

The Quantum Pair of Pants

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Abstract. We compute the spectrum of the operator of multiplication by the complex coordinate in a Hilbert space of holomorphic functions on a disk with two circular holes. Additionally we determine the structure of the C^* -algebra generated by that operator. The algebra can be considered as the quantum pair of pants.

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1 Introduction

In this paper we study the operator z of multiplication by the complex coordinate in Hilbert spaces of holomorphic functions on certain multiply connected domains in the complex plane. The domains we consider are disks with circular holes. The case of a disk with no holes is the classical one. In the Hardy space of the disk the multiplication operator z is the unilateral shift whose spectrum is the disk. The C^* -algebra generated by the unilateral shift, the Toeplitz algebra, is an extension of the algebra of compact operators by $C(S^1)$, S^1 being the boundary of the disk [4]. For the Bergman space the operator z is a weighted unilateral shift and its spectrum and the C^* -algebra it generates are the same as in the Hardy space [7]. Partially for those reasons the Toeplitz algebra is often considered as the quantum disk [7, 9, 10].

A disk with one hole is biholomorphic to an annulus. In the Bergman space for example, the z operator is a weighted bilateral shift with respect to the natural basis of (normalized) powers of the complex coordinate. Its spectrum is the annulus, and the C^* -algebra it generates is an extension of the algebra of compact operators by $C(S^1 \times S^1)$, where $S^1 \times S^1$ is the boundary of the annulus. The same is true for many other Hilbert spaces of holomorphic functions on an annulus. The resulting C^* -algebra is the quantum annulus of [10, 12].

In this paper we study in detail the two hole case: a pair of pants. Up to biholomorphism we can realize a disk with two holes as an annulus centered at zero with outer radius one, with an additional off centered hole. In the space of continuous functions on the closed pair of pants that are holomorphic in its interior, we consider a specific inner product with respect to which the operator of multiplication by the complex coordinate z has a particularly simple structure. The results we obtain are completely analogous to zero and one-hole cases: the spectrum of z is the domain of the corresponding pair of pants while the C^* -algebra generated by z is an extension of the algebra of compact operators by $C(S^1 \times S^1 \times S^1)$, where $S^1 \times S^1 \times S^1$ is the boundary of the pair of pants.

This work is part of an ongoing effort to understand the structure of quantum Riemann surfaces and their noncommutative differential geometry, see [7, 8, 10, 11, 12, 13, 14, 15]. Our

paper has many things in common with the work of Abrahamse [1] and Abrahamse–Douglas [2], who use different Hilbert spaces.

The paper is organized as follows. Section 2 contains an overview of the zero and one-hole cases, while Section 3 has a detailed discussion of the quantum pair of pants.

2 Preliminaries

In this section we describe in some detail, the zero and one-hole cases. Most of the material is well-known, however the treatment of the quantum annulus is somewhat new.

2.1 The quantum disk

In this subsection we look at the structure of the quantum disk. We review the tools and the relevant theorems that will be a motivation for the subsequent discussion of the quantum pair of pants.

Consider the closed unit disk $\mathbb{D} = \{\zeta \in \mathbb{C} : |\zeta| \leq 1\}$. We can represent any holomorphic function inside the disk as a convergent power series

$$f(\zeta) = \sum_{n=0}^{\infty} e_n \zeta^n.$$

The Hardy space on the disk is defined as

$$H^2(\mathbb{D}) = \left\{ f(\zeta) = \sum_{n=0}^{\infty} e_n \zeta^n : \sum_{n=0}^{\infty} |e_n|^2 < \infty \right\}.$$

We define the multiplication operator by the complex coordinate, $z : H^2(\mathbb{D}) \rightarrow H^2(\mathbb{D})$ by the formula $f(\zeta) \mapsto \zeta f(\zeta)$. If $E_n = \zeta^n$ is the orthonormal basis on $H^2(\mathbb{D})$, then applying z to the basis elements produces $zE_n = E_{n+1}$ for all $n \geq 0$, i.e., z is the unilateral shift; moreover, we have the following formula for the adjoint operator to z

$$z^* E_n = \begin{cases} E_{n-1} & \text{for } n \geq 1, \\ 0 & \text{for } n = 0. \end{cases}$$

Now we consider the C^* -algebra generated by z . This well-known algebra is called the Toeplitz algebra, denoted by \mathcal{T} , and has also been termed the quantum (noncommutative) disk. This is (partially) based on the following standard results collected here with sketches of proofs which serve as a guideline for considerations in the next section.

Theorem 2.1. *The norm of z is 1. The spectrum of z is all of \mathbb{D} , i.e., $\sigma(z) = \mathbb{D}$.*

Proof. The norm computation is straightforward. By the norm calculation it then follows that the spectrum is a closed subset of the unit disk. To illustrate that any λ in the interior of \mathbb{D} is an eigenvalue of z^* , take $f_\lambda(\zeta) = \sum_{n=0}^{\infty} \lambda^n \zeta^n$ and so

$$z^* f_\lambda(\zeta) = \sum_{n=1}^{\infty} \lambda^n \zeta^{n-1} = \lambda \sum_{n=1}^{\infty} \lambda^{n-1} \zeta^{n-1} = \lambda \sum_{n=0}^{\infty} \lambda^n \zeta^n = \lambda f_\lambda(\zeta). \quad \blacksquare$$

Let \mathcal{K} be the algebra of compact operators in $H^2(\mathbb{D})$. The next observation tells us how the commutator ideal of \mathcal{T} , and \mathcal{K} are related.

Theorem 2.2. *The commutator ideal of \mathcal{T} is the ideal of compact operators.*

Proof. Since \mathcal{T} is generated by z and z^* , the commutator ideal of \mathcal{T} is equal to the ideal generated by the commutator $[z^*, z]$. Note that $[z^*, z] = P_{E_0}$, the orthogonal projection onto the span of E_0 . Since this one-dimensional projection is a compact operator, it follows that the commutator ideal of \mathcal{T} is contained in \mathcal{K} . To prove the opposite inclusion we look at the following rank one operators: $E_{ij}(f) = \langle f, E_i \rangle E_j$. Notice that $E_{ij} = z^j P_{E_0} (z^*)^i$, hence those operators belong to the commutator ideal of \mathcal{T} . But every compact operator is a norm limit of finite rank operators, which in turn are finite linear combinations of E_{ij} 's. This verifies that the commutator ideal of \mathcal{T} contains \mathcal{K} . ■

In order to state the next result, first we introduce some more notation. We identify $H^2(\mathbb{D})$, the Hardy space on the unit disk, with the subspace of $L^2(S^1)$ spanned by $\{e^{inx}\}_{n \geq 0}$. Also given a continuous function f on the unit circle, we denote the multiplication operator by f as M_f . Let $P : L^2(S^1) \rightarrow H^2(\mathbb{D})$ be the orthogonal projection onto $\text{span}\{e^{inx}\}_{n \geq 0}$, then define the operator $T_f : H^2(\mathbb{D}) \rightarrow H^2(\mathbb{D})$ by $T_f = PM_f$. The operator T_f is known as a Toeplitz operator. Since $\|M_f\| = \|f\|_\infty$, $\|T_f\| \leq \|f\|_\infty$ and hence it is bounded. We have:

Theorem 2.3. *The quotient \mathcal{T}/\mathcal{K} is isomorphic to $C(S^1)$, the space of continuous functions on the unit circle.*

Proof. The usual proof constructs an isomorphism between the two algebras. Notice that for a continuous function f , we have $T_f \in \mathcal{T}$ and since $T_{e^{ix}}$ is the unilateral shift, $T_{e^{-ix}} = T_{e^{ix}}^*$. By the Stone–Weierstrass theorem, every continuous function can be approximated by trigonometric polynomials. Consequently we can define a map $\theta : C(S^1) \rightarrow \mathcal{T}/\mathcal{K}$ by $\theta : f \mapsto [T_f]$, the class of operators T_f .

Next we show that T_f is compact if and only if $f \equiv 0$. Suppose T_f is compact. Then for a continuous f with Fourier series $\sum_{n=-\infty}^{\infty} e_n e^{inx}$ we have

$$T_f(e^{ikx}) = \sum_{n=0}^{\infty} e_{n-k} e^{inx}.$$

Thus, the matrix coefficients $e_n = (E_{i+n}, T_f E_i)$ and since T_f is compact, we must have $(E_{i+n}, T_f E_i) \rightarrow 0$ as $i \rightarrow \infty$ for each fixed n . Therefore, $e_n = 0$ for all n and hence $f \equiv 0$. This result means that θ is injective.

Next we observe that $T_f T_g - T_{fg}$ is a compact operator for all continuous f, g . If f, g are trigonometric polynomials then a direct calculation shows that $T_f T_g - T_{fg}$ is a finite rank operator. The general case then follows by appealing to the Stone–Weierstrass theorem. As a consequence, the map θ above is a C^* -homomorphism.

The range of θ is dense since it contains (the classes of) polynomials in z and z^* . Then by general C^* -algebra theory (see [5] for example) θ is an isometry hence the range is closed. This means that $\text{Ran}(\theta) = \mathcal{T}/\mathcal{K}$ and therefore θ is a $*$ -isomorphism. ■

Note that from the last theorem we get a short exact sequence

$$0 \rightarrow \mathcal{K} \rightarrow \mathcal{T} \rightarrow C(S^1) \rightarrow 0.$$

We can compare this to the short exact sequence for the classical disk

$$0 \rightarrow C_0(\mathbb{D}) \rightarrow C(\mathbb{D}) \rightarrow C(S^1) \rightarrow 0,$$

where $C_0(\mathbb{D})$ are the continuous functions on the disk that vanish on the boundary.

2.2 The quantum annulus

Let $0 < r < 1$ and consider the annulus

$$A_r = \{\zeta \in \mathbb{C} : r \leq |\zeta| \leq 1\}.$$

The classical uniformization theory of Riemann surfaces implies that every open annulus is biholomorphically equivalent to an annulus of the above form.

We can write any holomorphic function $\varphi(\zeta)$ on the interior of A_r as the following convergent version of Laurent series

$$\varphi(\zeta) = \sum_{n=0}^{\infty} e_n \zeta^n + \sum_{n=-\infty}^{-1} f_n \left(\frac{\zeta}{r}\right)^n.$$

We label the basic monomials in the above expansion as

$$E_n = \zeta^n, \quad F_n = \left(\frac{\zeta}{r}\right)^n,$$

and define our specially convenient Hilbert space of holomorphic functions on A_r to be

$$H = \left\{ \varphi(\zeta) = \sum_{n=0}^{\infty} e_n E_n + \sum_{n=-\infty}^{-1} f_n F_n : \|\varphi\| < \infty \right\},$$

where

$$\|\varphi\|^2 = \sum_{n=0}^{\infty} |e_n|^2 + \sum_{n=-\infty}^{-1} |f_n|^2,$$

so that $\{E_n\}, \{F_m\}$ form an orthonormal basis. The operator $z : H \rightarrow H$ is defined by the formula $f(\zeta) \mapsto \zeta f(\zeta)$. With respect to the above basis, the operator z is a rather special weighted bilateral shift. We have

$$\begin{aligned} zE_n &= E_{n+1} & \text{for } n \geq 0, \\ zF_n &= rF_{n+1} & \text{for } n \leq -2, \\ zF_{-1} &= rE_0 \end{aligned}$$

and

$$\begin{aligned} z^*E_n &= E_{n-1} & \text{for } n \geq 1, \\ z^*E_0 &= rF_{-1}, \\ z^*F_n &= rF_{n-1} & \text{for } n \leq -1. \end{aligned}$$

In full analogy with the disk case, the operator z is a form of a noncommutative coordinate for what we call quantum annulus. First we look at the spectrum of z .

Theorem 2.4. *The norm of z is 1. The spectrum of z is all of A_r .*

Proof. The formulas above easily imply that $\|z\| \leq 1$, while the action of z on E_n shows that it is exactly 1. It is then straightforward to verify that for λ inside A_r the following is an eigenvector of z^* corresponding to the eigenvalue λ

$$\phi_\lambda = \sum_{n=0}^{\infty} \lambda^n E_n + \sum_{n=-\infty}^{-1} \left(\frac{\lambda}{r}\right)^n F_n.$$

Finally, using the techniques described in Lemma 3.7 below, we can prove that the operator $z - \lambda$ is invertible for $|\lambda| < r$. Put together those statements imply that the spectrum of z is A_r . ■

The operators z^*z and zz^* are diagonal. We have

$$\begin{aligned} zz^*E_n &= E_n & \text{for } n \geq 1, \\ zz^*F_n &= r^2F_n & \text{for } n \leq -1, \\ zz^*E_0 &= r^2E_0 \end{aligned}$$

and

$$\begin{aligned} z^*zE_n &= E_n & \text{for } n \geq 1, \\ z^*zF_n &= r^2F_n & \text{for } n \leq -1, \\ z^*zE_0 &= E_0. \end{aligned}$$

Thus the spectrum of those operators is $\sigma(zz^*) = \{1\} \cup \{r^2\} = \sigma(z^*z)$. Also notice that the spectral projections $P_{z^*z}(1)$ and $P_{z^*z}(r^2)$ of z^*z are orthogonal projections onto subspaces of H generated by E_n 's and F_n 's, respectively. By the continuous functional calculus applied to z^*z , both projections belong to $C^*(z)$, the C^* -algebra generated by z .

Remark 2.5. The above formulas also imply that the commutator $\frac{z^*z - zz^*}{1 - r^2} = \frac{[z^*, z]}{1 - r^2}$ is the orthogonal projection onto the one-dimensional subspace spanned by E_0 , hence a compact operator.

Theorem 2.6. *The commutator ideal of $C^*(z)$ is the ideal of compact operators.*

Proof. By the remark above the commutator ideal of $C^*(z)$ is contained in \mathcal{K} . Similar to the quantum disk case, the opposite inclusion follows from the easily verifiable fact that the rank one operators $f \mapsto \langle f, E_i \rangle E_j$, $f \mapsto \langle f, E_i \rangle F_j$, $f \mapsto \langle f, F_i \rangle E_j$, $f \mapsto \langle f, F_i \rangle F_j$ are in the commutator ideal of $C^*(z)$. ■

Theorem 2.7. *The quotient $C^*(z)/\mathcal{K}$ is isomorphic to $C(S^1) \oplus C(S^1)$, where $C(S^1)$ is the space of continuous functions on the unit circle. Thus we have a short exact sequence*

$$0 \rightarrow \mathcal{K} \rightarrow C^*(z) \rightarrow C(S^1) \oplus C(S^1) \rightarrow 0.$$

Proof. For details we refer to the proof of Theorem 3.15 in the next section. The key step is showing that the infinite-dimensional spectral projections $P_{z^*z}(1)$ and $P_{z^*z}(r^2)$ are in $C^*(z)$. They can be used together with Toeplitz operators on subspaces generated by E_n 's and F_n 's to construct an isomorphism between $C^*(z)/\mathcal{K}$ and $C(S^1) \oplus C(S^1)$ in a similar fashion to the Toeplitz algebra case. ■

3 The quantum pair of pants

Let $0 < a < 1$, $a + r_2 < 1$, $r_1 + r_2 < a$. We define the (closed) pair of pants as follows

$$PP_{(a, r_1, r_2)} = \{\zeta \in \mathbb{C} : |\zeta| \leq 1, |\zeta| \geq r_1, |\zeta - a| \geq r_2\}.$$

It is clear that every open disk with two nonintersecting circular holes is biholomorphically equivalent to the interior of the one of the above pair of pants. There are some technical advantages to having the holes located as above. To a pair of pants we associate a convenient Hilbert space of holomorphic functions on it and study the operator of multiplication by ζ on that Hilbert space. This is described more precisely in the following subsection.

3.1 Definitions

It follows from [16] that every holomorphic function on the interior of $PP_{(a,r_1,r_2)}$ can be approximated by rational functions with the only singularities at the centers of the smaller circles in $PP_{(a,r_1,r_2)}$ or at infinity. In fact we can do a little better.

Proposition 3.1. *Every holomorphic function $\varphi(\zeta)$ on the interior of $PP_{(a,r_1,r_2)}$ can be written as the following convergent series*

$$\varphi(\zeta) = \sum_{n=0}^{\infty} e_n \zeta^n + \sum_{n=-\infty}^{-1} f_n \left(\frac{\zeta}{r_1} \right)^n + \sum_{n=-\infty}^{-1} g_n \left(\frac{\zeta - a}{r_2} \right)^n.$$

Proof. In [3], it was shown that if $\varphi(\zeta)$ is holomorphic on an annulus $\{\zeta \in \mathbb{C} : R_1 < |\zeta - c| < R_2\}$, then $\varphi(\zeta) = \varphi_1(\zeta) + \varphi_2(\zeta)$ where $\varphi_1(\zeta)$ is holomorphic on $|\zeta - c| > R_1$ and $\varphi_2(\zeta)$ is holomorphic on $|\zeta - c| < R_2$. We apply this theorem twice. Let φ be a holomorphic function on the open pair of pants. Consider an annulus $A = \{\zeta \in \mathbb{C} : |\zeta - a| > r_2, |\zeta - c| < r\}$ around a with outer radius r and inner radius r_2 that does not intersect the hole around the origin with radius r_1 and let D be the disk with center a and radius r . Then $\varphi|_A$ is a holomorphic function and so from [3], $\varphi|_A = \varphi_1 + \varphi_2$ with φ_1 holomorphic outside the hole centered at a with radius r_2 , and φ_2 holomorphic on D . Consequently φ_1 has the following convergent series representation

$$\varphi_1 = \sum_{n=-\infty}^{-1} g_n \left(\frac{\zeta - a}{r_2} \right)^n.$$

Next consider the function $\varphi - \varphi_1$. This function is holomorphic on $PP_{(a,r_1,r_2)}$ and, because $\varphi - \varphi_1 = \varphi_2$ on A , it extends to a holomorphic function on D . This means that $\varphi - \varphi_1$ is holomorphic on the annulus $\{\zeta \in \mathbb{C} : |\zeta| > r_1, |\zeta - c| < 1\}$, and so by using [3] again, we have $\varphi - \varphi_1 = \varphi_3 + \varphi_4$ with φ_3 holomorphic in the unit disk \mathbb{D} and φ_4 holomorphic on $\{\zeta \in \mathbb{C} : |\zeta| > r_1\}$. Thus φ_3 and φ_4 have the following convergent series representation

$$\varphi_3 = \sum_{n=0}^{\infty} e_n \zeta^n \quad \text{and} \quad \varphi_4 = \sum_{n=-\infty}^{-1} f_n \left(\frac{\zeta}{r_1} \right)^n.$$

Combining these three series representations gives the desired result. ■

Similar to the annulus case we set

$$E_n = \zeta^n, \quad F_n = \left(\frac{\zeta}{r_1} \right)^n, \quad \text{and} \quad G_n = \left(\frac{\zeta - a}{r_2} \right)^n.$$

The Hilbert space H that we will use is defined as

$$H = \left\{ \varphi(\zeta) = \sum_{n=0}^{\infty} e_n E_n + \sum_{n=-\infty}^{-1} (f_n F_n + g_n G_n) : \|\varphi\| < \infty \right\}, \quad (3.1)$$

where

$$\|\varphi\|^2 = \sum_{n=0}^{\infty} |e_n|^2 + \sum_{n=-\infty}^{-1} (|f_n|^2 + |g_n|^2).$$

The advantage of working with the above Hilbert space of holomorphic functions on $PP_{(a,r_1,r_2)}$ is that there is a distinguished orthonormal basis in it, namely the basis consisting of $\{E_n\}$, $\{F_m\}$, $\{G_k\}$.

The object of study in this section is the operator $z : H \rightarrow H$ given by $z\varphi(\zeta) = M_\zeta \varphi(\zeta) = \zeta \varphi(\zeta)$, i.e., the multiplication operator by ζ . Straightforward calculations yields the following formulas.

Lemma 3.2. *The operators z and z^* act on the basis elements in the following way*

$$\begin{aligned} zE_n &= E_{n+1} && \text{for } n \geq 0, \\ zF_n &= r_1 F_{n+1} && \text{for } n \leq -2, \\ zF_{-1} &= r_1 E_0, \\ zG_n &= r_2 G_{n+1} + aG_n && \text{for } n \leq -2, \\ zG_{-1} &= r_2 E_0 + aG_{-1} \end{aligned}$$

and

$$\begin{aligned} z^*E_n &= E_{n-1} && \text{for } n \geq 1, \\ z^*E_0 &= r_1 F_{-1} + r_2 G_{-1}, \\ z^*F_n &= r_1 F_{n-1} && \text{for } n \leq -1, \\ z^*G_n &= r_2 G_{n-1} + aG_n && \text{for } n \leq -1. \end{aligned}$$

Lemma 3.3. *The operators z and z^* shift the coefficients of $\varphi(\zeta)$ in the series decomposition defined in equation (3.1) in the following way*

$$z\varphi = \sum_{n=0}^{\infty} \tilde{e}_n E_n + \sum_{n=-\infty}^{-1} (\tilde{f}_n F_n + \tilde{g}_n G_n) \quad \text{and} \quad z^*\varphi = \sum_{n=0}^{\infty} e'_n E_n + \sum_{n=-\infty}^{-1} (f'_n F_n + g'_n G_n),$$

where

$$\begin{aligned} \tilde{e}_n &= e_{n-1} && \text{for } n \geq 1, \\ \tilde{e}_0 &= r_1 f_{-1} + r_2 g_{-1}, \\ \tilde{f}_n &= r_1 f_{n-1} && \text{for } n \leq -1, \\ \tilde{g}_n &= r_2 g_{n-1} + a g_n && \text{for } n \leq -1 \end{aligned}$$

and

$$\begin{aligned} e'_n &= e_{n+1} && \text{for } n \geq 0, \\ f'_n &= r_1 f_{n+1} && \text{for } n \leq -2, \\ f'_{-1} &= r_1 e_0, \\ g'_n &= r_2 g_{n+1} + a g_n && \text{for } n \leq -2, \\ g'_{-1} &= r_2 e_0 + a g_{-1}. \end{aligned}$$

We can now define the quantum pair of pants.

Definition 3.4. The quantum pair of pants, denoted $QPP_{(a,r_1,r_2)}$, is defined to be the C^* -algebra generated by the operator z , i.e., $QPP_{(a,r_1,r_2)} = C^*(z)$.

3.2 The spectrum of z

In this subsection we study the spectrum of z , starting with a calculation of the norm of z .

Proposition 3.5. *With the above notation, we have: $\|z\| = 1$.*

Proof. Using the series representation of $\varphi(\zeta)$ in formula (3.1) above, and the coefficients of Lemma 3.3 we compute $\|z\varphi\|^2$:

$$\|z\varphi\|^2 = \sum_{n=1}^{\infty} |e_{n-1}|^2 + |r_1 f_{-1} + r_2 g_{-1}|^2 + r_1^2 \sum_{n=-\infty}^{-1} |f_{n-1}|^2 + \sum_{n=-\infty}^{-1} |r_2 g_{n-1} + a g_n|^2.$$

Using the triangle inequality and the fact that $a > r_1$ we obtain

$$\|z\varphi\|^2 \leq \sum_{n=0}^{\infty} |e_n|^2 + (a|f_{-1}| + r_2|g_{-1}|)^2 + r_1^2 \sum_{n=-\infty}^{-1} |f_{n-1}|^2 + \sum_{n=-\infty}^{-1} (r_2|g_{n-1}| + a|g_n|)^2.$$

Notice that by denoting $g_0 := f_{-1}$ we can write

$$\begin{aligned} (a|f_{-1}| + r_2|g_{-1}|)^2 + \sum_{n=-\infty}^{-1} (r_2|g_{n-1}| + a|g_n|)^2 &= \sum_{n=-\infty}^0 (r_2|g_{n-1}| + a|g_n|)^2 \\ &= r_2^2 \sum_{n=-\infty}^0 |g_{n-1}|^2 + a^2 \sum_{n=-\infty}^0 |g_n|^2 + 2ar_2 \sum_{n=-\infty}^0 |g_{n-1}||g_n|. \end{aligned}$$

The Cauchy–Schwartz inequality implies

$$\begin{aligned} \sum_{n=-\infty}^0 (r_2|g_{n-1}| + a|g_n|)^2 &\leq r_2^2 \sum_{n=-\infty}^0 |g_{n-1}|^2 + a^2 \sum_{n=-\infty}^0 |g_n|^2 \\ &\quad + 2ar_2 \left(\sum_{n=-\infty}^0 |g_{n-1}|^2 \right)^{1/2} \left(\sum_{n=-\infty}^0 |g_n|^2 \right)^{1/2} \\ &\leq (r_2^2 + a^2 + 2ar_2) \sum_{n=-\infty}^0 |g_n|^2 = (r_2 + a)^2 \left(|f_{-1}|^2 + \sum_{n=-\infty}^{-1} |g_n|^2 \right). \end{aligned}$$

Using the fact that $r_1, r_2 + a < 1$ in the above computations we see that

$$\begin{aligned} \|z\varphi\|^2 &\leq \sum_{n=0}^{\infty} |e_n|^2 + r_1^2 \sum_{n=-\infty}^{-1} |f_{n-1}|^2 + (r_2 + a)^2 |f_{-1}|^2 + (r_2 + a)^2 \sum_{n=-\infty}^{-1} |g_n|^2 \\ &\leq \sum_{n=0}^{\infty} |e_n|^2 + \sum_{n=-\infty}^{-1} |f_n|^2 + \sum_{n=-\infty}^{-1} |g_n|^2 = \|\varphi\|^2, \end{aligned}$$

showing that $\|z\| \leq 1$. On the other hand, $\|zE_1\| = \|E_2\| = \|E_1\|$. Thus $\|z\| = 1$. ■

Next we compute the spectrum of z . In estimating the norms of resolvents of z we use the following well known result.

Lemma 3.6 (Schur–Young inequality). *Let $T : L^2(Y) \rightarrow L^2(X)$ be an integral operator*

$$Tf(x) = \int K(x, y)f(y)dy.$$

Then one has

$$\|T\|^2 \leq \left(\sup_{x \in X} \int_Y |K(x, y)|dy \right) \left(\sup_{y \in Y} \int_X |K(x, y)|dx \right).$$

The details of the lemma and its proof can be found in [6].

Lemma 3.7. *The operator $z - \lambda$ has a bounded inverse for $|\lambda| < r_1$, $|\lambda - a| < r_2$, and $|\lambda| > 1$.*

Proof. Let

$$\varphi(\zeta) = \sum_{n=0}^{\infty} e_n E_n + \sum_{n=-\infty}^{-1} (f_n F_n + g_n G_n)$$

and

$$\tilde{\varphi}(\zeta) = \sum_{n=0}^{\infty} \tilde{e}_n E_n + \sum_{n=-\infty}^{-1} (\tilde{f}_n F_n + \tilde{g}_n G_n).$$

Consider the equation $(z - \lambda)\varphi(\zeta) = \tilde{\varphi}(\zeta)$. Using the above decompositions and Lemma 3.3 we obtain the following system of equations

$$\begin{aligned} r_1 f_{-1} + r_2 g_{-1} - \lambda e_0 &= \tilde{e}_0, \\ e_{n-1} - \lambda e_n &= \tilde{e}_n & \text{for } n \geq 1, \\ r_1 f_{n-1} - \lambda f_n &= \tilde{f}_n & \text{for } n \leq -1, \\ r_2 g_{n-1} + a g_n - \lambda g_n &= \tilde{g}_n & \text{for } n \leq -1. \end{aligned} \tag{3.2}$$

By Proposition 3.5, $\|z\| = 1$ and if $|\lambda| > 1 = \|z\|$ then by general functional analysis we know that $(z - \lambda)^{-1}$ is a bounded, invertible operator.

Next we consider three cases: the first case is for $0 < |\lambda| < r_1$, the second case is for $|\lambda - a| < r_2$, and the last case is for $\lambda = 0$.

If $0 < |\lambda| < r_1 < 1$, then $|\lambda - a| > r_2$. We can solve the system of equations (3.2) recursively. Rewriting the last equation and multiplying by $((\lambda - a)/r_2)^{n-1}$ yields

$$\left(\frac{\lambda - a}{r_2}\right)^{n-1} g_{n-1} - \left(\frac{\lambda - a}{r_2}\right)^n g_n = \left(\frac{\lambda - a}{r_2}\right)^{n-1} \frac{1}{r_2} \tilde{g}_n.$$

Letting $h_n = ((\lambda - a)/r_2)^n g_n$, we get

$$h_{n-1} - h_n = \left(\frac{\lambda - a}{r_2}\right)^{n-1} \frac{1}{r_2} \tilde{g}_n.$$

The requirement for a square summable solution forces $h_n = -\sum_{j=-\infty}^n ((\lambda - a)/r_2)^{j-1} \tilde{g}_j / r_2$ and hence for $n \leq -1$ we obtain

$$g_n = -\frac{1}{r_2} \sum_{j=-\infty}^n \left(\frac{\lambda - a}{r_2}\right)^{j-n-1} \tilde{g}_j.$$

Similar calculations show that

$$e_n = \sum_{j=n+1}^{\infty} \lambda^{j-n-1} \tilde{e}_j \quad \text{and} \quad f_n = \left(\frac{\lambda}{r_1}\right)^{-n-1} f_{-1} + \frac{1}{r_1} \sum_{j=n+1}^{-1} \left(\frac{\lambda}{r_1}\right)^{j-n-1} \tilde{f}_j$$

for $n \geq 0$ and $n \leq -2$ respectively. These formulas along with the first equation in system (3.2) give

$$f_{-1} = \frac{1}{r_1} \left(\sum_{j=0}^{\infty} \lambda^j \tilde{e}_j + \sum_{j=-\infty}^{-1} \left(\frac{\lambda - a}{r_2}\right)^j \tilde{g}_j \right).$$

We introduce some notation; first notice that we have a natural decomposition, $H \cong \ell^2(\mathbb{Z}_{\geq 0}) \oplus \ell^2(\mathbb{Z}_{< 0}) \oplus \ell^2(\mathbb{Z}_{< 0})$ given in the following way: for $\varphi \in H$ write $\varphi = e + f + g$ where $e = \sum_{n \geq 0} e_n E_n$, $f = \sum_{n \leq -1} f_n F_n$, and $g = \sum_{n \leq -1} g_n G_n$. Using this notation we see that $\|\varphi\|^2 = \|e\|^2 + \|f\|^2 + \|g\|^2$. Define the characteristic $\chi(t) = 1$ for $0 \leq t \leq 1$ and zero otherwise, then we can define seven different integral operators

$$\begin{aligned}
T_1 e &= \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} \lambda^{j-n-1} \chi\left(\frac{n+1}{j}\right) e_j E_n : \ell^2(\mathbb{Z}_{\geq 0}) \rightarrow \ell^2(\mathbb{Z}_{\geq 0}), \\
T_2(e, g) &= \sum_{n=-\infty}^{-1} \frac{1}{r_1} \left(\frac{\lambda}{r_1}\right)^{-n-1} \left(\sum_{j=0}^{\infty} \lambda^j e_j + \sum_{j=-\infty}^{-1} \left(\frac{\lambda-a}{r_2}\right)^j g_j \right) F_n : \\
&\quad \ell^2(\mathbb{Z}_{\geq 0}) \oplus \ell^2(\mathbb{Z}_{< 0}) \rightarrow \ell^2(\mathbb{Z}_{< 0}) \\
T_3 f &= \sum_{n=-\infty}^{-1} \frac{1}{r_1} \sum_{j=-\infty}^{-1} \left(\frac{\lambda}{r_1}\right)^{j-n-1} \chi\left(\frac{n+1}{j}\right) f_j F_n : \ell^2(\mathbb{Z}_{< 0}) \rightarrow \ell^2(\mathbb{Z}_{< 0}), \\
T_4 f &= \sum_{n=-\infty}^{-1} \frac{1}{r_1} \sum_{j=-\infty}^{-1} \left(\frac{\lambda}{r_1}\right)^{j-n-1} \chi\left(\frac{j}{n}\right) f_j F_n : \ell^2(\mathbb{Z}_{< 0}) \rightarrow \ell^2(\mathbb{Z}_{< 0}), \\
T_5(e, f) &= \sum_{n=-\infty}^{-1} \frac{1}{r_2} \left(\frac{\lambda-a}{r_2}\right)^{-n-1} \left(\sum_{j=0}^{\infty} \lambda^j e_j + \sum_{j=-\infty}^{-1} \left(\frac{\lambda}{r_1}\right)^j f_j \right) G_n : \\
&\quad \ell^2(\mathbb{Z}_{\geq 0}) \oplus \ell^2(\mathbb{Z}_{< 0}) \rightarrow \ell^2(\mathbb{Z}_{< 0}), \\
T_6 g &= \sum_{n=-\infty}^{-1} \frac{1}{r_2} \sum_{j=-\infty}^{-1} \left(\frac{\lambda-a}{r_2}\right)^{j-n-1} \chi\left(\frac{n+1}{j}\right) g_j G_n : \ell^2(\mathbb{Z}_{< 0}) \rightarrow \ell^2(\mathbb{Z}_{< 0}), \\
T_7 g &= \sum_{n=-\infty}^{-1} \frac{1}{r_2} \sum_{j=-\infty}^{-1} \left(\frac{\lambda-a}{r_2}\right)^{j-n-1} \chi\left(\frac{j}{n}\right) g_j G_n : \ell^2(\mathbb{Z}_{< 0}) \rightarrow \ell^2(\mathbb{Z}_{< 0}).
\end{aligned} \tag{3.3}$$

The operators from formula (3.3) can be used to represent $(z - \lambda)^{-1} \tilde{\varphi}$, for $\tilde{\varphi} = \tilde{e} + \tilde{f} + \tilde{g}$ where $\tilde{e} = \sum_{n \geq 0} \tilde{e}_n E_n$, $\tilde{f} = \sum_{n \leq -1} \tilde{f}_n F_n$ and $\tilde{g} = \sum_{n \leq -1} \tilde{g}_n G_n$, in the following way

$$(z - \lambda)^{-1} \tilde{\varphi} = T_1 \tilde{e} + T_2(\tilde{e}, \tilde{g}) + T_3 \tilde{f} - T_7 \tilde{g}.$$

Next we estimate the norm of $(z - \lambda)^{-1}$. We use Lemma 3.6 to estimate the norms of the operators T_1 , T_3 and T_7 and we directly estimate $\|T_2 \tilde{f}\|$. The first estimate is

$$\begin{aligned}
\|T_1\|^2 &\leq \left(\sup_{n \geq 0} |\lambda|^{-n-1} \sum_{j=n+1}^{\infty} |\lambda|^j \right) \left(\sup_{j \geq 1} |\lambda|^{j-1} \sum_{n=0}^{j-1} |\lambda|^{-n} \right) \\
&= \frac{1}{(1 - |\lambda|)} \left(\sup_{j \geq 1} \frac{1 - |\lambda|^j}{1 - |\lambda|} \right) = \frac{1}{(1 - |\lambda|)^2},
\end{aligned}$$

where we have used the fact that $|\lambda| < 1$. Similarly, we have

$$\|T_3\|^2 \leq \frac{1}{r_1^2} \left(\sup_{n \leq -2} \left(\frac{|\lambda|}{r_1}\right)^{-n-1} \sum_{j=n+1}^{-1} \left(\frac{|\lambda|}{r_1}\right)^j \right) \left(\sup_{j \leq -1} \left(\frac{|\lambda|}{r_1}\right)^{j-1} \sum_{n=-\infty}^{j-1} \left(\frac{|\lambda|}{r_1}\right)^{-n} \right)$$

$$\leq \frac{1}{r_1^2 \left(1 - \frac{|\lambda|}{r_1}\right)^2} \left(\sup_{n \leq -2} 1 - \left(\frac{|\lambda|}{r_1}\right)^{-n-1} \right).$$

Since $\frac{|\lambda|}{r_1} < 1$, it follows that $\|T_3\|^2 \leq \frac{1}{r_1^2 \left(1 - \frac{|\lambda|}{r_1}\right)^2}$.

Next,

$$\begin{aligned} \|T_7\|^2 &\leq \frac{1}{r_2^2} \left(\sup_{n \leq -1} \left| \frac{\lambda - a}{r_2} \right|^{-n-1} \sum_{j=-\infty}^n \left| \frac{\lambda - a}{r_2} \right|^j \right) \left(\sup_{j \leq -1} \left| \frac{\lambda - a}{r_2} \right|^{j-1} \sum_{n=j}^{-1} \left| \frac{\lambda - a}{r_2} \right|^{-n} \right) \\ &\leq \frac{1}{r_2^2} \frac{(|\lambda - a|/r_2)^{-2}}{\left(1 - (|\lambda - a|/r_2)^{-1}\right)^2} \left(\sup_{j \leq -1} 1 - \left(\frac{|\lambda - a|}{r_2}\right)^j \right). \end{aligned}$$

Because $\frac{|\lambda - a|}{r_2} > 1$, we have

$$\|T_7\|^2 \leq \frac{1}{r_2^2} \frac{(|\lambda - a|/r_2)^{-2}}{\left(1 - (|\lambda - a|/r_2)^{-1}\right)^2} = \frac{1}{r_1^2 \left(\frac{|\lambda - a|}{r_2} - 1\right)^2}.$$

The operator T_2 is a rank one operator and the norm $T_2 \tilde{f}$ can be estimated directly, using the Cauchy–Schwartz inequality

$$\begin{aligned} \|T_2(\tilde{e}, \tilde{g})\|^2 &= \frac{1}{r_1^2} \sum_{n=-\infty}^{-1} \left(\frac{|\lambda|}{r_1}\right)^{-2n-2} \left| \sum_{j=0}^{\infty} \lambda^j \tilde{e}_j + \sum_{j=-\infty}^{-1} \left(\frac{\lambda - a}{r_2}\right)^j \tilde{g}_j \right|^2 \\ &\leq \frac{1}{r_1^2 (1 - (|\lambda|/r_1)^2)} \left(\frac{1}{1 - |\lambda|^2} + \frac{1}{(|\lambda - a|/r_2)^2 - 1} \right) (\|\tilde{e}\|^2 + \|\tilde{g}\|^2). \end{aligned}$$

This shows that $(z - \lambda)^{-1}$ is bounded for $0 < |\lambda| < r_1$.

The second case is $|\lambda - a| < r_2$. This implies that $r_1 < |\lambda| < 1$. Under these constraints we solve system (3.2) using the same methods as those for the first case to obtain

$$\begin{aligned} e_n &= \sum_{j=n+1}^{\infty} \lambda^{j-n-1} \tilde{e}_j \quad \text{for } n \geq 0, \\ f_n &= -\frac{1}{r_1} \sum_{j=-\infty}^n \left(\frac{\lambda}{r_1}\right)^{j-n-1} \tilde{f}_j \quad \text{for } n \leq -1, \\ g_n &= -\left(\frac{\lambda - a}{r_2}\right)^{-n-1} g_{-1} + \frac{1}{r_2} \sum_{j=n+1}^{-1} \left(\frac{\lambda - a}{r_2}\right)^{j-n-1} \tilde{g}_j \quad \text{for } n \leq -2. \end{aligned}$$

Then the first equation of system (3.2) gives

$$g_{-1} = \frac{1}{r_2} \left(\sum_{j=0}^{\infty} \lambda^j \tilde{e}_j + \sum_{j=-\infty}^{-1} \left(\frac{\lambda}{r_1}\right)^j \tilde{f}_j \right).$$

Similar to the first case we can express $(z - \lambda)^{-1}$ using the operators defined in formula (3.3) to get

$$(z - \lambda)^{-1} \tilde{\varphi} = T_1 \tilde{e} - T_4 \tilde{f} + T_5(\tilde{e}, \tilde{f}) + T_6 \tilde{g}.$$

We omit the repetitive details of estimates of T_4 , T_5 , and T_6 norms. They imply that $(z - \lambda)^{-1}$ is bounded for $|\lambda - a| < r_2$.

The last case is when $\lambda = 0$. Solving system (3.2) we obtain

$$\begin{aligned} e_n &= \tilde{e}_{n+1} & \text{for } n \geq 0, \\ f_n &= \frac{1}{r_1} \tilde{f}_{n+1} & \text{for } n \leq -2, \\ g_n &= -\frac{1}{r_2} \sum_{j=-\infty}^n \left(-\frac{a}{r_2}\right)^{j-n-1} \tilde{g}_j & \text{for } n \leq -1. \end{aligned}$$

Using the first equation of system (3.2) we compute f_{-1}

$$f_{-1} = \frac{1}{r_1} \left(\tilde{e}_0 + \sum_{j=-\infty}^{-1} \left(-\frac{a}{r_2}\right)^j \tilde{g}_j \right).$$

As before the norm estimates hinge on convergent geometric series. This completes the proof. \blacksquare

Theorem 3.8. *The spectrum of z is the regular pair of pants, i.e., $\sigma(z) = PP_{(a,r_1,r_2)}$.*

Proof. By Proposition 3.5, $\sigma(z) \subset \mathbb{D}$. Let

$$\varphi_\lambda(\zeta) = \sum_{n=0}^{\infty} \lambda^n E_n + \sum_{n=-\infty}^{-1} \left(\frac{\lambda}{r_1}\right)^n F_n + \sum_{n=-\infty}^{-1} \left(\frac{\lambda - a}{r_2}\right)^n G_n.$$

It is easy to see that for any λ in the interior of $PP_{(a,r_1,r_2)}$, $\varphi_\lambda \in H$ and that λ is an eigenvalue with φ_λ as the associated eigenfunction for z^* . Therefore, $PP_{(a,r_1,r_2)} \subset \sigma(z)$. From Lemma 3.7 the operator $z - \lambda$ has a bounded inverse whenever $|\lambda| < r_1$ or $|\lambda - a| < r_2$. Hence the resolvent set is contained in the holes within the unit disk or outside the unit disk, and so $\sigma(z) \subset PP_{(a,r_1,r_2)}$. \blacksquare

In view of the above theorem we can think of the operator z as a form of a noncommutative complex coordinate for what we call quantum pair of pants.

3.3 Structure of $C^*(z)$

Next we study the commutator ideal of $C^*(z)$. A straightforward computation gives the following formulas.

Lemma 3.9. *The commutator of z^* and z act on the basis elements in the following way: $[z^*, z]E_n = 0$ for $n \geq 1$, $[z^*, z]F_n = 0$ for $n \leq -2$, $[z^*, z]G_n = 0$ for $n \leq -2$. Moreover on the initial elements we get:*

$$\begin{aligned} [z^*, z]E_0 &= (1 - r_1^2 - r_2^2)E_0 - ar_2G_{-1}, \\ [z^*, z]F_{-1} &= r_1r_2G_{-1}, \\ [z^*, z]G_{-1} &= r_1r_2F_{-1} - ar_2E_0. \end{aligned}$$

Let \mathcal{I} be the ideal generated by $[z^*, z]$. It is easy to see that \mathcal{I} is in fact the commutator ideal of $C^*(z)$ because that algebra is singly generated.

Theorem 3.10. *The commutator ideal \mathcal{I} of $C^*(z)$ is the C^* -algebra \mathcal{K} of compact operators in H .*

Proof. From Lemma 3.9 it is clear that the commutator $[z^*, z]$ is finite-rank and hence compact. Thus, $\mathcal{I} \subset \mathcal{K}$. On the other hand, to show that $\mathcal{K} \subset \mathcal{I}$ we will use the following step by step method building up to the conclusion that a large collection of rank one operators belong to \mathcal{I} and that the compact operators are exactly the norm limit of those.

Step 1. First we show that $P =$ orthogonal projection onto $\text{span}\{E_0, F_{-1}, G_{-1}\}$ belongs to the commutator ideal. Notice that the (self-adjoint) operator $[z^*, z]$ acting on $\text{span}\{E_0, F_{-1}, G_{-1}\}$ has the following matrix representation in the basis $\{E_0, F_{-1}, G_{-1}\}$:

$$A = \begin{pmatrix} 1 - r_1^2 - r_2^2 & 0 & -ar_2 \\ 0 & 0 & r_1r_2 \\ -ar_2 & r_1r_2 & 0 \end{pmatrix}.$$

This matrix has rank equal to 3 and the following characteristic polynomial

$$p_A(\lambda) = \lambda^3 - (1 - r_1^2 - r_2^2)\lambda^2 - (a^2r_2^2 + r_1^2r_2^2)\lambda + (1 - r_1^2 - r_2^2)r_1^2r_2^2.$$

Since $0 < r_1, r_2 < 1$ it is clear that zero is not an eigenvalue of A . If $\lambda_i, i = 1, 2, 3$ are the roots of $p_A(\lambda)$ then by functional calculus there exists a continuous function $f : \mathbb{R} \rightarrow \mathbb{R}$ such that $f(0) = 0, f(\lambda_i) = 1$ so that $f([z^*, z]) = P$. Consequently $P \in \mathcal{I} \subset C^*(z)$.

Step 2. The next step is showing that $P_{E_1} =$ orthogonal projection onto $\text{span}\{E_1\}$ belongs to \mathcal{I} . We first observe that the operator zPz^* acts on the basis elements in the following way

$$zpz^*B = \begin{cases} (r_1^2 + r_2^2)E_0 + ar_2G_{-1} & \text{if } B = E_0, \\ E_1 & \text{if } B = E_1, \\ ar_2E_0 + a^2G_{-1} & \text{if } B = G_{-1}, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, the operator zPz^* on $\text{span}\{E_0, G_{-1}\}$ is self-adjoint and has the following matrix representation in the basis $\{E_0, G_{-1}\}$:

$$C = \begin{pmatrix} r_1^2 + r_2^2 & ar_2 \\ ar_2 & a^2 \end{pmatrix}.$$

The characteristic polynomial for C is $p_C(\lambda) = \lambda^2 - (r_1^2 + r_2^2 + a^2)\lambda + a^2r_1^2$. First we need to show that $\lambda = 0, 1$ are not roots of $p_C(\lambda)$. Clearly $p_C(0) \neq 0$. Suppose $p_C(1) = 0$. Then solving the equation for r_2 we obtain, $r_2^2 = (1 - a^2)(1 - r_1^2)$. Since $r_2 < 1 - a$ and $r_2 < 1 - r_1$ we see that

$$(1 - a^2)(1 - r_1^2) < (1 - a)(1 - r_1),$$

implying $(1 + a)(1 + r_1) < 1$ which is clearly a contradiction since $a, r_1 > 0$. Thus, $p_C(1) \neq 0$.

Now we look at the discriminant Δ of $p_C(\lambda)$:

$$\Delta = (r_1^2 + r_2^2 + a^2)^2 - 4a^2r_1^2.$$

If $\Delta = 0$ this would imply that $a = r_1$ and $r_2 = 0$, which is a contradiction. Hence C has two distinct eigenvalues λ_1 and λ_2 . Thus, once again by functional calculus there exists a continuous real valued function f such that $f(0) = f(\lambda_1) = f(\lambda_2) = 0$ and $f(1) = 1$. Consequently, applying f to zPz^* we get that, $f(zPz^*) = p_{E_1} \in \mathcal{I}$.

Step 3. By similar functional calculus argument as above we also see that $P_{E_0, G_{-1}}$, the orthogonal projection onto $\text{span}\{E_0, G_{-1}\}$, belongs to \mathcal{I} . Consequently, if $P_{F_{-1}} =$ orthogonal projection onto $\text{span}\{F_{-1}\}$ then clearly, $P_{F_{-1}} = P - P_{E_0, G_{-1}} \in \mathcal{I}$.

Step 4. We will show that $P_{E_n} =$ orthogonal projection onto $\text{span}\{E_n\}$ belongs to \mathcal{I} for $n = 1, 2, \dots$. To this end, we compute the action of $z^{n-1}P_{E_1}(z^*)^{n-1}$ on the basis elements

$$z^{n-1}P_{E_1}(z^*)^{n-1}B = \begin{cases} E_n & \text{if } B = E_n, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, $z^{n-1}P_{E_1}(z^*)^{n-1} = P_{E_n} \in \mathcal{I}$ for $n \geq 1$.

Step 5. Now consider the action of $(z^*)^n P_{F_{-1}} z^n$, $n \geq 1$ on basis elements

$$(z^*)^n P_{F_{-1}} z^n B = \begin{cases} r_1^{2n} F_{-n-1} & \text{if } B = F_{-n-1}, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, $r_1^{2n}(z^*)^{-n}P_{F_{-1}}z^{-n} = P_{F_n}$, the projection onto F_n , for $n \leq -1$ and hence the projections P_{F_n} belong to \mathcal{I} .

Step 6. Since $z^*P_{E_1}zE_0 = E_0$ and zero elsewhere, it is clear that $z^*P_{E_1}z = P_{E_0} \in \mathcal{I}$. Hence $P_{G_{-1}} = P - P_{E_0} - P_{F_{-1}}$ also belongs to \mathcal{I} .

Step 7. It remains to show that for $n \leq -2$ the orthogonal projection P_{G_n} onto G_n belongs to \mathcal{I} . We consider the action of $z^*P_{G_{-1}}z$ on basis elements

$$z^*P_{G_{-1}}zB = \begin{cases} a^2G_{-1} + ar_2G_{-2} & \text{if } B = G_{-1}, \\ ar_2G_{-1} + r_2^2G_{-2} & \text{if } B = G_{-2}, \\ 0 & \text{otherwise.} \end{cases}$$

Thus it has the following matrix representation relative to the basis $\{G_{-1}, G_{-2}\}$:

$$D = \begin{pmatrix} a^2 & ar_2 \\ ar_2 & r_2^2 \end{pmatrix}.$$

The matrix D has eigenvalues $\lambda_1 = 0$ and $\lambda_2 = a^2 + r_2^2$ with $v = aG_{-1} + r_2G_{-2}$ being the eigenvector corresponding to λ_2 . Thus P_v , the one-dimensional orthogonal projection onto $v/\|v\|$, belongs to \mathcal{I} . Since $P_{G_{-1}}$ and P_v are not mutually orthogonal, simple matrix algebra shows that the set $\{I, P_{G_{-1}}, P_v, P_{G_{-1}}P_v\}$, where I is the 2×2 identity matrix, generates the set of all 2×2 matrices. Consequently, $P_{G_{-2}}$ can be written as a linear combination of these four projections, making it clear that $P_{G_{-2}} \in \mathcal{I}$.

Finally we use induction on n and follow a similar argument as above to show that $P_{G_n} \in \mathcal{I}$ for $n = -2, -3, -4, \dots$.

Step 8. Next we proceed to show that the one-dimensional operators $P_{B_i, B_j}(x) = \langle B_i, x \rangle B_j$, where B_i, B_j are basis elements, i.e., elements of the set $\{E_n, F_k, G_k : n \geq 0, k \leq -1\}$, also belong to \mathcal{I} .

Since $z^m P_{E_n} E_n = E_{n+m}$ we see that $P_{E_n, E_{n+m}} = z^m P_{E_n}$ for every $n, m \geq 0$. Similarly we observe that $P_{E_n, E_{n-m}} = (z^*)^m P_{E_n}$ for $m \leq n$. Together, this proves that all operators $P_{E_n, E_k} \in \mathcal{I}$ for any $n, k \geq 0$.

Next, we observe that

$$z^m P_{F_n} = \begin{cases} r_1^m F_{n+m} & \text{if } n + m < 0, \\ r_1^{-n} E_{n+m} & \text{if } n + m \geq 0. \end{cases}$$

Moreover, $(z^*)^m P_{F_n} = r_1^m F_{n-m}$ for all $n < 0, m \geq 0$. Consequently, $P_{F_n, F_{n+m}} = r_1^{-m} z^m P_{F_n}$ and $P_{F_n, F_{n-m}} = r_1^{-m} (z^*)^m P_{F_n}$. Hence, $P_{F_n, F_k} \in \mathcal{I}$ for all $n, k \leq -1$. Moreover, $P_{F_n, E_k} = r_1^n z^m P_{F_n}$ and so $P_{F_n, E_k} \in \mathcal{I}$ for $k \geq 0, n \leq -1$. In fact, it can be easily verified that, $P_{B_j, B_i}^* = P_{B_i, B_j}$. This would mean that P_{E_k, F_n} also belong to \mathcal{I} .

Similar calculations show that $P_{G_n, G_{n+m}} = r_2^{-m} P_{G_{n+m}} z^m P_{G_n}$ for $n+m < 0$ and $P_{G_n, G_{n-m}} = r_2^{-m} P_{G_{n-m}} (z^*)^m P_{G_n}$ for $n < 0, m \geq 0$. Collectively, these imply that $P_{G_n, G_k} \in \mathcal{I}$ for all $n, k \leq -1$. Also $P_{G_n, E_{n+m}} = r_2^m P_{E_{n+m}} z^m P_{G_n}$ for $n+m \geq 0$. Hence P_{G_n, E_k} and P_{E_k, G_n} belong to \mathcal{I} for all $k \geq 0, n \leq -1$.

Finally, we notice that $P_{G_n, F_{-m}} = r_1^{-m} r_2^n P_{F_{-m}} (z^*)^m P_{E_0} z^n P_{G_n}$, which shows that the operators $P_{G_n, F_k} \in \mathcal{I}$ for all $n, k \leq -1$.

Consider now finite rank operators which are finite linear combinations of the one-dimensional P_{B_i, B_j} for B_i, B_j in the basis for H . It is a simple exercise in functional analysis to show that all compact operators are norm limits of such finite rank operators. \blacksquare

To describe the structure of $C^*(z)$ we need to understand the commutative quotient $C^*(z)/\mathcal{K}$. For the case of quantum pair of pants that structure and the idea of proof is very similar to the case of quantum annulus.

Below we will show that the C^* -algebra $C^*(z)$ contains some infinite-dimensional projections. Those are obtained from the spectrum of zz^* . While tedious, the computation of the spectrum of zz^* is fairly straightforward and it amounts to studying a (multi parameter) system of two step difference equations with constant coefficients. The results of the computations are presented in the next three theorems.

We start with the computation of the pure-point spectrum.

Theorem 3.11. *The operator zz^* has three eigenvalues: 1 with eigenspace $\text{span}\{E_n\}_{n \geq 0}$, r_1^2 with eigenspace $\text{span}\{F_n\}_{n < 0}$, and the simple eigenvalue $\frac{r_1^2(a^2 - r_2^2 - r_1^2)}{a^2 - r_1^2}$.*

Proof. We study $(zz^* - \lambda)\varphi = 0$. Using Lemma 3.3 we get the following system of equations

$$\begin{aligned} (1 - \lambda)e_n &= 0 && \text{for } n \geq 1, \\ (r_1^2 + r_2^2 - \lambda)e_0 + ar_2g_{-1} &= 0, \\ (r_1^2 - \lambda)f_n &= 0 && \text{for } n \leq -1, \\ ar_2g_{n+1} + (a^2 + r_2^2 - \lambda)g_n + ar_2g_{n-1} &= 0 && \text{for } n \leq -2, \\ ar_2e_0 + (a^2 + r_2^2 - \lambda)g_{-1} + ar_2g_{-2} &= 0. \end{aligned} \tag{3.4}$$

The first and the third equations in this system yield the eigenvalues 1, r_1^2 and the eigenspaces $\text{span}\{E_n\}_{n \geq 0}$, $\text{span}\{F_n\}_{n < 0}$ respectively. For the fourth equation, which is a two step linear recurrence with constant coefficients, the characteristic equation is

$$ar_2x^2 + (r_2^2 + a^2 - \lambda)x + ar_2 = 0.$$

The discriminant Δ of this equation is

$$\Delta = (r_2^2 + a^2 - \lambda)^2 - 4a^2r_2^2 = ((a - r_2)^2 - \lambda)((a + r_2)^2 - \lambda),$$

and the roots are

$$x_{\pm} = \frac{\lambda - r_2^2 - a^2 \pm \sqrt{\Delta}}{2ar_2}.$$

Thus, the formal solution to the homogeneous equation is

$$g_n = c_+ x_+^{n+1} + c_- x_-^{n+1}, \tag{3.5}$$

where c_+ and c_- are arbitrary constants. Notice that if we adopt the convention $g_0 := e_0$ in the last equation of (3.4), then this formula holds for $n \leq 0$.

There are three separate cases to consider: $0 < \lambda < (r_2 - a)^2$, $(r_2 - a)^2 \leq \lambda \leq (r_2 + a)^2$ and $(r_2 + a)^2 < \lambda < 1$. If $0 < \lambda < (r_2 - a)^2$ we notice the following facts about the solutions. Since $\lambda < (r_2 - a)^2 = r_2^2 + a^2 - 2ar_2$ and both a and r_2 are positive, we have $\lambda - r_2^2 - a^2 < 0$, and since $\Delta > 0$, we have that $x_- < 0$. Also notice that $x_+x_- = 1$ from which it follows that $x_+ < 0$. Since $x_- < x_+$ we have $x_- < -1$ and $-1 < x_+ < 0$. Thus, $|x_-| > 1$ and $|x_+| < 1$.

Since we need $g_n \in \ell^2(\mathbb{Z}_{<0})$, and since $|x_+| < 1$, we must have $c_+ = 0$. In particular this implies that $g_0 = x_-g_{-1}$. The second equation of the system (3.4) then becomes

$$((r_1^2 + r_2^2 - \lambda)x_- + ar_2)g_{-1} = 0.$$

If $(r_1^2 + r_2^2 - \lambda)x_- + ar_2 = 0$, then using the relation $x_+x_- = 1$ we obtain

$$x_+ = \frac{\lambda - r_1^2 - r_2^2}{ar_2}.$$

On the other hand we also have

$$x_+ = \frac{\lambda - a^2 - r_2^2 + \sqrt{\Delta}}{ar_2}.$$

Setting these equal to each other and solving for Δ we get $\Delta = (\lambda - 2r_1^2 - r_2^2 + a^2)^2$. Solving for λ yields

$$\lambda = r_1^2 \left(1 - \frac{r_1^2}{a^2 - r_1^2} \right) = \frac{r_1^2(a^2 - r_2^2 - r_1^2)}{a^2 - r_1^2}.$$

First notice that due to the conditions on a , r_1 and r_2 we have that $\lambda > 0$ and $\lambda < r_1^2$ as $r_1^2/(a^2 - r_1^2) > 0$. Therefore this λ is in the interval $(0, (r_2 - a)^2)$, and $c_- = g_{-1}$ is arbitrary. Consequently $\lambda = \frac{r_1^2(a^2 - r_2^2 - r_1^2)}{a^2 - r_1^2}$ is a simple eigenvalue with an eigenvector

$$\varphi_\lambda = \sum_{n=-\infty}^0 x_-^{n+1} G_n.$$

In the case $(r_2 + a)^2 < \lambda < 1$, we have $\lambda - r_2^2 - a^2 = \lambda - (r_2 + a)^2 + 2r_2a > 0$, $\Delta > 0$ and hence $x_+ > 0$. Again from $x_+x_- = 1$ we see that $x_+ > 1$ and $0 < x_- < 1$. Consequently, in equation (3.5) we must have $c_- = 0$, which then implies that $g_0 = x_+g_{-1}$. The second equation of (3.4) then becomes

$$((r_1^2 + r_2^2 - \lambda)x_+ + ar_2)g_{-1} = 0.$$

Suppose there is λ such that $x_+(r_1^2 + r_2^2 - \lambda) + ar_2 = 0$, which would then imply that

$$x_+ = \frac{ar_2}{\lambda - r_1^2 - r_2^2}$$

and, since $x_+ > 1$, we must have

$$\lambda < r_1^2 + ar_2 + r_2^2 < a^2 + 2ar_2 + r_2^2 = (r_2 + a)^2.$$

This is a contradiction. Consequently $g_n = 0$ for every n and there are no eigenvectors in this case.

When $(r_2 - a)^2 \leq \lambda \leq (r_2 + a)^2$ the discriminant $\Delta \leq 0$ and the two solutions x_+ and x_- are complex numbers conjugate to each other with absolute value equal to one. Since we need $g_n \in \ell^2(\mathbb{Z}_{<0})$, equation (3.5) implies that $g_n = 0$ and there are again no eigenvectors in this case. ■

The second, and the most technical step in the calculation of the spectrum of zz^* is the calculation of the inverse of $zz^* - \lambda$. Norm estimates of the inverse provide insight about the resolvent set of zz^* .

Lemma 3.12. *The operator $(zz^* - \lambda)^{-1}$ is bounded for λ not an eigenvalue and $\lambda \in (0, (r_2 - a)^2) \cup ((r_2 + a)^2, 1)$.*

Proof. We study $(zz^* - \lambda)\varphi = \tilde{\varphi}$ using coordinates

$$\varphi = \sum_{n=0}^{\infty} e_n E_n + \sum_{n=-\infty}^{-1} (f_n F_n + g_n G_n) \quad \text{and} \quad \tilde{\varphi} = \sum_{n=0}^{\infty} \tilde{e}_n E_n + \sum_{n=-\infty}^{-1} (\tilde{f}_n F_n + \tilde{g}_n G_n).$$

Using Lemma 3.3 we get the following system of equations

$$\begin{aligned} (1 - \lambda)e_n &= \tilde{e}_n & \text{for } n \geq 1, \\ (r_1^2 + r_2^2 - \lambda)g_0 + ar_2g_{-1} &= \tilde{g}_0, \\ (r_1^2 - \lambda)f_n &= \tilde{f}_n & \text{for } n \leq -1, \\ ar_2g_{n+1} + (a^2 + r_2^2 - \lambda)g_n + ar_2g_{n-1} &= \tilde{g}_n & \text{for } n \leq -1. \end{aligned} \tag{3.6}$$

For the purpose of this proof we have introduced the notation $g_0 := e_0$ and $\tilde{g}_0 := \tilde{e}_0$.

The first and the third equations in system (3.6) can be solved directly

$$\begin{aligned} e_n &= \frac{1}{1 - \lambda} \tilde{e}_n & \text{for } n \geq 1 \quad \text{and} \quad \lambda \neq 1, \\ f_n &= \frac{1}{r_1^2 - \lambda} \tilde{f}_n & \text{for } n \leq -1 \quad \text{and} \quad \lambda \neq r_1^2. \end{aligned}$$

As for the fourth equation in system (3.6), we start with the case $0 < \lambda < (r_2 - a)^2$. From the proof of Theorem 3.11 we have that $|x_-| > 1$ and $|x_+| < 1$, where $x_{\pm} = \frac{\lambda - r_2^2 - a^2 \pm \sqrt{\Delta}}{2ar_2}$ are the solution of the characteristic equation $ar_2x^2 + (r_2^2 + a^2 - \lambda)x + ar_2 = 0$ with the discriminant $\Delta = ((a - r_2)^2 - \lambda)((a + r_2)^2 - \lambda)$.

We use the variation of parameters technique to solve this equation. For the homogeneous component of the solution, we use (3.5) to obtain $g_n = c_+x_+^{n+1} + c_-x_-^{n+1}$ with c_+ and c_- being constants to be determined. We look for the the solutions of the non-homogeneous equation in the form: $g_n = A_nx_+^{n+1} + B_nx_-^{n+1}$. Then the standard trick is to assume the first equation below to obtain the following system

$$\begin{aligned} (A_n - A_{n-1})x_+^{n+1} + (B_n - B_{n-1})x_-^{n+1} &= 0 & \text{for } n \leq 0, \\ (A_n - A_{n-1})x_+^n + (B_n - B_{n-1})x_-^n &= \frac{-\tilde{g}_n}{ar_2} & \text{for } n \leq -1. \end{aligned}$$

In particular,

$$(A_0 - A_{-1})x_+ + (B_0 - B_{-1})x_- = 0. \tag{3.7}$$

The solution of the above system is

$$A_n - A_{n-1} = \frac{\tilde{g}_n x_-^{n+1}}{\sqrt{\Delta}}, \quad B_n - B_{n-1} = \frac{-\tilde{g}_n x_+^{n+1}}{\sqrt{\Delta}}. \tag{3.8}$$

In solving these difference equations we pay attention to square summability, making sure we only consider convergent expressions in powers of x_{\pm} . This leads to the following special solution of the non-homogeneous equation

$$A_n = \frac{1}{\sqrt{\Delta}} \sum_{j=-\infty}^n \tilde{g}_j x_-^{j+1} \quad \text{and} \quad B_n = \frac{1}{\sqrt{\Delta}} \sum_{j=n+1}^{-1} \tilde{g}_j x_+^{j+1}$$

with $B_{-1} = 0$, $n \leq -1$. Consequently the general solution is

$$g_n = c_- x_-^{n+1} + \frac{1}{\sqrt{\Delta}} \left(\sum_{j=n+1}^{-1} \tilde{g}_j x_+^{j-n} + \sum_{j=-\infty}^n \tilde{g}_j x_-^{j-n} \right)$$

for $n < -1$, since we want $g_n \in \ell^2(\mathbb{Z}_{<0})$, and since $|x_+| < 1$ we must have $c_+ = 0$. For $n = -1$ we get

$$g_{-1} = c_- + \frac{1}{\sqrt{\Delta}} \sum_{j=-\infty}^{-1} \tilde{g}_j x_-^{j+1},$$

and for $n = 0$, using (3.7), we obtain

$$g_0 = c_- x_- + A_0 x_+ + B_0 x_- = c_- x_- + A_{-1} x_+ + B_{-1} x_- = c_- x_- + \frac{x_+}{\sqrt{\Delta}} \sum_{j=-\infty}^{-1} \tilde{g}_j x_-^{j+1}.$$

Next we study the second equation of system (3.6). Substituting the formulas for g_0 and g_{-1} we compute

$$c_- = \frac{\tilde{g}_0}{x_-(r_1^2 + r_2^2 - \lambda) + ar_2} - \left(\frac{x_+(r_1^2 + r_2^2 - \lambda) + ar_2}{x_-(r_1^2 + r_2^2 - \lambda) + ar_2} \right) \frac{1}{\sqrt{\Delta}} \sum_{j=-\infty}^{-1} \tilde{g}_j x_-^{j+1}.$$

The above formulas give the unique solution of the equation $(zz^* - \lambda)\varphi = \tilde{\varphi}$, and hence define the inverse operator $(zz^* - \lambda)^{-1}$. We need to verify that this operator is bounded. So we first define the following operator

$$Q\tilde{g} = \sum_{n=-\infty}^0 \left(\frac{\tilde{g}_0}{x_-(r_1^2 + r_2^2 - \lambda) + ar_2} - \left(\frac{x_+(r_1^2 + r_2^2 - \lambda) + ar_2}{x_-(r_1^2 + r_2^2 - \lambda) + ar_2} \right) \frac{1}{\sqrt{\Delta}} \sum_{j=-\infty}^{-1} \tilde{g}_j x_-^{j+1} \right) x_-^{n+1} G_n.$$

Notice that, since $|x_-| > 1$, Q is a bounded operator taking $\ell^2(\mathbb{Z}_{\leq 0})$ to itself. Here we have used the notation $G_0 := E_0$. Additionally we will need the following four operators, written in components:

$$(L_1 g)_n = \frac{1}{\sqrt{\Delta}} \sum_{j=-\infty}^{-1} x_-^{j-n} \chi \left(\frac{j}{n} \right) g_j : \ell^2(\mathbb{Z}_{<0}) \rightarrow \ell^2(\mathbb{Z}_{<0}),$$

$$(L_2 g)_n = \frac{1}{\sqrt{\Delta}} \sum_{j=-\infty}^{-1} x_+^{j-n} \chi \left(\frac{n+1}{j} \right) g_j : \ell^2(\mathbb{Z}_{<0}) \rightarrow \ell^2(\mathbb{Z}_{<0}),$$

$$(L_3 g)_n = \frac{1}{\sqrt{\Delta}} \sum_{j=-\infty}^{-1} x_+^{j-n} \chi \left(\frac{j}{n} \right) g_j : \ell^2(\mathbb{Z}_{<0}) \rightarrow \ell^2(\mathbb{Z}_{<0}),$$

$$(L_4 g)_n = \frac{1}{\sqrt{\Delta}} \sum_{j=-\infty}^{-1} x_-^{j-n} \chi \left(\frac{n+1}{j} \right) g_j : \ell^2(\mathbb{Z}_{<0}) \rightarrow \ell^2(\mathbb{Z}_{<0}).$$

Using these operators we write

$$g_n = (Q\tilde{g})_n + (L_1\tilde{g})_n + (L_2\tilde{g})_n \quad \text{for } n \leq -1$$

and

$$g_0 = (Q\tilde{g})_0 + x_+ (L_1\tilde{g})_{-1}.$$

Then we use Lemma 3.6 (Schur–Young inequality) to estimate the norms of L_1 and L_2 . We have

$$\begin{aligned} \|L_1\|^2 &\leq \frac{1}{\Delta} \left(\sup_{n \leq -1} |x_-|^{-n} \sum_{j=-\infty}^n |x_-|^j \right) \left(\sup_{j \leq -1} |x_-|^j \sum_{n=j}^{-1} |x_-|^{-n} \right) \\ &= \frac{1}{\Delta} \left(\frac{1}{1 - |x_-|^{-1}} \right) \left(\sup_{j \leq -1} \frac{1 - |x_-|^j}{1 - |x_-|^{-1}} \right) = \frac{1}{\Delta} \frac{1}{(1 - |x_-|^{-1})^2}. \end{aligned}$$

The computation of norm of L_2 is similar. Therefore $(zz^* - \lambda)^{-1}$ is bounded for $0 < \lambda < (r_2 - a)^2$.

The other case is when $(r_2 + a)^2 < \lambda < 1$. From the proof of Theorem 3.11 we have that $x_+ > 1$ and $x_- < 1$. Again using variation of parameters, we see that the particular solution of system (3.8) is given by:

$$B_n = \frac{-1}{\sqrt{\Delta}} \sum_{j=-\infty}^n \tilde{g}_j x_+^{j+1} \quad \text{and} \quad A_n = \frac{-1}{\sqrt{\Delta}} \sum_{j=n+1}^{-1} \tilde{g}_j x_-^{j+1}$$

with $A_{-1} = 0$, $n \leq -1$. Consequently the general solution is

$$g_n = c_+ x_+^{n+1} - \frac{1}{\sqrt{\Delta}} \left(\sum_{j=n+1}^{-1} \tilde{g}_j x_-^{j-n} + \sum_{j=-\infty}^n \tilde{g}_j x_+^{j-n} \right) \quad \text{for } n \leq -1.$$

Since we require $g_n \in \ell^2(\mathbb{Z}_{<0})$, we must have $c_- = 0$. For $n = -1$ we obtain

$$g_{-1} = c_+ - \frac{1}{\sqrt{\Delta}} \sum_{j=-\infty}^{-1} \tilde{g}_j x_+^{j+1},$$

and for $n = 0$, using (3.7), we have

$$g_0 = c_+ x_+ + A_0 x_+ + B_0 x_- = c_+ x_+ + A_{-1} x_+ + B_{-1} x_- = c_+ x_+ - \frac{x_-}{\sqrt{\Delta}} \sum_{j=-\infty}^{-1} \tilde{g}_j x_+^{j+1}.$$

Substituting the formulas for g_0 and g_{-1} into the second equation of system (3.6) we compute c_+

$$c_+ = \frac{\tilde{g}_0}{x_+(r_1^2 + r_2^2 - \lambda) + ar_2} + \left(\frac{x_-(r_1^2 + r_2^2 - \lambda) + ar_2}{x_+(r_1^2 + r_2^2 - \lambda) + ar_2} \right) \frac{1}{\sqrt{\Delta}} \sum_{j=-\infty}^{-1} \tilde{g}_j x_+^{j+1}.$$

The formulas above define the inverse operator $(zz^* - \lambda)^{-1}$. To verify that this operator is bounded we define the following operator

$$R\tilde{g} = \sum_{n=-\infty}^0 \left(\frac{\tilde{g}_0}{x_+(r_1^2 + r_2^2 - \lambda) + ar_2} - \left(\frac{x_-(r_1^2 + r_2^2 - \lambda) + ar_2}{x_+(r_1^2 + r_2^2 - \lambda) + ar_2} \right) \frac{1}{\sqrt{\Delta}} \sum_{j=-\infty}^{-1} \tilde{g}_j x_+^{j+1} \right) x_+^{n+1} G_n.$$

It is easy to see that R is a bounded operator taking $\ell^2(\mathbb{Z}_{\leq 0})$ to itself. Then we can write

$$g_n = (R\tilde{g})_n - (L_3\tilde{g})_n - (L_4\tilde{g})_n \quad \text{for } n \leq -1$$

and

$$g_0 = (R\tilde{g})_0 - x_- (L_3\tilde{g})_{-1}.$$

We use Lemma 3.6 to estimate the norms of L_3 and L_4 , in a manner similar to L_1 . We omit the repetitive details. This shows that $(zz^* - \lambda)^{-1}$ is bounded for $(r_2 + a)^2 < \lambda < 1$. ■

In the theorem below, we will see that the interval $[(r_2 - a)^2, (r_2 + a)^2]$ is the continuous part of the spectrum of zz^* , completing its full description.

Theorem 3.13. *The spectrum of zz^* is*

$$\sigma(zz^*) = \left\{ \frac{r_1^2(a^2 - r_2^2 - r_1^2)}{a^2 - r_1^2} \right\} \cup \{r_1^2\} \cup [(r_2 - a)^2, (r_2 + a)^2] \cup \{1\}.$$

Proof. Since zz^* is a positive operator with norm 1, its spectrum must be a closed subset of the interval $[0, 1]$. In Theorem 3.11 we computed the pure point spectrum of zz^* while Lemma 3.12 identified intervals belonging to the resolvent set of zz^* . So it remains to analyze the interval $[(r_2 - a)^2, (r_2 + a)^2]$. We will show that if $\lambda \in ((r_2 - a)^2, (r_2 + a)^2)$ then $\text{Ran}(zz^* - \lambda)$, the range of $(zz^* - \lambda)$, is not all of H .

In the notation of system (3.6) consider $(zz^* - \lambda)\varphi = \tilde{\varphi}$ with $\tilde{g}_n = 0$ for $n \leq -1$ but $\tilde{g}_0 = \tilde{e}_0 \neq 0$. This leads to the equation

$$ar_2g_{n+1} + (a^2 + r_2^2 - \lambda)g_n + ar_2g_{n-1} = 0 \quad \text{for } n \leq -1,$$

with the general solution $g_n = c_+x_+^{n+1} + c_-x_-^{n+1}$. Since for $(r_2 - a)^2 \leq \lambda \leq (r_2 + a)^2$, the two numbers x_+ and x_- are complex conjugates each with magnitude one; which follows from the arguments in the last part of Theorem 3.11; we must have $g_n = 0$ for $n \leq 0$ for φ to be in H . This however contradicts the second equation of (3.6) with $\tilde{g}_0 = \tilde{e}_0 \neq 0$. Consequently there is no $\varphi \in H$ satisfying $(zz^* - \lambda)\varphi = \tilde{\varphi}$ for such $\tilde{\varphi}$ and $\text{Ran}(zz^* - \lambda)$ is not H . ■

Let P_E , P_F , and P_G be the orthogonal projections onto the infinite-dimensional span of $\{E_n\}_{n \geq 0}$, $\{F_n\}_{n < 0}$ and $\{G_n\}_{n < 0}$ respectively.

Proposition 3.14. *The projections P_E , P_F , and P_G belong to the noncommutative pair of pants, i.e., they are all in $C^*(z)$.*

Proof. Since Lemma 3.12 implies that r_1^2 is an isolated eigenvalue of zz^* , there exists a continuous real valued function f so that $f(r_1^2) = 1$ and f is zero on the rest of the spectrum of zz^* . By functional calculus we have $f(zz^*) = P_F$ and so $P_F \in C^*(z)$. Similarly, since 1 is an isolated eigenvalue and we already know that $P_{E_0} \in C^*(z)$ we get $P_E \in C^*(z)$ as well. Since $P_G = I - P_E - P_F$, it then follows that $P_G \in C^*(z)$. ■

We can decompose the Hilbert space H into $H \cong H_E \oplus H_F \oplus H_G$, where $H_E \cong \ell^2(\mathbb{Z}_{\geq 0})$, $H_F \cong \ell^2(\mathbb{Z}_{< 0})$, and $H_G \cong \ell^2(\mathbb{Z}_{< 0})$ are the Hilbert spaces with basis elements E_n , F_n , and G_n respectively. Since $\ell^2(\mathbb{Z}_{\geq 0})$ is a subspace of $\ell^2(\mathbb{Z})$, which can be identified with $L^2(S^1)$ via the Fourier transform, we can view H_E as a subspace of $L^2(S^1)$. More precisely, if $B_n = \zeta^n$, $n \in \mathbb{Z}$ is the standard basis in $L^2(S^1)$, then H_E is identified with the subspace $\text{span}\{B_n\}_{n \geq 0}$ via

$$E_n \mapsto B_n.$$

Define $P_{\geq 0} : L^2(S^1) \rightarrow H_E$ to be the projection onto H_E . For a $\varphi \in C(S^1)$ we define $T_E(\varphi) : H_E \rightarrow H_E$ by $T_E(\varphi) = P_{\geq 0}M(\varphi)$ where $M(\varphi) : L^2(S^1) \rightarrow L^2(S^1)$ is the multiplication operator by φ . In particular for $\varphi(\zeta) = \zeta$ we have

$$T_E(\zeta)E_n = E_{n+1}, \tag{3.9}$$

the unilateral shift.

Similarly we identify H_F and H_G with the subspace $\text{span}\{B_n\}_{n<0}$ in $L^2(S^1)$, and let $P_{<0} : L^2(S^1) \rightarrow H_F$ and $P'_{<0} : L^2(S^1) \rightarrow H_G$ be the orthogonal projections onto H_F and H_G respectively. Then for a $\varphi \in C(S^1)$, we define $T_F(\varphi) : H_F \rightarrow H_F$ by $T_F(\varphi) = P_{<0}M(\varphi)$ and $T_G(\varphi) : H_G \rightarrow H_G$ by $T_G(\varphi) = P'_{<0}M(\varphi)$ respectively. We have

$$T_F(\zeta)F_n = F_{n+1}, \quad T_G(\zeta)G_n = G_{n+1} \quad \text{for } n < -1 \quad (3.10)$$

and

$$T_F(\zeta)F_{-1} = 0, \quad T_G(\zeta)G_{-1} = 0. \quad (3.11)$$

The operators $T_E(\varphi)$, $T_F(\varphi)$, and $T_G(\varphi)$ may be viewed as Toeplitz operators and they will be needed in proving the following result.

Theorem 3.15. *The quotient $C^*(z)/\mathcal{K}$ is isomorphic to $C(S^1) \oplus C(S^1) \oplus C(S^1)$.*

Proof. Using the above notation define

$$T : C(S^1) \oplus C(S^1) \oplus C(S^1) \rightarrow C^*(z)/\mathcal{K}$$

by

$$T(\varphi_1, \varphi_2, \varphi_3) = T_E(\varphi_1)P_E + T_F(\varphi_2)P_F + T_G(\varphi_3)P_G + \mathcal{K} \quad (3.12)$$

for continuous functions φ_1 , φ_2 , and φ_3 on the unit circle. To see that T is well defined we need to show that $T(\varphi_1, \varphi_2, \varphi_3)$ is in $C^*(z)$. We showed that $P_E \in C^*(z)$, and notice that $T_E(\zeta) \in C^*(z)$ because $T_E(\zeta) = zP_E$. But Toeplitz operators $T_E(\varphi)$ can be uniformly approximated by polynomials in $T_E(\zeta)$ and its adjoint, and so $T_E(\varphi_1)P_E \in C^*(z)$. Similar arguments work for $T_F(\varphi_2)P_F$ and $T_G(\varphi_3)P_G$.

We verify that T in (3.12) is a isomorphism between the two algebras. First notice that equation (3.12) implies that T is continuous and linear. Next we show that the kernel of T is trivial. Consider the equation $T(\varphi_1, \varphi_2, \varphi_3) = 0$ implying that $T_E(\varphi_1)P_E + T_F(\varphi_2)P_F + T_G(\varphi_3)P_G$ is compact. Since P_E , P_F and P_G are orthogonal, $T_E(\varphi_1)P_E$, $T_F(\varphi_2)P_F$ and $T_G(\varphi_3)P_G$ must be compact, and consequently $T_E(\varphi_1) : H_E \rightarrow H_E$, $T_F(\varphi_2) : H_F \rightarrow H_F$, and $T_G(\varphi_3) : H_G \rightarrow H_G$ are compact. By the proof of Theorem 2.3, it follows that $\varphi_1 = \varphi_2 = \varphi_3 = 0$ and thus the kernel of T is trivial.

Next we show that T is a homomorphism of algebras. Consider the difference:

$$\begin{aligned} & T(\varphi_1, \varphi_2, \varphi_3)T(\psi_1, \psi_2, \psi_3) - T(\varphi_1\psi_1, \varphi_2\psi_2, \varphi_3\psi_3) \\ &= (T_E(\varphi_1)T_E(\psi_1) - T_E(\varphi_1\psi_1))P_E \\ & \quad + (T_F(\varphi_2)T_F(\psi_2) - T_F(\varphi_2\psi_2))P_F + (T_G(\varphi_3)T_G(\psi_3) - T_G(\varphi_3\psi_3))P_G. \end{aligned} \quad (3.13)$$

Since T_E , T_F , and T_G are Toeplitz operators, the proof of Theorem 2.3 implies that all three differences on the right hand side of equation (3.13) are compact operators. Thus T is a homomorphism between the two algebras.

To show that the range of T is dense we consider the difference $T(\zeta, r_1\zeta, r_2\zeta + a) - z$. Using formulas (3.9), (3.10), and (3.11) we get $T_E(\zeta)E_n = E_{n+1}$, for $n \geq 0$, $T_F(r_1\zeta)F_n = r_1F_{n+1}$, for $n < -1$, and $T_G(r_2\zeta + a)G_n = r_2G_{n+1} + aG_n$, for $n < -1$. Observe that $T(\zeta, r_1\zeta, r_2\zeta + a) - z$ is not zero on F_{-1} and G_{-1} only and hence it is a compact operator. Thus we have constructed functions φ_1 , φ_2 , and φ_3 such that $T(\varphi_1, \varphi_2, \varphi_3) = z$ in $C^*(z)/\mathcal{K}$. Since the C^* -algebra $C^*(z)/\mathcal{K}$ is generated by (the class of) z , the range of T is dense and since the range of a C^* -morphism must be closed, T is an isomorphism of algebras. This completes the proof. \blacksquare

Note that from Theorem 3.15 we get a short exact sequence

$$0 \rightarrow \mathcal{K} \rightarrow C^*(z) \rightarrow C(S^1) \oplus C(S^1) \oplus C(S^1) \rightarrow 0.$$

We can compare this to the short exact sequence for the classical pair of pants

$$0 \rightarrow C_0(PP_{(a,r_1,r_2)}) \rightarrow C(PP_{(a,r_1,r_2)}) \rightarrow C(S^1) \oplus C(S^1) \oplus C(S^1) \rightarrow 0,$$

where $C_0(PP_{(a,r_1,r_2)})$ are the continuous functions on the pair of pants that vanish on the boundary.

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