

Metaplectic-c Qantomorphisms

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Abstract. In the classical Kostant–Souriau prequantization procedure, the Poisson algebra of a symplectic manifold (M, ω) is realized as the space of infinitesimal quantomorphisms of the prequantization circle bundle. Robinson and Rawnsley developed an alternative to the Kostant–Souriau quantization process in which the prequantization circle bundle and metaplectic structure for (M, ω) are replaced by a metaplectic-c prequantization. They proved that metaplectic-c quantization can be applied to a larger class of manifolds than the classical recipe. This paper presents a definition for a metaplectic-c quantomorphism, which is a diffeomorphism of metaplectic-c prequantizations that preserves all of their structures. Since the structure of a metaplectic-c prequantization is more complicated than that of a circle bundle, we find that the definition must include an extra condition that does not have an analogue in the Kostant–Souriau case. We then define an infinitesimal quantomorphism to be a vector field whose flow consists of metaplectic-c quantomorphisms, and prove that the space of infinitesimal metaplectic-c quantomorphisms exhibits all of the same properties that are seen for the infinitesimal quantomorphisms of a prequantization circle bundle. In particular, this space is isomorphic to the Poisson algebra $C^\infty(M)$.

Key words: geometric quantization; metaplectic-c prequantization; quantomorphism

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1 Introduction

Recall that a prequantization circle bundle for a symplectic manifold (M, ω) consists of a circle bundle $Y \rightarrow M$ and a connection one-form γ on Y such that $d\gamma = \frac{1}{i\hbar}\omega$. The Kostant–Souriau quantization recipe with half-form correction requires a prequantization circle bundle and a choice of metaplectic structure for (M, ω) .

Souriau [5] defined a *quantomorphism* between two prequantization circle bundles $(Y_1, \gamma_1) \rightarrow (M_1, \omega_1)$ and $(Y_2, \gamma_2) \rightarrow (M_2, \omega_2)$ to be a diffeomorphism $K : Y_1 \rightarrow Y_2$ such that $K^*\gamma_2 = \gamma_1$. This condition implies that K is equivariant with respect to the principal circle actions. Souriau then defined the *infinitesimal quantomorphisms* of a prequantization circle bundle (Y, γ) to be the vector fields on Y whose flows are quantomorphisms. Kostant [2] proved that the space of infinitesimal quantomorphisms, which we denote $\mathcal{Q}(Y, \gamma)$, is isomorphic to the Poisson algebra $C^\infty(M)$.

The metaplectic-c group is a circle extension of the symplectic group. Metaplectic-c quantization, which was developed by Robinson and Rawnsley [3], is a variant of Kostant–Souriau quantization in which the prequantization bundle and metaplectic structure are replaced by a metaplectic-c structure (P, Σ) and a prequantization one-form γ . Robinson and Rawnsley proved that metaplectic-c quantization can be applied to all systems that admit metaplectic quantizations, and to some where the Kostant–Souriau process fails.

In Section 2, we present an explicit construction of the isomorphism from $\mathcal{Q}(Y, \gamma)$ to $C^\infty(M)$. In Section 3, after describing the metaplectic-c prequantization (P, Σ, γ) , we define a metaplectic-c quantomorphism, which is a diffeomorphism of metaplectic-c prequantizations that preserves

all of their structures. Our definition is based on Souriau's, but includes a condition that is unique to the metaplectic-c context. We then use the metaplectic-c quantomorphisms to define $\mathcal{Q}(P, \Sigma, \gamma)$, the space of infinitesimal metaplectic-c quantomorphisms of (P, Σ, γ) . We show that every property that was proved for $\mathcal{Q}(Y, \gamma)$ has a parallel for $\mathcal{Q}(P, \Sigma, \gamma)$. In particular, $\mathcal{Q}(P, \Sigma, \gamma)$ is isomorphic to the Poisson algebra $C^\infty(M)$. The construction in Section 2 is used as a model for the proofs in Section 3. We indicate when the calculations are analogous, and when the metaplectic-c case requires additional steps.

Some global remarks concerning notation: for any vector field ξ , the Lie derivative with respect to ξ is written L_ξ . The space of smooth vector fields on a manifold P is denoted by $\mathcal{X}(P)$. Given a smooth map $F : P \rightarrow M$ and a vector field $\xi \in \mathcal{X}(P)$, we write $F_*\xi$ for the pushforward of ξ if and only if the result is a well-defined vector field on M . If P is a bundle over M , $\Gamma(P)$ denotes the space of smooth sections of P , where the base is always taken to be the symplectic manifold M . Planck's constant will only appear in the form \hbar .

2 Kostant–Souriau quantomorphisms

In this section, after reviewing the Kostant–Souriau prequantization of a symplectic manifold (M, ω) , we construct a Lie algebra isomorphism from $C^\infty(M)$ to the space of infinitesimal quantomorphisms. As we have already noted, the fact that these algebras are isomorphic was originally stated by Kostant [2] in the context of line bundles with connection. His proof can be reconstructed from several propositions across Sections 2–4 of [2]. Kostant's isomorphism is also stated by Śniatycki [4], but much of the proof is left as an exercise. We are not aware of a source in the literature for a self-contained proof that uses the language of principal bundles, and this is one of our reasons for performing an explicit construction here.

The other goal of this section is to motivate the analogous constructions for a metaplectic-c prequantization, which will be the subject of Section 3. Each result that we present for Kostant–Souriau prequantization will have a parallel in the metaplectic-c case. When the proofs are identical, we will simply refer back to the work shown here, thereby allowing Section 3 to focus on those features that are unique to metaplectic-c structures.

2.1 Basic definitions and notation

2.1.1 Hamiltonian vector fields and the Poisson algebra

Let (M, ω) be a symplectic manifold. Given $f \in C^\infty(M)$, define its Hamiltonian vector field $\xi_f \in \mathcal{X}(M)$ by

$$\xi_f \lrcorner \omega = df.$$

Define the Poisson bracket on $C^\infty(M)$ by

$$\{f, g\} = -\omega(\xi_f, \xi_g), \quad \forall f, g \in C^\infty(M).$$

These choices imply that

$$\xi_f g = \{f, g\}, \quad \forall f, g \in C^\infty(M).$$

A standard calculation establishes the following fact.

Lemma 2.1. *For all $f, g \in C^\infty(M)$, $[\xi_f, \xi_g] = \xi_{\{f, g\}}$.*

2.1.2 Circle bundles and connections

Let $Y \xrightarrow{p} M$ be a right principal $U(1)$ bundle over a manifold M .

- For any $\lambda \in U(1)$, let $R_\lambda : Y \rightarrow Y$ represent the right action by λ . That is, $R_\lambda(y) = y \cdot \lambda$ for all $y \in Y$.
- For any $\theta \in \mathfrak{u}(1)$, the Lie algebra of $U(1)$, let ∂_θ be the vector field on Y with flow $R_{\exp(t\theta)}$, where $t \in \mathbb{R}$. In particular, we will consider $\partial_{2\pi i}$.

Let γ be a connection one-form on Y . By definition, γ is invariant under the right principal action, and for all $\theta \in \mathfrak{u}(1)$, $\gamma(\partial_\theta) = \theta$. There is a two-form ϖ on M , called the *curvature* of γ , such that $d\gamma = p^*\varpi$.

For any $\xi \in \mathcal{X}(M)$, let $\tilde{\xi}$ be the lift of ξ to Y that is horizontal with respect to γ . That is, $p_*\tilde{\xi} = \xi$ and $\gamma(\tilde{\xi}) = 0$. For any $\theta \in \mathfrak{u}(1)$, note that $p_*\partial_\theta = 0$, which implies that $p_*[\tilde{\xi}, \partial_\theta] = [p_*\tilde{\xi}, p_*\partial_\theta] = 0$ and $\gamma([\tilde{\xi}, \partial_\theta]) = -(p^*\varpi)(\tilde{\xi}, \partial_\theta) = 0$. Therefore $[\tilde{\xi}, \partial_\theta] = 0$ for all θ .

Associated to Y is a complex line bundle L over M , given by $L = Y \times_{U(1)} \mathbb{C}$. We write an element of L as an equivalence class $[y, z]$ with $y \in Y$ and $z \in \mathbb{C}$. There is a connection ∇ on L that is constructed from the connection one-form γ through the following process.

- Given any $s \in \Gamma(L)$, define the map $\tilde{s} : Y \rightarrow \mathbb{C}$ so that $[y, \tilde{s}(y)] = s(p(y))$ for all $y \in Y$. Then \tilde{s} has the equivariance property

$$\tilde{s}(y \cdot \lambda) = \lambda^{-1}\tilde{s}(y), \quad \forall y \in Y, \lambda \in U(1).$$

- Conversely, any map $\tilde{s} : Y \rightarrow \mathbb{C}$ with the above equivariance property can be used to construct a section s of L by setting $s(m) = [y, \tilde{s}(y)]$ for all $m \in M$ and any $y \in Y$ such that $p(y) = m$.
- Let $\xi \in \mathcal{X}(M)$ be given, and let $\tilde{\xi}$ be its horizontal lift to Y . If $\tilde{s} : Y \rightarrow \mathbb{C}$ is an equivariant map, then so is $\tilde{\xi}\tilde{s}$. This follows from the fact that $[\tilde{\xi}, \partial_\theta] = 0$ for all $\theta \in \mathfrak{u}(1)$.
- Define the connection ∇ on L so that for any $\xi \in \mathcal{X}(M)$ and $s \in \Gamma(L)$, $\nabla_\xi s$ is the section of L that satisfies

$$\widetilde{\nabla_\xi s} = \tilde{\xi}\tilde{s}.$$

2.2 The prequantization circle bundle and its infinitesimal quantomorphisms

Definition 2.2. Let (M, ω) be a symplectic manifold. A *prequantization circle bundle* for (M, ω) is a right principal $U(1)$ bundle $Y \xrightarrow{p} M$, together with a connection one-form γ on Y satisfying $d\gamma = \frac{1}{i\hbar}p^*\omega$.

Definition 2.3. Let $(Y_1, \gamma_1) \xrightarrow{p_1} (M_1, \omega_1)$ and $(Y_2, \gamma_2) \xrightarrow{p_2} (M_2, \omega_2)$ be prequantization circle bundles for two symplectic manifolds. A diffeomorphism $K : Y_1 \rightarrow Y_2$ is called a *quantomorphism* if $K^*\gamma_2 = \gamma_1$.

Let $K : Y_1 \rightarrow Y_2$ be a quantomorphism. Notice that for any $\theta \in \mathfrak{u}(1)$, the vector field ∂_θ on Y_1 is completely specified by the conditions $\gamma_1(\partial_\theta) = \theta$ and $\partial_\theta \lrcorner d\gamma_1 = 0$, and the same is true on Y_2 . Since $K^*\gamma_2 = \gamma_1$, we see that $K_*\partial_\theta = \partial_\theta$ for all θ , and so K is equivariant with respect to the principal circle actions.

Definition 2.4. Let $(Y, \gamma) \xrightarrow{p} (M, \omega)$ be a prequantization circle bundle. An *infinitesimal quantomorphism* of (Y, γ) is a vector field $\zeta \in \mathcal{X}(Y)$ whose flow ϕ_t on Y is a quantomorphism from its domain to its range for each t . The space of infinitesimal quantomorphisms of (Y, γ) is denoted by $\mathcal{Q}(Y, \gamma)$.

Let $\zeta \in \mathcal{X}(Y)$ have flow ϕ_t . The connection form γ is preserved by ϕ_t if and only if $L_\zeta \gamma = 0$. Therefore the space of infinitesimal quantomorphisms of (Y, γ) is

$$\mathcal{Q}(Y, \gamma) = \{\zeta \in \mathcal{X}(Y) \mid L_\zeta \gamma = 0\}.$$

If $K : Y_1 \rightarrow Y_2$ is a quantomorphism, then it induces a diffeomorphism (in fact, a symplectomorphism) $K' : M_1 \rightarrow M_2$ such that the following diagram commutes:

$$\begin{array}{ccc} Y_1 & \xrightarrow{K} & Y_2 \\ \downarrow p_1 & & \downarrow p_2 \\ M_1 & \xrightarrow{K'} & M_2 \end{array}$$

This implies that for any $\zeta \in \mathcal{Q}(Y, \gamma)$ with flow ϕ_t , there is a flow ϕ'_t on M that satisfies $p \circ \phi_t = \phi'_t \circ p$. If ζ' is the vector field on M with flow ϕ'_t , then $p_* \zeta = \zeta'$. In other words, elements of $\mathcal{Q}(Y, \gamma)$ descend via p_* to well-defined vector fields on M .

2.3 The Lie algebra isomorphism

Let $(Y, \gamma) \xrightarrow{p} (M, \omega)$ be a prequantization circle bundle. We will now present an explicit construction of a Lie algebra isomorphism from $C^\infty(M)$ to $\mathcal{Q}(Y, \gamma)$. Recall that the vector field $\partial_{2\pi i}$ on Y satisfies $\gamma(\partial_{2\pi i}) = 2\pi i \in \mathfrak{u}(1)$ and $p_* \partial_{2\pi i} = 0$.

Lemma 2.5. *For all $f, g \in C^\infty(M)$,*

$$[\tilde{\xi}_f, \tilde{\xi}_g] = \tilde{\xi}_{\{f, g\}} - \frac{1}{2\pi\hbar} p^* \{f, g\} \partial_{2\pi i}.$$

Proof. It suffices to show that

$$p_* [\tilde{\xi}_f, \tilde{\xi}_g] = p_* \left(\tilde{\xi}_{\{f, g\}} - \frac{1}{2\pi\hbar} p^* \{f, g\} \partial_{2\pi i} \right)$$

and

$$\gamma([\tilde{\xi}_f, \tilde{\xi}_g]) = \gamma \left(\tilde{\xi}_{\{f, g\}} - \frac{1}{2\pi\hbar} p^* \{f, g\} \partial_{2\pi i} \right).$$

Using Lemma 2.1, we see that

$$p_* \left(\tilde{\xi}_{\{f, g\}} - \frac{1}{2\pi\hbar} p^* \{f, g\} \partial_{2\pi i} \right) = \xi_{\{f, g\}} = [\xi_f, \xi_g].$$

Since $p_* \tilde{\xi}_f = \xi_f$ and $p_* \tilde{\xi}_g = \xi_g$, it follows that $p_* [\tilde{\xi}_f, \tilde{\xi}_g] = [\xi_f, \xi_g]$. Thus the first equation is verified.

Next, note that

$$\gamma \left(\tilde{\xi}_{\{f, g\}} - \frac{1}{2\pi\hbar} p^* \{f, g\} \partial_{2\pi i} \right) = \frac{1}{i\hbar} p^* \{f, g\},$$

and

$$\gamma([\tilde{\xi}_f, \tilde{\xi}_g]) = -\frac{1}{i\hbar} (p^* \omega)(\tilde{\xi}_f, \tilde{\xi}_g) = \frac{1}{i\hbar} p^* \{f, g\}.$$

Therefore the second equation is also verified. ■

Lemma 2.6. *The map $E : C^\infty(M) \rightarrow \mathcal{X}(Y)$ given by*

$$E(f) = \tilde{\xi}_f + \frac{1}{2\pi\hbar} p^* f \partial_{2\pi i}, \quad \forall f \in C^\infty(M)$$

is a Lie algebra homomorphism.

Proof. Let $f, g \in C^\infty(M)$ be arbitrary. We need to show that

$$\tilde{\xi}_{\{f, g\}} + \frac{1}{2\pi\hbar} p^* \{f, g\} \partial_{2\pi i} = \left[\tilde{\xi}_f + \frac{1}{2\pi\hbar} p^* f \partial_{2\pi i}, \tilde{\xi}_g + \frac{1}{2\pi\hbar} p^* g \partial_{2\pi i} \right].$$

Using Lemma 2.5, the left-hand side becomes

$$[\tilde{\xi}_f, \tilde{\xi}_g] + 2 \frac{1}{2\pi\hbar} p^* \{f, g\} \partial_{2\pi i}.$$

Expanding the right-hand side yields

$$[\tilde{\xi}_f, \tilde{\xi}_g] + \left[\tilde{\xi}_f, \frac{1}{2\pi\hbar} p^* g \partial_{2\pi i} \right] + \left[\frac{1}{2\pi\hbar} p^* f \partial_{2\pi i}, \tilde{\xi}_g \right] + \left[\frac{1}{2\pi\hbar} p^* f \partial_{2\pi i}, \frac{1}{2\pi\hbar} p^* g \partial_{2\pi i} \right].$$

The fourth term vanishes because $\partial_\theta(p^* f) = \partial_\theta(p^* g) = 0$ for any $\theta \in \mathfrak{u}(1)$. To evaluate the third term, recall that $[\partial_\theta, \tilde{\xi}] = 0$ for any $\theta \in \mathfrak{u}(1)$ and $\xi \in \mathcal{X}(M)$. Therefore $[\partial_{2\pi i}, \tilde{\xi}_g] = 0$, so this term reduces to

$$-\frac{1}{2\pi\hbar} (\tilde{\xi}_g p^* f) \partial_{2\pi i} = \frac{1}{2\pi\hbar} p^* \{f, g\} \partial_{2\pi i}.$$

By the same argument, the second term also reduces to

$$\frac{1}{2\pi\hbar} p^* \{f, g\} \partial_{2\pi i}.$$

Combining these results, we find that the right-hand side of the desired equation is

$$[\tilde{\xi}_f, \tilde{\xi}_g] + 2 \frac{1}{i\hbar} p^* \{f, g\} \partial_{2\pi i},$$

which equals the left-hand side. ■

Lemma 2.7. *For all $f \in C^\infty(M)$, $E(f) \in \mathcal{Q}(Y, \gamma)$.*

Proof. We need to show that $L_{E(f)}\gamma = 0$. We calculate

$$L_{E(f)}\gamma = E(f) \lrcorner d\gamma + d(E(f) \lrcorner \gamma) = \frac{1}{i\hbar} p^* (\xi_f \lrcorner \omega) - \frac{1}{i\hbar} p^* df = 0. \quad \blacksquare$$

So far, we have shown that $E : C^\infty(M) \rightarrow \mathcal{Q}(Y, \gamma)$ is a Lie algebra homomorphism. We will now construct a map $F : \mathcal{Q}(Y, \gamma) \rightarrow C^\infty(M)$, and show that E and F are inverses. This will complete the proof that $C^\infty(M)$ and $\mathcal{Q}(Y, \gamma)$ are isomorphic.

Let $\zeta \in \mathcal{Q}(Y, \gamma)$ be arbitrary. Then $L_\zeta \gamma = \zeta \lrcorner d\gamma + d(\gamma(\zeta)) = 0$. This implies that $\partial_\theta \lrcorner (\zeta \lrcorner d\gamma + d(\gamma(\zeta))) = 0$ for any $\theta \in \mathfrak{u}(1)$. Since $d\gamma(\zeta, \partial_\theta) = \frac{1}{i\hbar} (p^* \omega)(\zeta, \partial_\theta) = 0$, it follows that $\partial_\theta \lrcorner d(\gamma(\zeta)) = L_{\partial_\theta} \gamma(\zeta) = 0$. We can therefore define the map $F : \mathcal{Q}(Y, \gamma) \rightarrow C^\infty(M)$ so that

$$-\frac{1}{i\hbar} p^* F(\zeta) = \gamma(\zeta), \quad \forall \zeta \in \mathcal{Q}(Y, \gamma).$$

Theorem 2.8. *The map $E : C^\infty(M) \rightarrow \mathcal{Q}(Y, \gamma)$ is a Lie algebra isomorphism with inverse F .*

Proof. Let $f \in C^\infty(M)$ and $\zeta \in \mathcal{Q}(Y, \gamma)$ be arbitrary. We will show that $F(E(f)) = f$ and $E(F(\zeta)) = \zeta$. Using the definitions of E and F , we have

$$-\frac{1}{i\hbar}p^*F(E(f)) = \gamma(E(f)) = \gamma\left(\tilde{\xi}_f + \frac{1}{2\pi\hbar}p^*f\partial_{2\pi i}\right) = -\frac{1}{i\hbar}p^*f.$$

This implies that $F(E(f)) = f$.

To show that $E(F(\zeta)) = \zeta$, it suffices to show that $\gamma(E(F(\zeta))) = \gamma(\zeta)$ and $p_*E(F(\zeta)) = p_*\zeta$. By definition,

$$E(F(\zeta)) = \tilde{\xi}_{F(\zeta)} + \frac{1}{2\pi\hbar}p^*F(\zeta)\partial_{2\pi i} = \tilde{\xi}_{F(\zeta)} + \frac{1}{2\pi i}\gamma(\zeta)\partial_{2\pi i}.$$

It is immediate that $\gamma(E(F(\zeta))) = \gamma(\zeta)$, and that $p_*E(F(\zeta)) = \xi_{F(\zeta)}$. Observe that

$$\zeta \lrcorner p^*\omega = i\hbar\zeta \lrcorner d\gamma = -i\hbar d(\gamma(\zeta)) = p^*(dF(\zeta)),$$

having used $L_\zeta\gamma = 0$. Therefore $(p_*\zeta) \lrcorner \omega = dF(\zeta)$, which implies that $p_*\zeta = \xi_{F(\zeta)}$. Thus $p_*E(F(\zeta)) = p_*\zeta$. This concludes the proof that $E(F(\zeta)) = \zeta$.

Since E and F are inverses, and we know from Lemma 2.7 that $E : C^\infty(M) \rightarrow \mathcal{Q}(Y, \gamma)$ is a Lie algebra homomorphism, it follows that E and F are the desired Lie algebra isomorphisms. ■

The primary goal of Section 3 is to duplicate the above construction for the infinitesimal quantomorphisms of a metaplectic- c prequantization. However, before moving on to the metaplectic- c case, we will show how the map E can be used to represent the elements of $C^\infty(M)$ as operators on the space of sections of the prequantization line bundle for (M, ω) . This result will also have an analogue in the metaplectic- c case, which we will discuss in Section 3.5.

2.4 An operator representation of $C^\infty(M)$

Let (L, ∇) be the complex line bundle with connection associated to (Y, γ) . One of the goals of the Kostant–Souriau prequantization process is to produce a representation $r : C^\infty(M) \rightarrow \text{End } \Gamma(L)$. To be consistent with quantum mechanics in the case of a physically realizable system, the map r is required to satisfy the following axioms:

- (1) $r(1)$ is the identity map on $\Gamma(L)$,
- (2) for all $f, g \in C^\infty(M)$, $[r(f), r(g)] = i\hbar r(\{f, g\})$ (up to sign convention).

These axioms are based on an analysis by Dirac [1] of the relationship between classical and quantum mechanical observables. For more detail in the context of geometric quantization, see, for example, Śniatycki [4] or Woodhouse [6].

Recall the association between a section s of L and an equivariant function $\tilde{s} : Y \rightarrow \mathbb{C}$. We note the following properties.

- For any $f \in C^\infty(M)$ and $s \in \Gamma(L)$, the equivariant function corresponding to the section fs is $\widetilde{fs} = p^*f\tilde{s}$.
- The vector field $\partial_{2\pi i}$ has flow $R_{\exp(2\pi it)}$. Thus, for all $y \in Y$,

$$(\partial_{2\pi i}\tilde{s})(y) = \left. \frac{d}{dt} \right|_{t=0} \tilde{s}(y \cdot e^{2\pi it}) = -2\pi i\tilde{s}(y).$$

The Kostant–Souriau representation $r : C^\infty(M) \rightarrow \text{End } \Gamma(L)$ is defined by

$$r(f)s = (i\hbar\nabla_{\xi_f} + f)s, \quad \forall f \in C^\infty(M), s \in \Gamma(L).$$

Using the preceding observations, we see that

$$\widetilde{r(f)s} = (i\hbar\tilde{\xi}_f + p^*f)\tilde{s} = \left(i\hbar\tilde{\xi}_f - \frac{1}{2\pi i} p^*f \partial_{2\pi i} \right) \tilde{s} = i\hbar E(f)\tilde{s}.$$

Since we proved in Lemma 2.6 that $E(\{f, g\}) = [E(f), E(g)]$ for all $f, g \in C^\infty(M)$, the following is immediate.

Theorem 2.9. *The map $r : C^\infty(M) \rightarrow \text{End } \Gamma(L)$ satisfies Dirac axioms (1) and (2).*

Thus the same map that provides the isomorphism from $C^\infty(M)$ to $\mathcal{Q}(Y, \gamma)$ also yields the usual Kostant–Souriau representation of $C^\infty(M)$ as a space of operators on $\Gamma(L)$. We will see a similar result in the case of metaplectic-c prequantization.

3 Metaplectic-c quantomorphisms

Having reviewed the properties of infinitesimal quantomorphisms in Kostant–Souriau prequantization, we will now explore their parallels in metaplectic-c prequantization. In Sections 3.1 and 3.2, we summarize the prequantization stage of the metaplectic-c quantization process developed by Robinson and Rawnsley [3]. In Section 3.3, we develop our definition for a metaplectic-c quantomorphism, and use it to define an infinitesimal metaplectic-c quantomorphism. The remainder of the paper is dedicated to proving the metaplectic-c analogues of the results presented in Section 2.

3.1 The metaplectic-c group

Fix a $2n$ -dimensional real vector space V , and equip it with a symplectic structure Ω . Let $\text{Sp}(V)$ be the symplectic group of (V, Ω) ; that is, $\text{Sp}(V)$ is the group of linear automorphisms of V that preserve Ω . The metaplectic group $\text{Mp}(V)$ is the unique connected double cover of $\text{Sp}(V)$. The metaplectic-c group $\text{Mp}^c(V)$ is defined by

$$\text{Mp}^c(V) = \text{Mp}(V) \times_{\mathbb{Z}_2} U(1),$$

where $\mathbb{Z}_2 \subset \text{Mp}(V)$ consists of the two preimages of $I \in \text{Sp}(V)$, and $\mathbb{Z}_2 \subset U(1)$ is the usual subgroup $\{1, -1\}$.

Two important group homomorphisms can be defined on $\text{Mp}^c(V)$. The first is the projection map $\sigma : \text{Mp}^c(V) \rightarrow \text{Sp}(V)$, which is part of the short exact sequence

$$1 \rightarrow U(1) \rightarrow \text{Mp}^c(V) \xrightarrow{\sigma} \text{Sp}(V) \rightarrow 1.$$

The second is the determinant map $\eta : \text{Mp}^c(V) \rightarrow U(1)$, which is part of the short exact sequence

$$1 \rightarrow \text{Mp}(V) \rightarrow \text{Mp}^c(V) \xrightarrow{\eta} U(1) \rightarrow 1.$$

This latter map has the property that if $\lambda \in U(1) \subset \text{Mp}^c(V)$, then $\eta(\lambda) = \lambda^2$. The Lie algebra $\mathfrak{mp}^c(V)$ is identified with $\mathfrak{sp}(V) \oplus \mathfrak{u}(1)$ under the map $\sigma_* \oplus \frac{1}{2}\eta_*$.

3.2 Metaplectic-c prequantization

Let (M, ω) be a $2n$ -dimensional symplectic manifold. The symplectic frame bundle for (M, ω) , modeled on (V, Ω) , is denoted $\mathrm{Sp}(M, \omega)$, and is defined fiberwise for each $m \in M$ by

$$\mathrm{Sp}(M, \omega)_m = \{b : V \rightarrow T_m M \mid b \text{ is an isomorphism and } b^* \omega_m = \Omega\}.$$

Then $\mathrm{Sp}(M, \omega)$ is a right principal $\mathrm{Sp}(V)$ bundle over M , where $g \in \mathrm{Sp}(V)$ acts by precomposition. Let $\mathrm{Sp}(M, \omega) \xrightarrow{\rho} M$ be the bundle projection map.

Definition 3.1. A *metaplectic-c structure* for (M, ω) is a pair (P, Σ) , where P is a right principal $\mathrm{Mp}^c(V)$ bundle $P \xrightarrow{\Pi} M$, and Σ is a map $P \xrightarrow{\Sigma} \mathrm{Sp}(M, \omega)$ that satisfies

$$\Sigma(q \cdot a) = \Sigma(q) \cdot \sigma(a), \quad \forall q \in P, a \in \mathrm{Mp}^c(V),$$

and $\Pi = \rho \circ \Sigma$.

We will need the following definitions and observations concerning Lie algebras and vector fields.

- Given $\kappa \in \mathfrak{sp}(V)$, let ∂_κ be the vector field on $\mathrm{Sp}(M, \omega)$ whose flow is $R_{\exp(t\kappa)}$.
- Given $\alpha \in \mathfrak{mp}^c(V)$, let $\hat{\partial}_\alpha$ be the vector field on P whose flow is $R_{\exp(t\alpha)}$. Under the identification $\mathfrak{mp}^c(V) = \mathfrak{sp}(V) \oplus \mathfrak{u}(1)$, we can write $\alpha = \kappa \oplus \tau$ for some $\kappa \in \mathfrak{sp}(V)$ and $\tau \in \mathfrak{u}(1)$. Naturality of the exponential map and equivariance of Σ with respect to σ ensure that $\Sigma_* \hat{\partial}_\alpha = \partial_\kappa$.

Definition 3.2. A *metaplectic-c prequantization* of (M, ω) is a triple (P, Σ, γ) , where (P, Σ) is a metaplectic-c structure for (M, ω) and γ is a $\mathfrak{u}(1)$ -valued one-form on P such that:

- (1) γ is invariant under the principal $\mathrm{Mp}^c(V)$ action,
- (2) $\gamma(\hat{\partial}_\alpha) = \frac{1}{2} \eta_* \alpha$ for all $\alpha \in \mathfrak{mp}^c(V)$,
- (3) $d\gamma = \frac{1}{i\hbar} \Pi^* \omega$.

When P is viewed as a bundle over $\mathrm{Sp}(M, \omega)$ with projection map Σ , it becomes a principal circle bundle with connection one-form γ . The circle that acts on the fibers of P is the center $U(1) \subset \mathrm{Mp}^c(V)$.

The space of infinitesimal quantomorphisms of (P, Σ, γ) consists of those vector fields on P whose flows preserve all of the structures on (P, Σ, γ) . Note that one of these structures is the map $P \xrightarrow{\Sigma} \mathrm{Sp}(M, \omega)$, which does not have a direct analogue in the Kostant–Souriau case. We will show how to incorporate this additional piece of information in the next section.

3.3 Infinitesimal metaplectic-c quantomorphisms

As in Section 2.2, we begin by developing the idea of a quantomorphism between metaplectic-c prequantizations. Let $(P_1, \Sigma_1, \gamma_1) \xrightarrow{\Sigma_1} \mathrm{Sp}(M_1, \omega_1) \xrightarrow{\rho_1} (M_1, \omega_1)$ and $(P_2, \Sigma_2, \gamma_2) \xrightarrow{\Sigma_2} \mathrm{Sp}(M_2, \omega_2) \xrightarrow{\rho_2} (M_2, \omega_2)$ be metaplectic-c prequantizations for two symplectic manifolds, and let $\Pi_j = \rho_j \circ \Sigma_j$ for $j = 1, 2$. Let $K : P_1 \rightarrow P_2$ be a diffeomorphism. We will determine the conditions that K must satisfy in order for it to preserve all of the structures of the metaplectic-c prequantizations. First, by analogy with the Kostant–Souriau definition, assume that K satisfies $K^* \gamma_2 = \gamma_1$.

Fix $m \in M_1$, and consider the fiber P_{1m} . For any $q \in P_{1m}$, notice that

$$T_q P_{1m} = \{\xi \in T_q P_1 \mid \Pi_{1*} \xi = 0\} = \ker d\gamma_{1q}.$$

The same property holds for a fiber of P_2 over a point in M_2 . By assumption, K_* is an isomorphism from $\ker d\gamma_{1q}$ to $\ker d\gamma_{2K(q)}$ for all $q \in P_1$. Therefore Π_2 is constant on $K(P_{1m})$. Moreover, since K is a diffeomorphism, $K(P_{1m})$ is in fact a fiber of P_2 over M_2 , and every fiber of P_2 is the image of a fiber of P_1 . Thus K induces a diffeomorphism $K'' : M_1 \rightarrow M_2$ such that the following diagram commutes:

$$\begin{array}{ccc} P_1 & \xrightarrow{K} & P_2 \\ \downarrow \Pi_1 & & \downarrow \Pi_2 \\ M_1 & \xrightarrow{K''} & M_2 \end{array}$$

Lemma 3.3. *The map $K'' : M_1 \rightarrow M_2$ is a symplectomorphism.*

Proof. It suffices to show that $K''^*\omega_2 = \omega_1$. Using the properties of K , γ_1 and γ_2 , we calculate

$$\Pi_1^*(K''^*\omega_2) = (K'' \circ \Pi_1)^*\omega_2 = (\Pi_2 \circ K)^*\omega_2 = K^*(i\hbar d\gamma_2) = i\hbar d\gamma_1 = \Pi_1^*\omega_1.$$

Therefore $K''^*\omega_2 = \omega_1$, as required. \blacksquare

Recall from the beginning of Section 3.2 that an element $b \in \mathrm{Sp}(M_1, \omega_1)_m$ is a map $b : V \rightarrow T_m M_1$ such that $b^*\omega_{1m} = \Omega$. Since K'' is a symplectomorphism, the composition $K''_* \circ b : V \rightarrow T_{K''(m)} M_2$ satisfies $(K''_* \circ b)^*\omega_{2K''(m)} = \Omega$, which implies that $K''_* \circ b \in \mathrm{Sp}(M_2, \omega_2)_{K''(m)}$. Let $\widetilde{K}'' : \mathrm{Sp}(M_1, \omega_1) \rightarrow \mathrm{Sp}(M_2, \omega_2)$ be the lift of K'' given by

$$\widetilde{K}''(b) = K''_* \circ b, \quad \forall b \in \mathrm{Sp}(M_1, \omega_1).$$

Then \widetilde{K}'' is a diffeomorphism, and it is equivariant with respect to the principal $\mathrm{Sp}(V)$ actions.

Thus, if we assume that $K^*\gamma_2 = \gamma_1$, we obtain the diffeomorphisms $K'' : M_1 \rightarrow M_2$ and $\widetilde{K}'' : \mathrm{Sp}(M_1, \omega_1) \rightarrow \mathrm{Sp}(M_2, \omega_2)$, where both K and \widetilde{K}'' are lifts of K'' . However, K is not necessarily a lift of \widetilde{K}'' . Indeed, there might not be any map $K' : \mathrm{Sp}(M_1, \omega_1) \rightarrow \mathrm{Sp}(M_2, \omega_2)$ of which K is a lift. A map K for which there is no corresponding K' is constructed in Appendix A, Example A.1. In Section 2.2, we showed that a diffeomorphism of prequantization circle bundles that preserves the connection forms must be equivariant with respect to the principal circle actions. By contrast, Example A.1 demonstrates that it is possible for K to preserve the prequantization one-forms without being equivariant with respect to the principal $\mathrm{Mp}^c(V)$ actions.

Suppose we make the additional assumption that $K(q \cdot a) = K(q) \cdot a$ for all $q \in P_1$ and $a \in \mathrm{Mp}^c(V)$. Then K induces a diffeomorphism $K' : \mathrm{Sp}(M_1, \omega_1) \rightarrow \mathrm{Sp}(M_2, \omega_2)$ that satisfies $K' \circ \Sigma_1 = \Sigma_2 \circ K$. Combining this with the map $K'' : M_1 \rightarrow M_2$ yields the following commutative diagram:

$$\begin{array}{ccc} P_1 & \xrightarrow{K} & P_2 \\ \downarrow \Sigma_1 & & \downarrow \Sigma_2 \\ \mathrm{Sp}(M_1, \omega_1) & \xrightarrow{K'} & \mathrm{Sp}(M_2, \omega_2) \\ \downarrow \rho_1 & & \downarrow \rho_2 \\ M_1 & \xrightarrow{K''} & M_2 \end{array}$$

We now have two maps, K' and \widetilde{K}'' , which are diffeomorphisms from $\mathrm{Sp}(M_1, \omega_1)$ to $\mathrm{Sp}(M_2, \omega_2)$. By construction, $\rho_2 \circ K' = \rho_2 \circ \widetilde{K}''$, and both K' and \widetilde{K}'' are equivariant with respect to the

principal $\mathrm{Sp}(V)$ actions. However, it is still possible for K' and \widetilde{K}'' to be different. A map K for which $K' \neq \widetilde{K}''$ is given in Example A.2.

As will be shown in Section 3.4, this potential discrepancy between K' and \widetilde{K}'' must be prevented in order to construct the desired isomorphism between $C^\infty(M)$ and the infinitesimal quantomorphisms. We therefore propose the following definition.

Definition 3.4. The diffeomorphism $K : P_1 \rightarrow P_2$ is a *metaplectic-c quantomorphism* if

- (1) $K^*\gamma_2 = \gamma_1$,
- (2) the induced diffeomorphism $K'' : M_1 \rightarrow M_2$ satisfies $\widetilde{K}'' \circ \Sigma_1 = \Sigma_2 \circ K$.

Let $K : P_1 \rightarrow P_2$ be a metaplectic-c quantomorphism. Given our concept of a quantomorphism as a diffeomorphism that preserves all of the structures of a metaplectic-c prequantization, we would expect that K is equivariant with respect to the $\mathrm{Mp}^c(V)$ actions. Let $\alpha \in \mathfrak{mp}^c(V)$ be arbitrary, and write $\alpha = \kappa \oplus \tau$ under the identification of $\mathfrak{mp}^c(V)$ with $\mathfrak{sp}(V) \oplus \mathfrak{u}(1)$. The vector field $\hat{\partial}_\alpha$ on P_1 is completely specified by the conditions $\gamma_1(\hat{\partial}_\alpha) = \tau$ and $\Sigma_{1*}\hat{\partial}_\alpha = \partial_\kappa$, and the same is true on P_2 . Notice that

$$\gamma_2(K_*\hat{\partial}_\alpha) = \gamma_1(\hat{\partial}_\alpha) = \tau,$$

and

$$\Sigma_{2*}K_*\hat{\partial}_\alpha = \widetilde{K}''_*\Sigma_{1*}\hat{\partial}_\alpha = \widetilde{K}''_*\partial_\kappa = \partial_\kappa,$$

where the final equality follows from the fact that \widetilde{K}'' is equivariant with respect to $\mathrm{Sp}(V)$. Thus $K_*\hat{\partial}_\alpha = \hat{\partial}_\alpha$ for all $\alpha \in \mathfrak{mp}^c(V)$, which implies that K is equivariant with respect to $\mathrm{Mp}^c(V)$, as desired.

Now consider a single metaplectic-c prequantized space $(P, \Sigma, \gamma) \xrightarrow{\Sigma} \mathrm{Sp}(M, \omega) \xrightarrow{\rho} (M, \omega)$ with $\Pi = \rho \circ \Sigma$.

Definition 3.5. A vector field $\zeta \in \mathcal{X}(P)$ is an *infinitesimal metaplectic-c quantomorphism* if its flow ϕ_t is a metaplectic-c quantomorphism from its domain to its range for each t .

Let $\zeta \in \mathcal{X}(P)$ have flow ϕ_t . Property (1) of a quantomorphism holds for ϕ_t if and only if $L_\zeta\gamma = 0$. If we assume that ϕ_t satisfies property (1), then we can make the following observations.

- There is a flow ϕ_t'' on M such that $\Pi \circ \phi_t = \phi_t'' \circ \Pi$. The vector field that it generates on M is $\Pi_*\zeta$.
- Lemma 3.3 shows that ϕ_t'' is a family of symplectomorphisms. Therefore we can lift ϕ_t'' to a flow on $\mathrm{Sp}(M, \omega)$, denoted by $\widetilde{\phi}_t''$, where $\widetilde{\phi}_t''(b) = (\phi_t'')_* \circ b$ for all $b \in \mathrm{Sp}(M, \omega)$. Let the vector field on $\mathrm{Sp}(M, \omega)$ that has flow $\widetilde{\phi}_t''$ be $\widetilde{\Pi_*\zeta}$.
- Property (2) of a quantomorphism holds for ϕ_t if and only if $\Sigma_*\zeta$ is a well-defined vector field on $\mathrm{Sp}(M, \omega)$ and $\Sigma_*\zeta = \widetilde{\Pi_*\zeta}$.

We conclude that the space of infinitesimal metaplectic-c quantomorphisms of (P, Σ, γ) is

$$\mathcal{Q}(P, \Sigma, \gamma) = \{\zeta \in \mathcal{X}(P) \mid L_\zeta\gamma = 0 \text{ and } \Sigma_*\zeta = \widetilde{\Pi_*\zeta}\},$$

where it is understood that the condition $\Sigma_*\zeta = \widetilde{\Pi_*\zeta}$ can only be satisfied if $\Sigma_*\zeta$ is well defined. In the next section, we will construct a Lie algebra isomorphism from $C^\infty(M)$ to $\mathcal{Q}(P, \Sigma, \gamma)$.

3.4 The Lie algebra isomorphism

We begin with a procedure, given by Robinson and Rawnsley in [3, Section 7], for lifting a Hamiltonian vector field on M to $\text{Sp}(M, \omega)$ and then to P . These steps will be used in constructing the isomorphism $E : C^\infty(M) \rightarrow \mathcal{Q}(P, \Sigma, \gamma)$.

Fix $f \in C^\infty(M)$, and let its Hamiltonian vector field ξ_f have flow φ_t on M . We know that φ_{t*} preserves ω because $L_{\xi_f}\omega = 0$. Let $\tilde{\varphi}_t$ be the lift of φ_t to $\text{Sp}(M, \omega)$ given by

$$\tilde{\varphi}_t(b) = \varphi_{t*} \circ b, \quad \forall b \in \text{Sp}(M, \omega),$$

and let the vector field on $\text{Sp}(M, \omega)$ with flow $\tilde{\varphi}_t$ be $\tilde{\xi}_f$. We have $\rho_*\tilde{\xi}_f = \xi_f$ by construction. Also, $\tilde{\varphi}_t$ commutes with the right principal $\text{Sp}(V)$ action on $\text{Sp}(M, \omega)$, so $[\tilde{\xi}_f, \partial_\kappa] = 0$ for all $\kappa \in \mathfrak{sp}(V)$. Now let $\hat{\xi}_f$ be the lift of $\tilde{\xi}_f$ to P that is horizontal with respect to γ . Then $\Sigma_*\hat{\xi}_f = \tilde{\xi}_f$ and $\gamma(\hat{\xi}_f) = 0$. A summary of the key properties of ξ_f , $\tilde{\xi}_f$ and $\hat{\xi}_f$ is below:

$$\begin{array}{ccc} (P, \Sigma, \gamma) & \hat{\xi}_f & \gamma(\hat{\xi}_f) = 0, \quad \Sigma_*\hat{\xi}_f = \tilde{\xi}_f, \quad \Pi_*\hat{\xi}_f = \xi_f \\ \downarrow \Sigma & & \\ \text{Sp}(M, \omega) & \tilde{\xi}_f & [\tilde{\xi}_f, \partial_\kappa] = 0 \quad \forall \kappa \in \mathfrak{sp}(V), \quad \rho_*\tilde{\xi}_f = \xi_f \\ \downarrow \rho & & \\ (M, \omega) & \xi_f & \end{array}$$

The following is a consequence of Lemma 2.1.

Lemma 3.6. *For all $f, g \in C^\infty(M)$, $\tilde{\xi}_{\{f, g\}} = [\tilde{\xi}_f, \tilde{\xi}_g]$.*

In Section 2.3, we made use of the vector field $\partial_{2\pi i}$ on Y . The corresponding object in this context is the vector field $\hat{\partial}_{2\pi i}$ on P , which satisfies $\gamma(\hat{\partial}_{2\pi i}) = 2\pi i$ and $\Sigma_*(\hat{\partial}_{2\pi i}) = 0$.

Lemma 3.7. *For all $f, g \in C^\infty(M)$,*

$$[\hat{\xi}_f, \hat{\xi}_g] = \hat{\xi}_{\{f, g\}} - \frac{1}{2\pi\hbar}\Pi^*\{f, g\}\hat{\partial}_{2\pi i}.$$

Proof. It suffices to show that

$$\Sigma_*[\hat{\xi}_f, \hat{\xi}_g] = \Sigma_*\left(\hat{\xi}_{\{f, g\}} - \frac{1}{2\pi\hbar}\Pi^*\{f, g\}\hat{\partial}_{2\pi i}\right)$$

and

$$\gamma([\hat{\xi}_f, \hat{\xi}_g]) = \gamma\left(\hat{\xi}_{\{f, g\}} - \frac{1}{2\pi\hbar}\Pi^*\{f, g\}\hat{\partial}_{2\pi i}\right).$$

The proof proceeds identically to that of Lemma 2.5. ■

Lemma 3.8. *The map $E : C^\infty(M) \rightarrow \mathcal{X}(P)$ given by*

$$E(f) = \hat{\xi}_f + \frac{1}{2\pi\hbar}\Pi^*f\hat{\partial}_{2\pi i}, \quad \forall f \in C^\infty(M)$$

is a Lie algebra homomorphism.

Proof. Precisely analogous to Lemma 2.6. ■

Lemma 3.9. *For all $f \in C^\infty(M)$, $E(f) \in \mathcal{Q}(P, \Sigma, \gamma)$.*

Proof. We need to show that $L_{E(f)}\gamma = 0$ and $\Sigma_*E(f) = \widetilde{\Pi_*E(f)}$. The verification that $L_{E(f)}\gamma = 0$ is the same as that in Lemma 2.7. Note that $\Sigma_*E(f) = \tilde{\xi}_f$ and $\Pi_*E(f) = \xi_f$, so $\widetilde{\Pi_*E(f)} = \tilde{\xi}_f = \Sigma_*E(f)$. Thus the necessary conditions are satisfied, and $E(f) \in \mathcal{Q}(P, \Sigma, \gamma)$. ■

As before, we will construct an inverse for E , and conclude that E is a Lie algebra isomorphism. Let $\zeta \in \mathcal{Q}(P, \Sigma, \gamma)$ and $\alpha \in \mathfrak{mp}^c(V)$ be arbitrary. Using an identical argument to the one that precedes Theorem 2.8, the fact that $\hat{\partial}_{\alpha \lrcorner}(L_\zeta\gamma) = 0$ implies that $L_{\hat{\partial}_\alpha}\gamma(\zeta) = 0$. Therefore we can define $F : \mathcal{Q}(P, \Sigma, \gamma) \rightarrow C^\infty(M)$ so that

$$-\frac{1}{i\hbar}\Pi^*F(\zeta) = \gamma(\zeta), \quad \forall \zeta \in \mathcal{Q}(P, \Sigma, \gamma).$$

Theorem 3.10. *The map $E : C^\infty(M) \rightarrow \mathcal{Q}(P, \Sigma, \gamma)$ is a Lie algebra isomorphism with inverse F .*

Proof. Let $f \in C^\infty(M)$ and $\zeta \in \mathcal{Q}(P, \Sigma, \gamma)$ be arbitrary. From the definitions of E and F , $F(E(f))$ satisfies

$$-\frac{1}{i\hbar}\Pi^*F(E(f)) = \gamma(E(f)) = \gamma\left(\hat{\xi}_f + \frac{1}{2\pi\hbar}\Pi^*f\hat{\partial}_{2\pi i}\right) = -\frac{1}{i\hbar}\Pi^*f.$$

Thus $F(E(f)) = f$.

Next, we claim that $\gamma(E(F(\zeta))) = \gamma(\zeta)$ and $\Sigma_*E(F(\zeta)) = \Sigma_*\zeta$. Observe that

$$E(F(\zeta)) = \hat{\xi}_{F(\zeta)} + \frac{1}{2\pi\hbar}\Pi^*F(\zeta)\hat{\partial}_{2\pi i} = \hat{\xi}_{F(\zeta)} + \frac{1}{2\pi i}\gamma(\zeta)\hat{\partial}_{2\pi i}.$$

It is immediate that $\gamma(E(F(\zeta))) = \gamma(\zeta)$ and $\Sigma_*E(F(\zeta)) = \tilde{\xi}_{F(\zeta)}$. From the definition of $\mathcal{Q}(P, \Sigma, \gamma)$, we know that $\Sigma_*\zeta = \widetilde{\Pi_*\zeta}$. It remains to show that $\Pi_*\zeta = \xi_{F(\zeta)}$. We calculate

$$\zeta \lrcorner \Pi^*\omega = \zeta \lrcorner i\hbar d\gamma = -i\hbar d(\gamma(\zeta)) = \Pi^*dF(\zeta).$$

This demonstrates that $(\Pi_*\zeta) \lrcorner \omega = dF(\zeta)$, which implies that $\Pi_*\zeta = \xi_{F(\zeta)}$ as needed. Thus we have shown that $E(F(\zeta)) = \zeta$, and this completes the proof that E and F are inverses. ■

If the definition of $\mathcal{Q}(P, \Sigma, \gamma)$ did not include the condition that $\Sigma_*\zeta = \widetilde{\Pi_*\zeta}$, this proof would fail in the final step. We would be able to show that $\Sigma_*E(F(\zeta)) = \tilde{\xi}_{F(\zeta)} = \widetilde{\Pi_*\zeta}$, but this vector field would not necessarily equal $\Sigma_*\zeta$, and so F would not be the inverse of E . This explains why property (2) of a metaplectic-c quantomorphism is necessary in order to obtain a subalgebra of $\mathcal{X}(P)$ that is isomorphic to $C^\infty(M)$.

3.5 An operator representation of $C^\infty(M)$

In [3], Robinson and Rawnsley construct an infinite-dimensional Hilbert space $\mathcal{E}'(V)$ of holomorphic functions on $V \cong \mathbb{C}^n$, on which the group $\text{Mp}^c(V)$ acts via the metaplectic representation. They then define the bundle of symplectic spinors for the prequantized system $(P, \Sigma, \gamma) \xrightarrow{\Pi} (M, \omega)$ to be

$$\mathcal{E}'(P) = P \times_{\text{Mp}^c(V)} \mathcal{E}'(V).$$

We omit the details of the metaplectic representation here; the only fact we need is that the subgroup $U(1) \subset \text{Mp}^c(V)$ acts on elements of $\mathcal{E}'(V)$ by scalar multiplication. We write an element of $\mathcal{E}'(P)$ as an equivalence class $[q, \psi]$ with $q \in P$ and $\psi \in \mathcal{E}'(V)$.

Section 7 of [3] contains the following construction.

- Let $s \in \Gamma(\mathcal{E}'(P))$ be given, and define the map $\tilde{s} : P \rightarrow \mathcal{E}'(V)$ so that $[q, \tilde{s}(q)] = s(\Pi(q))$ for all $q \in P$. This map \tilde{s} satisfies the equivariance condition

$$\tilde{s}(q \cdot a) = a^{-1}\tilde{s}(q), \quad \forall q \in P, a \in \text{Mp}^c(V),$$

where the action on the right-hand side is that of the metaplectic representation.

- Conversely, if $\tilde{s} : P \rightarrow \mathcal{E}'(V)$ is any map with the equivariance property above, it can be used to define a section $s \in \Gamma(\mathcal{E}'(P))$ by setting $s(m) = [q, \tilde{s}(q)]$ for each $m \in M$ and any $q \in P$ such that $\Pi(q) = m$.
- Let $f \in C^\infty(M)$ be arbitrary, and recall the lifting $\xi_f \rightarrow \tilde{\xi}_f \rightarrow \hat{\xi}_f$ of ξ_f to P . A standard calculation establishes that $[\hat{\xi}_f, \hat{\partial}_\alpha] = 0$ for all $\alpha \in \mathfrak{mp}^c(V)$. Thus, if $\tilde{s} : P \rightarrow \mathcal{E}'(V)$ is an equivariant map, then so is $\hat{\xi}_f \tilde{s}$.
- Define the map $D : C^\infty(M) \rightarrow \text{End} \Gamma(\mathcal{E}'(P))$ such that for all $f \in C^\infty(M)$ and $s \in \Gamma(\mathcal{E}'(P))$, $D_f s$ is the section of $\mathcal{E}'(P)$ that satisfies

$$\widetilde{D_f s} = \hat{\xi}_f \tilde{s}.$$

Further, define $\delta : C^\infty(M) \rightarrow \text{End} \Gamma(\mathcal{E}'(P))$ by

$$\delta_f s = D_f s + \frac{1}{i\hbar} f s, \quad \forall f \in C^\infty(M), s \in \Gamma(\mathcal{E}'(P)).$$

Theorem 7.8 of [3] states that δ is a Lie algebra homomorphism.

We see that the construction of D precisely parallels the construction of the connection ∇ on the prequantization line bundle L associated to a prequantization circle bundle (Y, γ) . As in Section 2.4, we make two observations.

- For any $s \in \Gamma(\mathcal{E}'(P))$ and $f \in C^\infty(M)$, $\widetilde{f s} = \Pi^* f \tilde{s}$.
- For any equivariant map $\tilde{s} : P \rightarrow \mathcal{E}'(V)$, $\hat{\partial}_{2\pi i} \tilde{s} = -2\pi i \tilde{s}$.

Therefore

$$\widetilde{\delta_f s} = \left(\hat{\xi}_f + \frac{1}{2\pi\hbar} \Pi^* f \hat{\partial}_{2\pi i} \right) \tilde{s} = E(f) \tilde{s}.$$

The fact that δ is a Lie algebra homomorphism then follows immediately from Lemma 3.8. This construction would apply equally well to any associated bundle where the subgroup $U(1) \subset \text{Mp}^c(V)$ acts on the fiber by scalar multiplication.

A Example of a metaplectic-c prequantization

Recall from Section 3.3 that a metaplectic-c quantomorphism K between two metaplectic-c prequantizations $(P_1, \Sigma_1, \gamma_1) \xrightarrow{\Sigma_1} \text{Sp}(M_1, \omega_1) \xrightarrow{\rho_1} (M_1, \omega_1)$ and $(P_2, \Sigma_2, \gamma_2) \xrightarrow{\Sigma_2} \text{Sp}(M_2, \omega_2) \xrightarrow{\rho_2} (M_2, \omega_2)$ is a diffeomorphism $K : P_1 \rightarrow P_2$ such that

- (1) $K^* \gamma_2 = \gamma_1$,
- (2) the induced diffeomorphism $K'' : M_1 \rightarrow M_2$ satisfies $\widetilde{K''} \circ \Sigma_1 = \Sigma_2 \circ K$,

where $\widetilde{K''} : \text{Sp}(M_1, \omega_1) \rightarrow \text{Sp}(M_2, \omega_2)$ is the lift of K'' given by

$$\widetilde{K''}(b) = K''_* b, \quad \forall b \in \text{Sp}(M_1, \omega_1).$$

We claimed that condition (1) is insufficient to guarantee that K is the lift of some map $K' : \mathrm{Sp}(M_1, \omega_1) \rightarrow \mathrm{Sp}(M_2, \omega_2)$. In particular, a diffeomorphism K that only satisfies condition (1) might not be equivariant with respect to the principal $\mathrm{Mp}^c(V)$ actions. We further claimed that K might be equivariant and satisfy condition (1), yet fail to satisfy condition (2). We will now construct examples to support these claims.

Let $M = \mathbb{R}^2 \setminus \{(0, 0)\}$ with Cartesian coordinates (p, q) and polar coordinates (r, θ) . Equip M with the symplectic form $\omega = dp \wedge dq = r dr \wedge d\theta$, and observe that the one-form $\beta = \frac{1}{2}r^2 d\theta$ satisfies $d\beta = \omega$. Let $V = \mathbb{R}^2$ with basis $\{\hat{x}, \hat{y}\}$ and symplectic form $\Omega = \hat{x}^* \wedge \hat{y}^*$, and consider the global trivialization of the tangent bundle TM such that for all $m \in M$, $T_m M$ is identified with V by mapping $\hat{x} \rightarrow \left. \frac{\partial}{\partial p} \right|_m$ and $\hat{y} \rightarrow \left. \frac{\partial}{\partial q} \right|_m$. Identify $\mathrm{Sp}(M, \omega)$ with $M \times \mathrm{Sp}(V)$ using this trivialization.

Let $P = M \times \mathrm{Mp}^c(V)$, and define the map $\Sigma : P \rightarrow \mathrm{Sp}(M, \omega)$ by $\Sigma(m, a) = (m, \sigma(a))$ for all $m \in M$ and $a \in \mathrm{Mp}^c(V)$. Let ϑ_0 be the trivial connection on the product bundle $M \times \mathrm{Mp}^c(V)$, and let $\gamma = \frac{1}{i\hbar}\beta + \frac{1}{2}\eta_*\vartheta_0$. Then (P, Σ, γ) is a metaplectic- c prequantization of (M, ω) . In both of the examples below, we will give a diffeomorphism $K : P \rightarrow P$.

To facilitate the construction in Example A.1, we introduce a more explicit representation for elements of the metaplectic- c group. By definition, $\mathrm{Mp}^c(V) = \mathrm{Mp}(V) \times_{\mathbb{Z}_2} U(1)$. The restriction of the projection map $\mathrm{Mp}^c(V) \xrightarrow{\sigma} \mathrm{Sp}(V)$ to $\mathrm{Mp}(V)$ yields the double covering $\mathrm{Mp}(V) \xrightarrow{\sigma} \mathrm{Sp}(V)$. Write an element of $\mathrm{Mp}^c(V)$ as an equivalence class $[h, e^{2\pi it}]$ with $h \in \mathrm{Mp}(V)$ and $t \in \mathbb{R}$. In terms of this parametrization, the projection map is given by

$$\sigma[h, e^{2\pi it}] = \sigma(h),$$

and the determinant map $\mathrm{Mp}^c(V) \xrightarrow{\eta} U(1)$ is given by

$$\eta[h, e^{2\pi it}] = (e^{2\pi it})^2.$$

Example A.1. We will define a diffeomorphism $K : P \rightarrow P$ that preserves γ , but that does not descend through Σ to a well-defined map on $\mathrm{Sp}(M, \omega)$.

Let $\mu : \mathbb{R} \rightarrow \mathrm{Mp}(V)$ be any smooth nonconstant path such that $\mu(t+1) = \mu(t)$ for all $t \in \mathbb{R}$. Note that the composition $\sigma \circ \mu : \mathbb{R} \rightarrow \mathrm{Sp}(V)$ is also nonconstant. Now define $F : \mathrm{Mp}(V) \times U(1) \rightarrow \mathrm{Mp}(V) \times U(1)$ by

$$F(h, e^{2\pi it}) = (h\mu(2t), e^{2\pi it}), \quad \forall h \in \mathrm{Mp}(V), t \in \mathbb{R}.$$

This map is a diffeomorphism of $\mathrm{Mp}(V) \times U(1)$, and it descends to a diffeomorphism of $\mathrm{Mp}^c(V)$, which we also denote F . For any $[h, e^{2\pi it}] \in \mathrm{Mp}^c(V)$, observe that

$$\eta(F[h, e^{2\pi it}]) = \eta[h\mu(2t), e^{2\pi it}] = (e^{2\pi it})^2 = \eta[h, e^{2\pi it}].$$

This implies that for any $\alpha \in \mathfrak{mp}^c(V)$, $\frac{1}{2}\eta_*F_*\alpha = \frac{1}{2}\eta_*\alpha$.

Define the diffeomorphism $K : P \rightarrow P$ by $K(m, a) = (m, F(a))$ for all $m \in M$ and $a \in \mathrm{Mp}^c(V)$. Since K is the identity on M , it preserves β . From the property of F shown above, K also preserves $\frac{1}{2}\eta_*\vartheta_0$, and thus it preserves γ . Fix $(m, g) \in \mathrm{Sp}(M, \omega)$, and let $h \in \mathrm{Mp}(V)$ be such that $\sigma(h) = g$. Then the fiber of P over (m, g) is $P_{(m, g)} = \{(m, [h, e^{2\pi it}]) \mid t \in \mathbb{R}\}$. Notice that

$$\Sigma \circ K(m, [h, e^{2\pi it}]) = \Sigma(m, [h\mu(2t), e^{2\pi it}]) = (m, g\sigma(\mu(2t))),$$

which is not constant with respect to t . Thus $K(P_{(m, g)})$ is not contained within a single fiber of P over $\mathrm{Sp}(M, \omega)$, which shows that there is no map $K' : \mathrm{Sp}(M, \omega) \rightarrow \mathrm{Sp}(M, \omega)$ such that $K' \circ \Sigma = \Sigma \circ K$.

If $K : P \rightarrow P$ is equivariant with respect to $\text{Mp}^c(V)$, then it induces a diffeomorphism $K' : \text{Sp}(M, \omega) \rightarrow \text{Sp}(M, \omega)$ that satisfies $K' \circ \Sigma = \Sigma \circ K$. This map and \widetilde{K}'' are both lifts of $K'' : M \rightarrow M$, but they might not be the same map.

Example A.2. We will define a diffeomorphism $K : P \rightarrow P$ that preserves γ and is equivariant with respect to $\text{Mp}^c(V)$, but where $K' \neq \widetilde{K}''$.

Let $T_\lambda : M \rightarrow M$ be the map that rotates M about the origin by the angle λ , where λ is not an integer multiple of 2π . Define $K : P \rightarrow P$ by

$$K(m, a) = (T_\lambda(m), a), \quad \forall m \in M, a \in \text{Mp}^c(V).$$

Then $K^*\gamma = \gamma$, and $K(q \cdot a) = K(q) \cdot a$ for all $q \in P$ and $a \in \text{Mp}^c(V)$. The map $K' : \text{Sp}(M, \omega) \rightarrow \text{Sp}(M, \omega)$ is given by

$$K'(m, g) = (T_\lambda(m), g), \quad \forall m \in M, g \in \text{Sp}(V),$$

and the map $K'' : M \rightarrow M$ is simply T_λ . If we let T_λ also denote the automorphism of V given by rotation about the origin by λ , then under our chosen identification of TM with $M \times V$, we have

$$K''_*(m, v) = (T_\lambda(m), T_\lambda(v)), \quad \forall m \in M, v \in V.$$

Therefore $\widetilde{K}'' : \text{Sp}(M, \omega) \rightarrow \text{Sp}(M, \omega)$ is given by

$$\widetilde{K}''(m, g) = (T_\lambda(m), T_\lambda \circ g), \quad \forall m \in M, g \in \text{Sp}(V).$$

Hence $K' \neq \widetilde{K}''$.

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