaces; Askey scheme

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1 Introduction

We define an abstract *nondegenerate (quantum) quadratic algebra* as a noncommutative associative algebra generated by linearly independent operators H , L_1 , L_2 such that H is in the

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center, $R = [L_1, L_2] \neq 0$ and the following relations hold:

$$[L_j, R] = \sum_{0 \leq e_1 + e_2 + e_3 \leq 2} M_{e_1, e_2, e_3}^{(j)} \{L_1^{e_1}, L_2^{e_2}\} H^{e_3}, \quad e_k \geq 0, \quad L_k^0 = I,$$

for some $M_{e_1, e_2, e_3}^{(j)} \in \mathbb{C}$, and where $[A, B] = AB - BA$ is the commutator and $\{L_1, L_2\} = L_1 L_2 + L_2 L_1$ is the symmetrizer. Also the operator R^2 is contained in the algebra of symmetrized products:

$$R^2 - F \equiv R^2 - \sum_{0 \leq e_1 + e_2 + e_3 \leq 3} N_{e_1, e_2, e_3} \{L_1^{e_1}, L_2^{e_2}\} H^{e_3} = 0$$

for some $N_{e_1, e_2, e_3} \in \mathbb{C}$.

An abstract *degenerate (quantum) quadratic algebra* is a noncommutative associative algebra generated by linearly independent operators X, H, L_1, L_2 such that H is in the center and the following relations hold:

$$[X, L_j] = \sum_{0 \leq e_1 + e_2 + e_3 + e_4 \leq 1} P_{e_1, e_2, e_3, e_4}^{(j)} L_1^{e_1} L_2^{e_2} H^{e_3} X^{e_4}, \quad j = 1, 2,$$

for some $P_{e_1, e_2, e_3, e_4}^{(j)} \in \mathbb{C}$. The commutator $[L_1, L_2]$ is expressed as

$$[L_1, L_2] = \sum_{0 \leq e_1 + e_2 + e_3 + e_4 \leq 1} Q_{e_1, e_2, e_3, e_4} \{L_1^{e_1} L_2^{e_2}, X\} H^{e_3} X^{2e_4}$$

for some $Q_{e_1, e_2, e_3, e_4} \in \mathbb{C}$. Finally, there is the relation:

$$G \equiv \sum_{0 \leq e_1 + e_2 + e_3 + e_4 \leq 2} S_{e_1, e_2, e_3, e_4} \{L_1^{e_1}, L_2^{e_2}, X^{2e_4}\} H^{e_3} = 0, \quad X^0 = H^0 = I,$$

for some $S_{e_1, e_2, e_3, e_4} \in \mathbb{C}$ and where $\{L_1^{e_1}, L_2^{e_2}, X^{2e_4}\}$ is the 6-term symmetrizer of three operators.

For both quantum quadratic algebras there is a natural grading such that the operators H, L_j are 2nd order, X is 1st order and

$$\begin{aligned} \text{ord}([A, B]) &\leq \text{ord}(A) + \text{ord}(B) - 1, & \text{ord}(AB) &= \text{ord}(A) + \text{ord}(B), \\ \text{ord}(I) &= 0, & \text{ord}(A + B) &= \max\{\text{ord}(A), \text{ord}(B)\}, & \text{ord}(cA) &= \text{ord}(A), \end{aligned} \quad (1.1)$$

for operators A, B , identity operator I and scalar c , with $A \neq -B$, $A, B \neq 0$, and $c \neq 0$. Thus R is usually 3rd order, expression G is 4th order and F is 6th order. The field of scalars can be either \mathbb{R} or \mathbb{C} .

There is an analogous quadratic algebra structure for Poisson algebras. An abstract *nondegenerate (classical) quadratic algebra* is a Poisson algebra with functionally independent generators $\mathcal{H}, \mathcal{L}_1, \mathcal{L}_2$ such that all generators are in involution with \mathcal{H} and the following relations hold:

$$\begin{aligned} \{\mathcal{L}_j, \mathcal{R}\} &= \sum_{0 \leq e_1 + e_2 + e_3 \leq 2} M_{e_1, e_2, e_3}^{(j)} \mathcal{L}_1^{e_1} \mathcal{L}_2^{e_2} \mathcal{H}^{e_3}, & e_k &\geq 0, & \mathcal{L}_k^0 &= 1, \\ \mathcal{R}^2 - \mathcal{F} &\equiv \mathcal{R}^2 - \sum_{0 \leq e_1 + e_2 + e_3 \leq 3} N_{e_1, e_2, e_3} \mathcal{L}_1^{e_1} \mathcal{L}_2^{e_2} \mathcal{H}^{e_3} = 0 \end{aligned}$$

for some $M_{e_1, e_2, e_3}^{(j)}, N_{e_1, e_2, e_3} \in \mathbb{C}$. An abstract *degenerate (classical) quadratic algebra* is a Poisson algebra with linearly independent generators $\mathcal{X}, \mathcal{H}, \mathcal{L}_1, \mathcal{L}_2$ such that all generators are in involution with \mathcal{H} and obey structure equations

$$\{\mathcal{X}, \mathcal{L}_j\} = \sum_{0 \leq e_1 + e_2 + e_3 + e_4 \leq 1} P_{e_1, e_2, e_3, e_4}^{(j)} \mathcal{L}_1^{e_1} \mathcal{L}_2^{e_2} \mathcal{H}^{e_3} \mathcal{X}^{e_4}, \quad j = 1, 2,$$

$$\begin{aligned} \{\mathcal{L}_1, \mathcal{L}_2\} &= \sum_{0 \leq e_1 + e_2 + e_3 + e_4 \leq 1} Q_{e_1, e_2, e_3, e_4} \mathcal{L}_1^{e_1} \mathcal{L}_2^{e_2} \mathcal{X} \mathcal{H}^{e_3} \mathcal{X}^{2e_4}, \\ \mathcal{G} \equiv \sum_{0 \leq e_1 + e_2 + e_3 + e_4 \leq 2} S_{e_1, e_2, e_3, e_4} \mathcal{L}_1^{e_1} \mathcal{L}_2^{e_2} \mathcal{X}^{2e_4} \mathcal{H}^{e_3} &= 0, \quad \mathcal{X}^0 = \mathcal{H}^0 = 1 \end{aligned} \quad (1.2)$$

for some $P_{e_1, e_2, e_3, e_4}^{(j)}$, Q_{e_1, e_2, e_3, e_4} , $S_{e_1, e_2, e_3, e_4} \in \mathbb{C}$. There is a grading for these quadratic algebras with properties analogous to (1.1), but with the Poisson bracket instead of the commutator.

These quadratic algebra structures arise naturally in the study of classical and quantum superintegrable systems in two dimensions and are key to the exact solvability of these systems, e.g., [2, 3, 6, 7, 9, 10, 25, 28, 32, 33]. A quantum 2D superintegrable system is an integrable Hamiltonian system on an 2-dimensional Riemannian/pseudo-Riemannian manifold with potential that admits 3 algebraically independent partial differential operators commuting with H , the maximum possible.

$$H = \Delta + V, \quad [H, L_j] = 0, \quad L_3 = H, \quad j = 1, 2, 3.$$

(In 2 dimensions we can always find Cartesian-like coordinates x_1, x_2 such that

$$H = \frac{1}{\lambda(x_1, x_2)} (\partial_{x_1}^2 + \partial_{x_2}^2) + V(x_1, x_2)$$

and we adopt these coordinates in the following.) A system is of order k if the maximum order of the symmetry operators L_j (other than H) is k ; all such systems are known for $k = 1, 2$ [14, 15, 16, 17, 23]. Superintegrability captures the properties of quantum Hamiltonian systems that allow the Schrödinger eigenvalue problem $H\Psi = E\Psi$ to be solved exactly, analytically and algebraically. A classical 2D superintegrable system is an integrable Hamiltonian system on an 2-dimensional Riemannian/pseudo-Riemannian manifold with potential that admit 3 functionally independent phase space functions $\mathcal{H}, \mathcal{L}_1, \mathcal{L}_2$ in involution with \mathcal{H} , the maximum possible:

$$\mathcal{H} = \frac{p_1^2 + p_2^2}{\lambda(\mathbf{x})} + V(\mathbf{x}), \quad \{\mathcal{H}, \mathcal{L}_j\} = 0, \quad \mathcal{L}_3 = \mathcal{H}, \quad j = 1, 2, 3,$$

expressed in local Cartesian-like coordinates x_1, x_2, p_1, p_2 . A system is of order k if the maximum order of the constants of the motion L_j , $j \neq 3$, as polynomials in p_1, p_2 is k . Again all such systems are known for $k = 1, 2$, and there is a 1-1 relationship between classical and quantum 2nd order 2D superintegrable systems [14, 15, 16, 17].

The possible superintegrable systems divide into four classes:

1. *First order systems.* These are the (zero-potential) Laplace–Beltrami eigenvalue equations on constant curvature spaces. The symmetry algebras close under commutation to form the Lie algebras $e(2, \mathbb{R})$, $e(1, 1)$, $o(3, \mathbb{R})$ or $o(2, 1)$. Such systems have been studied in detail, using group theory methods, e.g., [27, 31].
2. *Free triplets.* These are superintegrable systems with zero potential and all generators of 2nd order. The possible spaces for which these systems can occur were classified by Koenigs [23]. They are: constant curvature spaces (6 linearly independent 2nd order symmetries, 3 1st order), the four Darboux spaces (4 linearly independent 2nd order symmetries, 1 1st order), and eleven 4-parameter Koenigs spaces (3 linearly independent 2nd order symmetries, 0 1st order). In most cases the symmetry operators will not generate a quadratic algebra, i.e., the algebra will not close. If the system generates a nondegenerate quadratic algebra we call it a *free quadratic triplet*.

3. *Nondegenerate systems* [14, 15, 16, 17]. Here all symmetries are of 2nd order and the space of potentials is 4-dimensional:

$$V(\mathbf{x}) = a_1 V_{(1)}(\mathbf{x}) + a_2 V_{(2)}(\mathbf{x}) + a_3 V_{(3)}(\mathbf{x}) + a_4.$$

The symmetry operators generate a nondegenerate quadratic algebra with parameters a_j .

4. *Degenerate systems* [18]. There are 4 generators: one 1st order X and 3 second order H , L_1 , L_2 . Here, X^2 is not contained in the span of H , L_1 , L_2 . The space of potentials is 2-dimensional: $V(\mathbf{x}) = a_1 V_{(1)}(\mathbf{x}) + a_2$. The symmetry operators generate a degenerate quadratic algebra with parameters a_j . Relation (1.2) is an expression of the fact that 4 symmetry operators cannot be algebraically independent.

Every degenerate superintegrable system occurs as a restriction of the 3-parameter potentials (i.e., 4-dimensional potential space) to 1-parameter ones, such that one of the symmetries becomes a perfect square: $L = X^2$. Here X is a first order symmetry and a new 2nd order symmetry appears so that this restriction admits more symmetries than the original system.

Strictly speaking, a nondegenerate 2D superintegrable system, both classical and quantum, is not a single system but in fact a family of superintegrable systems parameterized by three parameters, a_1 , a_2 , a_3 . Similarly a degenerate 2D superintegrable system, both classical and quantum, is a family of superintegrable systems parameterized by one parameter, a_1 .

For a quadratic algebra that comes from a nondegenerate 2D superintegrable system (classical and quantum) the constants $M_{e_1, e_2, e_3}^{(j)}$ and N_{e_1, e_2, e_3} are polynomials in the parameters a_1 , a_2 , a_3 of degree $2 - e_1 - e_2 - e_3$ and $3 - e_1 - e_2 - e_3$, respectively. If all parameters $a_j = 0$ the algebra is *free*. For a quadratic algebra that comes from a degenerate 2D superintegrable system (classical and quantum) the constants $P_{e_1, e_2, e_3, e_4}^{(j)}$, Q_{e_1, e_2, e_3, e_4} and S_{e_1, e_2, e_3, e_4} are polynomials in a_1 of degrees $1 - e_1 - e_2 - e_3 - e_4$, $1 - e_1 - e_2 - e_3 - e_4$ and $2 - e_1 - e_2 - e_3 - e_4$, respectively. If all parameters $a_j = 0$ these algebras are *free*.

Basic results that relate these superintegrable systems are the closure theorems:

Theorem 1.1. *A free triplet, classical or quantum, extends to a superintegrable system with potential if and only if it generates a free quadratic algebra \tilde{Q} .*

Theorem 1.2. *A superintegrable system, degenerate or nondegenerate, classical or quantum, with quadratic algebra Q , is uniquely determined by its free quadratic algebra \tilde{Q} .*

These theorems were proved in [21], except for systems on the Darboux spaces which will be proved in this paper. The proofs are constructive: Given a free quadratic algebra \tilde{Q} one can compute the potential V and the symmetries of the quadratic algebra Q . Thus as far as superintegrable systems on specific spaces are concerned, all information about the systems is contained in the free quadratic algebras.

We will refer to quadratic algebras associated with superintegrable systems as *geometric*, a subset of *abstract* quadratic algebras. For quadratic algebras associated with quantum superintegrable systems the order of A is its order as a partial differential operator. For quadratic algebras associated with classical superintegrable systems the order of A is its order as a polynomial in the momenta. Although there is a 1-1 relationship between classical and quantum geometric systems the corresponding classical and quantum geometric quadratic algebras are not the same; only the highest order terms in the structure equations agree.

2 Contractions

The notion of contractions for quadratic algebras is based on that for Lie algebras, e.g., [13, 29, 34]:

Definition 2.1. Let \mathfrak{g} be a complex Lie algebra with an underlying vector space V and Lie brackets $[\cdot, \cdot]$. In the following we simply write it as $\mathfrak{g} = (V, [\cdot, \cdot])$. Suppose that for any $\epsilon \in (0, 1]$, $t_\epsilon: V \rightarrow V$ is a linear invertible operator and that $\lim_{\epsilon \rightarrow 0^+} t_\epsilon^{-1}[t_\epsilon X, t_\epsilon Y]$ converges for any $X, Y \in V$. We use the notation

$$\lim_{\epsilon \rightarrow 0^+} t_\epsilon^{-1}[t_\epsilon X, t_\epsilon Y] = [X, Y]_0.$$

Then $[\cdot, \cdot]_0$ are in fact Lie brackets on V and we denote this Lie algebra by $\mathfrak{g}_0 = (V, [\cdot, \cdot]_0)$. We say that \mathfrak{g}_0 is a contraction of \mathfrak{g} (that is realized by the family of linear maps $\{t_\epsilon\}_{\epsilon \in (0,1]}$) and we denote it by $\mathfrak{g} \rightarrow \mathfrak{g}_0$.

Thus, as $\epsilon \rightarrow 0$ the 1-parameter family of basis transformations can become singular but the structure constants go to a finite limit.

Note. In this paper all of the Lie algebra contractions needed are of standard Inönü–Wigner type [29], but for our theory on contractions of quadratic algebras and second order superintegrable systems on constant curvature spaces [21] IW-contractions are not enough. In most cases we employ a generalized Inönü–Wigner contraction (Doebner–Melsheimer type) [4, 29], but in some specific cases we are forced to use a general contraction of Lie algebras in the sense of Saletan [30]. In a paper under preparation we shall demonstrate that all of these generalizations of Inönü–Wigner contractions are induced by a well defined family of contractions of the conformal Lie algebra $so(4, \mathbb{C})$ to itself that follow from limiting processes for R -separable coordinate systems for wave equations, introduced by Bôcher in his famous 1894 thesis [1]. We call these Bôcher contractions.

Of particular interest to us are contractions that are induced by ϵ -dependent local analytic coordinate transformations $x_j(\epsilon, x'_1, \dots, x'_n)$, $j = 1, \dots, n$ on a manifold \mathcal{M} such that the Jacobian $\det(\frac{\partial \mathbf{x}}{\partial \mathbf{x}'}) \neq 0$ for $\epsilon \in (0, 1]$, but the Jacobian is undefined or nonsingular in the limit as $\epsilon \rightarrow 0$.

Definition 2.2. Suppose that $\mathfrak{g} = (V, [\cdot, \cdot]_0)$ is a contraction of $\mathfrak{g} = (V, [\cdot, \cdot])$ that is realized by the family of linear maps $\{t_\epsilon\}_{\epsilon \in (0,1]}$. Let \mathcal{M} be a smooth manifold with a local coordinate system (x_1, x_2, \dots, x_n) and let $\psi: \mathfrak{g} \rightarrow C^\infty(\mathcal{T}\mathcal{M})$ be an embedding of Lie algebras, where $C^\infty(\mathcal{T}\mathcal{M})$ is the space of smooth functions on the cotangent bundle (the phase space) of \mathcal{M} which is equipped with its canonical Poisson brackets as Lie brackets. Suppose that $x_j(\epsilon, x'_1, \dots, x'_n)$, $j = 1, \dots, n$ are ϵ -dependent local analytic coordinate transformations such that the Jacobian $\det(\frac{\partial \mathbf{x}}{\partial \mathbf{x}'}) \neq 0$ for $\epsilon \in (0, 1]$. If the limit

$$\lim_{\epsilon \rightarrow 0^+} \psi(t_\epsilon(v))(x_1(\epsilon, x'_1, \dots, x'_n), \dots, x_n(\epsilon, x'_1, \dots, x'_n))$$

converges for any $v \in V$ and defines a Lie algebra homomorphism from \mathfrak{g}_0 into a local expression of a certain space of $C^\infty(\mathcal{T}\mathcal{M}')$ for some smooth manifold \mathcal{M}' . Then we say that the contraction $\mathfrak{g} \rightarrow \mathfrak{g}_0$ is implemented by $x_j(\epsilon, x'_1, \dots, x'_n)$, $j = 1, \dots, n$ and call this procedure a *geometric Lie algebra contraction*. This distinction between abstract contractions of Lie algebras and geometric contractions has been recognized from the earliest days of the theory, e.g., [13].

We give some pertinent examples.

Example 2.1. Consider the complex three dimensional Lie algebra $\mathcal{G}\mathbf{3}$ defined by basis elements $\{\mathcal{P}_1, \mathcal{P}_2, \mathcal{D}\}$ that satisfy $[\mathcal{P}_1, \mathcal{P}_2] = 0$, $[\mathcal{P}_1, \mathcal{D}] = \mathcal{P}_1$, $[\mathcal{P}_2, \mathcal{D}] = \mathcal{P}_2$. This algebra admits an abstract Inönü–Wigner contraction defined by $t_\epsilon(\mathcal{P}_1) = \epsilon \mathcal{P}_1$, $t_\epsilon(\mathcal{P}_2) = \epsilon \mathcal{P}_2$, $t_\epsilon(\mathcal{D}) = \mathcal{D}$. In this case the contracted Lie algebra, $\mathcal{G}\mathbf{3}_0$, coincides with $\mathcal{G}\mathbf{3}$ and $[\mathcal{P}_1, \mathcal{P}_2]_0 = 0$, $[\mathcal{P}_1, \mathcal{D}]_0 = \mathcal{P}_1$, $[\mathcal{P}_2, \mathcal{D}]_0 = \mathcal{P}_2$.

Now considering the complex analytic manifold \mathbb{C}^2 with coordinates (x_1, x_2) we can realize $\mathcal{G}\mathbf{3}$ by

$$\begin{aligned}\psi(\mathcal{P}_1)(x_1, x_2, p_{x_1}, p_{x_2}) &= p_{x_1}, & \psi(\mathcal{P}_2)(x_1, x_2, p_{x_1}, p_{x_2}) &= p_{x_2}, \\ \psi(\mathcal{D})(x_1, x_2, p_{x_1}, p_{x_2}) &= x_1 p_{x_1} + x_2 p_{x_2}.\end{aligned}$$

A geometric implementation of the contraction is obtained by the substitution $x_1 = \epsilon x'_1$, $x_2 = \epsilon x'_2$. Then, taking the limit, we find

$$\begin{aligned}\lim_{\epsilon \rightarrow 0^+} \psi(t_\epsilon(\mathcal{P}_1))(\epsilon x'_1, \epsilon x'_2, p_{\epsilon x'_1}, p_{\epsilon x'_2}) &= \lim_{\epsilon \rightarrow 0^+} \epsilon p_{\epsilon x'_1} = \lim_{\epsilon \rightarrow 0^+} \frac{1}{\epsilon} p_{x'_1} = p_{x'_1}, \\ \lim_{\epsilon \rightarrow 0^+} \psi(t_\epsilon(\mathcal{P}_2))(\epsilon x'_1, \epsilon x'_2, p_{\epsilon x'_1}, p_{\epsilon x'_2}) &= \lim_{\epsilon \rightarrow 0^+} \epsilon p_{\epsilon x'_2} = \lim_{\epsilon \rightarrow 0^+} \frac{1}{\epsilon} p_{x'_2} = p_{x'_2}, \\ \lim_{\epsilon \rightarrow 0^+} \psi(t_\epsilon(\mathcal{D}))(\epsilon x'_1, \epsilon x'_2, p_{\epsilon x'_1}, p_{\epsilon x'_2}) &= \lim_{\epsilon \rightarrow 0^+} \epsilon x'_1 \frac{1}{\epsilon} p_{x'_1} + \epsilon x'_2 \frac{1}{\epsilon} p_{x'_2} = x'_1 p_{x'_1} + x'_2 p_{x'_2}.\end{aligned}$$

Though this Lie algebra contraction acts like the identity map here, we shall see that its action on Darboux quadratic algebras is nontrivial.

Example 2.2. We again consider the Lie algebra $\mathcal{G}\mathbf{3}$ with the same geometric identification. This algebra admits another abstract Inönü–Wigner contraction defined by $t_\epsilon(\mathcal{P}_1) = \mathcal{P}_1$, $t_\epsilon(\mathcal{P}_2) = \mathcal{P}_2$, $t_\epsilon(\mathcal{D}) = \epsilon \mathcal{D}$. The contracted Lie algebra, $\mathcal{G}\mathbf{3}_0$, is given by $[\mathcal{P}_1, \mathcal{P}_2]_0 = 0$, $[\mathcal{P}_1, \mathcal{D}]_0 = 0$, $[\mathcal{P}_2, \mathcal{D}]_0 = 0$. A geometric implementation of the contraction is obtained by the substitution $x_1 = x'_1 + \frac{1}{\epsilon}$, $x_2 = x'_2$. Then, taking the limit, we find

$$\begin{aligned}\lim_{\epsilon \rightarrow 0^+} \psi(t_\epsilon(\mathcal{P}_1))(x'_1 + \frac{1}{\epsilon}, x'_2, p_{x'_1 + \frac{1}{\epsilon}}, p_{x'_2}) &= \lim_{\epsilon \rightarrow 0^+} p_{x'_1 + \frac{1}{\epsilon}} = \lim_{\epsilon \rightarrow 0^+} p_{x'_1} = p_{x'_1}, \\ \lim_{\epsilon \rightarrow 0^+} \psi(t_\epsilon(\mathcal{P}_2))(x'_1 + \frac{1}{\epsilon}, x'_2, p_{x'_1 + \frac{1}{\epsilon}}, p_{x'_2}) &= \lim_{\epsilon \rightarrow 0^+} p_{x'_2} = p_{x'_2}, \\ \lim_{\epsilon \rightarrow 0^+} \psi(t_\epsilon(\mathcal{D}))(\mathcal{P}_2) \left(x'_1 + \frac{1}{\epsilon}, x'_2, p_{x'_1 + \frac{1}{\epsilon}}, p_{x'_2} \right) &= \lim_{\epsilon \rightarrow 0^+} \epsilon \left(\left(x'_1 + \frac{1}{\epsilon} \right) p_{x'_1 + \frac{1}{\epsilon}} + x'_2 p_{x'_2} \right) = p_{x'_1}.\end{aligned}$$

Hence the resulting map from $\mathcal{G}\mathbf{3}_0$ that is given by $\mathcal{P}_1 \mapsto p_{x'_1}$, $\mathcal{P}_2 \mapsto p_{x'_2}$, $\mathcal{D}_1 \mapsto p_{x'_1}$ is a Lie algebra homomorphism with kernel spanned by $\mathcal{P}_1 - \mathcal{D}$. Though this geometric implementation is not an isomorphism of the contracted Lie algebra we shall see that its action on Darboux quadratic algebras is isomorphic.

There are exactly analogous implementations of geometric contractions in the quantum case.

Definition 2.3 (algebraic contraction of quadratic algebras). Let \mathcal{A} be a classical nondegenerate quadratic algebra with a generating set which consist of a Hamiltonian, \mathcal{H} (second order element which lies in the center of \mathcal{A}) and two second order constants of motion $\mathcal{L}_1, \mathcal{L}_2$. Let $\{\mathcal{L}_1, \mathcal{L}_2\}^2 = \mathcal{R}^2 = \mathcal{F}(\mathcal{H}, \mathcal{L}_1, \mathcal{L}_2)$ be the Casimir of \mathcal{A} . For any $\epsilon \in (0, 1]$ consider a matrix $A_\epsilon \in \text{GL}(3, \mathbb{C})$ of the form

$$A_\epsilon = \begin{pmatrix} A_{1,1}(\epsilon) & A_{1,2}(\epsilon) & A_{1,3}(\epsilon) \\ A_{2,1}(\epsilon) & A_{2,2}(\epsilon) & A_{2,3}(\epsilon) \\ 0 & 0 & A_{3,3}(\epsilon) \end{pmatrix} \in \text{GL}(3).$$

Assume that the map from $(0, 1]$ to $\text{GL}(3, \mathbb{C})$ that is given by $\epsilon \mapsto A_\epsilon$ is continuous. For any $\epsilon \in (0, 1]$ we have another set of generators, $\{\mathcal{L}_1^\epsilon, \mathcal{L}_2^\epsilon, \mathcal{H}^\epsilon\}$, for \mathcal{A} that is defined by

$$\begin{pmatrix} \mathcal{L}_1 \\ \mathcal{L}_2 \\ \mathcal{H} \end{pmatrix} = A_\epsilon \begin{pmatrix} \mathcal{L}_1^\epsilon \\ \mathcal{L}_2^\epsilon \\ \mathcal{H}^\epsilon \end{pmatrix}$$

and satisfies the Casimir relation

$$\begin{aligned} \{\mathcal{L}_1^\epsilon, \mathcal{L}_2^\epsilon\}^2 &= (\mathcal{R}^\epsilon)^2 = \mathcal{R}^2(A_{11}(\epsilon)A_{22}(\epsilon) - A_{12}(\epsilon)A_{21}(\epsilon))^{-2} \\ &= (A_{11}(\epsilon)A_{22}(\epsilon) - A_{12}(\epsilon)A_{21}(\epsilon))^{-2} \mathcal{F} \left(A_\epsilon \begin{pmatrix} \mathcal{L}_1^\epsilon \\ \mathcal{L}_2^\epsilon \\ \mathcal{H}^\epsilon \end{pmatrix} \right) \\ &= \sum_{i+j+k=3, 0 \leq i,j,k} \alpha_{ijk}(\epsilon) (\mathcal{L}_1^\epsilon)^i (\mathcal{L}_2^\epsilon)^j (\mathcal{H}^\epsilon)^k. \end{aligned}$$

If $\lim_{\epsilon \rightarrow 0^+} \alpha_{ijk}(\epsilon)$ exists for any i, j, k we denote it by $\alpha_{ijk}(0)$. Then there exists a quadratic algebra, \mathcal{A}_0 with a set of generators $\{\mathcal{L}_1^0, \mathcal{L}_2^0, \mathcal{H}^0\}$ that satisfy

$$\{\mathcal{L}_1^0, \mathcal{L}_2^0\}^2 = (\mathcal{R}^0)^2 = \sum_{i+j+k=3, 0 \leq i,j,k} \alpha_{ijk}(0) (\mathcal{L}_1^0)^i (\mathcal{L}_2^0)^j (\mathcal{H}^0)^k$$

we call \mathcal{A}_0 the contraction of \mathcal{A} with respect to $\{A_\epsilon\}_{\epsilon \in (0,1]}$.

Note that we can expand $\{\mathcal{R}^\epsilon, \mathcal{L}_1^\epsilon\}$, $\{\mathcal{R}^\epsilon, \mathcal{L}_2^\epsilon\}$ as quadratic expressions in $\mathcal{L}_1^\epsilon, \mathcal{L}_2^\epsilon, \mathcal{H}^\epsilon$. For a contraction it might seem that we must also require these expansion coefficients to have finite limits as $\epsilon \rightarrow 0$. However from the results of Section 3.1 the convergence of these other structure equations follows from the convergence of the Casimir. There is a completely analogous definition of contraction for quantum quadratic algebras.

Just as for abstract classical and quantum Lie algebra contractions there are abstract classical and quantum quadratic algebra contractions that are induced by ϵ -dependent local analytic coordinate transformations $x_j(\epsilon, x'_1, \dots, x'_n)$, $j = 1, \dots, n$ on a manifold \mathcal{M} such that the Jacobian $\det(\frac{\partial \mathbf{x}}{\partial \mathbf{x}'}) \neq 0$ for $\epsilon \in (0, 1]$, but the Jacobian is undefined or nonsingular in the limit as $\epsilon \rightarrow 0$. If an algebraic contraction $A \rightarrow B$ can be implemented by some coordinate transformation, we say that it is a *geometric quadratic algebra contraction*. In fact, all of the quadratic algebra contractions for Darboux systems discussed in this paper are geometric implementations. We will give many examples in the following sections.

The notion of contraction applied to structures other than Lie algebras is not new, see for example [11] (and references there-in) and [26].

In [21] Lie algebra and quadratic algebra contractions for superintegrable systems on constant curvature spaces were related:

Theorem 2.1. *Every Lie algebra contraction of $\mathcal{G} = e(2, \mathbb{C})$ or $\mathcal{G} = o(3, \mathbb{C})$ induces a geometric contraction of a free geometric quadratic algebra \tilde{Q} based on \mathcal{G} , which in turn induces uniquely a contraction of the quadratic algebra Q with potential. This is true for both classical and quantum algebras.*

Here we will demonstrate the analogous result for Darboux spaces, using the conformal symmetry algebra \mathcal{G}_3 with basis $\{\partial_x, \partial_y, x\partial_x + y\partial_y\}$.

3 Structure relations

Although the full sets of classical structure equations can be rather complicated, the function \mathcal{F} contains all of the structure information for nondegenerate systems and \mathcal{G} (only unique up to a nonzero scalar multiple) most of the information for degenerate systems. In particular, it is easy to show that [8, 21], $\{\mathcal{L}_1, \mathcal{R}\} = \frac{1}{2} \frac{\partial \mathcal{F}}{\partial \mathcal{L}_2}$, $\{\mathcal{L}_2, \mathcal{R}\} = -\frac{1}{2} \frac{\partial \mathcal{F}}{\partial \mathcal{L}_1}$, so the Casimir contains within itself all of the structure equations. For degenerate systems we have [21]

$$\{\mathcal{X}, \mathcal{L}_1\} = K \frac{\partial \mathcal{G}}{\partial \mathcal{L}_2}, \quad \{\mathcal{X}, \mathcal{L}_2\} = -K \frac{\partial \mathcal{G}}{\partial \mathcal{L}_1}, \quad \{\mathcal{L}_1, \mathcal{L}_2\} = K \frac{\partial \mathcal{G}}{\partial \mathcal{X}},$$

where $\{\mathcal{H}, K\} = 0$. Here, K is a scalar, unless $\{\mathcal{X}, \mathcal{L}_1\}$ and $\{\mathcal{X}, \mathcal{L}_2\}$ are linearly dependent. In the latter case there would exist 3 algebraically independent elements of the algebra in involution, including the Hamiltonian. This is impossible for a Hamiltonian system. Thus, except for some abstract quadratic algebras unrelated to geometric superintegrable systems, K will always be a scalar that can be normalized to 1.

3.1 The quantum operators F and G

The quantum case is similar to the classical case, but more complicated. From the Casimir relation

$$\begin{aligned} R^2 - F \equiv & R^2 - (b_1 L_1^3 + b_2 L_2^3 + b_3 H^3 + b_4 \{L_1^2, L_2\} + b_5 \{L_1, L_2^2\} + b_6 L_1 L_2 L_1 + b_7 L_2 L_1 L_2 \\ & + b_8 H \{L_1, L_2\} + b_9 H L_1^2 + b_{10} H L_2^2 + b_{11} H^2 L_1 + b_{12} H^2 L_2 + b_{13} L_1^2 + b_{14} L_2^2 \\ & + b_{15} \{L_1, L_2\} + b_{16} H L_1 + b_{17} H L_2 + b_{18} H^2 + b_{19} L_1 + b_{20} L_2 + b_{21} H + b_{22}) = 0, \end{aligned}$$

we want to determine the structure relations. Noting that $R = [L_1, L_2]$ and using operator identities

$$\begin{aligned} L_2 R L_2 &= \frac{1}{2} [[L_2, R], L_2] + \frac{1}{2} \{R, L_2^2\}, & L_1 R L_1 &= \frac{1}{2} [[L_1, R], L_1] + \frac{1}{2} \{R, L_1^2\}, \\ L_1 R L_2 + L_2 R L_1 &= -\frac{1}{2} [L_2, [L_1, R]] - \frac{1}{2} [L_1, [L_2, R]] + \frac{1}{2} \{R, \{L_1, L_2\}\}, \\ [L_1, R^2] &= \{R, [L_1, R]\}, & [L_2, R^2] &= \{R, [L_2, R]\}, \\ [L_1, L_2 L_1 L_2] &= \{[L_1, L_2], \frac{1}{2} \{L_1, L_2\}\}, \\ X L_1 L_2 + L_2 L_1 X &= \frac{1}{2} \{X, \{L_1, L_2\}\} + \frac{1}{2} [X, [L_1, L_2]], \end{aligned} \tag{3.1}$$

(true formally for all operators L_1, L_2, R , not just for R the commutator), and setting

$$\begin{aligned} [L_1, R] &= A_1 L_1^2 + A_2 L_2^2 + A_3 H^2 + A_4 \{L_1, L_2\} \\ &+ A_5 H L_1 + A_6 H L_2 + A_7 L_1 + A_8 L_2 + A_9 H + A_{10}, \\ [L_2, R] &= B_1 L_1^2 + B_2 L_2^2 + B_3 H^2 + B_4 \{L_1, L_2\} \\ &+ B_5 H L_1 + B_6 H L_2 + B_7 L_1 + B_8 L_2 + B_9 H + B_{10}, \end{aligned}$$

we can write $[L_j, R^2 - F] = 0$ in the form $\{R, Q_j\} = 0$ for some explicit differential operator Q_j . This can only hold if $Q_j = 0$. As a result we find:

$$\begin{aligned} A_1 &= b_4 + \frac{1}{2} b_6, & A_2 &= \frac{3}{2} b_2, & A_3 &= \frac{1}{2} b_{12}, & A_4 &= b_5 + \frac{1}{2} b_7, \\ A_5 &= b_8, & A_6 &= b_{10}, & A_7 &= b_{15} - \frac{3}{4} b_1 b_2 + b_4 b_5 - \frac{1}{4} b_6 b_7, \\ A_8 &= b_{14} - \frac{1}{2} b_2 b_4 + b_5^2 - b_2 b_6 + \frac{1}{2} b_5 b_7, & A_9 &= \frac{1}{2} b_{17} - \frac{1}{4} b_2 b_9 + \frac{1}{2} b_5 b_8 - \frac{1}{4} b_6 b_{10}, \\ A_{10} &= \frac{1}{2} b_{20} - \frac{1}{4} b_2 b_{13} - \frac{1}{4} b_1 b_2 b_5 \\ &\quad - \frac{1}{4} b_2 b_4^2 + \frac{1}{2} b_5 b_{15} + \frac{1}{2} b_4 b_5^2 + \frac{1}{4} b_1 b_2 b_7 + \frac{1}{4} b_2 b_6^2 - \frac{1}{4} b_5^2 b_6 - \frac{1}{4} b_5 b_6 b_7 - \frac{1}{4} b_6 b_{14}, \\ B_1 &= -\frac{3}{2} b_1, & B_2 &= -b_5 - \frac{1}{2} b_7, & B_3 &= -\frac{1}{2} b_{11}, & B_4 &= -b_4 - \frac{1}{2} b_6, \\ B_5 &= -b_9, & B_6 &= -b_8, & B_7 &= -b_{13} + \frac{1}{2} b_1 b_5 - b_4^2 + b_1 b_7 - \frac{1}{2} b_4 b_6, \\ B_8 &= -b_{15} + \frac{3}{4} b_1 b_2 - b_4 b_5 + \frac{1}{4} b_6 b_7, & B_9 &= -\frac{1}{2} b_{16} + \frac{1}{4} b_1 b_{10} - \frac{1}{2} b_4 b_8 + \frac{1}{4} b_7 b_9, \\ B_{10} &= -\frac{1}{2} b_{19} + \frac{1}{4} b_1 b_{14} + \frac{1}{4} b_1 b_2 b_4 + \frac{1}{4} b_1 b_5^2 - \frac{1}{2} b_4 b_{15} \\ &\quad - \frac{1}{2} b_4^2 b_5 - \frac{1}{4} b_1 b_2 b_6 - \frac{1}{4} b_1 b_7^2 + \frac{1}{4} b_4^2 b_7 + \frac{1}{4} b_4 b_6 b_7 + \frac{1}{4} b_7 b_{13}. \end{aligned}$$

For quantum degenerate systems, in the Casimir relation

$$G(L_1, L_2, H, X, \alpha) = 0$$

for 2nd order superintegrable systems with degenerate potentials we assume that G is given, up to a multiplicative factor, and set

$$\begin{aligned}
 G &= c_1 L_1^2 + c_2 L_2^2 + c_3 H^2 + c_4 \{L_1, L_2\} + c_5 H L_1 + c_6 H L_2 + c_7 X^4 + c_8 \{X^2, L_1\} \\
 &\quad + c_9 \{X^2, L_2\} + c_{10} H X^2 + c_{11} X L_1 X + c_{12} X L_2 X + c_{13} L_1 + c_{14} L_2 \\
 &\quad + c_{15} H + c_{16} X^2 + c_{17}, \\
 [X, L_1] &= C_1 L_1 + C_2 L_2 + C_3 H + C_4 X^2 + C_5, \\
 [X, L_2] &= D_1 L_1 + D_2 L_2 + D_3 H + D_4 X^2 + D_5, \\
 [L_1, L_2] &= E_1 \{L_1, X\} + E_2 \{L_2, X\} + E_3 H X + E_4 X^3 + E_5 X.
 \end{aligned}$$

Using identities (3.1) we find

$$\begin{aligned}
 [X, G] &= \left\{ [X, L_1], A_1 - \frac{c_{11}}{4} (C_1^2 + C_2 D_1) - \frac{c_{12}}{4} (C_1 D_1 + D_1 D_2) + \frac{c_{13}}{2} \right\} \\
 &\quad + \left\{ [X, L_2], A_2 + \frac{c_{11}}{4} (C_1 C_2 + C_2 D_2) - \frac{c_{12}}{4} (C_2 D_1 + D_2^2) + \frac{c_{14}}{2} \right\} = 0, \quad (3.2)
 \end{aligned}$$

$$\begin{aligned}
 [L_1, G] &= \left\{ [L_1, L_2], A_2 + \frac{c_8}{4} (C_1 C_2 + C_2 D_2) - \frac{c_9}{4} (C_1^2 + C_1 D_2) + \frac{c_{12}}{4} (C_1 D_2 - C_2 D_1) \right. \\
 &\quad \left. + \frac{c_{14}}{2} \right\} + \{[L_1, X], A_3 + k_1 X\} + \{[L_2, X], k_2 X\} = 0, \quad (3.3)
 \end{aligned}$$

$$\begin{aligned}
 [L_2, G] &= \left\{ [L_1, L_2], -A_1 + \frac{c_8}{4} (C_1 D_2 + D_2^2) - \frac{c_9}{4} (C_1 D_1 + D_1 D_2) + \frac{c_{11}}{4} (C_2 D_1 - C_1 D_2) \right. \\
 &\quad \left. - \frac{c_{13}}{2} \right\} + \{[L_1, X], k_3 X\} + \{[L_2, X], A_3 + k_4 X\} = 0, \quad (3.4)
 \end{aligned}$$

where

$$\begin{aligned}
 A_1 &= c_1 L_1 + c_4 L_2 + \frac{c_5}{2} H + \left(c_8 + \frac{c_{11}}{2} \right) X^2, \\
 A_2 &= c_2 L_2 + c_4 L_1 + \frac{c_6}{2} H + \left(c_9 + \frac{c_{12}}{2} \right) X^2, \\
 A_3 &= 2c_7 X^3 + \left(c_8 + \frac{c_{11}}{2} \right) \{L_1, X\} + \left(c_9 + \frac{c_{12}}{2} \right) \{L_2, X\} + c_{10} H X, \\
 k_1 &= -c_7 (C_1^2 + C_2 D_1) + \frac{c_8}{2} (2C_1 C_4 + C_2 D_4 - C_2 E_1) + \frac{c_9}{2} (2C_4 D_1 + C_1 E_1 - C_1 D_4) \\
 &\quad + \frac{c_{12}}{2} (C_1 E_1 + C_1 D_4 + D_1 E_2 - C_4 D_1) + c_{16}, \\
 k_2 &= -c_7 (C_1 C_2 + C_2 D_2) + \frac{c_8}{2} (C_2 C_4 - C_2 E_2) + \frac{c_9}{2} (2C_4 D_2 + C_1 C_4 + C_1 E_2) \\
 &\quad + \frac{c_{12}}{2} (C_2 D_4 + C_2 E_1 + D_2 E_2 - C_4 D_2), \\
 k_3 &= -c_7 (C_1 D_1 + D_1 D_2) + \frac{c_8}{2} (2C_1 D_4 + D_2 D_4 - D_1 E_1) + \frac{c_9}{2} (D_1 E_1 + D_1 D_4) \\
 &\quad + \frac{c_{11}}{2} (C_4 D_1 - C_1 D_4 - C_1 E_1 - D_1 E_2), \\
 k_4 &= -c_7 (C_2 D_1 + D_2^2) + \frac{c_8}{2} (2C_2 D_4 - C_4 D_2 - D_2 E_2) + \frac{c_9}{2} (2D_2 D_4 + C_4 D_1 + D_1 E_2) \\
 &\quad + \frac{c_{11}}{2} (C_4 D_2 - C_2 D_4 - C_2 E_1 - D_2 E_2) + c_{16}.
 \end{aligned}$$

Equating the coefficients of the 4th order terms in (3.2) and the coefficients of the 5th order terms in (3.3) and (3.4) we find

$$\begin{aligned}
 [X, L_1] &= K A_2 + C_5, & [X, L_2] &= -K A_1 + D_5, & [L_1, L_2] &= K A_3 + E_5 X, \\
 C_1 &= K c_4, & C_2 &= K c_2, & C_3 &= K \frac{c_6}{2}, & C_4 &= K \left(c_9 + \frac{c_{12}}{2} \right),
 \end{aligned}$$

$$\begin{aligned}
D_1 &= -Kc_1, & D_2 &= -Kc_4, & D_3 &= -K\frac{c_5}{2}, & D_4 &= -K\left(c_8 + \frac{c_{11}}{2}\right), \\
E_1 &= K\left(c_8 + \frac{c_{11}}{2}\right), & E_2 &= K\left(c_9 + \frac{c_{12}}{2}\right), & E_3 &= Kc_{10}, & E_4 &= 2Kc_7,
\end{aligned}$$

for some constant K . Now we substitute these values back into (3.2), (3.3), and (3.4). We immediately see that $k_2 = k_3 = 0$ and

$$\begin{aligned}
k_1 = k_4 = c_{16} - K^2c_7(c_4^2 - c_1c_2) + K^2(c_4c_9 - c_2c_8)\left(c_8 + \frac{c_{11}}{2}\right) \\
+ K^2(c_4c_8 - c_1c_9)\left(c_9 + \frac{c_{12}}{2}\right)
\end{aligned}$$

and we obtain

$$\begin{aligned}
C_5A_1 + D_5A_2 + \left(\frac{c_{13}}{2} - K^2\frac{c_{11}}{4}(c_4^2 - c_1c_2)\right)(KA_2 + C_5) \\
+ \left(\frac{c_{14}}{2} - K^2\frac{c_{12}}{4}(c_4^2 - c_1c_2)\right)(-KA_1 + D_5) = 0, \tag{3.5}
\end{aligned}$$

$$\begin{aligned}
2\left(\frac{c_{14}}{2} - K^2\frac{c_{12}}{4}(c_4^2 - c_1c_2)\right)(KA_3 + E_5X) - 2C_5(A_3 + k_1X) \\
+ (E_5 - Kk_1)\{X, A_2\} = 0, \tag{3.6}
\end{aligned}$$

$$\begin{aligned}
-2\left(\frac{c_{13}}{2} - K^2\frac{c_{11}}{4}(c_4^2 - c_1c_2)\right)(KA_3 + E_5X) - 2D_5(A_3 + k_1X) \\
- (E_5 - Kk_1)\{X, A_1\} = 0. \tag{3.7}
\end{aligned}$$

These equations split into terms of order 3,2,1 and 0. From equation (3.5) we find

$$C_5 = \frac{c_{14}}{2}K - \frac{c_{12}}{4}K^3(c_4^2 - c_1c_2), \quad D_5 = -\frac{c_{13}}{2}K + \frac{c_{11}}{4}K^3(c_4^2 - c_1c_2) \tag{3.8}$$

except, possibly, for some degenerate cases. The condition that (3.8) is the unique solution of (3.5) is exactly that the set $([X, S_1], [X, S_2])$ is linearly independent. Otherwise the solution, though it always exists, is not unique.

Substituting (3.8) into (3.6) and (3.7), we have

$$\begin{aligned}
\frac{2C_5(E_5 - Kk_1)}{K}X + (E_5 - Kk_1)\{X, A_2\} &= 0, \\
\frac{2D_5(E_5 - Kk_1)}{K}X - (E_5 - Kk_1)\{X, A_1\} &= 0,
\end{aligned}$$

whence we find

$$\begin{aligned}
E_5 &= Kc_{16} - K^3c_7(c_4^2 - c_1c_2) + K^3(c_4c_9 - c_2c_8)\left(c_8 + \frac{c_{11}}{2}\right) \\
&+ K^3(c_4c_8 - c_1c_9)\left(c_9 + \frac{c_{12}}{2}\right).
\end{aligned}$$

We conclude in both the classical and quantum cases that the Casimirs of superintegrable systems determine the structure equations.

4 Free 2D 2nd order superintegrable systems

As was shown in [14, 15, 16, 17] the “free” 2nd order superintegrable system obtained by setting all the parameters in a nondegenerate potential equal to zero retains all of the information needed to reconstruct the potential. Thus we can, in principle, restrict our attention to free systems. First we review from [14, 15, 16, 17, 21] how the structure equations for 2D 2nd

order nondegenerate classical superintegrable systems are determined. Such a system admits a symmetry $\mathcal{L} = \sum a^{ij} p_i p_j + W$ if and only if $\{\mathcal{H}, \mathcal{L}\} = 0$, i.e., the Killing equations are satisfied and $W_i = \lambda \sum_{j=1}^2 a^{ij} V_j$. Here $W_i = \partial_{x_i} W$ with a similar convention for subscripts on V .

The equations for W can be solved provided the Bertrand–Darboux equation $\partial_{x_1} W_2 = \partial_{x_2} W_1$ holds. For a superintegrable system with independent symmetries $\mathcal{L}_1 = \sum a^{ij} p_i p_j + W^{(1)}$, $\mathcal{L}_2 = \sum b^{ij} p_i p_j + W^{(2)}$, we can solve the two independent Bertrand–Darboux equations for the potential to obtain the canonical system

$$V_{22} - V_{11} = A^{22} V_1 + B^{22} V_2, \quad V_{12} = A^{12} V_1 + B^{12} V_2. \quad (4.1)$$

Here,

$$\begin{aligned} A^{12} &= -G_2 + \frac{D^{(2)}}{D}, & A^{22} &= 2G_1 + \frac{D^{(3)}}{D}, \\ B^{12} &= -G_1 - \frac{D^{(0)}}{D}, & B^{22} &= -2G_2 - \frac{D^{(1)}}{D}, \\ D &= \det \begin{pmatrix} a^{11} - a^{22} & a^{12} \\ b^{11} - b^{22} & b^{12} \end{pmatrix}, & D^{(0)} &= \det \begin{pmatrix} 3a_2^{12} & -a^{12} \\ 3b_2^{12} & -b^{12} \end{pmatrix}, \\ D^{(1)} &= \det \begin{pmatrix} 3a_2^{12} & a^{11} - a^{22} \\ 3b_2^{12} & b^{11} - b^{22} \end{pmatrix}, & D^{(2)} &= \det \begin{pmatrix} 3a_1^{12} & a^{12} \\ 3b_1^{12} & b^{12} \end{pmatrix}, \\ D^{(3)} &= \det \begin{pmatrix} 3a_1^{12} & a^{11} - a^{22} \\ 3b_1^{12} & b^{11} - b^{22} \end{pmatrix}, \end{aligned}$$

where $\lambda = \exp G$. If the integrability equations for (4.1) are satisfied identically then the solution space is 4-dimensional and we can always express the solution in the form $V(\mathbf{x}) = \sum_{j=1}^3 a_j V_{(j)}(\mathbf{x}) + a_4$, where a_4 is a trivial additive constant. In this case the potential is *nondegenerate* and 3-parameter. Another possibility is that the solution space is 2-dimensional with general solution $V(\mathbf{x}) = a_1 V_{(1)}(\mathbf{x}) + a_2$. For nondegenerate superintegrability, the integrability conditions for the canonical equations must be satisfied identically, so that V , V_1 , V_2 , V_{11} can be prescribed arbitrarily at a fixed regular point.

To obtain the integrability conditions for equations (4.1) we introduce the dependent variables $W^{(1)} = V_1$, $W^{(2)} = V_2$, $W^{(3)} = V_{11}$, and matrices

$$\begin{aligned} \mathbf{w} &= \begin{pmatrix} W^{(1)} \\ W^{(2)} \\ W^{(3)} \end{pmatrix}, & \mathbf{A}^{(1)} &= \begin{pmatrix} 0 & 0 & 1 \\ A^{12} & B^{12} & 0 \\ A^{13} & B^{13} & B^{12} - A^{22} \end{pmatrix}, & \mathbf{A}^{(2)} &= \begin{pmatrix} A^{12} & B^{12} & 0 \\ A^{22} & B^{22} & 1 \\ A^{23} & B^{23} & A^{12} \end{pmatrix}, \\ A^{13} &= A_2^{12} - A_1^{22} + B^{12} A^{22} + A^{12} A^{12} - B^{22} A^{12}, \\ B^{13} &= B_2^{12} - B_1^{22} + A^{12} B^{12}, & A^{23} &= A_1^{12} + B^{12} A^{12}, & B^{23} &= B_1^{12} + B^{12} B^{12}. \end{aligned}$$

Then the integrability conditions for system $\partial_{x_j} \mathbf{w} = \mathbf{A}^{(j)} \mathbf{w}$, $j = 1, 2$, must hold:

$$A_1^{(2)} - A_2^{(1)} = A^{(1)} A^{(2)} - A^{(2)} A^{(1)} \equiv [A^{(1)}, A^{(2)}]. \quad (4.2)$$

If and only if (4.2) holds, the system has a 4D vector space of solutions V .

There is a similar analysis for a “free” 2nd order superintegrable system obtained by setting the parameter in a *degenerate* potential equal to zero, [18]: The free system retains all of the information needed to reconstruct the potential. All such degenerate superintegrable systems with potential are restrictions of nondegenerate systems obtained by restricting the parameters

so that one 2nd order symmetry becomes a perfect square, e.g., $\mathcal{L}_1 = \mathcal{X}^2$. Then \mathcal{X} is a 1st order constant, necessarily of the form $\mathcal{X} = \xi_1 p_1 + \xi_2 p_2$, without a function term. Since the degenerate systems are obtained by restriction, the potential function must satisfy the equations (4.1) inherited from the nondegenerate system, with the same functions A^{ij} , B^{ij} . In addition the relation $\{\mathcal{X}, \mathcal{H}\} = 0$ imposes the condition $\xi_1 V_1 + \xi_2 V_2 = 0$. By relabeling the coordinates, we can always assume $\xi_2 \neq 0$ and write the system of equations for the potential in the form $V_2 = C^2 V_1$, $V_{22} = V_{11} + C^{22} V_1$, $V_{12} = C^{12} V_1$, where

$$C^2(x_1, x_2) = -\frac{\xi_1}{\xi_2}, \quad C^{22}(x_1, x_2) = A^{22} - \frac{\xi_1}{\xi_2} B^{22}, \quad C^{12}(x_1, x_2) = A^{12} - \frac{\xi_1}{\xi_2} B^{12}.$$

To find integrability conditions for these equations we introduce matrices

$$\mathbf{v} = \begin{pmatrix} V \\ V_1 \end{pmatrix}, \quad \mathbf{B}^{(1)} = \begin{pmatrix} 0 & 1 \\ 0 & \partial_2 C^2 + C^{22} C^{12} - C^{22} \end{pmatrix}, \quad \mathbf{B}^{(2)} = \begin{pmatrix} 0 & C^2 \\ 0 & C^{12} \end{pmatrix}.$$

Then integrability conditions for system $\partial_{x_j} \mathbf{v} = \mathbf{B}^{(j)} \mathbf{v}$, $j = 1, 2$, must hold:

$$B_1^{(2)} - B_2^{(1)} = B^{(1)} B^{(2)} - B^{(2)} B^{(1)} \equiv [B^{(1)}, B^{(2)}]. \quad (4.3)$$

If and only if (4.3) holds, the system has a 2D space of solutions V . Since $V = \text{constant}$ is always a solution, (4.3) is necessary and sufficient for the existence of a nonzero 1-parameter potential system. In this case we can prescribe the values V , V_2 at any regular point \mathbf{x}_0 ; there will exist a unique $V(\mathbf{x})$ taking these values.

4.1 Free triplets

Here we review information about free triplets that was presented and proved in [21]. A 2nd order classical free triplet is a 2D system without potential, $\mathcal{H}_0 = \frac{p_1^2 + p_2^2}{\lambda(x, y)}$ and with a basis of 3 functionally independent second-order constants of the motion $\mathcal{L}_{(s)} = \sum_{i,j=1}^2 a_{(s)}^{ij} p_i p_j$, $a_{(s)}^{ij} = a_{(s)}^{ji}$, $s = 1, 2, 3$, $\mathcal{L}_{(3)} = \mathcal{H}_0$. Since the duals of these constants of the motion are 2nd order Killing tensors, the spaces associated with free triplets can be characterized as 2D manifolds that admit 3 functionally independent 2nd order Killing tensors. As mentioned above, they have been classified in [23]. Since the vectors $\{\mathbf{h}_{(s)}\}$, $\mathbf{h}_{(s)}^{\text{tr}}(x, y, z) = (a_{(s)}^{11}, a_{(s)}^{12}, a_{(s)}^{22})$ form a linearly independent set, there exist unique 3×3 matrices $\mathcal{C}^{(j)}$ such that $\partial_{x_j} \mathbf{h}_{(s)} = \mathcal{C}^{(j)} \mathbf{h}_{(s)}$, $j, s = 1, 2$. By linearity, any element $\mathcal{L} = \sum_{i,j=1}^2 a^{ij} p_i p_j$ of the space of 2nd order symmetries spanned by the basis triplet is characterized by matrix equations

$$\partial_{x_j} \mathbf{h} = \mathcal{C}^{(j)} \mathbf{h}, \quad j = 1, 2, \quad \mathbf{h}^{\text{tr}}(x, y, z) = (a^{11}, a^{12}, a^{22}). \quad (4.4)$$

In particular, at any regular point \mathbf{x}_0 we can arbitrarily choose the value of the 3-vector \mathbf{h}_0 and solve (4.4) to find the unique symmetry \mathcal{L} of \mathcal{H}_0 such that $\mathbf{h}(\mathbf{x}_0) = \mathbf{h}_0$. A normalization condition for the $\mathcal{C}^{(j)}$: (4.4) is valid for $a^{11} = a^{22} = 1/\lambda$, $a^{12} = 0$, i.e., for \mathcal{H}_0 . From this and the requirement that the \mathcal{L} are free constants of the motion we find

$$\mathcal{C}^{(1)} = \begin{pmatrix} -G_1 & -G_2 & 0 \\ -\frac{1}{2} \mathcal{C}_{11}^{(2)} & -\frac{1}{2} G_1 - \frac{1}{2} \mathcal{C}_{12}^{(2)} & \frac{1}{2} \mathcal{C}_{11}^{(2)} \\ -G_1 - 2\mathcal{C}_{21}^{(2)} & -G_2 - 2\mathcal{C}_{22}^{(2)} & 2\mathcal{C}_{21}^{(2)} \end{pmatrix}, \quad \mathcal{C}^{(2)} = \begin{pmatrix} \mathcal{C}_{11}^{(2)} & \mathcal{C}_{12}^{(2)} & -G_2 - \mathcal{C}_{11}^{(2)} \\ \mathcal{C}_{21}^{(2)} & \mathcal{C}_{22}^{(2)} & -\mathcal{C}_{21}^{(2)} \\ 0 & -G_1 & -G_2 \end{pmatrix},$$

with the 4 functions $\mathcal{C}_{11}^{(2)}$, $\mathcal{C}_{12}^{(2)}$, $\mathcal{C}_{21}^{(2)}$, $\mathcal{C}_{22}^{(2)}$ free. If we define the functions A^{12} , B^{12} , A^{22} , B^{22} by the requirement

$$\begin{aligned}\mathcal{C}_{11}^{(2)} &= -\frac{2}{3}G_2 - \frac{2}{3}A^{12}, & \mathcal{C}_{12}^{(2)} &= \frac{1}{3}G_1 - \frac{2}{3}A^{22}, \\ \mathcal{C}_{21}^{(2)} &= -\frac{1}{3}G_1 - \frac{1}{3}B^{12}, & \mathcal{C}_{22}^{(2)} &= -\frac{2}{3}G_2 - \frac{1}{3}B^{22},\end{aligned}$$

then equations (4.4) agree with the equations that are obtained from a superintegrable system with nondegenerate potential satisfying (4.1). Thus, for a free system there always exist unique functions A^{ij} , B^{ij} . Then necessary and sufficient conditions for extension to a system with nondegenerate potential V satisfying equations (4.1) are that conditions (4.2) hold identically.

This analysis also extends, via restriction, to superintegrable systems with degenerate potential. A free triplet that corresponds to a degenerate superintegrable system is one that corresponds to a nondegenerate system but such that one of the free generators can be chosen as a perfect square. For these systems conditions (4.3) for the potential are satisfied identically.

Similarly, we define a *2nd order quantum free triplet* as a 2D quantum system without potential, $H_0 = \frac{1}{\lambda(\mathbf{x})}(\partial_{11} + \partial_{22})$, and with a basis of 3 algebraically independent second-order symmetry operators

$$L_k = \frac{1}{\lambda} \sum_{i,j=1}^2 \partial_i (\lambda a_{(k)}^{ij} \partial_j)(\mathbf{x}), \quad k = 1, 2, 3, \quad a_{(k)}^{ij} = a_{(k)}^{ji}, \quad L_3 = H_0.$$

There is a 1-1 relationship between classical and quantum free triplets.

5 Free Darboux systems

The Darboux spaces admit a 1-dimensional space of Killing vectors and a 4-dimensional space of 2nd order Killing tensors. Thus each space can admit at most one superintegrable system with degenerate potential, and each space does so. We merely need to check that equations (4.2) are satisfied. Then we can compute the degenerate potential. Turning off the 1-parameter potential produces a single free degenerate quadratic algebra which we list below. There are no more possibilities. There are a number of possibilities for free triplets to define a nondegenerate quadratic algebra for a Darboux space, however. We classify the possibilities up to conjugacy under the action of the 1-parameter symmetry group of the manifold. Note that the 4-dimensional space of free constants of the motion is not obtained from the enveloping algebra of an underlying symmetry group. We shall see that there is a 1-1 relationship between free quadratic algebras and restrictions of quadratic algebras of nondegenerate superintegrable systems. We adopt the labeling of superintegrable systems and constants of the motion on Darboux spaces introduced in [19], using a tilde to differentiate between a free triplet and its associated superintegrable system. In the following sections, with the aid of MAPLE, we classify all possible free quadratic algebras generated by the 2nd order Killing tensors, up to conjugacy. Then, using MAPLE again, we verify for each quadratic algebra that the integrability conditions (4.3) are satisfied and we compute the nondegenerate potentials. Most of the results are presented in lists but in Section 5.2 we give more details on the construction of the superintegrable system with potential in one case.

Each of the Darboux spaces can be embedded as a surface in 3 dimensions if we regard the ignorable variable as an angle, i.e., $X = f(x) \cos(y)$, $Y = f(x) \sin(y)$, $Z = h(x)$, and this is not unique [5]. We give an illustrative example for each case.

5.0.1 Free Darboux 1 systems

The space Darboux 1 (D_1) has free degenerate Hamiltonian

$$\tilde{\mathbf{D1D}}: \quad \mathcal{H} = \frac{1}{4x}(p_x^2 + p_y^2)$$

with a single Killing vector $\mathcal{K} = p_y$ and a basis, $\{\mathcal{H}, \mathcal{K}^2, \mathcal{X}_1, \mathcal{X}_2\}$ for the 4-dimensional space of 2nd order Killing tensors. Here,

$$\mathcal{X}_1 = p_x p_y - \frac{y}{2x}(p_x^2 + p_y^2), \quad \mathcal{X}_2 = p_y(y p_x - x p_y) - \frac{y^2}{4x}(p_x^2 + p_y^2).$$

The commutation relations are

$$\{\mathcal{K}, \mathcal{X}_1\} = -2\mathcal{H}, \quad \{\mathcal{K}, \mathcal{X}_2\} = \mathcal{X}_1, \quad \{\mathcal{X}_1, \mathcal{X}_2\} = -2\mathcal{K}^3,$$

and there is the functional relation $4\mathcal{H}\mathcal{X}_2 + \mathcal{X}_1^2 + \mathcal{K}^4 = 0$. The degenerate potential is $V(x, y) = \frac{b_1}{x} + b_2$.

As shown in [20], a possible embedding of this system in 3-dimensional Euclidean space with Cartesian coordinates X, Y, Z is

$$X = 2\sqrt{x} \cos y, \quad Y = 2\sqrt{x} \sin y, \quad Z = \frac{2}{3} \left(F \left(\phi, \frac{1}{\sqrt{2}} \right) + \sqrt{4x^3 - x} \right),$$

where $x \geq \frac{1}{2}$, y is 2π -periodic and $\sin \phi = \sqrt{2x + 1}$. Here $F(\phi, k)$ is an incomplete elliptic integral of the 1st kind. Then $ds^2 = 4x(dx^2 + dy^2) = dX^2 + dY^2 + dZ^2$.

A general 2nd order symmetry, mod \mathcal{H} , can be written as $\mathcal{L}_1 = a_1\mathcal{X}_2 + a_2\mathcal{X}_1 + a_3\mathcal{K}^2$, and the translation group generated by \mathcal{K} : $x \rightarrow x, y \rightarrow y + \alpha$, leaves \mathcal{K}^2 and \mathcal{H} invariant, but $\mathcal{X}_1 \rightarrow \mathcal{X}_1 - 2\alpha\mathcal{H}, \mathcal{X}_2 \rightarrow \mathcal{X}_2 + \alpha\mathcal{X}_1 - \alpha^2\mathcal{H}$.

We classify the distinct free nondegenerate systems under this conjugacy action. We choose one generator \mathcal{L}_1 and determine the possibilities for \mathcal{L}_2 such that $\mathcal{L}_1, \mathcal{L}_2, \mathcal{H}$ form a quadratic algebra. Then we eliminate redundancies. Under conjugacy we can assume that \mathcal{L}_1 takes one of the 3 forms $\mathcal{X}_1 + a\mathcal{K}^2, \mathcal{X}_2 + a\mathcal{K}^2, \mathcal{K}^2$.

1st case. We choose $\mathcal{L}_1 = \mathcal{X}_2 + a\mathcal{K}^2$ and try to determine the possibilities for \mathcal{L}_2 , up to conjugacy under $e^{\alpha\mathcal{K}}$, such that $\mathcal{L}_1, \mathcal{L}_2, \mathcal{H}$ form a quadratic algebra. (As we go through the cases step by step, we ignore systems that have already been exhibited in earlier steps.) In general $\mathcal{L}_2 = c_1\mathcal{X}_1 + c_3\mathcal{K}^2$ and c_1, c_3 are to be determined. We must require that

$$\begin{aligned} \mathcal{R}^2 = & b_1\mathcal{L}_1^3 + b_2\mathcal{L}_2^3 + b_3\mathcal{H}^3 + b_4\mathcal{L}_1^2\mathcal{L}_2 + b_5\mathcal{L}_1\mathcal{L}_2^2 + b_6\mathcal{L}_1\mathcal{L}_2\mathcal{H} \\ & + b_7\mathcal{L}_1^2\mathcal{H} + b_8\mathcal{L}_2^2\mathcal{H} + b_9\mathcal{H}^2\mathcal{L}_1 + b_{10}\mathcal{H}^2\mathcal{L}_2, \end{aligned}$$

for some constants b_1, \dots, b_{10} . There are 2 possible classes:

- $\tilde{\mathbf{D1A}}$:** $\mathcal{L}_1 = \mathcal{X}_2 + b\mathcal{K}^2, \mathcal{L}_2 = \mathcal{X}_1 + i\mathcal{K}^2, \mathcal{R}^2 = 2i\mathcal{L}_2^3 + 8i\mathcal{L}_1\mathcal{L}_2\mathcal{H} + 4b\mathcal{L}_2^2\mathcal{H} + 16b\mathcal{H}^2\mathcal{L}_1$. This class is superintegrable with

$$\begin{aligned} A^{12} = 0, \quad A^{22} = \frac{2}{x}, \quad B^{12} = -\frac{1}{2} \frac{5x - 2b + 2iy}{x(x - b + iy)}, \quad B^{22} = \frac{-3i}{x - b + iy}, \\ G(x, y) = \ln(4x), \quad D = -\frac{1}{2}(x - b + iy). \end{aligned}$$

The potential of the superintegrable system is

$$V(x, y) = \frac{b_1(2x - 2b + iy)}{x\sqrt{x - b + iy}} + \frac{b_2}{x\sqrt{x - b + iy}} + \frac{b_3}{x} + b_4.$$

(This is missing in the tabulation in [20], but pointed out in [19] and [24].)

2. $\tilde{\mathbf{D1B}}$: $\mathcal{L}_1 = \mathcal{X}_2$, $\mathcal{L}_2 = \mathcal{K}^2$, $\mathcal{R}^2 = -4\mathcal{L}_2^3 - 16\mathcal{L}_1\mathcal{L}_2\mathcal{H}$,

$$A^{12} = 0, \quad A^{22} = \frac{2}{x}, \quad B^{12} = -\frac{1}{x}, \quad B^{22} = -\frac{3}{y},$$

$$G(x, y) = \ln(4x), \quad D = -\frac{y}{2}.$$

(Listed as a superintegrable system in the tabulation in [20].) The potential of the superintegrable system is

$$V(x, y) = \frac{b_1(4x^2 + y^2)}{x} + \frac{b_2}{x} + \frac{b_3}{xy^2} + b_4.$$

2nd case. We choose $\mathcal{L}_1 = \mathcal{X}_1$. Then the only possibility is

3. $\tilde{\mathbf{D1C}}$: $\mathcal{L}_1 = \mathcal{X}_1$, $\mathcal{L}_2 = \mathcal{K}^2$, $\mathcal{R}^2 = 16\mathcal{L}_2\mathcal{H}^2$,

$$A^{12} = 0, \quad A^{22} = \frac{2}{x}, \quad B^{12} = -\frac{1}{x}, \quad B^{22} = 0,$$

$$G(x, y) = \ln(4x), \quad D = -\frac{1}{2}.$$

(Listed in the tabulation in [20].) The potential of the superintegrable system is

$$V(x, y) = \frac{b_1(x^2 + y^2)}{x} + \frac{b_2}{x} + \frac{b_3y}{x} + b_4.$$

5.0.2 Free Darboux 2 systems

The space Darboux 2 (D_2) has free degenerate Hamiltonian

$$\tilde{\mathbf{D2D}}: \quad \mathcal{H} = \frac{x^2}{x^2 + 1}(p_x^2 + p_y^2)$$

with a single Killing vector $\mathcal{K} = p_y$ and a basis, $\{\mathcal{H}, \mathcal{K}^2, \mathcal{X}_1, \mathcal{X}_2\}$ for the 4-dimensional space of 2nd order Killing tensors. Here,

$$\mathcal{X}_1 = 2xp_xp_y + \frac{2y}{x^2 + 1}(p_y^2 - x^2p_x^2), \quad \mathcal{X}_2 = 2xyp_xp_y + \frac{(y^2 - x^4)p_y^2 + x^2(1 - y^2)p_x^2}{x^2 + 1}.$$

The commutation relations are

$$\{\mathcal{K}, \mathcal{X}_1\} = 2(\mathcal{K}^2 - \mathcal{H}), \quad \{\mathcal{K}, \mathcal{X}_2\} = \mathcal{X}_1, \quad \{\mathcal{X}_1, \mathcal{X}_2\} = 4\mathcal{K}\mathcal{X}_2,$$

and there is the functional relation $4\mathcal{H}\mathcal{X}_2 + \mathcal{X}_1^2 - 4\mathcal{K}^2\mathcal{X}_2 - 4\mathcal{H}^2 = 0$. The degenerate potential is

$$V(x, y) = \frac{b_1}{x^2 + 1} + b_2.$$

As shown in [19] The line element ds^2 can be realized as a two-dimensional surface embedded in three dimensions by

$$X = \frac{y\sqrt{x^2 + 1}}{x}, \quad Y - Z = \frac{\sqrt{x^2 + 1}}{x},$$

$$Y + Z = -\frac{(2x^4 + 5x^2 + 8y^2)\sqrt{x^2 + 1}}{8x} - \frac{3}{8} \operatorname{arcsinh} x,$$

in which case,

$$ds^2 = dX^2 + dY^2 - dZ^2 = \frac{x^2 + 1}{x^2} (dx^2 + dy^2).$$

A general 2nd order symmetry, mod \mathcal{H} , can be written as $\mathcal{L}_1 = a_1\mathcal{X}_2 + a_2\mathcal{X}_1 + a_3\mathcal{K}^2$ and the translation group generated by \mathcal{K} : $x \rightarrow x$, $y \rightarrow y + \alpha$, leaves \mathcal{K}^2 and \mathcal{H} invariant, but

$$\mathcal{X}_1 \rightarrow \mathcal{X}_1 - 2\alpha\mathcal{H} + 2\alpha\mathcal{K}^2, \quad \mathcal{X}_2 \rightarrow \mathcal{X}_2 + \alpha\mathcal{X}_1 + \alpha^2(\mathcal{K}^2 - \mathcal{H}).$$

We classify the distinct free nondegenerate superintegrable systems under this conjugacy action. We choose one generator \mathcal{L}_1 and determine the possibilities for \mathcal{L}_2 such that $\mathcal{L}_1, \mathcal{L}_2, \mathcal{H}$ form a quadratic algebra. Under conjugacy there are 3 possible choices: $\mathcal{L}_1 = \mathcal{X}_2 + a\mathcal{K}^2, \mathcal{X}_1, \mathcal{K}^2$.

1st case. We choose $\mathcal{L}_1 = \mathcal{X}_2 + a\mathcal{K}^2$ and try to determine the possibilities for \mathcal{L}_2 , up to conjugacy under $e^{\alpha\mathcal{K}}$ such that $\mathcal{L}_1, \mathcal{L}_2, \mathcal{H}$ form a quadratic algebra. (As we go through the cases step by step, we ignore systems that have already been exhibited in earlier steps.) In general $\mathcal{L}_2 = c_1\mathcal{X}_1 + c_3\mathcal{K}^2$ and c_1, c_3 are to be determined. We must require that

$$\begin{aligned} \mathcal{R}^2 = & b_1\mathcal{L}_1^3 + b_2\mathcal{L}_2^3 + b_3\mathcal{H}^3 + b_4\mathcal{L}_1^2\mathcal{L}_2 + b_5\mathcal{L}_1\mathcal{L}_2^2 + b_6\mathcal{L}_1\mathcal{L}_2\mathcal{H} \\ & + b_7\mathcal{L}_1^2\mathcal{H} + b_8\mathcal{L}_2^2\mathcal{H} + b_9\mathcal{H}^2\mathcal{L}_1 + b_{10}\mathcal{H}^2\mathcal{L}_2, \end{aligned}$$

for some constants b_1, \dots, b_{10} . There are 2 possible classes:

$$1. \ \tilde{\mathbf{D2C}}: \mathcal{L}_1 = \mathcal{X}_2, \mathcal{L}_2 = \mathcal{X}_1, \mathcal{R}^2 = 4\mathcal{L}_1\mathcal{L}_2^2 + 16\mathcal{L}_1^2\mathcal{H} - 16\mathcal{L}_1\mathcal{H}^2,$$

$$\begin{aligned} A^{12} = 0, \quad A^{22} = \frac{3x^2 - 1}{x(x^2 + 1)}, \quad B^{12} = -\frac{(3x^4 + x^2 - 2y^2)}{x(x^2 + 1)(x^2 + y^2)}, \\ B^{22} = -\frac{6y}{x^2 + y^2}, \quad G(x, y) = \ln\left(\frac{x^2 + 1}{x^2}\right), \quad D = -x(x^2 + y^2). \end{aligned}$$

The potential of the superintegrable system is

$$V(x, y) = \frac{x^2}{\sqrt{x^2 + y^2}(x^2 + 1)} \left(b_1 + \frac{b_2}{y + \sqrt{x^2 + y^2}} + \frac{b_3}{y - \sqrt{x^2 + y^2}} \right) + b_4.$$

(Listed as a superintegrable system in [19].)

$$2. \ \tilde{\mathbf{D2B}}: \mathcal{L}_1 = \mathcal{X}_2, \mathcal{L}_2 = \mathcal{K}^2, \mathcal{R}^2 = 16\mathcal{L}_1\mathcal{L}_2^2 - 16\mathcal{L}_1\mathcal{L}_2\mathcal{H} + 16\mathcal{L}_2\mathcal{H}^2,$$

$$\begin{aligned} A^{12} = 0, \quad A^{22} = \frac{3x^2 - 1}{x(x^2 + 1)}, \quad B^{12} = -\frac{2}{x(x^2 + 1)}, \quad B^{22} = -\frac{3}{y}, \\ G(x, y) = \ln\left(\frac{x^2 + 1}{x^2}\right), \quad D = -xy. \end{aligned}$$

The potential of the superintegrable system is

$$V(x, y) = \frac{x^2}{x^2 + 1} \left(b_1(x^2 + y^2) + \frac{b_2}{x^2} + \frac{b_3}{y^2} \right) + b_4.$$

(Listed as a superintegrable system in [19].)

2nd case. We choose $\mathcal{L}_1 = \mathcal{X}_1$. Then the only possibility is

$$3. \tilde{\mathbf{D}}2\mathbf{A}: \mathcal{L}_1 = \mathcal{X}_1, \mathcal{L}_2 = \mathcal{K}^2, \mathcal{R}^2 = 16\mathcal{L}_2^3 - 32\mathcal{L}_2^2\mathcal{H} + 16\mathcal{L}_2\mathcal{H}^2,$$

$$A^{12} = 0, \quad A^{22} = \frac{3x^2 - 1}{x(x^2 + 1)}, \quad B^{12} = \frac{2}{x(x^2 + 1)}, \quad B^{22} = 0,$$

$$G(x, y) = \ln\left(\frac{x^2 + 1}{x^2}\right), \quad D = -x.$$

The potential of the superintegrable system is

$$V(x, y) = \frac{x^2}{x^2 + 1} \left(b_1(x^2 + 4y^2) + \frac{b_2}{x^2} + b_3y \right) + b_4.$$

(Listed as a superintegrable system in [19].)

5.0.3 Free Darboux 3 systems

The space Darboux 3 (D_3) has free degenerate Hamiltonian

$$\tilde{\mathbf{D}}3\mathbf{E}: \quad \mathcal{H} = \frac{1}{2} \frac{e^{2x}}{e^x + 1} (p_x^2 + p_y^2)$$

with a single Killing vector $\mathcal{K} = p_y$ and a basis, $\{\mathcal{H}, \mathcal{K}^2, \mathcal{X}_1, \mathcal{X}_2\}$ for the 4-dimensional space of 2nd order Killing tensors. Here,

$$\begin{aligned} \mathcal{X}_1 &= \frac{1}{2} e^x \sin y p_x p_y + \frac{1}{4} \frac{e^{2x}}{e^x + 1} \cos y p_x^2 - \frac{1}{4} \frac{e^x(e^x + 2)}{e^x + 1} \cos y p_y^2, \\ \mathcal{X}_2 &= -\frac{1}{2} e^x \cos y p_x p_y + \frac{1}{4} \frac{e^{2x}}{e^x + 1} \sin y p_x^2 - \frac{1}{4} \frac{e^x(e^x + 2)}{e^x + 1} \sin y p_y^2. \end{aligned}$$

The commutation relations are

$$\{\mathcal{K}, \mathcal{X}_1\} = -\mathcal{X}_2, \quad \{\mathcal{K}, \mathcal{X}_2\} = \mathcal{X}_1, \quad \{\mathcal{X}_1, \mathcal{X}_2\} = \frac{1}{2} \mathcal{K} \mathcal{H},$$

and there is the functional relation $\mathcal{X}_1^2 + \mathcal{X}_2^2 - \frac{1}{4} \mathcal{H}^2 - \frac{1}{2} \mathcal{K}^2 \mathcal{H} = 0$. The degenerate potential is

$$V(x, y) = \frac{b_1}{e^x + 1} + b_2.$$

As shown in [19], we can embed D_3 as a surface in 3D Minkowski space with coordinates X , Y , Z in such a way as to preserve rotational symmetry. Let

$$\begin{aligned} X &= 2\sqrt{2}e^{-\frac{x}{2}} \sqrt{1 + e^{-x}} \cos \frac{y}{2}, \quad Y = 2\sqrt{2}e^{-\frac{x}{2}} \sqrt{1 + e^{-x}} \sin \frac{y}{2}, \\ Z &= \frac{\sqrt{6}}{12} \ln \left(\frac{\frac{\sqrt{3}}{6}(6 + 5e^x)\sqrt{3 + 2e^{2x} + 5e^x} + 1}{\frac{\sqrt{3}}{6}(6 + 5e^x)\sqrt{3 + 2e^{2x} + 5e^x} - 1} \right) - e^{-x} \sqrt{2} \sqrt{3 + 2e^{2x} + 5e^x}. \end{aligned}$$

Then

$$dX^2 + dY^2 - dZ^2 = \frac{2(e^x + 1)}{e^{2x}} (dx^2 + dy^2).$$

An alternate basis is $\{\mathcal{H}, \mathcal{K}^2, \mathcal{Y}_1, \mathcal{Y}_2\}$, where

$$\mathcal{Y}_1 = \left(e^x p_x p_y + \frac{i}{2} \frac{e^{2x}}{e^x + 1} p_x^2 - \frac{i}{2} \frac{e^x(e^x + 2)}{e^x + 1} p_y^2 \right) e^{iy},$$

$$\mathcal{Y}_2 = \left(e^x p_x p_y - \frac{i}{2} \frac{e^{2x}}{e^x + 1} p_x^2 + \frac{i}{2} \frac{e^x(e^x + 2)}{e^x + 1} p_y^2 \right) e^{-iy},$$

and $\mathcal{X}_1 = -\frac{i}{4}(\mathcal{Y}_1 - \mathcal{Y}_2)$, $\mathcal{X}_2 = -\frac{1}{4}(\mathcal{Y}_1 + \mathcal{Y}_2)$. The new commutation relations are

$$\{\mathcal{K}, \mathcal{Y}_1\} = i\mathcal{Y}_1, \quad \{\mathcal{K}, \mathcal{Y}_2\} = -i\mathcal{Y}_2, \quad \{\mathcal{Y}_1, \mathcal{Y}_2\} = -4i\mathcal{K}\mathcal{H},$$

and the functional relation is $\mathcal{Y}_1\mathcal{Y}_2 - \mathcal{H}^2 - 2\mathcal{K}^2\mathcal{H} = 0$.

Returning to the first basis, we note that a general 2nd order symmetry, mod \mathcal{H} , can be written as $\mathcal{L}_1 = a_1\mathcal{X}_2 + a_2\mathcal{X}_1 + a_3\mathcal{K}^2$. and the translation group generated by \mathcal{K} : $x \rightarrow x$, $y \rightarrow y + \alpha$, leaves \mathcal{K}^2 and \mathcal{H} invariant, but $\mathcal{X}_1 \rightarrow \cos \alpha \mathcal{X}_1 - \sin \alpha \mathcal{X}_2$, $\mathcal{X}_2 \rightarrow \sin \alpha \mathcal{X}_1 + \cos \alpha \mathcal{X}_2$. We classify the distinct free superintegrable systems under this conjugacy action. We choose one generator \mathcal{L}_1 and determine the possibilities for \mathcal{L}_2 such that $\mathcal{L}_1, \mathcal{L}_2, \mathcal{H}$ form a quadratic algebra. Under conjugacy the choices are $\mathcal{L}_1 = \mathcal{X}_1 + a\mathcal{K}^2$, $\mathcal{X}_1 + i\mathcal{X}_2$, $\mathcal{X}_1 + i\mathcal{X}_2 - \mathcal{K}^2$, $\mathcal{X}_1, \mathcal{K}^2$.

1st case. We choose $\mathcal{L}_1 = \mathcal{X}_1 + a\mathcal{K}^2$ and try to determine the possibilities for a, \mathcal{L}_2 , up to conjugacy under $e^{\alpha\mathcal{K}}$ such that $\mathcal{L}_1, \mathcal{L}_2, \mathcal{H}$ form a quadratic algebra. (As we go through the cases step by step, we ignore systems that have already been exhibited in earlier steps.)

There are 3 possible classes:

1st class.

1. **$\tilde{\mathbf{D3A}}$:** $\mathcal{L}_1 = \mathcal{X}_1 - \frac{1}{2}\mathcal{K}^2$, $\mathcal{L}_2 = \mathcal{X}_2 + \frac{i}{2}\mathcal{K}^2$. $\mathcal{R}^2 = -\mathcal{L}_1^3 + i\mathcal{L}_2^3 - \frac{1}{8}\mathcal{H}^3 + i\mathcal{L}_1^2\mathcal{L}_2 - \mathcal{L}_1\mathcal{L}_2^2 + \frac{1}{2}(\mathcal{L}_1^2 + \mathcal{L}_2^2)\mathcal{H} + \frac{1}{2}\mathcal{L}_1\mathcal{H}^2 - \frac{i}{4}\mathcal{L}_2\mathcal{H}^2$. Here,

$$\begin{aligned} A^{12} &= 0, & A^{22} &= -\frac{e^x(-e^x + e^{-2x-iy} - e^{-iy} + e^{-x})}{(1+e^{-x})(e^x+1)(e^x+e^{-iy})}, & B^{22} &= \frac{3ie^{-iy}}{e^x+e^{-iy}}, \\ B^{12} &= \frac{1}{2} \frac{e^x(-1+2e^{-x-iy}+e^{-x}+4e^{-2x-iy})}{(1+e^{-x})(e^x+e^{-iy})}, \\ G(x, y) &= \ln(2(e^{-x}+e^{-2x})), & D &= \frac{e^x}{8}(e^x-2e^{iy}). \end{aligned}$$

The potential of the superintegrable system is

$$V(x, y) = \frac{b_1}{1+e^x} + \frac{b_2 e^x}{\sqrt{1+2e^{x+iy}}(1+e^x)} + \frac{b_3 e^{x+iy}}{\sqrt{1+2e^{x+iy}}(1+e^x)} + b_4.$$

(This is a superintegrable system listed in [19].)

2. **$\tilde{\mathbf{D3B}}$:** $\mathcal{L}_1 = \mathcal{X}_1$, $\mathcal{L}_2 = \mathcal{X}_2$, $\mathcal{R}^2 = -\frac{1}{8}\mathcal{H}^3 + \frac{1}{2}(\mathcal{L}_1^2 + \mathcal{L}_2^2)\mathcal{H}$,

$$\begin{aligned} A^{12} &= 0, & A^{22} &= \frac{1-e^{-x}}{1+e^{-x}}, & B^{22} &= 0, & B^{12} &= -\frac{1(1-e^{-x})}{2(1+e^{-x})}, \\ G(x, y) &= \ln(2(e^{-x}+e^{-2x})), & D &= \frac{e^{2x}}{8}. \end{aligned}$$

The potential of the superintegrable system is

$$V(x, y) = \frac{e^x}{e^x+1} \left(b_1 + e^{-\frac{x}{2}} \left(b_2 \cos \frac{y}{2} + b_3 \sin \frac{y}{2} \right) \right) + b_4.$$

(Listed as a superintegrable system in [19].)

3. **$\tilde{\mathbf{D3C}}$:** $\mathcal{L}_1 = \mathcal{X}_1$, $\mathcal{L}_2 = \mathcal{K}^2$, $\mathcal{R}^2 = -4\mathcal{L}_1^2\mathcal{L}_2 + 2\mathcal{L}_2^2\mathcal{H} + \mathcal{L}_2\mathcal{H}^2$,

$$A^{12} = 0, \quad A^{22} = \frac{1-e^{-x}}{1+e^{-x}}, \quad B^{12} = \frac{1+2e^{-x}}{1+e^{-x}}, \quad B^{22} = -3 \cot y,$$

$$G(x, y) = \ln(2(e^{-x} + e^{-2x})), \quad D = -\frac{e^x}{4} \sin y.$$

The potential of the superintegrable system is

$$V(x, y) = \frac{e^x}{e^x + 1} \left(b_1 + e^x \left(\frac{b_2}{\cos^2 \frac{y}{2}} + \frac{b_3}{\sin^2 \frac{y}{2}} \right) \right) + b_4.$$

(Listed as a superintegrable system in [19].)

2nd class. We choose $\mathcal{L}_1 = \mathcal{X}_1 + i\mathcal{X}_2$. Then the only new possibility is

$$4. \tilde{\mathbf{D3D}}: \mathcal{L}_1 = \mathcal{X}_1 + i\mathcal{X}_2, \mathcal{L}_2 = \mathcal{K}^2, \mathcal{R}^2 = -4\mathcal{L}_1^2\mathcal{L}_2,$$

$$A^{12} = 0, \quad A^{22} = \frac{1 - e^{-x}}{1 + e^x}, \quad B^{12} = \frac{1 + 2e^{-x}}{1 + e^{-x}}, \quad B^{22} = -3i,$$

$$G(x, y) = \ln(2(e^{-x} + e^{-2x})), \quad D = \frac{i}{4}e^{x+iy}.$$

The potential of the superintegrable system is

$$V(x, y) = \frac{e^{2x}}{1 + e^x} (b_1 e^{-iy} + b_2 e^{-2iy}) + \frac{b_3}{1 + e^x} + b_4.$$

(Listed as a superintegrable system in [19].)

3rd class. We choose $\mathcal{L}_1 = \mathcal{X}_1 + i\mathcal{X}_2 - \mathcal{K}^2$. There are no new possibilities.

5.0.4 Free Darboux 4 systems

The spaces Darboux 4 ($D_4(b)$) have free degenerate Hamiltonian

$$\tilde{\mathbf{D4(b)D}}: \quad \mathcal{H} = -\frac{\sin^2 2x}{2 \cos 2x + b} (p_x^2 + p_y^2), \quad (5.1)$$

$b \neq \pm 2$, with a single Killing vector $\mathcal{K} = p_y$ and a basis, $\{\mathcal{H}, \mathcal{K}^2, \mathcal{X}_1, \mathcal{X}_2\}$ for the 4-dimensional space of 2nd order Killing tensors. Here,

$$\mathcal{X}_1 = e^{2y} (-\mathcal{H} + \cos 2x p_y^2 + \sin 2x p_x p_y), \quad \mathcal{X}_2 = e^{-2y} (-\mathcal{H} + \cos 2x p_y^2 - \sin 2x p_x p_y).$$

The commutation relations are

$$\{\mathcal{K}, \mathcal{X}_1\} = 2\mathcal{X}_1, \quad \{\mathcal{K}, \mathcal{X}_2\} = -2\mathcal{X}_2, \quad \{\mathcal{X}_1, \mathcal{X}_2\} = -8\mathcal{K}^3 - 4b\mathcal{K}\mathcal{H},$$

and there is the functional relation $\mathcal{X}_1\mathcal{X}_2 - \mathcal{K}^4 - b\mathcal{K}^2\mathcal{H} - \mathcal{H}^2 = 0$. The degenerate potential is

$$V(x, y) = \frac{b_1}{2 \cos 2x + b} + b_2.$$

A general 2nd order symmetry, mod \mathcal{H} , can be written as $\mathcal{L}_1 = a_1\mathcal{X}_2 + a_2\mathcal{X}_1 + a_3\mathcal{K}^2$, and the translation group generated by \mathcal{K} : $x \rightarrow x, y \rightarrow y + \alpha$, leaves \mathcal{K}^2 and \mathcal{H} invariant, but $\mathcal{X}_1 \rightarrow e^{-2\alpha}\mathcal{X}_1, \mathcal{X}_2 \rightarrow e^{2\alpha}\mathcal{X}_2$. Also the reflection $y \rightarrow -y, x \rightarrow x$ leaves \mathcal{H} and \mathcal{K}^2 fixed but $\mathcal{X}_1 \leftrightarrow \mathcal{X}_2$. We classify the distinct free superintegrable systems under this conjugacy action. We choose one generator \mathcal{L}_1 and determine the possibilities for \mathcal{L}_2 such that $\mathcal{L}_1, \mathcal{L}_2, \mathcal{H}$ form a quadratic algebra. Under conjugacy the choices are $\mathcal{L}_1 = \mathcal{K}^2 + a\mathcal{X}_2, \mathcal{X}_1 + \mathcal{X}_2 + a\mathcal{K}^2$ ($a = 0, 2$, or $a \neq 0, 2$), $\mathcal{X}_2, \mathcal{K}^2$.

1st case. We choose $\mathcal{L}_1 = \mathcal{K}^2 + a\mathcal{X}_2$ and try to determine the possibilities for \mathcal{L}_2 , up to conjugacy under $e^{\alpha\mathcal{K}}$ such that $\mathcal{L}_1, \mathcal{L}_2, \mathcal{H}$ form a quadratic algebra. (As we go through the cases step by step, we ignore systems that have already been exhibited in earlier steps.) We first try $\mathcal{L}_2 = \mathcal{X}_2 + c\mathcal{X}_1$. The class $ac \neq 0$ doesn't yield a quadratic algebra. There are 3 other possible classes:

1. $\tilde{\mathbf{D4}}(\mathbf{b})\mathbf{A}$: $\mathcal{L}_1 = \mathcal{K}^2$, $\mathcal{L}_2 = \mathcal{X}_2 + \mathcal{X}_1$, $\mathcal{R}^2 = -64\mathcal{L}_1^3 + 16\mathcal{L}_1\mathcal{L}_2^2 - 64b\mathcal{L}_1^2\mathcal{H} - 64\mathcal{L}_1\mathcal{H}^2$,

$$\begin{aligned} A^{12} &= 0, & A^{22} &= \frac{2(-2\sin^2 2x - b\cos 2x - 2)}{\sin 2x(2\cos 2x + b)}, \\ B^{12} &= \frac{4(-\sin^2 2x + b\cos 2x + 2)}{\sin 2x(2\cos 2x + b)}, & B^{22} &= 6\left(\frac{1 + e^{4y}}{1 - e^{4y}}\right), \\ G(x, y) &= \ln\left(\frac{-b - 2\cos 2x}{\sin^2 2x}\right), & D &= \frac{\sin 2x}{2}(e^{2y} + e^{-2y}). \end{aligned}$$

The potential of the superintegrable system is

$$V(x, y) = \frac{\sin^2 2x}{2\cos 2x + b} \left(\frac{b_1}{\sinh^2 y} + \frac{b_2}{\sinh^2 2y} \right) + \frac{b_3}{2\cos 2x + b} + b_4.$$

(Listed as a superintegrable system in [19].)

2. $\tilde{\mathbf{D4}}(\mathbf{b})\mathbf{B}$: $\mathcal{L}_1 = \mathcal{K}^2$, $\mathcal{L}_2 = \mathcal{X}_2$, $\mathcal{R}^2 = 16\mathcal{L}_1\mathcal{L}_2^2$,

$$\begin{aligned} A^{12} &= 0, & A^{22} &= \frac{2(-2\sin^2 2x - b\cos 2x - 2)}{\sin 2x(2\cos 2x + b)}, \\ B^{12} &= \frac{4(-\sin^2 2x + b\cos 2x + 2)}{\sin 2x(2\cos 2x + b)}, & B^{22} &= 6, \\ G(x, y) &= \ln\left(\frac{-b - 2\cos 2x}{\sin^2 2x}\right), & D &= -\frac{\sin 2x}{2}e^{-2y}. \end{aligned}$$

The potential of the superintegrable system is

$$V(x, y) = \frac{\sin^2 2x}{2\cos 2x + b} \left(\frac{b_1}{\sin^2 2x} + b_2e^{4y} + b_3e^{2y} \right) + b_4.$$

(Listed as a superintegrable system in [19].)

2nd case. We choose $\mathcal{L}_1 = \mathcal{K}^2 + a\mathcal{X}_2$, $\mathcal{L}_2 = \mathcal{X}_1$. Then the only possibility is $a = 0$, which is redundant.

3rd case. We choose $\mathcal{L}_1 = \mathcal{X}_1 + \mathcal{X}_2 + a\mathcal{K}^2$, $\mathcal{L}_2 = \mathcal{X}_2 + c\mathcal{K}^2$. We generate a quadratic algebra for the system

3. $\tilde{\mathbf{D4}}(\mathbf{b})\mathbf{C}$: $\mathcal{L}_1 = \mathcal{X}_1 + \mathcal{X}_2 + 2\mathcal{K}^2$, $\mathcal{L}_2 = \mathcal{X}_2 + \mathcal{K}^2$. $\mathcal{R}^2 = -16b\mathcal{H}^3 + 16\mathcal{L}_1^2\mathcal{L}_2 - 16\mathcal{L}_1\mathcal{L}_2^2 + 16b\mathcal{L}_1\mathcal{L}_2\mathcal{H} - 16b\mathcal{L}_2^2\mathcal{H} - 16\mathcal{L}_1\mathcal{H}^2$,

$$\begin{aligned} A^{12} &= 0, & A^{22} &= -\frac{4\sin^2 2x + 2b\cos 2x + 4}{\sin 2x(2\cos 2x + b)}, & B^{22} &= -\frac{6(e^{4y} - 1)}{2e^{2y}\cos 2x + e^{4y} + 1}, \\ B^{12} &= \frac{2be^{2y}\cos^2 2x + 6be^{2y} + 4b\cos 2x + 4be^{4y}\cos 2x - 8\cos 2x + 16\cos^4 x}{\sin 2x(e^{4y} + 2e^{2y}\cos 2x + 1)(2\cos 2x + b)}, \\ &+ \frac{-8e^{4y}\cos 2x + 16e^{4y}\cos^4 x - 4e^{2y}\cos^3 2x + 16e^{2y}\cos 2x}{\sin 2x(e^{4y} + 2e^{2y}\cos 2x + 1)(2\cos 2x + b)}, \\ G(x, y) &= \ln\left(\frac{-b - 2\cos 2x}{\sin^2 2x}\right), & D &= -\frac{\sin 2x}{2}(2\cos 2x + e^{-2y} + e^{2y}). \end{aligned}$$

The potential of the superintegrable system is

$$V(x, y) = \frac{e^{2y}}{\frac{b+2}{\sin^2 x} + \frac{b-2}{\cos^2 x}} \left(\frac{b_1}{Z + (1 - e^{2y})\sqrt{Z}} + \frac{b_2}{Z + (1 + e^{2y})\sqrt{Z}} + \frac{b_3 e^{-2y}}{\cos^2 x} \right) + b_4,$$

where $Z = (1 - e^{2y})^2 + 4e^{2y}\cos^2 x$. (Listed as a superintegrable system in [19].)

4th case. We choose $\mathcal{L}_1 = \mathcal{X}_1$, $\mathcal{L}_2 = \mathcal{X}_2$. We do not generate a quadratic algebra.

5.1 Alternate free Darboux 4 systems

There is an alternate form of $\tilde{D}4(b)$ that we shall employ. Set

$$x = iX, \quad y = iY, \quad \mathcal{J} = i\mathcal{K} = p_Y, \quad \mathcal{Y}_1 = \frac{\mathcal{X}_1 + \mathcal{X}_2}{2}, \quad \mathcal{Y}_2 = \frac{\mathcal{X}_1 - \mathcal{X}_2}{2i}.$$

Then for $b \neq \pm 2$ we have

$$\begin{aligned} \mathcal{H} &= \frac{\sinh^2 2X}{2 \cosh 2X + b} (p_X^2 + p_Y^2), \\ \mathcal{Y}_1 &= -\cos(2Y)(\mathcal{H} - \cosh 2X p_Y^2) - \sin(2Y) \sinh 2X p_X p_Y, \\ \mathcal{Y}_2 &= -\sin(2Y)(\mathcal{H} - \cosh 2X p_Y^2) + \cos(2Y) \sinh 2X p_X p_Y. \end{aligned} \quad (5.2)$$

The structure equations are

$$\begin{aligned} \{\mathcal{J}, \mathcal{Y}_1\} &= -2\mathcal{Y}_2, & \{\mathcal{J}, \mathcal{Y}_2\} &= 2\mathcal{Y}_1, & \{\mathcal{Y}_1, \mathcal{Y}_2\} &= 4\mathcal{J}^3 + 2b\mathcal{J}\mathcal{H}, \\ \mathcal{Y}_1^2 + \mathcal{Y}_2^2 - \mathcal{J}^4 - b\mathcal{J}^2\mathcal{H} - \mathcal{H}^2 &= 0. \end{aligned}$$

We can embed $D4(b)$ as a surface in 3D Minkowski space with Cartesian coordinates X, Y, Z . Let

$$\begin{aligned} X &= y\sqrt{b + 2 \cosh 2x}, & Y - Z &= \sqrt{b + 2 \cosh 2x}, \\ Y + Z &= \frac{y^2 \sqrt{b + 2 \cosh 2x}}{\sinh 2x} + \int \frac{2 \cosh 4x + 2 + 4b \cosh 2x + b^2}{\sqrt{b + 2 \cosh 2x}(\cosh 4x + 3 + 2b \cosh 2x)} dx. \end{aligned}$$

Then

$$ds^2 = \frac{2 \cosh 2x + b}{\sinh^2 2x} (dx^2 + dy^2) = -dX^2 + dY^2 - dZ^2.$$

The change of variable $u = e^x$ converts the integral into an elliptic integral in u that can be evaluated as a rather complicated sum of elementary functions and the elliptic integrals of types one, two and three.

In terms of this alternate form the superintegrable systems can be expressed as:

$$1. \quad \tilde{\mathbf{D}}4(\mathbf{b})\mathbf{A}': \quad \mathcal{L}_1 = \mathcal{J}^2, \quad \mathcal{L}_2 = \mathcal{Y}_1, \quad \mathcal{R}^2 = 16\mathcal{L}_1^3 - 16\mathcal{L}_1\mathcal{L}_2^2 + 16b\mathcal{L}_1^2\mathcal{H} + 16\mathcal{L}_1\mathcal{H}^2,$$

$$\begin{aligned} A^{12} &= 0, & A^{22} &= -\frac{2(-2 \sinh^2 2X + b \cosh 2X + 2)}{\sinh 2X(2 \cosh 2X + b)}, & B^{22} &= -6 \left(\frac{\cos 2Y}{\sin 2Y} \right), \\ B^{12} &= \frac{4(\sinh^2 2X + b \cosh 2X + 2)}{\sinh 2X(2 \cosh 2X + b)}, & G(x, y) &= \ln \left(\frac{b + 2 \cosh 2X}{\sinh^2 2X} \right). \end{aligned}$$

$$2. \quad \tilde{\mathbf{D}}4(\mathbf{b})\mathbf{B}': \quad \mathcal{L}_1 = \mathcal{J}^2, \quad \mathcal{L}_2 = \mathcal{Y}_1 - i\mathcal{Y}_2, \quad \mathcal{R}^2 = -16\mathcal{L}_1\mathcal{L}_2^2,$$

$$\begin{aligned} A^{12} &= 0, & A^{22} &= \frac{2(-2 \sinh^2 2X + b \cosh 2X + 2)}{\sinh 2X(2 \cosh 2X + b)}, & B^{22} &= 6i, \\ B^{12} &= \frac{4(+ \sinh^2 2X + b \cosh 2X + 2)}{\sinh 2X(2 \cosh 2X + b)}, & G(x, y) &= \ln \left(\frac{b + 2 \cosh 2X}{\sinh^2 2X} \right). \end{aligned}$$

$$3. \quad \tilde{\mathbf{D}}4(\mathbf{b})\mathbf{C}': \quad \mathcal{L}_1 = \mathcal{Y}_1 - \mathcal{J}^2, \quad \mathcal{L}_2 = \mathcal{Y}_2. \quad \mathcal{R}^2 = -4b\mathcal{H}^3 - 8\mathcal{L}_1^3 - 8\mathcal{L}_1\mathcal{L}_2^2 + 4b(\mathcal{L}_1^2 + \mathcal{L}_2^2)\mathcal{H} + 8\mathcal{L}_1\mathcal{H}^2.$$

$$A^{12} = 0, \quad A^{22} = -\frac{4 \sinh^2 2X + 2b \cosh 2X + 4}{\sinh 2X(2 \cosh 2X + b)}, \quad B^{22} = \frac{6 \sin 2Y}{-\cosh 2X + \cos 2Y},$$

$$B^{12} = -\frac{-2b \cosh^2 2X - 6b + (8b + 16) \cosh 2X \cos 2Y}{\sinh 2X (2 \cos 2Y - 2 \cosh 2X)(2 \cosh 2X + b)} - \frac{32 \cosh^4 2X \cos 2Y + 4 \cosh^3 2X - 16 \cosh 2X}{\sinh 2X (2 \cos 2Y - 2 \cosh 2X)(2 \cosh 2X + b)},$$

$$G(x, y) = \ln \left(\frac{b + 2 \cosh 2X}{\sinh^2 2X} \right).$$

4. $\tilde{\mathbf{D4}}(\mathbf{b})\mathbf{D}'$ (free degenerate): $\mathcal{J}, \mathcal{L}_1 = \mathcal{Y}_1, \mathcal{L}_2 = \mathcal{Y}_2, \mathcal{Y}_1^2 + \mathcal{Y}_2^2 - \mathcal{J}^4 - b\mathcal{J}^2\mathcal{H} - \mathcal{H}^2 = 0$.

5.2 Summary and an example

From the results of [21] and the calculations of the preceding sections we see that Theorems 1.1 and 1.2 are valid for Darboux spaces.

We use $\mathbf{D2C}$ as an example to give more details about how a nondegenerate superintegrable system with potential is induced from a free system: From Section 5.0.2 we have the free Darboux 2 system

$$\mathcal{L}_1 = \mathcal{X}_2, \quad \mathcal{L}_2 = \mathcal{X}_1, \quad \mathcal{R}^2 = 4\mathcal{L}_1\mathcal{L}_2^2 + 16\mathcal{L}_1^2\mathcal{H} - 16\mathcal{L}_1\mathcal{H}^2.$$

The potential equations are determined by

$$A^{12} = 0, \quad A^{22} = \frac{3x^2 - 1}{x(x^2 + 1)},$$

$$B^{12} = -\frac{(3x^4 + x^2 - 2y^2)}{x(x^2 + 1)(x^2 + y^2)}, \quad B^{22} = -\frac{6y}{x^2 + y^2}, \quad (5.3)$$

and the general solution is

$$V(x, y) = \frac{x^2}{2\sqrt{x^2 + y^2}(x^2 + 1)} \left(b_1 + \frac{b_2}{y + \sqrt{x^2 + y^2}} + \frac{b_3}{-y + \sqrt{x^2 + y^2}} \right) + b_4. \quad (5.4)$$

The induced classical system has a basis of symmetries

$$\mathcal{H} = \frac{x^2}{x^2 + 1} (p_x^2 + p_y^2) + V(x, y), \quad \mathcal{L}_1 = 2xyp_y p_x + \frac{(y^2 - x^4)p_y^2 + x^2(1 - y^2)p_x^2}{x^2 + 1} + W^{(1)},$$

$$\mathcal{L}_2 = 2xp_x p_y + \frac{2y(p_y^2 - x^2 p_x^2)}{x^2 + 1} + W^{(2)}, \quad (5.5)$$

where

$$W^{(1)} = \frac{1}{2} \frac{b_1 y (1 - x^2) + b_2 ((-y + \sqrt{x^2 + y^2})^2 + 1) - b_3 ((y + \sqrt{x^2 + y^2})^2 + 1)}{(x^2 + 1) \sqrt{x^2 + y^2}},$$

$$W^{(2)} = \frac{2b_1 \sqrt{x^2 + y^2} - b_2 ((-y + \sqrt{x^2 + y^2})^2 - 1) - b_3 ((y + \sqrt{x^2 + y^2})^2 - 1)}{4x^2 + 4}.$$

The Casimir is

$$\mathcal{R}^2 - 4\mathcal{L}_1\mathcal{L}_2^2 - 16\mathcal{L}_1^2\mathcal{H} + 16\mathcal{L}_1\mathcal{H}^2 - 4(b_2 + b_3)\mathcal{H}^2 + 16b_4\mathcal{L}_1^2 + (b_2 + b_3)\mathcal{L}_2^2 - 32b_4\mathcal{H}\mathcal{L}_1$$

$$+ (8b_2b_4 - cb_1^2 + 4cb_2b_3 + 8b_3b_4)\mathcal{H} + (16b_4^2 + b_1^2)\mathcal{L}_1 + b_1(-b_3 + b_2)\mathcal{L}_2$$

$$+ \left(\frac{1}{4}b_1^2b_2 + \frac{1}{4}b_1^2b_3 + b_1^2b_4 - 4b_3b_4^2 - 4b_2b_3b_4 - 4b_2b_4^2 \right). \quad (5.6)$$

We will not work out the details of the quantum case but merely note that the potential terms of the symmetries remain unchanged as do the 2nd order kinetic energy terms, but there are now 1st order terms: the Hamiltonian kinetic energy is replaced by the Laplace–Beltrami operator on D2 and the other generating symmetry operators are formally self-adjoint with respect to the D2 volume measure $x^2 dx dy/(x^2 + 1)$.

6 Contractions of Darboux systems

Recall that the scalar curvature of a space with metric $ds^2 = e^{G(x,y)}(dx^2 + dy^2)$, where $\lambda(x, y) = e^{G(x,y)}$ is $C = -e^{-G}(\partial_{xx}G + \partial_{yy}G)$. Constant curvature spaces are just those for which C is constant; flat spaces are those for which $C = 0$.

Theorem 6.1. *A Darboux or Koenigs superintegrable system cannot be obtained as a geometric contraction of a nonzero constant curvature or flat space superintegrable system.*

Proof. Suppose we have a contraction of a nonzero constant curvature system with Hamiltonian $\mathcal{H} = \frac{p_x^2 + p_y^2}{e^G}$. Then there is a 1-parameter family of Hamiltonians $\mathcal{H}' = \frac{p_{x'}^2 + p_{y'}^2}{e^{G'(\epsilon)}}$, where $G'(1) = G$, G' depends smoothly on ϵ in the interval $0 \leq \epsilon \leq 1$, and $G'(0)$ defines the metric of the target manifold for the contraction. Further, for $\epsilon \neq 0$ the metric defined by G' will be a scalar multiple of a metric on the original constant curvature system. Thus we have $C(\epsilon) = -e^{-G'}(\partial_{x'x'}G' + \partial_{y'y'}G')$ for $\epsilon \neq 0$, where $C(\epsilon)$ is nonzero and independent of x', y' . In the limit we obtain the constant $C(0)$, so the target manifold must be flat or of nonzero constant curvature, hence not a Darboux or Koenigs manifold. Similarly, if the original manifold is flat the target manifold must also be flat. ■

Now we investigate contractions involving Darboux superintegrable systems. From Theorem 6.1 such systems cannot be obtained as contractions of constant curvature systems. (However, they are all Stäckel equivalent to constant curvature space systems.) Thus we limit ourselves to the search for contractions such that the originating manifold is a Darboux space. In distinction to the case of constant curvature spaces as originating manifolds (where all quadratic algebra contractions were induced by Lie algebra contractions of $e(2, \mathbb{C})$ and $o(3, \mathbb{C})$) here the Darboux spaces have only a one-dimensional Lie symmetry algebra, so we have to search for new contraction mechanisms. However, we can restrict our search to free systems and then induce the superintegrable system contractions automatically.

To induce the contractions through Lie algebra methods we will employ the conformal symmetry algebra for each free Darboux system generated by functions $\mathcal{Q}(x, y, p_x, p_y) = A(x, y)p_x + B(x, y)p_y$ that satisfy the relations

$$\{\mathcal{H}, \mathcal{Q}\} = R_Q(x, y)\mathcal{H}, \quad \{\mathcal{K}, \mathcal{Q}\} = K_Q(x, y)\mathcal{K}, \quad \mathcal{K} = p_y.$$

A straightforward calculation gives the same algebra in each case:

$$\begin{aligned} \mathbf{G3}: \quad \mathcal{P}_x &= p_x, & \mathcal{P}_y &= p_y, & \mathcal{M} &= xp_x + yp_y, \\ \{\mathcal{P}_x, \mathcal{P}_y\} &= 0, & \{\mathcal{P}_x, \mathcal{M}\} &= \mathcal{P}_x, & \{\mathcal{P}_y, \mathcal{M}\} &= \mathcal{P}_y. \end{aligned}$$

The Inönü–Wigner contractions and their geometric implementations are:

- 1) $\mathcal{P}_x, \mathcal{P}_y, \epsilon\mathcal{M}; \quad x = x' + \frac{1}{\epsilon}, y = y', \text{ or } x = x', y = y' + \frac{1}{\epsilon},$
- 2) $\epsilon\mathcal{P}_x, \epsilon\mathcal{P}_y, \mathcal{M}; \quad x = \epsilon x', y = \epsilon y'.$

Up to an isomorphism of the Lie algebra that is obtained by contraction of G3 there are no continuous one parametric contractions besides these two, [29]. Each of these contractions does induce a corresponding contraction of each free Darboux system and we have found no contractions of Darboux systems other than these. This approach is compatible with the use of generalizations of Inönü–Wigner contractions for the symmetry algebras $o(3, \mathbb{C})$ and $e(2, \mathbb{C})$ of constant curvature spaces. Indeed if we compute the conformal symmetry algebra for each free constant curvature system generated by functions $\mathcal{Q}(x, y, p_x, p_y) = A(x, y)p_x + B(x, y)p_y$ that satisfy the relations

$$\{\mathcal{H}, \mathcal{Q}\} = R_{\mathcal{Q}}(x, y)\mathcal{H}, \quad \{\mathcal{L}_j, \mathcal{Q}\} = \sum_{\ell=1}^3 K_{\mathcal{Q},j}^{(\ell)}(x, y)\mathcal{L}_\ell,$$

where the \mathcal{L}_j form a basis for either $o(3, \mathbb{C})$ or $e(2, \mathbb{C})$ we find $o(3, \mathbb{C})$ in the case of the sphere and the affine algebra, the semidirect product of $e(2, \mathbb{C})$ and the dilation \mathcal{M} , for flat space. One can show that the geometrical implementations of generalized Inönü–Wigner contractions of the affine algebra either agree with those of $e(2, \mathbb{C})$ or the contractions cannot be implemented geometrically. See for example [12, Table XXIX] for the ordinary Inönü–Wigner contractions of the affine algebra.

6.1 D1 contractions

We list approaches to finding the contractions (not the contractions themselves).

1. Let $x = x' + \frac{1}{\epsilon}$, $y = y'$, $\mathcal{H}' = \frac{4}{\epsilon}\mathcal{H}$. Then as $\epsilon \rightarrow 0$ we have

$$\mathcal{H}' = p_{x'}^2 + p_{y'}^2, \quad \mathcal{X}_1 \rightarrow p_{x'}p_{y'}, \quad \mathcal{X}_2 \approx p_{y'}(y'p_{x'} - x'p_{y'}) - \frac{p_{y'}^2}{\epsilon}.$$

Contractions constructed from such limits would have flat space as the target manifold.

2. Let $x = x'$, $y = y' + \frac{1}{\epsilon}$, $\mathcal{H}' = \mathcal{H}$. Then as $\epsilon \rightarrow 0$ we have

$$\begin{aligned} \mathcal{H}' &= \frac{p_{x'}^2 + p_{y'}^2}{4x'}, & \mathcal{X}_1 &\approx p_{x'}p_{y'} - \frac{y'}{2x'}(p_{x'}^2 + p_{y'}^2) - \frac{2}{\epsilon}\mathcal{H}', \\ \mathcal{X}_2 &\approx p_{y'}(y'p_{x'} - x'p_{y'}) + \frac{p_{x'}p_{y'}}{\epsilon} - \frac{y'^2}{4x'}(p_{x'}^2 + p_{y'}^2) - \left(\frac{y'}{2\epsilon x'} + \frac{1}{4\epsilon^2 x'}\right)(p_{x'}^2 + p_{y'}^2). \end{aligned}$$

Contractions constructed from such limits would have D1 itself as the target manifold.

3. Let $x = \epsilon x'$, $y = \epsilon y'$, $\mathcal{H}' = 4\epsilon\mathcal{H}$. Then as $\epsilon \rightarrow 0$ we have $\mathcal{H}' = \frac{p_{x'}^2 + p_{y'}^2}{x'}$ and

$$\begin{aligned} \epsilon^2 \mathcal{X}_1 &= p_{x'}p_{y'} - \frac{y'}{2x'}(p_{x'}^2 + p_{y'}^2), \\ \epsilon \mathcal{X}_2 &= p_{y'}(y'p_{x'} - x'p_{y'}) - \frac{y'^2}{4x'}(p_{x'}^2 + p_{y'}^2), \quad \epsilon \mathcal{K} = p_{y'}. \end{aligned}$$

Contractions constructed from such limits would have D1 again as the target manifold.

6.2 D2 contractions

We list approaches to finding the contractions (not the contractions themselves).

1. Let $x = x' + \frac{1}{\epsilon}$, $y = y'$, $\mathcal{H}' = \mathcal{H}$. Then as $\epsilon \rightarrow 0$ we have

$$\mathcal{H}' = p_{x'}^2 + p_{y'}^2, \quad \epsilon \mathcal{X}_1 \approx 2p_{x'}p_{y'}, \quad \epsilon^2 \mathcal{X}_2 \approx -p_{y'}^2.$$

Contractions constructed from such limits would have flat space as the target manifold.

2. Let $x = \epsilon x'$, $y = \epsilon y'$, $\mathcal{H}' = \mathcal{H}$. Then as $\epsilon \rightarrow 0$ we have $\mathcal{H}' = x'^2(p_{x'}^2 + p_{y'}^2)$ and

$$\epsilon \mathcal{X}_1 \approx 2x' p_{x'} p_{y'} + 2y' p_{y'}^2, \quad \epsilon \mathcal{X}_2 = 2x' y' p_{x'} p_{y'} + y'^2 p_{y'}^2 + x'^2 p_{x'}^2, \quad \epsilon \mathcal{K} = p_{y'}.$$

Contractions constructed from such limits would have the complex 2-sphere as the target manifold.

3. Let $x = x'$, $y = y' + \frac{1}{\epsilon}$, $\mathcal{H}' = \mathcal{H}$. Then as $\epsilon \rightarrow 0$ we have $\mathcal{H}' = \frac{x'^2}{x'^2+1}(p_{x'}^2 + p_{y'}^2)$, $\mathcal{K}' = \mathcal{K}$,

$$\begin{aligned} \mathcal{X}'_1(x', y', p_{x'}, p_{y'}) &= \mathcal{X}_1 + \frac{2}{\epsilon}(\mathcal{K}^2 - \mathcal{H}), \\ \mathcal{X}'_2(x', y', p_{x'}, p_{y'}) &= \mathcal{X}_2 + \frac{1}{\epsilon^2}(\mathcal{K}^2 - \mathcal{H}) + \frac{1}{\epsilon} \mathcal{X}_1. \end{aligned}$$

Contractions constructed from such limits would have $D2$ as the target manifold.

6.3 D3 contractions

We list approaches to finding the contractions (not the contractions themselves).

1. Let $x = x' + \ln(\frac{1}{\epsilon})$, $y = y'$, $\mathcal{H}' = 2\epsilon \mathcal{H}$. Then as $\epsilon \rightarrow 0$ we have $\mathcal{K} = p_{y'}$

$$\begin{aligned} \mathcal{H}' &= e^{x'}(p_{x'}^2 + p_{y'}^2), \quad \epsilon \mathcal{X}_1 \approx \frac{e^{x'}}{2} \left(\sin y' p_{x'} p_{y'} + \frac{1}{2} \cos y' (p_{x'}^2 - p_{y'}^2) \right), \\ \epsilon \mathcal{X}_2 &\approx \frac{e^{x'}}{2} \left(-\cos y' p_{x'} p_{y'} + \frac{1}{2} \sin y' (p_{x'}^2 - p_{y'}^2) \right). \end{aligned}$$

Contractions constructed from such limits would have flat space as the target manifold. In terms of flat space Cartesian coordinates $X = r \cos \theta$, $Y = r \sin \theta$ we have

$$\begin{aligned} e^{x'} &= \frac{4}{r^2}, \quad y' = 2\theta, \quad \mathcal{H}' = p_X^2 + p_Y^2, \quad \epsilon \mathcal{X}_1 \approx \frac{1}{4}(p_X^2 - p_Y^2), \\ \epsilon \mathcal{X}_2 &\approx \frac{1}{2} p_X p_Y, \quad \mathcal{K}' = \frac{1}{2}(X p_Y - Y p_X) = \frac{1}{2} \mathcal{J}. \end{aligned}$$

2. Let $x = \epsilon x'$, $y = \epsilon y'$, $\mathcal{H}' = 4\epsilon^2 \mathcal{H}$. Then as $\epsilon \rightarrow 0$ we have $\mathcal{H}' = p_{x'}^2 + p_{y'}^2$ and

$$\epsilon^2 \mathcal{X}_1 \approx \frac{1}{8}(p_{x'}^2 - 3p_{y'}^2), \quad \epsilon^2 \mathcal{X}_2 \approx -\frac{1}{2} p_{x'} p_{y'}, \quad \epsilon \mathcal{K} = p_{y'}.$$

Contractions constructed from such limits would have flat space as the target manifold.

3. Let $x = x'$, $y = y' + i \ln \epsilon$, $\mathcal{H}' = \mathcal{H}'$. Then as $\epsilon \rightarrow 0$ we have

$$\mathcal{Y}'_1(x', y', p_{x'}, p_{y'}) \approx \epsilon \mathcal{Y}_1, \quad \mathcal{Y}'_2(x', y', p_{x'}, p_{y'}) \approx \frac{1}{\epsilon} \mathcal{Y}_2, \quad \mathcal{K} = p_{y'}.$$

Contractions constructed from such limits would have $D3$ as the target manifold.

6.4 D4 contractions

We list approaches to finding the contractions (not the contractions themselves).

1. Let $X = X' + \ln \epsilon$, $Y = Y'$, $\mathcal{H}' = 4\epsilon^2 \mathcal{H}$, with b fixed. Then as $\epsilon \rightarrow 0$ we have $\mathcal{J} = p_{Y'}$ and

$$\begin{aligned} \mathcal{H}' &= e^{-2X'}(p_{X'}^2 + p_{Y'}^2), \quad \mathcal{J}' = p_{Y'}, \\ 4\epsilon^2 \mathcal{Y}_1 &\approx -\cos(2Y')(\mathcal{H}' - 2e^{-2X'} p_{Y'}^2) + 2\sin(2Y')e^{-2X'} p_{X'} p_{Y'} \approx p_y^2 - p_x^2, \\ 4\epsilon^2 \mathcal{Y}_2 &\approx -\sin(2Y')(\mathcal{H}' - 2e^{-2X'} p_{Y'}^2) - 2\cos(2Y')e^{-2X'} p_{X'} p_{Y'} \approx -2p_x p_y, \end{aligned}$$

where x, y are standard Cartesian coordinates. Contractions constructed from such limits would have flat space as the target manifold.

2. Let $X = \epsilon X'$, $Y = \epsilon Y'$, $\mathcal{H}' = \mathcal{H}$, with b fixed. Then as $\epsilon \rightarrow 0$ we have $\mathcal{H}' = \frac{4X'^2}{2+b}(p_{X'}^2 + p_{Y'}^2)$ and

$$\begin{aligned}\mathcal{Y}_1 - p_{Y'}^2 + \mathcal{H} &\approx 2(X'^2 - Y'^2)p_{Y'}^2 - 4X'Y'p_{X'}p_{Y'}, \\ \epsilon\mathcal{Y}_2 &\approx 2Y'p_{Y'}^2 + 2X'p_{X'}p_{Y'}, \quad \epsilon\mathcal{J} = p_{Y'}.\end{aligned}$$

Contractions constructed from such limits would have the complex 2-sphere as the target manifold.

3. For this case it is most convenient to take the Hamiltonian in the form (5.1). Let $x = x'$, $y = y' + \frac{1}{2} \ln \epsilon$, $\mathcal{H}' = \mathcal{H}$, with b fixed. Then we have

$$X'_1(x', y', p_{x'}, p_{y'}) = \frac{1}{\epsilon} X'_1, \quad X'_2(x', y', p_{x'}, p_{y'}) = \epsilon X'_2.$$

Contractions constructed from such limits would have $D4(b)$ as the target manifold.

The manifold $D4(b)$ depends on a parameter, so we can extend the contractions that we consider by allowing the parameter to vary. We find the following new contractions:

4. Let $X = \frac{X'}{2} - \frac{1}{2} \ln(\epsilon)$, $Y = \frac{Y'}{2}$, $b = \frac{1}{\epsilon}$, $\mathcal{H}' = \frac{\epsilon}{2} \mathcal{H}$. Then as $\epsilon \rightarrow 0$ we have $\mathcal{H}' = \frac{1}{2} \frac{e^{2X'}}{e^{X'}+1}(p_{X'}^2 + p_{Y'}^2)$, $\mathcal{J}' = \frac{1}{2} \mathcal{J} = p_{Y'}$, and

$$\begin{aligned}\frac{\epsilon}{2} \mathcal{Y}_1 &\approx -\cos(Y')(\mathcal{H}' - e^{X'} p_{Y'}^2) - \sin(Y') e^{X'} p_{X'}^2 p_{Y'}^2, \\ \frac{\epsilon}{2} \mathcal{Y}_2 &\approx -\sin(Y')(\mathcal{H}' - e^{X'} p_{Y'}^2) + \cos(Y') e^{X'} p_{X'}^2 p_{Y'}^2.\end{aligned}$$

Contractions constructed from such limits would have D3 as the target manifold.

5. Let $X = \frac{\epsilon X'}{2}$, $Y = \frac{\epsilon Y'}{2}$, $b = -2 + \epsilon^2$, $\mathcal{H}' = \frac{\epsilon^2}{4} \mathcal{H}$. Then as $\epsilon \rightarrow 0$ we have $\mathcal{H}' \rightarrow \frac{X'^2}{X'^2+1}(p_{X'}^2 + p_{Y'}^2)$, $\mathcal{J}' = \frac{\epsilon}{2} \mathcal{J} = p_{Y'}$, and

$$\begin{aligned}\mathcal{Y}_1 + \mathcal{H} - \mathcal{J}^2 - \frac{\epsilon^2}{2} \mathcal{H} &\approx -2 \left(2X'Y'p_{X'}p_{Y'} + \frac{(Y'^2 - X'^4)p_{Y'}^2 + X'^2(1 - Y'^2)p_{X'}^2}{X'^2 + 1} \right), \\ \mathcal{Y}_2 &\approx \frac{2}{\epsilon} \left(2X'p_{X'}p_{Y'} + \frac{2Y'}{X'^2 + 1}(p_{Y'}^2 - X'^2 p_{X'}^2) \right).\end{aligned}$$

Contractions constructed from such limits would have D2 as the target manifold.

6. Let $X = \epsilon X' + \frac{\ln(\epsilon)}{2}$, $Y = \epsilon Y'$, $b = -\frac{1}{\epsilon}$, $\mathcal{H}' = -2\epsilon^4 \mathcal{H}$. Then as $\epsilon \rightarrow 0$ we have $\mathcal{H}' \rightarrow \frac{1}{4X'}(p_{X'}^2 + p_{Y'}^2)$, $\mathcal{J}' = \epsilon \mathcal{J} = p_{Y'}$, and

$$\begin{aligned}\mathcal{Y}_1 + \mathcal{H} - \frac{1}{2\epsilon} \mathcal{J}^2 &\approx \frac{1}{\epsilon^2} \left(p_{Y'}(Y'p_{X'} - X'p_{Y'}) - \frac{Y'^2}{4X'}(p_{X'}^2 + p_{Y'}^2) \right), \\ \mathcal{Y}_2 &\approx -\frac{1}{2\epsilon^3} \left(p_{X'}p_{Y'} - \frac{Y'}{2X'}(p_{X'}^2 + p_{Y'}^2) \right).\end{aligned}$$

Contractions constructed from such limits would have D1 as the target manifold.

7. Using Hamiltonian (5.2) we let $b = -2 + \epsilon^2$, $X = X'$, $Y = Y'$, $\mathcal{H} \rightarrow \mathcal{H}'$. Then

$$\begin{aligned}\mathcal{H}' &= \cosh^2 X'(p_{X'}^2 + p_{Y'}^2), \quad \mathcal{J}' = \mathcal{J} = p_{Y'}, \\ \mathcal{Y}'_1 &= -\cos(2Y')(\mathcal{H}' - \cosh 2X' p_{Y'}^2) - \sin(2Y') \sinh 2X' p_{X'} p_{Y'}, \\ \mathcal{Y}'_2 &= -\sin(2Y')(\mathcal{H}' - \cosh 2X' p_{Y'}^2) + \cos(2Y') \sinh 2X' p_{X'} p_{Y'}.\end{aligned}$$

Contractions constructed from such limits would have the complex 2-sphere as the target manifold. Expressed in terms of the symmetries of the 2-sphere we have

$$\mathcal{Y}'_1 = \mathcal{J}'^2 - \mathcal{J}_2^2, \quad \mathcal{Y}'_2 = 2\mathcal{J}_1\mathcal{J}_2, \quad \mathcal{J}' = \mathcal{J}_3$$

6.5 Summary and examples of Darboux contractions

From the results of [21] and the preceding sections we have an analog of Theorem 2.1 for Darboux spaces:

Theorem 6.2. *Every Lie algebra contraction of $\mathcal{G}3$ induces a contraction of a free geometric quadratic algebra \tilde{Q} on a Darboux space, which in turn induces a contraction of the quadratic algebra Q with potential. This is true for both classical and quantum Darboux algebras.*

Some examples follow.

1. We describe how a Lie algebra contraction induces the contraction of $D2C$ to $S4$, including the potential terms. Recall for $D2C$ we have the symmetries (5.5), potential (5.4), canonical equations (5.3) and Casimir (5.6). The coordinate implementation is defined by $x = \epsilon y'$, $y = \epsilon x'$. In terms of the coordinates x' , y' the canonical equations become

$$A'^{12} = 0, \quad A'^{22} = -\frac{1}{y'}, \quad B'^{12} = \frac{2x'^2 - y'^2}{y'(x'^2 + y'^2)}, \quad b'^{22} = -\frac{6x'}{(x'^2 + y'^2)}.$$

The contraction is defined by

$$\mathcal{L}'_1 = \mathcal{L}_1, \quad \mathcal{L}'_2 = \frac{\epsilon}{2}\mathcal{L}_1, \quad \mathcal{H} = \mathcal{H}', \quad \mathcal{R}' = \frac{\epsilon}{2}\mathcal{R}.$$

In the limit we find

$$\begin{aligned} \mathcal{H}' &= y'^2(p_{x'}^2 + p_{y'}^2) + V' = \mathcal{S}_2^2 - 2\mathcal{S}_1\mathcal{S}_3 + V', \\ \mathcal{L}'_1 &= \mathcal{S}_2^2 + W'_1, \quad \mathcal{L}'_2 = \mathcal{S}_1\mathcal{S}_2 + W'_2, \end{aligned}$$

where

$$V' = \frac{y'^2}{2\sqrt{x'^2 + y'^2}} \left(c_1 + \frac{c_2}{\sqrt{x'^2 + y'^2} + x'} + \frac{c_3}{\sqrt{x'^2 + y'^2} - x'} \right) + c_4$$

and $b_1 = \frac{c_1}{\epsilon}$, $b_2 = c_2$, $b_3 = c_3$, $b_4 = c_4$. Here

$$\mathcal{S}_1 = p_{x'}, \quad \mathcal{S}_2 = x'p_{x'} + y'p_{y'}, \quad \mathcal{S}_3 = \frac{1}{2}(x'^2 - y'^2)p_{x'}^2 + x'y'p_{y'},$$

are the 1st order generators of the symmetry algebra for the free Hamiltonian $\mathcal{H}'_0 = y'^2(p_{x'}^2 + p_{y'}^2)$ on the Poincaré upper half plane: x real, $y > 0$. The Casimir becomes in the limit:

$$\begin{aligned} \mathcal{R}'^2 - 4\mathcal{L}'_1\mathcal{L}'_2{}^2 + (c_2 + c_3)\mathcal{L}'_2 - \frac{c_1^2}{4}\mathcal{H}' + \frac{c_1^2}{4}\mathcal{L}'_1 \\ + \frac{c_1}{2}(c_2 - c_3)\mathcal{L}'_2 + \frac{c_1^2}{16}(c_2 + c_3 + 4c_4) = 0. \end{aligned}$$

2. $D4(b)A' \rightarrow D3E$: We again give more details in our 2nd example, which involves changing the parameter b . For the degenerate system

$$\mathcal{H} = \frac{\sinh^2(2x)}{2\cosh(2x) + b}(p_x^2 + p_y^2) + \frac{\alpha}{2\cosh(2x) + b},$$

we can get

$$\mathcal{H}' = \frac{1}{2} \frac{e^{2x'}}{e^{x'} + 1} (p_{x'}^2 + p_{y'}^2) + \frac{\beta}{e^{x'} + 1}$$

as a contraction case by taking

$$x = \frac{x'}{2} - \frac{1}{2} \ln(\epsilon), \quad y = \frac{y'}{2}, \quad b = \frac{1}{\epsilon}, \quad \mathcal{H}' = 8\epsilon\mathcal{H}, \quad \alpha = \frac{\beta}{8\epsilon^2}.$$

Then we have $\mathcal{J}' = 2\mathcal{J} = p_{y'}$, and $\mathcal{Y}'_1 \approx -4\epsilon\mathcal{Y}_1$, $\mathcal{Y}'_2 \approx -4\epsilon\mathcal{Y}_2$.

3. $\tilde{D}1A \rightarrow \tilde{E}9(a = -1)$: For system $\tilde{D}1A$ we have

$$\begin{aligned} \mathcal{L}_1 &= \mathcal{X}_2 + b\mathcal{K}^2, & \mathcal{L}_2 &= \mathcal{X}_1 + b\mathcal{K}^2, \\ \mathcal{R}^2 &= 2i\mathcal{L}_2^3 + 8i\mathcal{L}_1\mathcal{L}_2\mathcal{H} + 4b\mathcal{L}_2^2\mathcal{H} + 16b\mathcal{H}^2\mathcal{L}_1. \end{aligned}$$

We choose contraction 1) for $\tilde{D}1A$: $x = x' + \frac{1}{\epsilon}$, $y = y'$, $\mathcal{H} = \frac{\epsilon}{4}\mathcal{H}'$ and let $b = \frac{1}{\epsilon}$. Then as $\epsilon \rightarrow 0$ we find

$$\mathcal{L}_1 \rightarrow \mathcal{L}'_1 = -p_{y'}(y'p_{x'} - x'p_{y'}), \quad \mathcal{L}_2 \rightarrow \mathcal{L}'_2 = p_{y'}(p_{x'} + ip_{y'}), \quad \mathcal{H}' = p_{x'}^2 + p_{y'}^2,$$

and with $\mathcal{R}' = \{\mathcal{L}'_1, \mathcal{L}'_2\}$ the Casimir becomes $\mathcal{R}'^2 = 2i\mathcal{L}'_2{}^3 + \mathcal{L}'_2{}^2\mathcal{H}'$.

4. $\tilde{D}1B \rightarrow \tilde{E}2$: For system $\tilde{D}1B$ we have

$$\mathcal{L}_1 = \mathcal{X}_2, \quad \mathcal{L}_2 = \mathcal{K}^2, \quad \mathcal{R}^2 = -4\mathcal{L}_2^3 - 16\mathcal{L}_1\mathcal{L}_2\mathcal{H}.$$

We choose contraction 1) for $\tilde{D}1B$: $x = x' + \frac{1}{\epsilon}$, $y = y'$, $\mathcal{H} = \frac{\epsilon}{4}\mathcal{H}'$. Then as $\epsilon \rightarrow 0$ we find

$$\mathcal{L}_1 \approx \mathcal{L}'_1 - \frac{1}{\epsilon}\mathcal{L}'_2, \quad \mathcal{L}'_1 = -p_{y'}(x'p_{y'} - y'p_{x'}), \quad \mathcal{L}_2 = \mathcal{L}'_2 = p_{y'}^2, \quad \mathcal{H}' = p_{x'}^2 + p_{y'}^2,$$

and with $\mathcal{R}' = \{\mathcal{L}'_1, \mathcal{L}'_2\}$ the Casimir becomes $\mathcal{R}'^2 = -4\mathcal{L}'_2{}^3 + 4\mathcal{L}'_2{}^2\mathcal{H}'$.

5. $\tilde{D}1C \rightarrow \tilde{E}3'$: For system $\tilde{D}1C$ we have $\mathcal{L}_1 = \mathcal{X}_1$, $\mathcal{L}_2 = \mathcal{K}^2$, $\mathcal{R}^2 = 16\mathcal{L}_2\mathcal{H}^2$. We choose contraction 1) for $\tilde{D}1C$: $x = x' + \frac{1}{\epsilon}$, $y = y'$, $\mathcal{H} = \frac{\epsilon}{4}\mathcal{H}'$. Then as $\epsilon \rightarrow 0$ we find

$$\mathcal{L}_1 \rightarrow \mathcal{L}'_1 = p'_{x'}p_{y'}, \quad \mathcal{L}_2 = \mathcal{L}'_2 = p_{y'}^2, \quad \mathcal{H}' = p_{x'}^2 + p_{y'}^2,$$

and with $\mathcal{R}' = \{\mathcal{L}'_1, \mathcal{L}'_2\}$ the Casimir becomes $\mathcal{R}'^2 = 0$.

6. $\tilde{D}4(b)C' \rightarrow \tilde{S}7$ as $b \rightarrow -2$: For system $\tilde{D}4(b)C'$ we have

$$\begin{aligned} \mathcal{L}_1 &= \mathcal{Y}_1 - \mathcal{J}^2, & \mathcal{L}_2 &= \mathcal{Y}_2, \\ \mathcal{R}^2 &= 4b\mathcal{H}^3 - 8\mathcal{L}_1\mathcal{J}^3 - 8\mathcal{L}_1\mathcal{L}_2^2 + 4b(\mathcal{L}_1^2 + \mathcal{L}_2^2)\mathcal{H} + 8\mathcal{L}_1\mathcal{H}^2. \end{aligned}$$

In the limit as $b \rightarrow -2$ we have

$$\mathcal{L}'_1 = 2\mathcal{J}_1^2 - \mathcal{H}', \quad \mathcal{L}'_2 = 2\mathcal{J}_1\mathcal{J}_2,$$

where $\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3$ are the symmetries of the sphere and $\mathcal{H}' = \mathcal{J}_1^2 + \mathcal{J}_2^2 + \mathcal{J}_3^2$. The limit system is $\tilde{S}7$.

7 Tables/figures describing the contractions of the nondegenerate and degenerate Darboux systems

We now tabulate the contractions of Darboux systems. The system on the left is the starting system, and an asterisk indicates that a system is degenerate

7.1 D1 contraction table

	1: $D1$ to flat space	2: $D1$ to $D1$	3: $D1$ to $D1$
$\tilde{D}1A$	$\tilde{E}3', \tilde{E}9$	$\tilde{D}1C, \tilde{D}1A$	$\tilde{D}1C, \tilde{D}1A$
$\tilde{D}1B$	$\tilde{E}2$	$\tilde{D}1C$	$\tilde{D}1B$
$\tilde{D}1C$	$\tilde{E}3'$	$\tilde{D}1C$	$\tilde{D}1C$
$\tilde{D}1D^*$	$\tilde{E}5^*$	$\tilde{D}1D^*$	$\tilde{D}1D^*$

7.2 D2 contraction table

	1: $D2$ to flat space	2: $D2$ to 2-sphere	3: $D2$ to $D2$
$\tilde{D}2A$	$\tilde{E}3'$	$\tilde{S}1$	$\tilde{D}2A$
$\tilde{D}2B$	$\tilde{E}2$	$\tilde{S}2$	$\tilde{D}2A$
$\tilde{D}2C$	$\tilde{E}3'$	$\tilde{S}4$	$\tilde{D}2A$
$\tilde{D}2D^*$	$\tilde{E}5^*$	$\tilde{S}5^*$	$\tilde{D}2D^*$

7.3 D3 contraction table

	1: $D3$ to flat space	2: $D3$ to flat space	3: $D3$ to $D3$
$\tilde{D}3A$	$\tilde{E}3'$	$\tilde{E}3'$	$\tilde{D}3D$
$\tilde{D}3B$	$\tilde{E}3'$	$\tilde{E}3'$	$\tilde{D}3B$
$\tilde{D}3C$	$\tilde{E}1$	$\tilde{E}2$	$\tilde{D}3D$
$\tilde{D}3D$	$\tilde{E}8$	$\tilde{E}3'$	$\tilde{D}3D$
$\tilde{D}3E^*$	$\tilde{E}3^*$	$\tilde{E}5^*$	$\tilde{D}3E^*$

7.4 D4 contraction table

	1: $D4$ to flat space	2: $D4$ to 2-sphere	3: $D4$ to $D4$
$\tilde{D}4(b)A$	$\tilde{E}1$	$\tilde{S}2$	$D4(b)B$
$\tilde{D}4(b)B$	$\tilde{E}8$	$\tilde{S}1$	$D4(b)B$
$\tilde{D}4(b)C$	$\tilde{E}3'$	$\tilde{S}4$	$D4(b)B$
$\tilde{D}4(b)D^*$	$\tilde{E}3^*$	$\tilde{S}5^*$	$D4(b)D^*$

	4: $D4$ to $D3$	5: $D4$ to $D2$	6: $D4$ to $D1$	7: $D4$ to 2-sphere
$\tilde{D}4(b)A$	$\tilde{D}3C$	$\tilde{D}2B$	$\tilde{D}1B$	$\tilde{S}9$
$\tilde{D}4(b)B$	$\tilde{D}3D$	$\tilde{D}2A$	$\tilde{D}1C$	$\tilde{S}2$
$\tilde{D}4(b)C$	$\tilde{D}3B$	$\tilde{D}2C$	$\tilde{D}1C$	$\tilde{S}7$
$\tilde{D}4(b)D^*$	$\tilde{D}3E^*$	$\tilde{D}2D^*$	$\tilde{D}1D^*$	$\tilde{S}3^*$

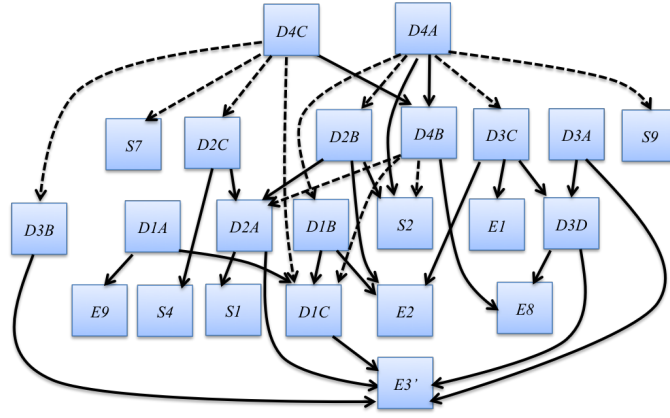


Figure 1. Diagram indicating the contractions of nondegenerate Darboux systems.

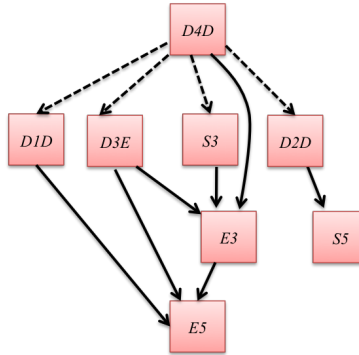


Figure 2. Diagram indicating the contractions of degenerate Darboux systems.

7.5 Free nondegenerate Darboux systems to degenerate systems

The following are not contractions in the usual sense, but we show which nondegenerate systems restrict to degenerate ones.

nondegenerate	→ degenerate
$\tilde{D}1B$	$\tilde{D}1D^*$
$\tilde{D}1C$	$\tilde{D}1D^*$
$\tilde{D}2A$	$\tilde{D}2D^*$
$\tilde{D}2B$	$\tilde{D}2D^*$
$\tilde{D}3C$	$\tilde{D}3E^*$
$\tilde{D}3D$	$\tilde{D}3E^*$
$\tilde{D}4(b)A$	$\tilde{D}4(b)D^*$
$\tilde{D}4(b)B$	$\tilde{D}4(b)D^*$

7.6 Figures

What follows are diagrams illustrating the contractions of Darboux systems. Boxes represent systems (blue boxes being nondegenerate and red boxes being degenerate), and arrows represent contractions. In these diagrams, certain contractions that do not affect the overall hierarchy have been omitted for aesthetic reasons, and a dotted arrow indicates that the limiting process inducing the contraction changes the free Hamiltonian. A diagram illustrating the nondegenerate Darboux systems which restrict to degenerate ones is also presented.

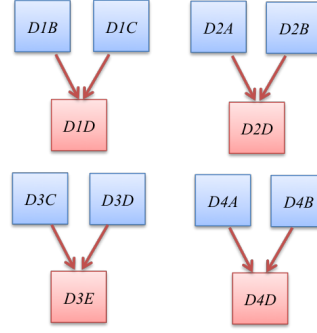


Figure 3. Diagram indicating restrictions of nondegenerate Darboux systems to degenerate ones.

For completeness we give the definitions of the constant curvature superintegrable systems that are targets of Darboux contractions. The following lists contain the defining relations for the free systems and the potentials of the superintegrable systems.

Degenerate Euclidean targets: Here the coordinates are x , y and $p_1 = p_x$, $p_2 = p_y$, and $\mathcal{J} = xp_2 - yp_1$.

1. $\tilde{E}3$: $\mathcal{H} = p_1^2 + p_2^2$, $\mathcal{X} = \mathcal{J}$, $\mathcal{L}_1 = p_1^2$, $\mathcal{L}_2 = p_1p_2$,
Casimir: $-\mathcal{L}_2^2 - \mathcal{L}_1(\mathcal{L}_1 - \mathcal{H}) = 0$, potential: $V = \alpha(x^2 + y^2)$.
2. $\tilde{E}5$: $\mathcal{H} = p_1^2 + p_2^2$, $\mathcal{X} = p_1$, $\mathcal{L}_1 = \mathcal{J}p_1$, $\mathcal{L}_2 = p_2p_1$,
Casimir: $\frac{1}{2}(\mathcal{L}_2^2 + \mathcal{X}^4 - \mathcal{H}\mathcal{X}^2) = 0$, potential: $V = \alpha x$.

Nondegenerate Euclidean targets:

1. $\tilde{E}1$: $\mathcal{L}_1 = \mathcal{J}^2$, $\mathcal{L}_2 = p_1^2$, $\mathcal{R}^2 = 16\mathcal{L}_1\mathcal{L}_2(\mathcal{H} - \mathcal{L}_2)$, $V = \alpha(x^2 + y^2) + \frac{\beta}{x^2} + \frac{\gamma}{y^2}$,
2. $\tilde{E}2$: $\mathcal{L}_1 = p_2^2$, $\mathcal{L}_2 = p_2\mathcal{J}$, $\mathcal{R}^2 = 4\mathcal{L}_1^2(\mathcal{H} - \mathcal{L}_1)$, $V = \alpha(4x^2 + y^2) + \beta x + \frac{\gamma}{y^2}$,
3. $\tilde{E}3'$: $\mathcal{L}_1 = p_1^2$, $\mathcal{L}_2 = p_1p_2$, $\mathcal{R}^2 = 0$, $V = \alpha(x^2 + y^2) + \beta x + \gamma y$,
4. $\tilde{E}8$: $\mathcal{L}_1 = \mathcal{J}^2$, $\mathcal{L}_2 = (p_1 + ip_2)^2$, $\mathcal{R}^2 = -16\mathcal{L}_1\mathcal{L}_2^2$, $V = \frac{\alpha(x-iy)}{(x+iy)^3} + \frac{\beta}{(x+iy)^2} + \gamma(x^2 + y^2)$,
5. $\tilde{E}9$: $\mathcal{L}_1 = (p_1 + ip_2)^2$, $\mathcal{L}_2 = p_1\mathcal{J}$, $\mathcal{R}^2 = -2\mathcal{L}_1(2\mathcal{L}_1 + \mathcal{H})^2$, $V = \frac{\alpha}{\sqrt{x+iy}} + \beta y + \frac{\gamma(x+2iy)}{\sqrt{x+iy}}$,

Degenerate targets on the 2-sphere: Here $\mathcal{J}_1 = yp_3 - zp_2$, $\mathcal{J}_2 = zp_1 - xp_3$, $\mathcal{J}_3 = xp_2 - yp_1$.

1. $\tilde{S}3$: $\mathcal{H} = \mathcal{J}_1^2 + \mathcal{J}_2^2 + \mathcal{J}_3^2$, $\mathcal{X} = \mathcal{J}_3$, $\mathcal{L}_1 = (\mathcal{J}_1 + i\mathcal{J}_2)^2$, $\mathcal{L}_2 = (\mathcal{J}_1 - i\mathcal{J}_2)^2$,
Casimir: $-2i((\mathcal{H} - \mathcal{X}^2)^2 - \mathcal{L}_1\mathcal{L}_2) = 0$, potential: $V = \frac{\alpha}{z^2}$,
2. $\tilde{S}5$: $\mathcal{H} = \mathcal{J}_1^2 + \mathcal{J}_2^2 + \mathcal{J}_3^2$, $\mathcal{X} = \mathcal{J}_1 + i\mathcal{J}_2$, $\mathcal{L}_1 = \mathcal{J}_3^2$, $\mathcal{L}_2 = (\mathcal{J}_1 + i\mathcal{J}_2)\mathcal{J}_3$,
Casimir: $-i(\mathcal{L}_2^2 - \mathcal{X}^2\mathcal{L}_1) = 0$, potential: $V = \frac{\alpha}{(x+iy)^2}$.
3. $\tilde{S}6$: $\mathcal{H} = \mathcal{J}_1^2 + \mathcal{J}_2^2 + \mathcal{J}_3^2$, $\mathcal{X} = \mathcal{J}_3$, $\mathcal{L}_1 = \mathcal{J}_3\mathcal{J}_1$, $\mathcal{L}_2 = \mathcal{J}_3\mathcal{J}_2$,
Casimir: $-\frac{1}{2}(\mathcal{L}_1^2 + \mathcal{L}_2^2 + \mathcal{X}^2(\mathcal{X}^2 - \mathcal{H})) = 0$, potential: $V = \frac{\alpha z}{\sqrt{x^2 + y^2}}$,

Nondegenerate targets on the 2-sphere:

1. $\tilde{S}1$: $\mathcal{L}_1 = (\mathcal{J}_1 + i\mathcal{J}_2)\mathcal{J}_3$, $\mathcal{L}_2 = (\mathcal{J}_1 + i\mathcal{J}_2)^2$, $\mathcal{R}^2 = -4\mathcal{L}_2^3$, $V = \frac{\alpha}{(x+iy)^2} + \frac{\beta z}{(x+iy)^2} + \frac{\gamma(1-4z^2)}{(x+iy)^4}$,
2. $\tilde{S}2$: $\mathcal{L}_1 = (\mathcal{J}_1 + i\mathcal{J}_2)^2$, $\mathcal{L}_2 = \mathcal{J}_3^2$, $\mathcal{R}^2 = -16\mathcal{L}_1^2\mathcal{L}_2$, $V = \frac{\alpha}{z^2} + \frac{\beta}{(x+iy)^2} + \frac{\gamma(x-iy)}{(x+iy)^3}$,
3. $\tilde{S}4$: $\mathcal{L}_1 = \mathcal{J}_3^2$, $\mathcal{L}_2 = (\mathcal{J}_1 + i\mathcal{J}_2)\mathcal{J}_3$, $\mathcal{R}^2 = -4\mathcal{L}_1\mathcal{L}_2^2$, $V = \frac{\alpha}{(x+iy)^2} + \frac{\beta z}{\sqrt{x^2 + y^2}} + \frac{\gamma}{(x+iy)\sqrt{x^2 + y^2}}$,
4. $\tilde{S}7$: $\mathcal{L}_1 = \mathcal{J}_3^2$, $\mathcal{L}_2 = \mathcal{J}_1\mathcal{J}_3$, $\mathcal{R}^2 = -4\mathcal{L}_1^3 - 4\mathcal{L}_2^2\mathcal{L}_1 + 4\mathcal{L}_1^2\mathcal{H}$, $V = \frac{\alpha z}{\sqrt{x^2 + y^2}} + \frac{\beta x}{y^2\sqrt{x^2 + y^2}} + \frac{\gamma}{y^2}$,
5. $\tilde{S}9$: $\mathcal{L}_1 = \mathcal{J}_3^2$, $\mathcal{L}_2 = \mathcal{J}_1^2$, $\mathcal{R}^2 = -16\mathcal{L}_1^2\mathcal{L}_2 - 16\mathcal{L}_1\mathcal{L}_2^2 + 16\mathcal{L}_1\mathcal{L}_2\mathcal{H}$, $V = \frac{\alpha}{x^2} + \frac{\beta}{y^2} + \frac{\gamma}{z^2}$,

8 Conclusions and discussion

This paper is part of a series on 2D quadratic algebras, their classification, structure, representations, and especially, contractions as they relate to 2nd order 2D superintegrable systems. Of special interest are contractions that correspond to geometrical pointwise limiting processes in the physical superintegrable systems. As shown in [22], one of the consequences of contracting between superintegrable systems is a series of limiting relations between special functions associated with the superintegrable systems, a special case of which is the Askey scheme for hypergeometric orthogonal polynomials. In [21] we studied quadratic algebras related to 2nd order superintegrable systems on constant curvature spaces and showed that there is a one-to-one correspondence between conjugacy classes of quadratic algebras in the enveloping algebras of $e(2, \mathbb{C})$ and $o(3, \mathbb{C})$, and isomorphism classes of superintegrable systems with potential. Further, we showed for constant curvature spaces that generalizations of Inönü–Wigner Lie algebra contractions of $e(2, \mathbb{C})$ and $o(3, \mathbb{C})$, induce quadratic algebra contractions that correspond to geometrical pointwise limiting processes in the physical systems. The procedure is rigid and deterministic. The present paper extends these results and shows that Darboux superintegrable systems are also characterized by free quadratic algebras contained in the symmetry algebras of these spaces and that their contractions are also induced by Inönü–Wigner contractions. Thus our basic results hold for all 2nd order 2D superintegrable systems. In follow-up papers, in preparation, we will classify abstract quadratic algebras and their contractions, including those not induced from Lie algebras, and study which of these relate to superintegrable systems.

We intend to conclude this series by relating contractions of 2nd order superintegrable systems to limiting processes for R -separable coordinate systems for wave equations, introduced by Bôcher in his famous 1894 thesis [1]. We will show that in 2D, Bôcher’s limiting processes for cyclides and ellipses induce generalizations of Inönü–Wigner contractions of the $so(4, \mathbb{C})$ conformal symmetry algebra of the 2D wave equation with potential and that these contractions explain the full contraction pattern for 2nd order superintegrable systems. Bôcher’s limits, which we term Bôcher contractions, apply to all dimensions $n \geq 2$, so this should provide a useful guide to the analysis of 2nd order superintegrable systems in higher dimensions.

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References

- [1] Bôcher M., Über die Reihenentwicklungen der Potentialtheory, B.G. Teubner, Leipzig, 1894.
- [2] Bonatsos D., Daskaloyannis C., Kokkotas K., Deformed oscillator algebras for two-dimensional quantum superintegrable systems, *Phys. Rev. A* **50** (1994), 3700–3709, [hep-th/9309088](#).
- [3] Daskaloyannis C., Tanoudis Y., Quantum superintegrable systems with quadratic integrals on a two dimensional manifold, *J. Math. Phys.* **48** (2007), 072108, 22 pages, [math-ph/0607058](#).
- [4] Doebner H.D., Melsheimer O., On a class of generalized group contractions, *Nuovo Cimento A* **49** (1967), 306–311.
- [5] Eisenhart L.P., Transformations of surfaces, 2nd ed., Chelsea, New York, 1966.
- [6] Evans N.W., Super-integrability of the Winternitz system, *Phys. Lett. A* **147** (1990), 483–486.
- [7] Fordy A.P., Quantum super-integrable systems as exactly solvable models, *SIGMA* **3** (2007), 025, 10 pages, [math-ph/0702048](#).
- [8] Grabowski J., Marmo G., Perelomov A.M., Poisson structures: towards a classification, *Modern Phys. Lett. A* **8** (1993), 1719–1733.

- [9] Granovskii Ya.I., Zhedanov A.S., Lutsenko I.M., Quadratic algebras and dynamics in curved space. I. Oscillator, *Theoret. and Math. Phys.* **91** (1992), 474–480.
- [10] Granovskii Ya.I., Zhedanov A.S., Lutsenko I.M., Quadratic algebras and dynamics in curved space. II. The Kepler problem, *Theoret. and Math. Phys.* **91** (1992), 604–612.
- [11] Gromov N.A., From Wigner–Inönü group contraction to contractions of algebraic structures, *Acta Phys. Hung. A* **19** (2004), 209–212, [hep-th/0210304](#).
- [12] Huddleston P.L., Inönü–Wigner contractions of the real four-dimensional Lie algebras, *J. Math. Phys.* **19** (1978), 1645–1649.
- [13] Inönü E., Wigner E.P., On the contraction of groups and their representations, *Proc. Nat. Acad. Sci. USA* **39** (1953), 510–524.
- [14] Kalnins E.G., Kress J.M., Miller Jr. W., Second-order superintegrable systems in conformally flat spaces. I. Two-dimensional classical structure theory, *J. Math. Phys.* **46** (2005), 053509, 28 pages.
- [15] Kalnins E.G., Kress J.M., Miller Jr. W., Second order superintegrable systems in conformally flat spaces. II. The classical two-dimensional Stäckel transform, *J. Math. Phys.* **46** (2005), 053510, 15 pages.
- [16] Kalnins E.G., Kress J.M., Miller Jr. W., Second order superintegrable systems in conformally flat spaces. V. Two- and three-dimensional quantum systems, *J. Math. Phys.* **47** (2006), 093501, 25 pages.
- [17] Kalnins E.G., Kress J.M., Miller Jr. W., Nondegenerate 2D complex Euclidean superintegrable systems and algebraic varieties, *J. Phys. A: Math. Theor.* **40** (2007), 3399–3411, [arXiv:0708.3044](#).
- [18] Kalnins E.G., Kress J.M., Miller Jr. W., Post S., Structure theory for second order 2D superintegrable systems with 1-parameter potentials, *SIGMA* **5** (2009), 008, 24 pages, [arXiv:0901.3081](#).
- [19] Kalnins E.G., Kress J.M., Miller Jr. W., Winternitz P., Superintegrable systems in Darboux spaces, *J. Math. Phys.* **44** (2003), 5811–5848, [math-ph/0307039](#).
- [20] Kalnins E.G., Kress J.M., Winternitz P., Superintegrability in a two-dimensional space of nonconstant curvature, *J. Math. Phys.* **43** (2002), 970–983, [math-ph/0108015](#).
- [21] Kalnins E.G., Miller Jr. W., Quadratic algebra contractions and second-order superintegrable systems, *Anal. Appl. (Singap.)* **12** (2014), 583–612, [arXiv:1401.0830](#).
- [22] Kalnins E.G., Miller Jr. W., Post S., Contractions of 2D 2nd order quantum superintegrable systems and the Askey scheme for hypergeometric orthogonal polynomials, *SIGMA* **9** (2013), 057, 28 pages, [arXiv:1212.4766](#).
- [23] Koenigs G., Sur les géodésiques a intégrales quadratiques, in Darboux G., Lecons sur la théorie générale des surfaces et les applications géométriques du calcul infinitesimal, Vol. 4, Chelsea, New York, 1972, 368–404.
- [24] Kress J.M., Equivalence of superintegrable systems in two dimensions, *Phys. Atomic Nuclei* **70** (2007), 560–566.
- [25] Létourneau P., Vinet L., Superintegrable systems: polynomial algebras and quasi-exactly solvable Hamiltonians, *Ann. Physics* **243** (1995), 144–168.
- [26] Maduemezia A., On hyper-relativistic quantum systems, *J. Math. Phys.* **28** (1987), 79–84.
- [27] Miller Jr. W., Symmetry and separation of variables, *Encyclopedia of Mathematics and its Applications*, Vol. 4, Addison-Wesley Publishing Co., Reading, Mass. – London – Amsterdam, 1977.
- [28] Miller Jr. W., Post S., Winternitz P., Classical and quantum superintegrability with applications, *J. Phys. A: Math. Theor.* **46** (2013), 423001, 97 pages, [arXiv:1309.2694](#).
- [29] Nesterenko M., Popovych R., Contractions of low-dimensional Lie algebras, *J. Math. Phys.* **47** (2006), 123515, 45 pages, [math-ph/0608018](#).
- [30] Saletan E.J., Contraction of Lie groups, *J. Math. Phys.* **2** (1961), 1–21.
- [31] Talman J.D., Special functions: a group theoretic approach (based on lectures by Eugene P. Wigner), W. A. Benjamin, Inc., New York – Amsterdam, 1968.
- [32] Tempesta P., Turbiner A.V., Winternitz P., Exact solvability of superintegrable systems, *J. Math. Phys.* **42** (2001), 4248–4257, [hep-th/0011209](#).
- [33] Tempesta P., Winternitz P., Harnad J., Miller W., Pogosyan G., Rodriguez M. (Editors), Superintegrability in classical and quantum systems, *CRM Proceedings and Lecture Notes*, Vol. 37, American Mathematical Society, Providence, RI, 2004.
- [34] Weimar-Woods E., The three-dimensional real Lie algebras and their contractions, *J. Math. Phys.* **32** (1991), 2028–2033.