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A Characterization of Invariant Connections

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Abstract. Given a principal fibre bundle with structure group S, and a fibre transitive Lie group G of automorphisms thereon, Wang's theorem identifies the invariant connections with certain linear maps $\psi\colon \mathfrak{g} \to \mathfrak{s}$. In the present paper, we prove an extension of this theorem which applies to the general situation where G acts non-transitively on the base manifold. We consider several special cases of the general theorem, including the result of Harnad, Shnider and Vinet which applies to the situation where G admits only one orbit type. Along the way, we give applications to loop quantum gravity.

Key words: invariant connections; principal fibre bundles; loop quantum gravity; symmetry reduction

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1 Introduction

The set of connections on a principal fibre bundle (P,π,M,S) is closed under pullback by automorphisms, and it is natural to search for connections that do not change under this operation. Especially, connections invariant under a Lie group (G,Φ) of automorphisms are of particular interest as they reflect the symmetry of the whole group and, for this reason, find their applications in the symmetry reduction of (quantum) gauge field theories [1, 4, 5]. The first classification theorem for such connections was given by Wang [8], cf. Case 5.7. This applies to the case where the induced action[1] φ acts transitively on the base manifold and states that each point in the bundle gives rise to a bijection between the set of Φ-invariant connections and certain linear maps $\psi\colon \mathfrak{g} \to \mathfrak{s}$. In [6] the authors generalize this to the situation where φ admits only one orbit type. More precisely, they discuss a variation[2] of the case where the bundle admits a submanifold P_0 with $\pi(P_0)$ intersecting each φ-orbit in a unique point, see Case 4.5 and Example 4.6. Here, the Φ-invariant connections are in bijection with such smooth maps $\psi\colon \mathfrak{g}\times P_0 \to \mathfrak{s}$ for which the restrictions $\psi|_{\mathfrak{g}\times T_{p_0}P_0}$ are linear for all $p_0\in P_0$, and that fulfil additional consistency conditions.

Now, in the general case we consider Φ-*coverings* of P. These are families $\{P_\alpha\}_{\alpha\in I}$ of immersed submanifolds[3] P_α of P such that each φ-orbit has non-empty intersection with $\bigcup_{\alpha\in I}\pi(P_\alpha)$ and for which

$$T_pP = T_pP_\alpha + \mathrm{d}_e\Phi_p(\mathfrak{g}) + Tv_pP$$

holds whenever $p\in P_\alpha$ for some $\alpha\in I$. Here, $Tv_pP\subseteq T_pP$ denotes the vertical tangent space at $p\in P$ and e the identity in G. Observe that the intersection properties of the sets $\pi(P_\alpha)$

[1] Each Lie group of automorphisms of a bundle induces a smooth action on the base manifold.

[2] Amongst others, they assume the φ-stabilizer of $\pi(p_0)$ to be the same for all $p_0\in P_0$.

[3] For the moment, assume that $P_\alpha\subseteq P$ is a subset which, at the same time, is a manifold such that the inclusion map $\iota_\alpha\colon P_\alpha\to P$ is an immersion. Here, we tacitly identify $T_{p_\alpha}P_\alpha$ with $\mathrm{im}[\mathrm{d}_{p_\alpha}\iota_\alpha]$. Note that we do not require P_α to be an embedded submanifold of P. For details, see Convention 3.1.

with the φ- orbits in the base manifold need not to be convenient in any sense. Indeed, here one might think of situations in which φ admits dense orbits, or of the almost-fibre transitive case, cf. Case 5.4.

Let $\Theta\colon (G\times S)\times P\to P$ be defined by $((g,s),p)\mapsto \Phi(g,p)\cdot s^{-1}$ for (G,Φ) a Lie group of automorphisms of (P,π,M,S). Then, the main result of the present paper can be stated as follows:

Theorem. *Each Φ-covering $\{P_\alpha\}_{\alpha\in I}$ of P gives rise to a bijection between the Φ-invariant connections on P and the families $\{\psi_\alpha\}_{\alpha\in I}$ of smooth maps $\psi_\alpha\colon \mathfrak{g}\times TP_\alpha\to\mathfrak{s}$ for which $\psi_\alpha|_{\mathfrak{g}\times T_{p_\alpha}P_\alpha}$ is linear for all $p_\alpha\in P_\alpha$, and that fulfil the following two (generalized Wang) conditions:*

- $\widetilde{g}(p_\beta)+\vec{w}_{p_\beta}-\widetilde{s}(p_\beta)=\mathrm{d}L_q\vec{w}_{p_\alpha} \quad\Longrightarrow\quad \psi_\beta(\vec{g},\vec{w}_{p_\beta})-\vec{s}=\rho(q)\circ\psi_\alpha\big(\vec{0}_{\mathfrak{g}},\vec{w}_{p_\alpha}\big),$
- $\psi_\beta\big(\mathrm{Ad}_q(\vec{g}),\vec{0}_{p_\beta}\big)=\rho(q)\circ\psi_\alpha\big(\vec{g},\vec{0}_{p_\alpha}\big)$

with $\rho(q):=\mathrm{Ad}_s$ and $\mathrm{Ad}_q(\vec{g}):=\mathrm{Ad}_g(\vec{g})$ for $q=(g,s)\in Q$.

Here, $\widetilde{g}$ and $\widetilde{s}$ denote the fundamental vector fields that correspond to the elements $\vec{g}\in\mathfrak{g}$ and $\vec{s}\in\mathfrak{s}$, respectively; and of course we have $\vec{0}_{p_\alpha},\vec{w}_{p_\alpha}\in T_{p_\alpha}P_\alpha$, $\vec{0}_{p_\beta},\vec{w}_{p_\beta}\in T_{p_\beta}P_\beta$ as well as $p_\beta=q\cdot p_\alpha$ for $p_\alpha\in P_\alpha, p_\beta\in P_\beta$.

Using this theorem, the calculation of invariant connections reduces to identifying a Φ-covering which makes the above conditions as easy as possible. Here, one basically has to find the balance between quantity and complexity of these conditions. Of course, the more submanifolds there are, the more conditions we have, so that usually it is convenient to use as few of them as possible. For instance, in the situation where φ is transitive, it suggests itself to choose a Φ-covering that consists of one single point; which, in turn, has to be chosen appropriately. Also if there is some $m\in M$ contained in the closure of each φ-orbit, one single submanifold is sufficient, see Case 5.4 and Example 5.5. The same example also shows that sometimes pointwise[4] evaluation of the above conditions proves non-existence of Φ-invariant connections.

In any case, one can use the inverse function theorem to construct a Φ-covering $\{P_\alpha\}_{\alpha\in I}$ of P such that the submanifolds P_α have minimal dimension in a certain sense, see Lemma 3.4 and Corollary 5.1. This reproduces the description of connections by means of local 1-forms on M provided that G acts trivially or, more generally, via gauge transformations on P, see Case 5.2.

Finally, since orbit structures can depend very sensitively on the action or the group, one cannot expect to have a general concept for finding the Φ-covering optimal for calculations. Indeed, sometimes these calculations become easier if one uses coverings that seem less optimal at a first sight (as, e.g., if they have no minimal dimension, cf. calculations in Appendix B.2).

The present paper is organized as follows: In Section 2, we fix the notations. In Section 3, we introduce the notion of a Φ-covering, the central object of this paper. In Section 4, we prove the main theorem and deduce a slightly more general version of the result from [6]. In Section 5, we show how to construct Φ-coverings to be used in special situations. In particular, we consider the (almost) fibre transitive case, trivial principal fibre bundles and Lie groups of gauge transformations. Along the way, we give applications to loop quantum gravity.

2 Preliminaries

We start with fixing the notations.

[4]Here, pointwise means to consider such elements $q\in G\times S$ that are contained in the Θ-stabilizer of some fixed $p_\alpha\in P_\alpha$ for $\alpha\in I$.

2.1 Notations

Manifolds are always assumed to be smooth. If M, N are manifolds and $f\colon M \to N$ is a smooth map, then $\mathrm{d}f\colon TM \to TN$ denotes the differential map between their tangent manifolds. The map f is said to be an immersion iff for each $x \in M$ the restriction $\mathrm{d}_x f := \mathrm{d}f|_{T_xM}\colon T_xM \to T_{f(x)}N$ is injective.

Let V be a finite dimensional vector space. A V-valued 1-form ω on the manifold N is a smooth map $\omega\colon TN \to V$ whose restriction $\omega_y := \omega|_{T_yN}$ is linear for all $y \in N$. The pullback of ω by f is the V-valued 1-form $f^*\omega\colon TM \to V$, $\vec{v}_x \to \omega_{f(x)}(\mathrm{d}_x f(\vec{v}_x))$.

Let G be a Lie group with Lie algebra $\mathfrak{g}$. For $g \in G$, we define the corresponding conjugation map by $\alpha_g\colon G \to G$, $h \mapsto ghg^{-1}$. Its differential $\mathrm{d}_e\alpha_g\colon \mathfrak{g} \to \mathfrak{g}$ at the unit element $e \in G$ is denoted by Ad_g in the following.

Let Ψ be a (left) action of the Lie group G on the manifold M. For $g \in G$ and $x \in M$, we define $\Psi_g\colon M \to M$, $\Psi_g\colon y \mapsto \Psi(g,y)$ and $\Psi_x\colon G \to M$, $h \mapsto \Psi(h,x)$, respectively. If it is clear which action is meant, we will often write L_g instead of Ψ_g as well as $g\cdot y$ or gy instead of $\Psi_g(y)$. For $\vec{g} \in \mathfrak{g}$ and $x \in M$, the map

$$\widetilde{g}(x) := \tfrac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}\Psi_x(\exp(t\vec{g}))$$

is called the *fundamental vector field of* $\vec{g}$. The Lie subgroup $G_x := \{g \in G \mid g \cdot x = x\}$ is called the *stabilizer* of $x \in M$ (w.r.t. Ψ), and its Lie algebra $\mathfrak{g}_x$ equals $\ker[\mathrm{d}_x\Psi]$, see e.g. [3]. The *orbit* of x under G is the set $Gx := \mathrm{im}[\Psi_x]$. Ψ is said to be *transitive* iff $Gx = M$ holds for one (and then each) $x \in M$. Analogous conventions we also use for right actions.

2.2 Invariant connections

Let $\pi\colon P \to M$ be a smooth map between manifolds P and M, and denote by $F_x := \pi^{-1}(x) \subseteq P$ the fibre over $x \in M$ in P. Moreover, let S be a a Lie group that acts via $R\colon P \times S \to P$ from the right on P. If there is an open covering $\{U_\alpha\}_{\alpha\in I}$ of M and a family $\{\phi_\alpha\}_{\alpha\in I}$ of diffeomorphisms $\phi_\alpha\colon \pi^{-1}(U_\alpha) \to U_\alpha \times S$ with

$$\phi_\alpha(p\cdot s) = \big(\pi(p), [\mathrm{pr}_2 \circ \phi_\alpha](p)\cdot s\big) \qquad \forall\, p \in \pi^{-1}(U_\alpha), \qquad \forall\, s \in S, \tag{2.1}$$

then (P, π, M, S) is called *principal fibre bundle* with total space P, projection map π, base manifold M and structure group S. Here, pr_2 denotes the projection onto the second factor. It follows from (2.1) that π is surjective, and that:

- $R_s(F_x) \subseteq F_x$ for all $x \in M$ and all $s \in S$,
- for each $x \in M$ the map $R_x\colon F_x \times S \to F_x$, $(p,s) \mapsto p\cdot s$ is transitive and free.

The subspace $Tv_pP := \ker[d_p\pi] \subseteq T_pP$ is called *vertical tangent space* at $p \in P$ and

$$\widetilde{s}(p) := \tfrac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}\, p\cdot\exp(t\vec{s}) \in Tv_pP \qquad \forall\, p \in P,$$

denotes the fundamental vector field of $\vec{s}$ w.r.t. the right action of S on P. The map $\mathfrak{s} \ni \vec{s} \to \widetilde{s}(p) \in Tv_pP$ is a vector space isomorphism for all $p \in P$.

Complementary to that, a *connection* ω is an $\mathfrak{s}$-valued 1-form on P with

- $R_s^*\omega = \mathrm{Ad}_{s^{-1}} \circ \omega \qquad \forall\, s \in S$,
- $\omega_p(\widetilde{s}(p)) = \vec{s} \qquad \forall\, \vec{s} \in \mathfrak{s}$.

The subspace $Th_pP := \ker[\omega_p] \subseteq T_pP$ is called the *horizontal tangent space* at p (w.r.t. ω). We have $\mathrm{d}R_s(Th_pP) = Th_{p\cdot s}P$ for all $s \in S$, and one can show that $T_pP = Tv_pP \oplus Th_pP$ holds for all $p \in P$.

A diffeomorphism $\kappa\colon P \to P$ is said to be an *automorphism* iff $\kappa(p\cdot s) = \kappa(p)\cdot s$ holds for all $p \in P$ and all $s \in S$. It is straightforward to see that an $\mathfrak{s}$-valued 1-form ω on P is a connection iff this is true for the pullback $\kappa^*\omega$. A *Lie group of automorphisms* (G, Φ) *of* P is a Lie group G together with a left action Φ of G on P such that the map Φ_g is an automorphism for each $g \in G$. This is equivalent to say that $\Phi(g, p\cdot s) = \Phi(g,p)\cdot s$ holds for all $p \in P$, $g \in G$ and all $s \in S$. In this situation, we will often write *gps* instead of $(g\cdot p)\cdot s = g\cdot(p\cdot s)$. Each such a left action Φ gives rise to two further actions:

- The induced action φ is defined by

$$\begin{aligned} \varphi\colon\quad & G\times M \to M,\\ & (g,m) \mapsto (\pi\circ\Phi)(g,p_m), \end{aligned} \tag{2.2}$$

  where $p_m \in \pi^{-1}(m)$ is arbitrary. Φ is called *fibre transitive* iff φ is transitive.

- We equip $Q = G\times S$ with the canonical Lie group structure and define [8]

$$\begin{aligned} \Theta\colon\quad & Q\times P \to P,\\ & ((g,s),p) \mapsto \Phi\left(g, p\cdot s^{-1}\right). \end{aligned} \tag{2.3}$$

A connection ω is said to be Φ-invariant iff $\Phi_g^*\omega = \omega$ holds for all $g \in G$. This is equivalent to require that for each $p \in P$ and $g \in G$ the differential d_pL_g induces an isomorphism between the horizontal tangent spaces Th_pP and $Th_{gp}P$.[5]

We conclude this subsection with the following straightforward facts, see also [8]:

- Consider the representation $\rho\colon Q \to \mathsf{Aut}(\mathfrak{s})$, $(g,s) \mapsto \mathrm{Ad}_s$. Then, it is straightforward to see that each Φ-invariant connection ω is of type ρ, i.e., ω is an $\mathfrak{s}$-valued 1-form on P with $L_q^*\omega = \rho(q)\circ\omega$ for all $q \in Q$.
- An $\mathfrak{s}$-valued 1-form ω on P with $\omega(\widetilde{s}(p)) = \vec{s}$ for all $\vec{s} \in \mathfrak{s}$ is a Φ-invariant connection iff it is of type ρ.
- Let Q_p denote the stabilizer of $p \in P$ w.r.t. Θ, and $G_{\pi(p)}$ the stabilizer of $\pi(p)$ w.r.t. φ. Then, $G_{\pi(p)} = \left\{h \in G \mid L_h\colon F_{\pi(p)} \to F_{\pi(p)}\right\}$, and we obtain a Lie group homomorphism

$$\phi_p\colon G_{\pi(p)} \to S \quad \text{by requiring that} \quad \Phi(h,p) = p\cdot\phi_p(h) \quad \text{for all} \quad h \in G_{\pi(p)}.$$

  If $\mathfrak{q}_p$ and $\mathfrak{g}_{\pi(p)}$ denote the Lie algebras of Q_p and $G_{\pi(p)}$, respectively, then

$$Q_p = \{(h,\phi_p(h)) \mid h \in G_{\pi(p)}\} \qquad \text{and} \qquad \mathfrak{q}_p = \left\{\left(\vec{h}, \mathrm{d}_e\phi_p\left(\vec{h}\right)\right) \,\middle|\, \vec{h} \in \mathfrak{g}_{\pi(p)}\right\}. \tag{2.4}$$

3 Φ-coverings

We start this section with some facts and conventions concerning submanifolds. Then, we provide the definition of a Φ-covering and discuss some its properties.

Convention 3.1. Let M be a manifold.

1. A pair (N, τ_N) consisting of a manifold N and an injective immersion $\tau\colon N \to M$ is called submanifold of M.

[5] In literature sometimes the latter condition is used to define Φ-invariance of connections.

2. If (N, τ_N) is a submanifold of M, we tacitly identify N and TN with their images $\tau_N(N) \subseteq M$ and $\mathrm{d}\tau_N(TN) \subseteq TM$, respectively. In particular, this means that:

   - If M' is a manifold and $\kappa\colon M \to M'$ a smooth map, then for $x \in N$ and $\vec{v} \in TN$ we write $\kappa(x)$ and $\mathrm{d}\kappa(\vec{v})$ instead of $\kappa(\tau_N(x))$ and $\mathrm{d}\kappa(\mathrm{d}\tau(\vec{v}))$, respectively.
   - If $\Psi\colon G \times M \to M$ is a left action of the Lie group G and (H, τ_H) a submanifold of G, the restriction of Ψ to $H \times N$ is defined by

   $$\Psi|_{H\times N}(h, x) := \Psi(\tau_H(h), \tau_N(x)) \qquad \forall\, (h, x) \in H \times N.$$

   - If $\omega\colon TM \to V$ is a V-valued 1-form on M, we let

   $$(\Psi^*\omega)|_{TG\times TN}(\vec{m}, \vec{v}) := (\Psi^*\omega)(\vec{m}, \mathrm{d}\tau(\vec{v})) \qquad \forall\, (\vec{m}, \vec{v}) \in TG \times TN.$$

   - We will not explicitly refer to the maps τ_N and τ_H in the following.

3. Open subsets $U \subseteq M$ are equipped with the canonical manifold structure making the inclusion map an embedding.

4. If L is a submanifold of N, and N is a submanifold of M, we consider L as a submanifold of M in the canonical way.

Definition 3.2. A submanifold $N \subseteq M$ is called Ψ-patch iff for each $x \in N$ we find an open neighbourhood $N' \subseteq N$ of x and a submanifold H of G through e, such that the restriction $\Psi|_{H\times N'}$ is a diffeomorphism to an open subset $U \subseteq M$.

Remark 3.3.

1. It follows from the inverse function theorem and[6]

$$\mathrm{d}_{(e,x)}\Psi(\mathfrak{g} \times T_xN) = \mathrm{d}_e\Psi_x(\mathfrak{g}) + \mathrm{d}_x\Psi_e(T_xN) = \mathrm{d}_e\Psi_x(\mathfrak{g}) + T_xN \qquad \forall\, x \in N$$

   that N is a Ψ-patch iff $T_xM = \mathrm{d}_e\Psi_x(\mathfrak{g}) + T_xN$ holds for all $x \in N$.[7]

2. Open subsets $U \subseteq M$ are always Ψ-patches. They are of maximal dimension, which, for instance, is necessary if there is a point in U whose stabilizer equals G, see Lemma 3.4.1.

3. We allow zero-dimensional patches, i.e., $N = \{x\}$ for some $x \in M$. Necessarily, then we have $\mathrm{d}_e\Psi_x(\mathfrak{g}) = T_xM$ as well as $\Psi|_{H\times N} = \Psi_x|_H$ for each submanifold H of G.

The second part of the following elementary lemma equals Lemma 2.1.1 in [3].

Lemma 3.4. *Let (G, Ψ) be a Lie group that acts on the manifold M, and let $x \in M$.*

1. *If N is a Ψ-patch with $x \in N$, then* $\dim[N] \geq \dim[M] - \dim[G] + \dim[G_x]$.

2. *Let V and W be algebraic complements of $\mathrm{d}_e\Psi_x(\mathfrak{g})$ in T_xM and of $\mathfrak{g}_x$ in $\mathfrak{g}$, respectively. Then there are submanifolds N of M through x and H of G through e such that $T_xN = V$, $T_eH = W$. In particular, N is a Ψ-patch and* $\dim[N] = \dim[M] - \dim[G] + \dim[G_x]$.

[6] The sum is not necessarily direct.

[7] In fact, let $V \subseteq \mathrm{d}_e\Psi_x(\mathfrak{g})$ be an algebraic complement of T_xN in T_xM and $V' \subseteq \mathfrak{g}$ a linear subspace with $\dim[V'] = \dim[V]$ and $\mathrm{d}_e\Psi_x(V') = V$. Then, we find a submanifold H of G through e with $T_eH = V'$, so that $\mathrm{d}_{(e,x)}\Psi\colon T_eH \times T_xN \to T_xM$ is bijective.

Proof. 1. By Remark 3.3.1 and since $\ker[\mathrm{d}_e\Psi_x] = \mathfrak{g}_x$, we have

$$\dim[M] \leq \dim[\mathrm{d}_e\Psi_x(\mathfrak{g})] + \dim[T_xN] = \dim[G] - \dim[G_x] + \dim[N]. \tag{3.1}$$

2. Of course, we find submanifolds N' of M through x and H' of G through e such that $T_xN' = V$ and $T_eH' = W$. So, if $\vec{g} \in \mathfrak{g}$ and $\vec{v}_x \in T_xN'$, then $0 = \mathrm{d}_{(e,x)}\Psi(\vec{g}, \vec{v}_x) = \mathrm{d}_e\Psi_x(\vec{g}) + \vec{v}_x$ implies $\mathrm{d}_e\Psi_x(\vec{g}) = 0$ and $\vec{v}_x = 0$. Hence, $\vec{g} \in \ker[\mathrm{d}_e\Psi_x] = \mathfrak{g}_x$, so that[8] $\mathrm{d}_{(e,x)}\Psi|_{T_eH'\times T_eN'}$ is injective. It is immediate from the definitions that this map is surjective, so that by the inverse function theorem we find open neighbourhoods $N \subseteq N'$ of x and $H \subseteq G$ of e such that $\Psi|_{H\times N}$ is a diffeomorphism to an open subset $U \subseteq M$. Then N is a Ψ-patch, and since in (3.1) equality holds, also the last claim is clear. ■

Definition 3.5. Let (G, Φ) be a Lie group of automorphisms of the principal fibre bundle P, and recall the actions φ and Θ defined by (2.2) and (2.3), respectively. A family of Θ-patches $\{P_\alpha\}_{\alpha\in I}$ is said to be a Φ-covering of P iff each φ-orbit intersects at least one of the sets $\pi(P_\alpha)$.

Remark 3.6.

1. If $O \subseteq P$ is a Θ-patch, Lemma 3.4.1 and (2.4) yield

$$\dim[O] \geq \dim[P] - \dim[Q] + \dim[Q_p] \overset{(2.4)}{=} \dim[M] - \dim[G] + \dim[G_{\pi(p)}].$$

2. It follows from Remark 3.3.1 and $\mathrm{d}_e\Theta_p(\mathfrak{q}) = \mathrm{d}_e\Phi_p(\mathfrak{g}) + Tv_pP$ that O is a Θ-patch iff

$$T_pP = T_pO + \mathrm{d}_e\Phi_p(\mathfrak{g}) + Tv_pP \qquad \forall\, p \in O. \tag{3.2}$$

   As a consequence,

   - each Φ-patch is a Θ-patch,
   - P is always a Φ-covering by itself. Moreover, if $P = M \times S$ is trivial, then $M \times \{e\}$ is a Φ-covering.

3. If N is a φ-patch and $s_0\colon N \to P$ a smooth section (i.e., $\pi \circ s_0 = \mathrm{id}_N$), then $s_0(N)$ is a Θ-patch by Lemma 3.7.2.

   Conversely, if $N \subseteq M$ is a submanifold such that $s_0(N)$ is a Θ-patch for s_0 as above, then N is a φ-patch. In fact, applying $\mathrm{d}\pi$ to (3.2), this is immediate from Remark 3.3.1 and the definition of φ.

Lemma 3.7. *Let (G, Φ) be a Lie group of automorphisms of the principal bundle (P, π, M, S).*

1. *If $O \subseteq P$ is a Θ-patch, then for each $p \in O$ and $q \in Q$ the differential $\mathrm{d}_{(q,p)}\Theta\colon T_qQ \times T_pO \to T_{q\cdot p}P$ is surjective.*
2. *If N is a φ-patch and $s_0\colon N \to P$ a smooth section, then $s_0(N)$ is a Θ-patch.*

Proof. 1. Since O is a Θ-patch, the claim is clear for $q = e$. If q is arbitrary, then for each $\vec{m}_q \in T_qQ$ we find some $\vec{q} \in \mathfrak{q}$ such that $\vec{m}_q = \mathrm{d}L_q\vec{q}$. Consequently, for $\vec{w}_p \in T_pP$ we have

$$\mathrm{d}_{(q,p)}\Theta\left(\vec{m}_q, \vec{w}_p\right) = \mathrm{d}_{(q,p)}\Theta(\mathrm{d}L_q\vec{q}, \vec{w}_p) = \mathrm{d}_pL_q\left(\mathrm{d}_{(e,p)}\Theta(\vec{q}, \vec{w}_p)\right).$$

So, since left translation w.r.t. Θ is a diffeomorphism, d_pL_q is surjective.

[8] Recall that $\mathrm{d}_{(e,x)}\Psi|_{T_eH'\times T_eN'}\colon \big(\vec{h}, \vec{v}_x\big) \mapsto \mathrm{d}_{(e,x)}\Psi\big(\mathrm{d}_e\tau_H(\vec{h}), \mathrm{d}_x\tau_N(\vec{v}_x)\big)$.

2. $O := s_0(N)$ is a submanifold of P because s_0 is an injective immersion. Thus, by Remark 3.6.2 it suffices to show that

$$\dim\left[T_{s_0(x)}O + \mathrm{d}_e\Phi_{s_0(x)}(\mathfrak{g}) + Tv_{s_0(x)}P\right] \geq \dim[T_{s_0(x)}P] \qquad \forall\, x \in N.$$

For this, let $x \in N$ and $V' \subseteq \mathfrak{g}$ be a linear subspace with $V' \oplus \mathfrak{g}_x$ and $T_xM = T_xN \oplus \mathrm{d}_e\varphi_x(V')$. Then, we have $T_{s_0(x)}O \oplus \mathrm{d}_e\Phi_{s_0(x)}(V') \oplus Tv_{s_0(x)}P$ because if $\mathrm{d}_x s_0(\vec{v}_x) + \mathrm{d}_e\Phi_{s_0(x)}(\vec{g}') + \vec{v}_v = 0$ for $\vec{v}_x \in T_xN$, $\vec{g}' \in V'$ and $\vec{v}_v \in Tv_{s_0(x)}P$,

$$0 = \mathrm{d}_{s_0(x)}\pi\big(\mathrm{d}_x s_0(\vec{v}_x) + \mathrm{d}_e\Phi_{s_0(x)}(\vec{g}') + \vec{v}_v\big) = \vec{v}_x \oplus \mathrm{d}_e\varphi_x(\vec{g}')$$

shows $\vec{v}_x = 0$ and $\mathrm{d}_e\phi_x(\vec{g}') = 0$, hence $\vec{g}' = 0$ by the choice of V', i.e., $\vec{v}_v = 0$ by assumption. In particular, $\mathrm{d}_e\phi_x(\vec{g}') = 0$ if $\mathrm{d}_e\Phi_{s_0(x)}(\vec{g}') = 0$, hence $\dim[\mathrm{d}_e\Phi_{s_0(x)}(V')] \geq \dim[\mathrm{d}_e\varphi_x(V')]$, from which we obtain

$$\begin{aligned}\dim&\left[T_{s_0(x)}O + \mathrm{d}_e\Phi_{s_0(x)}(\mathfrak{g}) + Tv_{s_0(x)}P\right] \geq \dim\left[T_{s_0(x)}O \oplus \mathrm{d}_e\Phi_{s_0(x)}(V') \oplus Tv_{s_0(x)}P\right]\\ &= \dim[T_xN] + \dim[\mathrm{d}_e\Phi_{s_0(x)}(V')] + \dim[S] \geq \dim[T_xN] + \dim[\mathrm{d}_e\varphi_x(V')] + \dim[S]\\ &= \dim[P]. \qquad \blacksquare\end{aligned}$$

4 Characterization of invariant connections

In this section, we will use Φ-coverings $\{P_\alpha\}_{\alpha\in I}$ of the bundle P in order to characterize the set of Φ-invariant connections by families $\{\psi_\alpha\}_{\alpha\in I}$ of smooth maps $\psi_\alpha\colon \mathfrak{g} \times TP_\alpha \to \mathfrak{s}$ whose restrictions $\psi_\alpha|_{\mathfrak{g}\times T_{p_\alpha}P_\alpha}$ are linear and that fulfil two additional compatibility conditions. Here, we will follow the lines of Wang's original approach, which basically means that we generalize the proofs from [8] to the non-transitive case. We will proceed in two steps, the first one being performed in Subsection 4.1. There, we show that a Φ-invariant connection gives rise to a consistent family $\{\psi_\alpha\}_{\alpha\in I}$ of smooth maps as described above. We also discuss the situation in [6] in order to make the two conditions more intuitive. Then, in Subsection 4.2, we will verify that such families $\{\psi_\alpha\}_{\alpha\in I}$ glue together to a Φ-invariant connection on P.

4.1 Reduction of invariant connections

In the following, let $\{P_\alpha\}_{\alpha\in I}$ be a fixed Φ-covering of P and ω a Φ-invariant connection on P. We define

$$\omega_\alpha := (\Theta^*\omega)|_{TQ\times TP_\alpha} \qquad \text{as well as} \qquad \psi_\alpha := \omega_\alpha|_{\mathfrak{g}\times TP_\alpha},$$

and for $q' \in Q$ we let $\alpha_{q'}\colon Q\times P \to Q \times P$, $(q,p) \mapsto \big(\alpha_{q'}(q), p\big)$. Finally, we define

$$\mathrm{Ad}_q(\vec{g}) := \mathrm{Ad}_g(\vec{g}) \qquad \forall\, q = (g,s) \in Q, \qquad \forall\, \vec{g} \in \mathfrak{g}.$$

Lemma 4.1. *Let $q \in Q$, $p_\alpha \in P_\alpha$, $p_\beta \in P_\beta$ with*[9] *$p_\beta = q\cdot p_\alpha$ and $\vec{w}_{p_\alpha} \in T_{p_\alpha}P_\alpha$. Then*

1) $\omega_\beta(\vec{\eta}) = \rho(q)\circ\omega_\alpha(\vec{0}_{\mathfrak{q}}, \vec{w}_{p_\alpha})$ *for all* $\vec{\eta} \in TQ\times TP_\beta$ *with* $\mathrm{d}\Theta(\vec{\eta}) = \mathrm{d}L_q\vec{w}_{p_\alpha}$,
2) $\big(\alpha_q^*\omega_\beta\big)\big(\vec{m}, \vec{0}_{p_\beta}\big) = \rho(q)\circ\omega_\alpha\big(\vec{m}, \vec{0}_{p_\alpha}\big)$ *for all* $\vec{m} \in TQ$.

Proof. 1. Let $\vec{\eta} \in T_{q'}Q \times T_pP_\beta$ for $q' \in Q$. Then, since[10] $L_q^*\omega = \rho(q)\circ\omega$ for each $q \in Q$ and $q'\cdot p = q\cdot p_\alpha = p_\beta$, we have

$$\begin{aligned}\omega_\beta(\vec{\eta}) &= \omega_{q'\cdot p}(\mathrm{d}_{(q',p)}\Theta(\vec{\eta})) = \omega_{p_\beta}(\mathrm{d}L_q\vec{w}_{p_\alpha}) = (L_q^*\omega)_{p_\alpha}(\vec{w}_{p_\alpha})\\ &= \rho(q)\circ\omega_{p_\alpha}(\vec{w}_{p_\alpha}) = \rho(q)\circ\omega_{p_\alpha}\big(\mathrm{d}_{(e,p_\alpha)}\Theta\big(\vec{0}_{\mathfrak{q}}, \vec{w}_{p_\alpha}\big)\big) = \rho(q)\circ\omega_\alpha\big(\vec{0}_{\mathfrak{q}}, \vec{w}_{p_\alpha}\big).\end{aligned}$$

[9] Recall that, by Convention 3.1, this actually means $\tau_{P_\beta}(p_\beta) = q\cdot\tau_{P_\alpha}(p_\alpha)$.
[10] See end of Subsection 2.2.

2. For $\vec{m}_{q'} \in T_{q'}Q$ let $\gamma\colon (-\epsilon,\epsilon) \to Q$ be smooth with $\dot\gamma(0) = \vec{m}_{q'}$. Then

$$\begin{aligned}\big(\alpha_q^*\omega_\beta\big)_{(q',p_\beta)}\big(\vec{m}_{q'},\vec{0}_{p_\beta}\big) &= \omega_{\beta(\alpha_q(q'),p_\beta)}\big(\mathrm{Ad}_q(\vec{m}_{q'}),\vec{0}_{p_\beta}\big) = \omega_{qq'q^{-1}q\cdot p_\alpha}\left(\tfrac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} q\gamma(t)q^{-1}q\cdot p_\alpha\right)\\ &= \big(L_q^*\omega\big)_{q'\cdot p_\alpha}\left(\tfrac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}\gamma(t)\cdot p_\alpha\right) = \rho(q)\circ\omega_{q'\cdot p_\alpha}\left(\mathrm{d}_{(q',p_\alpha)}\Theta\left(\vec{m}_{q'}\right)\right)\\ &= \rho(q)\circ\omega_{\alpha(q',p_\alpha)}\big(\vec{m}_{q'},\vec{0}_{p_\alpha}\big). \qquad \blacksquare\end{aligned}$$

Corollary 4.2. *Let $q\in Q$, $p_\alpha \in P_\alpha$, $p_\beta \in P_\beta$ with $p_\beta = q\cdot p_\alpha$ and $\vec{w}_{p_\alpha} \in T_{p_\alpha}P_\alpha$. Then, for $\vec{w}_{p_\beta} \in T_{p_\beta}P_\beta$, $\vec{g}\in\mathfrak{g}$ and $\vec{s}\in\mathfrak{s}$ we have*

i) $\widetilde{g}(p_\beta) + \vec{w}_{p_\beta} - \widetilde{s}(p_\beta) = \mathrm{d}L_q\vec{w}_{p_\alpha} \quad\Longrightarrow\quad \psi_\beta(\vec{g},\vec{w}_{p_\beta}) - \vec{s} = \rho(q)\circ\psi_\alpha\big(\vec{0}_{\mathfrak{g}},\vec{w}_{p_\alpha}\big)$,

ii) $\psi_\beta\big(\mathrm{Ad}_q(\vec{g}),\vec{0}_{p_\beta}\big) = \rho(q)\circ\psi_\alpha\big(\vec{g},\vec{0}_{p_\alpha}\big)$.

Proof. *i)* In general, for $\vec{w}_p\in T_pP$, $\vec{g}\in\mathfrak{g}$ and $\vec{s}\in\mathfrak{s}$ we have

$$\mathrm{d}_{(e,p)}\Theta((\vec{g},\vec{s}),\vec{w}_p) = \mathrm{d}_{(e,p)}\Phi(\vec{g},\vec{w}_p) - \widetilde{s}(p) = \widetilde{g}(p) + \vec{w}_p - \widetilde{s}(p) \tag{4.1}$$

and, since ω is a connection, for $((\vec{g},\vec{s}),\vec{w}_{p_\alpha})\in\mathfrak{q}\times TP_\alpha$ we obtain

$$\begin{aligned}\omega_\alpha((\vec{g},\vec{s}),\vec{w}_{p_\alpha}) &= \omega\big(\mathrm{d}_{(e,p_\alpha)}\Phi(\vec{g},\vec{w}_{p_\alpha}) - \widetilde{s}(p_\alpha)\big) = \omega\big(\mathrm{d}_{(e,p_\alpha)}\Phi(\vec{g},\vec{w}_{p_\alpha})\big) - \vec{s}\\ &= \omega_\alpha\left(\vec{g},\vec{w}_{p_\alpha}\right) - \vec{s} = \psi_\alpha\left(\vec{g},\vec{w}_{p_\alpha}\right) - \vec{s}.\end{aligned} \tag{4.2}$$

Now, assume that $\mathrm{d}_e\Phi_{p_\beta}(\vec{g}) + \vec{w}_{p_\beta} - \widetilde{s}(p) = \mathrm{d}L_q\vec{w}_{p_\alpha}$. Then $\mathrm{d}_{(e,p_\beta)}\Theta((\vec{g},\vec{s}),\vec{w}_{p_\beta}) = \mathrm{d}L_q\vec{w}_{p_\alpha}$ by (4.1) so that $\omega_\beta((\vec{g},\vec{s}),\vec{w}_{p_\beta}) = \rho(q)\circ\omega_\alpha\big(\vec{0}_{\mathfrak{g}},\vec{w}_{p_\alpha}\big)$ by Lemma 4.1.1. Consequently,

$$\psi_\beta\left(\vec{g},\vec{w}_{p_\beta}\right) - \vec{s} \overset{(4.2)}{=} \omega_\beta((\vec{g},\vec{s}),\vec{w}_{p_\beta}) = \rho(q)\circ\omega_\alpha\big(\vec{0}_{\mathfrak{q}},\vec{w}_{p_\alpha}\big) \overset{(4.2)}{=} \rho(q)\circ\psi_\alpha\big(\vec{0}_{\mathfrak{g}},\vec{w}_{p_\alpha}\big).$$

ii) Lemma 4.1.2 yields

$$\psi_\beta\big(\mathrm{Ad}_q(\vec{g}),\vec{0}_{p_\beta}\big) = (\alpha_q^*\omega_\beta)_{(e,p_\beta)}\big(\vec{g},\vec{0}_{p_\beta}\big) = \rho(q)\circ(\omega_\alpha)_{(e,p_\alpha)}\big(\vec{g},\vec{0}_{p_\alpha}\big) = \rho(q)\circ\psi_\alpha\big(\vec{g},\vec{0}_{p_\alpha}\big). \qquad \blacksquare$$

Definition 4.3 (reduced connection)**.** A family $\{\psi_\alpha\}_{\alpha\in I}$ of smooth maps $\psi_\alpha\colon \mathfrak{g}\times TP_\alpha \to \mathfrak{s}$ which are linear in the sense that $\psi_\alpha|_{\mathfrak{g}\times T_{p_\alpha}P_\alpha}$ is linear for all $p_\alpha\in P_\alpha$ is called reduced connection w.r.t. $\{P_\alpha\}_{\alpha\in I}$ iff it fulfils the conditions $i)$ and $ii)$ from Corollary 4.2.

Remark 4.4.

1) In particular, Corollary 4.2.i) encodes the following condition

   a) For all $\beta\in I$, $(\vec{g},\vec{s})\in\mathfrak{q}$ and $\vec{w}_{p_\beta}\in T_{p_\beta}P_\beta$ we have

   $$\widetilde{g}(p_\beta) + \vec{w}_{p_\beta} - \widetilde{s}(p_\beta) = 0 \quad\Longrightarrow\quad \psi_\beta(\vec{g},\vec{w}_{p_\beta}) - \vec{s} = 0.$$

2) Assume that a) is true and let $q\in Q$, $p_\alpha\in P_\alpha$, $p_\beta\in P_\beta$ with $p_\beta = q\cdot p_\alpha$. Moreover, assume that we find elements $\vec{w}_{p_\alpha}\in T_{p_\alpha}P_\alpha$ and $((\vec{g},\vec{s}),\vec{w}_{p_\beta})\in\mathfrak{q}\times T_{p_\beta}P_\beta$ such that

   $$\mathrm{d}_{(e,p_\beta)}\Theta((\vec{g},\vec{s}),\vec{w}_{p_\beta}) = \mathrm{d}L_q\vec{w}_{p_\alpha} \qquad\text{and}\qquad \psi_\beta(\vec{g},\vec{w}_{p_\beta}) - \vec{s} = \rho(q)\circ\psi_\alpha(\vec{0}_{\mathfrak{g}},\vec{w}_{p_\alpha})$$

   holds. Then $\psi_\beta\big(\vec{g}\,',\vec{w}\,'_{p_\beta}\big) - \vec{s}\,' = \rho(q)\circ\psi_\alpha\big(\vec{0}_{\mathfrak{g}},\vec{w}_{p_\alpha}\big)$ holds for each element[11] $\big((\vec{g}\,',\vec{s}\,'),\vec{w}\,'_{p_\beta}\big)\in\mathfrak{q}\times T_{p_\beta}P_\beta$ with[12] $\mathrm{d}_{(e,p_\beta)}\Theta\big((\vec{g}\,',\vec{s}\,'),\vec{w}\,'_{p_\beta}\big) = \mathrm{d}L_q\vec{w}_{p_\alpha}$. In fact, we have

   $$\mathrm{d}_{(e,p_\beta)}\Theta\big((\vec{g}-\vec{g}\,',\vec{s}-\vec{s}\,'),\vec{w}_{p_\beta}-\vec{w}\,'_{p_\beta}\big) = 0,$$

   so that by (4.1) condition a) gives

   $$\begin{aligned}0 &\overset{a)}{=} \psi_\beta(\vec{g}-\vec{g}\,',\vec{w}_{p_\beta}-\vec{w}\,'_{p_\beta}) - (\vec{s}-\vec{s}\,')) = \big[\psi_\beta(\vec{g},\vec{w}_{p_\beta}) - \vec{s}\big] - \big[\psi_\beta(\vec{g}\,',\vec{w}\,'_{p_\beta}) - \vec{s}\,'\big]\\ &= \rho(q)\circ\psi_\alpha\big(\vec{0}_{\mathfrak{g}},\vec{w}_{p_\alpha}\big) - \big[\psi_\beta(\vec{g}\,',\vec{w}\,'_{p_\beta}) - \vec{s}\,'\big].\end{aligned}$$

[11] Observe that due to surjectivity of $\mathrm{d}_{(e,p_\beta)}\Phi$ such elements always exist.

[12] Recall equation (4.1).

3) Assume that $\mathrm{d}L_q\vec{w}_{p_\alpha} \in T_{p_\beta}P_\beta$ holds for all $q \in Q$, $p_\alpha \in P_\alpha$, $p_\beta \in P_\beta$ with $p_\beta = q \cdot p_\alpha$ and all $\vec{w}_{p_\alpha} \in T_{p_\alpha}P_\alpha$. Then $\mathrm{d}_{(e,p_\beta)}\Theta\left(\mathrm{d}L_q\vec{w}_{p_\alpha}\right) = \mathrm{d}L_q\vec{w}_{p_\alpha}$ so that it follows from 2) that in this case we can substitute *i*) by *a*) and condition

   b) Let $q \in Q$, $p_\alpha \in P_\alpha$, $p_\beta \in P_\beta$ with $p_\beta = q \cdot p_\alpha$. Then

$$\psi_\beta\big(\vec{0}_{\mathfrak{g}}, \mathrm{d}L_q\vec{w}_{p_\alpha}\big) = \rho(q) \circ \psi_\alpha\big(\vec{0}_{\mathfrak{g}}, \vec{w}_{p_\alpha}\big) \qquad \forall\, \vec{w}_{p_\alpha} \in T_{p_\alpha}P_\alpha.$$

   Now, *b*) looks similar to *ii*) and makes it plausible that the conditions *i*) and *ii*) from Corollary 4.2 together encode the ρ-invariance of the corresponding connection ω. However, usually there is no reason for $\mathrm{d}L_q\vec{w}_{p_\alpha}$ to be an element of $T_{p_\beta}P_\beta$. Even for $p_\alpha = p_\beta$ and $q \in Q_{p_\alpha}$ this is usually not true. Thus, typically there is no way to split up *i*) into parts whose meaning is more intuitive.

Remark 4.4 immediately proves

Case 4.5 (gauge fixing)**.** *Let P_0 be a Θ-patch of the bundle P such that $\pi(P_0)$ intersects each φ-orbit in a unique point, and that $\mathrm{d}L_q(T_pP_0) \subseteq T_pP_0$ holds for all $p \in P_0$ and all $q \in Q_p$. Then, a corresponding reduced connection consists of one single smooth map $\psi\colon \mathfrak{g} \times TP_0 \rightarrow \mathfrak{s}$, and we have $p = q \cdot p'$ for $q \in Q$, $p, p' \in P_0$ iff $p = p'$ and $q \in Q_p$ holds. Thus, by Remark 4.4 the two conditions from Corollary 4.2 are equivalent to:*

Let $p \in P_0$, $q = (h, \phi_p(h)) \in Q_p$, $\vec{w}_p \in T_pP_0$ and $\vec{g} \in \mathfrak{g}$, $\vec{s} \in \mathfrak{s}$. Then

i') $\widetilde{g}(p) + \vec{w}_p - \widetilde{s}(p) = 0 \quad\Longrightarrow\quad \psi(\vec{g}, \vec{w}_p) - \vec{s} = 0,$

ii') $\psi\big(\vec{0}_{\mathfrak{g}}, \mathrm{d}L_q\vec{w}_p\big) = \rho(q) \circ \psi\big(\vec{0}_{\mathfrak{g}}, \vec{w}_p\big),$

iii') $\psi\big(\mathrm{Ad}_h(\vec{g}), \vec{0}_p\big) = \mathrm{Ad}_{\phi_p(h)} \circ \psi\big(\vec{g}, \vec{0}_p\big).$

The next example is a slight generalization of Theorem 2 in [6]. There, the authors assume that φ admits only one orbit type so that $\dim[G_x] = l$ holds for all $x \in M$. Then, they restrict to the situation where one finds a triple (U_0, τ_0, s_0) consisting of an open subset $U_0 \subseteq \mathbb{R}^k$ for $k = \dim[M] - [\dim[G] - l]$, an embedding $\tau_0\colon U_0 \rightarrow M$, and a smooth map $s_0\colon U_0 \rightarrow P$ with $\pi \circ s_0 = \tau_0$ and the addition property that Q_p is the same for all $p \in \mathrm{im}[s_0]$. More precisely, they assume that G_x and the structure group of the bundle are compact. Then they show the non-trivial fact that s_0 can be modified in such a way that in addition Q_p is the same for all $p \in \mathrm{im}[s_0]$.

Observe that the authors forgot to require that $\mathrm{im}[\mathrm{d}_x\tau_0] + \mathrm{im}\left[\mathrm{d}_e\varphi_{\tau_0(x)}\right] = T_{\tau_0(x)}M$ holds for all $x \in U_0$, i.e., that $\tau_0(U_0)$ is a φ-patch (so that $s_0(U_0)$ is a Θ-patch). Indeed, Example 4.10.2 shows that this additional condition is crucial. The next example is a slight modification of the result [6] in the sense that we do not assume G_x and the structure group to be compact but make the ad hoc requirement that Q_p is the same for all $p \in P_0$.

Example 4.6 (Harnad, Shnider, Vinet)**.** Let P_0 be a Θ-patch of the bundle P such that $\pi(P_0)$ intersects each φ-orbit in a unique point. Moreover, assume that the Θ-stabilizer $L := Q_p$ is the same for all $p \in P_0$. Then, it is clear from (2.4) that $H := G_{\pi(p)}$ and $\phi := \phi_p\colon H \rightarrow S$ are independent of the choice of $p \in P_0$. Finally, we require that

$$\dim[P_0] = \dim[M] - [\dim[G] - \dim[H]] \equiv \dim[P] - [\dim[Q] - \dim[H]] \tag{4.3}$$

holds. Now, let $p \in P_0$ and $q = (h, \phi(h)) \in Q_p$. Then, for $\vec{w}_p \in T_pP_0$ we have

$$\begin{aligned}\mathrm{d}L_q\vec{w}_p &= \tfrac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}\Phi(h, \gamma(t)) \cdot \phi_p^{-1}(h) = \tfrac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}[\gamma(t) \cdot \phi_{\gamma(t)}(h)] \cdot \phi_p^{-1}(h)\\ &= \tfrac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}[\gamma(t) \cdot \phi_p(h)] \cdot \phi_p^{-1}(h) = \vec{w}_p\end{aligned}$$

for $\gamma\colon(-\epsilon,\epsilon)\rightarrow P_0$ some smooth curve with $\dot{\gamma}(0)=\vec{w}_p$. Consequently, $\mathrm{d}L_q(T_pP_0)\subseteq T_pP_0$ so that we are in the situation of Case 4.5. Here, $ii')$ now reads $\psi\big(\vec{0}_{\mathfrak{g}},\vec{w}_p\big)=\mathrm{Ad}_{\phi(h)}\circ\psi\big(\vec{0}_{\mathfrak{g}},\vec{w}_p\big)$ for all $h\in H$ and $iii')$ does not change. For $i')$, observe that the Lie algebra $\mathfrak{l}$ of L is contained in the kernel of $\mathrm{d}_{(e,p_0)}\Theta$; denoting the differential of the restriction of Θ to $Q\times P_0$ for the moment. Then, $\mathrm{d}_{(e,p_0)}\Theta$ is surjective by Lemma 3.7.1 since P_0 is a Θ-patch, so that

$$\dim\left[\ker\left[\mathrm{d}_{(e,p_0)}\Theta\right]\right]=\dim[Q]+\dim[P_0]-\dim[P]\stackrel{(4.3)}{=}\dim[H],$$

hence $\ker[\mathrm{d}_{(e,p)}\Theta]=\mathfrak{l}$ holds for all $p\in P_0$. Altogether it follows that a reduced connection w.r.t. P_0 is a smooth, linear[13] map $\psi\colon\mathfrak{g}\times TP_0\rightarrow\mathfrak{s}$ which fulfils the following three conditions:

$$\begin{array}{llll} i'') & \psi\big(\vec{h},\vec{0}_p\big)\stackrel{(4.1)}{=}\mathrm{d}_e\phi\big(\vec{h}\big) & \forall\,\vec{h}\in\mathfrak{h}, & \forall\,p\in P_0,\\ ii'') & \psi\big(\vec{0}_{\mathfrak{g}},\vec{w}\big)=\mathrm{Ad}_{\phi(h)}\circ\psi\big(\vec{0}_{\mathfrak{g}},\vec{w}\big) & \forall\,h\in H, & \forall\,\vec{w}\in TP_0,\\ iii'') & \psi\big(\mathrm{Ad}_h(\vec{g}),\vec{0}_p\big)=\mathrm{Ad}_{\phi(h)}\circ\psi\big(\vec{g},\vec{0}_p\big) & \forall\,h\in H, & \forall\,\vec{g}\in\mathfrak{g},\quad \forall\,p\in P_0. \end{array}$$

Then, $\mu:=\psi|_{TP_0}$ and $A_{p_0}(\vec{g}):=\psi\big(\vec{g},\vec{0}_{p_0}\big)$ are the maps that are used for the characterization in Theorem 2 in [6].

4.2 Reconstruction of invariant connections

Let $\{P_\alpha\}_{\alpha\in I}$ be some fixed Φ-covering of P. We are going to show that each respective reduced connection $\{\psi_\alpha\}_{\alpha\in I}$ gives rise to a unique Φ-invariant connection on P. To this end, for each $\alpha\in I$ we define the maps $\lambda_\alpha\colon\mathfrak{q}\times TP_\alpha\rightarrow\mathfrak{s},((\vec{g},\vec{s}),\vec{w})\mapsto\psi_\alpha(\vec{g},\vec{w})-\vec{s}$ and

$$\begin{aligned}\omega_\alpha\colon\quad & TQ\times TP_\alpha\rightarrow\mathfrak{s},\\ & \big(\vec{m}_q,\vec{w}_{p_\alpha}\big)\mapsto\rho(q)\circ\lambda_\alpha\left(\mathrm{d}L_{q^{-1}}\vec{m}_q,\vec{w}_{p_\alpha}\right)\end{aligned}$$

for $\vec{m}_q\in T_qQ$ and $\vec{w}_{p_\alpha}\in T_{p_\alpha}P_\alpha$.

Lemma 4.7. *Let $q\in Q$, $p_\alpha\in P_\alpha$, $p_\beta\in P_\beta$ with $p_\beta=q\cdot p_\alpha$ and $\vec{w}_{p_\alpha}\in T_{p_\alpha}P_\alpha$. Then*

1) $\lambda_\beta(\vec{\eta})=\rho(q)\circ\lambda_\alpha\big(\vec{0}_{\mathfrak{q}},\vec{w}_{p_\alpha}\big)$ *for all* $\vec{\eta}\in\mathfrak{q}\times T_{p_\beta}P$ *with* $\mathrm{d}\Theta_{(e,p_\beta)}(\vec{\eta})=\mathrm{d}L_q\vec{w}_{p_\alpha}$,
2) $\lambda_\beta\big(\mathrm{Ad}_q(\vec{q}),\vec{0}_{p_\beta}\big)=\rho(q)\circ\lambda_\alpha\big(\vec{q},\vec{0}_{p_\alpha}\big)$ *for all* $\vec{q}\in\mathfrak{q}$.

For each $\alpha\in I$ we have

3) $\ker\left[\lambda_\alpha|_{\mathfrak{q}\times T_{p_\alpha}P_\alpha}\right]\subseteq\ker\left[\mathrm{d}_{(e,p_\alpha)}\Theta\right]$ *for all* $p_\alpha\in P_\alpha$,
4) *the map ω_α is the unique $\mathfrak{s}$-valued 1-form on $Q\times P_\alpha$ which extends λ_α and for which we have $L_q^*\omega_\alpha=\rho(q)\circ\omega_\alpha$ for all $q\in Q$.*

Proof. 1. Write $\vec{\eta}=((\vec{g},\vec{s}),\vec{w}_{p_\beta})$ for $\vec{g}\in\mathfrak{g}$, $\vec{s}\in\mathfrak{s}$ and $\vec{w}_{p_\beta}\in T_{p_\beta}P_\beta$. Then

$$\widetilde{g}(p_\beta)+\vec{w}_{p_\beta}-\widetilde{s}(p_\beta)\stackrel{(4.1)}{=}\mathrm{d}\Theta_{(e,p_\beta)}(\vec{\eta})=\mathrm{d}L_q\vec{w}_{p_\alpha}$$

so that from condition $i)$ in Corollary 4.2 we obtain

$$\lambda_\beta(\vec{\eta})=\psi_\beta(\vec{g},\vec{w}_{p_\beta})-\vec{s}=\rho(q)\circ\psi_\alpha\big(\vec{0}_{\mathfrak{g}},\vec{w}_{p_\alpha}\big)=\rho(q)\circ\lambda_\alpha\big(\vec{0}_{\mathfrak{q}},\vec{w}_{p_\alpha}\big).$$

2. Let $\vec{q}=(\vec{g},\vec{s})$ for $\vec{g}\in\mathfrak{g}$ and $\vec{s}\in\mathfrak{s}$. Then, by Corollary 4.2.$ii)$ we have

$$\lambda_\beta\big(\mathrm{Ad}_q(\vec{q}),\vec{0}_{p_\beta}\big)=\psi_\beta\big(\mathrm{Ad}_q(\vec{g}),\vec{0}_{p_\beta}\big)-\mathrm{Ad}_q(\vec{s})=\rho(q)\circ\left[\psi_\alpha\big(\vec{g},\vec{0}_{p_\alpha}\big)-\vec{s}\right]=\rho(q)\circ\lambda_\alpha\big(\vec{q},\vec{0}_{p_\alpha}\big).$$

[13] In the sense that $\psi|_{\mathfrak{g}\times T_pP_0}$ is linear for all $p\in P_0$.

3. This follows from the first part for $\alpha=\beta$, $q=e$ and $\vec{w}_{p_\alpha}=\vec{0}_{p_\alpha}$.
4. By definition we have $\omega_\alpha|_{\mathfrak{q}\times TP_\alpha}=\lambda_\alpha$, and for the pullback property we calculate

$$\begin{aligned}\left(L^*_{q'}\omega_\alpha\right)_{(q,p_\alpha)}\left(\vec{m}_q,\vec{w}_{p_\alpha}\right)&=\omega_{\alpha(q'q,p_\alpha)}\left(\mathrm{d}L_{q'}\vec{m}_q,\vec{w}_{p_\alpha}\right)=\rho\left(q'q\right)\circ\lambda_\alpha\left(\mathrm{d}L_{q^{-1}q'^{-1}}\mathrm{d}L_{q'}\vec{m}_q,\vec{w}_{p_\alpha}\right)\\&=\rho\left(q'\right)\circ\rho(q)\circ\lambda_\alpha\left(\mathrm{d}L_{q^{-1}}\vec{m}_q,\vec{w}_{p_\alpha}\right)=\rho\left(q'\right)\circ\omega_{\alpha(q,p_\alpha)}(\vec{m}_q,\vec{w}_{p_\alpha}),\end{aligned}$$

where $q,q'\in Q$ and $\vec{m}_q\in T_qQ$. For uniqueness, let ω be another $\mathfrak{s}$-valued 1-form on $Q\times P_\alpha$ whose restriction to $\mathfrak{q}\times TP_\alpha$ is λ_α and that fulfils $L^*_q\omega=\rho(q)\circ\omega$ for all $q\in Q$. Then

$$\begin{aligned}\omega_{(q,p_\alpha)}\left(\vec{m}_q,\vec{w}_{p_\alpha}\right)&=\omega_{(q,p_\alpha)}\left(\mathrm{d}L_q\circ\mathrm{d}L_{q^{-1}}\vec{m}_q,\vec{w}_{p_\alpha}\right)=(L^*_q\omega)_{(e,p_\alpha)}\left(\mathrm{d}L_{q^{-1}}\vec{m}_q,\vec{w}_{p_\alpha}\right)\\&=\rho(q)\circ\omega_{(e,p_\alpha)}(\mathrm{d}L_{q^{-1}}\vec{m}_q,\vec{w}_{p_\alpha})=\rho(q)\circ\lambda_\alpha\left(\mathrm{d}L_{q^{-1}}\vec{m}_q,\vec{w}_{p_\alpha}\right)\\&=\omega_\alpha(\mathrm{d}L_{q^{-1}}\vec{m}_q,\vec{w}_{p_\alpha}).\end{aligned}$$

Finally, smoothness of ω_α is an easy consequence of smoothness of the maps ρ, λ_α and $\mu\colon TQ\to\mathfrak{q}$, $\vec{m}_q\mapsto\mathrm{d}L_{q^{-1}}\vec{m}_q$ with $\vec{m}_q\in T_qQ$. For this, observe that $\mu=\mathrm{d}\tau\circ\kappa$ for $\tau\colon Q\times Q\to Q$, $(q,q')\mapsto q^{-1}q'$ and $\kappa\colon TQ\to TQ\times TQ$, $\vec{m}_q\mapsto\left(\vec{0}_q,\vec{m}_q\right)$ for $\vec{m}_q\in T_qQ$. ■

So far, we have shown that each reduced connection $\{\psi_\alpha\}_{\alpha\in I}$ gives rise to uniquely determined maps $\{\lambda_\alpha\}_{\alpha\in I}$ and $\{\omega_\alpha\}_{\alpha\in I}$. In the final step, we will construct a unique Φ-invariant connection ω from the data $\{(P_\alpha,\lambda_\alpha)\}_{\alpha\in I}$. Here, uniqueness and smoothness of ω will follow from uniqueness and smoothness of the maps ω_α.

Proposition 4.8. *There is one and only one $\mathfrak{s}$-valued 1-form ω on P with $\omega_\alpha=(\Theta^*\omega)|_{TQ\times TP_\alpha}$ for all $\alpha\in I$. This 1-form is a Φ-invariant connection on P.*

Proof. For uniqueness, we have to show that the values of such an ω are uniquely determined by the maps ω_α. To this end, let $p\in P$, $\alpha\in I$ and $p_\alpha\in P_\alpha$ be such that $p=q\cdot p_\alpha$ holds for some $q\in Q$. By Lemma 3.7.1 for $\vec{w}_p\in T_pP$ we find some $\vec{\eta}\in T_qQ\times T_{p_\alpha}P_\alpha$ with $\vec{w}_p=\mathrm{d}_{(q,p_\alpha)}\Theta(\vec{\eta})$, so that uniqueness follows from

$$\omega_p(\vec{w}_p)=\omega_{q\cdot p_\alpha}\left(\mathrm{d}_{(q,p_\alpha)}\Theta(\vec{\eta})\right)=(\Theta^*\omega)_{(q,p_\alpha)}(\vec{\eta})=\omega_\alpha(\vec{\eta}).$$

For existence, let $\alpha\in I$ and $p_\alpha\in P_\alpha$. Due to surjectivity of $\mathrm{d}_{(e,p_\alpha)}\Theta$ and Lemma 4.7.3, there is a (unique) map $\widehat{\lambda}_{p_\alpha}\colon T_{p_\alpha}P\to\mathfrak{s}$ with

$$\widehat{\lambda}_{p_\alpha}\circ\mathrm{d}_{(e,p_\alpha)}\Theta=\lambda_\alpha\big|_{\mathfrak{q}\times T_{p_\alpha}P_\alpha}. \tag{4.4}$$

Let $\widehat{\lambda}_\alpha\colon\bigsqcup_{p_\alpha\in P_\alpha}T_{p_\alpha}P\to\mathfrak{s}$ denote the (unique) map whose restriction to $T_{p_\alpha}P$ is $\widehat{\lambda}_{p_\alpha}$ for each $p_\alpha\in P_\alpha$. Then $\lambda_\alpha=\widehat{\lambda}_\alpha\circ\mathrm{d}\Theta|_{\mathfrak{q}\times TP_\alpha}$ and we construct the connection ω as follows. For $p\in P$ we choose some $\alpha\in I$ and $(q,p_\alpha)\in Q\times P_\alpha$ such that $q\cdot p_\alpha=p$ and define

$$\omega_p\big(\vec{w}_p\big):=\rho(q)\circ\widehat{\lambda}_\alpha\left(\mathrm{d}L_{q^{-1}}\big(\vec{w}_p\big)\right)\qquad\forall\,\vec{w}_p\in T_pP. \tag{4.5}$$

We have to show that this depends neither on $\alpha\in I$ nor on the choice of $(q,p_\alpha)\in Q\times P_\alpha$. For this, let $p_\alpha\in P_\alpha$, $p_\beta\in P_\beta$ and $q\in Q$ with $p_\beta=q\cdot p_\alpha$. Then for $\vec{w}\in T_{p_\alpha}P$ we have $\vec{w}=\mathrm{d}\Theta(\vec{q},\vec{w}_{p_\alpha})$ for some $(\vec{q},\vec{w}_{p_\alpha})\in\mathfrak{q}\times T_{p_\alpha}P_\alpha$, and since $\mathrm{d}L_q\vec{w}_{p_\alpha}\in T_{p_\beta}P$, there is $\vec{\eta}\in\mathfrak{q}\times T_{p_\beta}P_\beta$ such that $\mathrm{d}_{(e,p_\beta)}\Theta(\vec{\eta})=\mathrm{d}L_q\vec{w}_{p_\alpha}$ holds. It follows from the conditions 1 and 2 in Lemma 4.7 that

$$\begin{aligned}\widehat{\lambda}_\beta(\mathrm{d}L_q\vec{w})&=\widehat{\lambda}_\beta((\mathrm{d}L_q\circ\mathrm{d}\Theta)(\vec{q},\vec{w}_{p_\alpha}))=\widehat{\lambda}_\beta\big((\mathrm{d}L_q\circ\mathrm{d}\Theta)\big(\vec{q},\vec{0}_{p_\alpha}\big)\big)+\widehat{\lambda}_\beta\big(\mathrm{d}L_q\vec{w}_{p_\alpha}\big)\\&\overset{(4.7)}{=}\widehat{\lambda}_\beta\circ\mathrm{d}\Theta\big(\mathrm{Ad}_q(\vec{q}),\vec{0}_{p_\beta}\big)+\widehat{\lambda}_\beta\circ\mathrm{d}\Theta(\vec{\eta})\\&\overset{(4.4)}{=}\lambda_\beta\big(\mathrm{Ad}_q(\vec{q}),\vec{0}_{p_\beta}\big)+\lambda_\beta(\vec{\eta})=\rho(q)\circ\lambda_\alpha\big(\vec{q},\vec{0}_{p_\alpha}\big)+\rho(q)\circ\lambda_\alpha\big(\vec{0}_{\mathfrak{q}},\vec{w}_{p_\alpha}\big)\\&=\rho(q)\circ\lambda_\alpha(\vec{q},\vec{w}_{p_\alpha})=\rho(q)\circ\widehat{\lambda}_\alpha\circ\mathrm{d}\Theta(\vec{q},\vec{w}_{p_\alpha})=\rho(q)\circ\widehat{\lambda}_\alpha(\vec{w}),\end{aligned} \tag{4.6}$$

where for the third equality we have used that

$$\begin{aligned}(\mathrm{d}L_q \circ \mathrm{d}\Theta)\left(\vec{q}, \vec{0}_{p_\alpha}\right) &= \tfrac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} q \cdot (\exp(t\vec{q}) \cdot p_\alpha) \\ &= \tfrac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} \alpha_q(\exp(t\vec{q})) \cdot p_\beta = \mathrm{d}\Theta\big(\mathrm{Ad}_q(\vec{q}), \vec{0}_{p_\beta}\big).\end{aligned} \tag{4.7}$$

Consequently, if $\widetilde{q} \cdot p_\beta = p$ with $(\widetilde{q}, p_\beta) \in Q \times P_\beta$ for some $\beta \in I$, then $p_\beta = (q^{-1}\widetilde{q})^{-1} \cdot p_\alpha$ and well-definedness follows from

$$\begin{aligned}\rho(\widetilde{q}) \circ \widehat{\lambda}_\beta\left(\mathrm{d}L_{\widetilde{q}^{-1}}(\vec{w}_p)\right) &= \rho(q) \circ \rho\big(q^{-1}\widetilde{q}\big) \circ \widehat{\lambda}_\beta\left(\mathrm{d}L_{(q^{-1}\widetilde{q})^{-1}}\big(\mathrm{d}L_{q^{-1}}\vec{w}_p\big)\right) \\ &= \rho(q) \circ \widehat{\lambda}_\alpha\big(\mathrm{d}L_{q^{-1}}\vec{w}_p\big),\end{aligned}$$

where the last step is due to (4.6) with $\vec{w} = \mathrm{d}L_{q^{-1}}\vec{w}_p \in T_{p_\alpha}P$. Next, we show that ω fulfils the pullback property. For this, let $(\vec{m}, \vec{w}_{p_\alpha}) \in T_qQ \times T_{p_\alpha}P_\alpha$. Then

$$\begin{aligned}(\Theta^*\omega)\left(\vec{m}_q, \vec{w}_{p_\alpha}\right) &= \omega_{q\cdot p_\alpha}\left(\mathrm{d}\Theta(\vec{m}_q, \vec{w}_{p_\alpha})\right) \overset{(4.5)}{=} \rho(q) \circ \widehat{\lambda}_\alpha\left(\mathrm{d}L_{q^{-1}}\mathrm{d}\Theta(\vec{m}_q, \vec{w}_{p_\alpha})\right) \\ &= \rho(q) \circ \widehat{\lambda}_\alpha \circ \mathrm{d}\Theta\left(\mathrm{d}L_{q^{-1}}\vec{m}_q, \vec{w}_{p_\alpha}\right) \overset{(4.4)}{=} \rho(q) \circ \lambda_\alpha\left(\mathrm{d}L_{q^{-1}}\vec{m}_q, \vec{w}_{p_\alpha}\right) \\ &= \omega_\alpha(\vec{m}_q, \vec{w}_{p_\alpha}).\end{aligned}$$

In the third step, we have used that $L_{q^{-1}} \circ \Theta = \Theta(L_{q^{-1}}(\cdot), \cdot)$. Finally, we have to verify that ω is a Φ-invariant smooth connection. For this, let $p \in P$ and $(\widetilde{q}, p_\alpha) \in Q \times P_\alpha$ with $p = \widetilde{q} \cdot p_\alpha$. Then, for $q \in Q$ and $\vec{w}_p \in T_pP$ we have

$$\begin{aligned}\left(L_q^*\omega\right)_p(\vec{w}_p) &= \omega_{q\cdot p}\left(\mathrm{d}L_q\vec{w}_p\right) = \omega_{(q\widetilde{q})\cdot p_\alpha}\left(\mathrm{d}L_q\vec{w}_p\right) \\ &= \rho(q) \circ \rho\left(\widetilde{q}\right) \circ \widehat{\lambda}_\alpha\left(\mathrm{d}L_{\widetilde{q}^{-1}}\vec{w}_p\right) = \rho(q) \circ \omega_p(\vec{w}_p),\end{aligned}$$

hence

$$\begin{aligned}R_s^*\omega &= L_{(e,s^{-1})}^*\omega = \rho\big((e, s^{-1})\big) \circ \omega = \mathrm{Ad}_{s^{-1}} \circ \omega, \\ L_g^*\omega &= L_{(g,e)}^*\omega = \rho((g, e)) \circ \omega = \omega.\end{aligned}$$

Thus, it remains to show smoothness of ω, and that $\omega_p(\widetilde{s}(p)) = \vec{s}$ holds for all $p \in P$ and all $\vec{s} \in \mathfrak{s}$. For the second property, let $p = q \cdot p_\alpha$ for $(q, p_\alpha) \in Q \times P_\alpha$. Then $q = (g, s)$ for some $g \in G$ and $s \in S$ and we obtain

$$\begin{aligned}\omega_p(\widetilde{s}(p)) &= \rho(q) \circ \widehat{\lambda}_\alpha\left(\mathrm{d}L_{q^{-1}}\widetilde{s}(q \cdot p_\alpha)\right) = \rho(q) \circ \widehat{\lambda}_\alpha\left(\tfrac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} p_\alpha \cdot (\alpha_{s^{-1}}(\exp(t\vec{s})))\right) \\ &= \rho(q) \circ \widehat{\lambda}_\alpha\big(\mathrm{d}\Theta\big(\mathrm{Ad}_{s^{-1}}(\vec{s}), \vec{0}_{p_\alpha}\big)\big) = \mathrm{Ad}_s \circ \lambda_\alpha\big(\mathrm{Ad}_{s^{-1}}(\vec{s}), \vec{0}_{p_\alpha}\big) = \mathrm{Ad}_s \circ \mathrm{Ad}_{s^{-1}}(\vec{s}) = \vec{s}.\end{aligned}$$

For smoothness, let $p_\alpha \in P_\alpha$ and choose a submanifold Q' of Q through e, an open neighbourhood $P'_\alpha \subseteq P_\alpha$ of p_α, and an open subset $U \subseteq P$ such that the restriction $\widehat{\Theta} := \Theta|_{Q' \times P'_\alpha}$ is a diffeomorphism to U. Then $p_\alpha \in U$ because $e \in Q'$, hence

$$\omega|_U = \widehat{\Theta}^{-1*}\big[\widehat{\Theta}^*\omega\big] = \widehat{\Theta}^{-1*}\big[(\Theta^*\omega)|_{TQ \times TP_\alpha}\big] = \widehat{\Theta}^{-1*}\omega_\alpha.$$

Since ω_α is smooth and $\widehat{\Theta}$ is a diffeomorphism, $\omega|_U$ is smooth as well. Finally, if $p = q \cdot p_\alpha$ holds for $q \in Q$, then $L_q(U)$ is an open neighbourhood of p and

$$\omega|_{L_q(U)} = \left(L_{q^{-1}}^*\left(L_q^*\omega\right)\right)\big|_{L_q(U)} = \rho(q) \circ \left(L_{q^{-1}}^*\omega\right)\big|_{L_q(U)} = \rho(q) \circ L_{q^{-1}}^*\left(\omega|_U\right)$$

is smooth because $\omega|_U$ and $L_{q^{-1}}$ are smooth. ■

Corollary 4.2 and Proposition 4.8 now prove

Theorem 4.9. *Let G be a Lie group of automorphisms of the principal fibre bundle P. Then, for each Φ-covering $\{P_\alpha\}_{\alpha\in I}$ of P the assignment*

$$\omega \mapsto \{(\Phi^*\omega)|_{\mathfrak{g}\times TP_\alpha}\}_{\alpha\in I}$$

is a bijection between the Φ-invariant connections on P and the reduced connections that correspond to $\{P_\alpha\}_{\alpha\in I}$.

As already mentioned in the remarks following Case 4.5, the second part of the next example shows the importance of the transversality condition

$$\mathrm{im}[\mathrm{d}_x\tau_0] + \mathrm{im}\left[\mathrm{d}_e\varphi_{\tau_0(x)}\right] = T_{\tau_0(x)}M \qquad \forall\, x\in U_0$$

for the formulation in [6].

Example 4.10 ((semi-)homogeneous connections)**.**

1. Let $P = X\times S$ for an n-dimensional $\mathbb{R}$-vector space X and an arbitrary structure group S. Moreover, let $G\subseteq X$ be a linear subspace of dimension $1\leq k\leq n$ acting via

   $$\Phi\colon G\times P\to P, \quad (g,(x,\sigma))\mapsto (g+x,\sigma).$$

   If W is an algebraic complement of G in X and $P_0 := W\times\{e_S\}\subseteq P$, then P_0 is a Φ-covering because $\Theta\colon (G\times S)\times P_0\to P$ is a diffeomorphism and each φ-orbit intersects W in a unique point. Consequently, identifying G with its Lie algebra $\mathfrak{g}$, the Φ-invariant connections on P are in bijection with the smooth maps $\psi\colon G\times TW\to\mathfrak{s}$ for which $\psi_w := \psi|_{G\times T_wW}$ is linear for all $w\in W$. This is because the conditions *i)* and *ii)* from Corollary 4.2 give no further restrictions in this case. It is straightforward to see[14] that the Φ-invariant connection that corresponds to ψ is given by

   $$\omega^\psi_{(x,s)}(\vec{v}_x,\vec{\sigma}_s) = \mathrm{Ad}_{s^{-1}}\circ\psi_{\mathrm{pr}_W(x)}\big(\mathrm{pr}_G(\vec{v}_x),\mathrm{pr}_W(\vec{v}_x)\big) + \mathrm{d}L_{s^{-1}}(\vec{\sigma}_s) \tag{4.8}$$

   for $(\vec{v}_x,\vec{\sigma}_s)\in T_{(x,s)}P$.

2. Let (P,π,M,S) be a principal fibre bundle with Lie group of automorphisms (G,Φ). Then, for (U_0,τ_0,s_0) a triple as in [6][15], Theorem 2 in [6] states that each smooth $\overline{\psi}\colon\mathfrak{g}\times TU_0\to\mathfrak{s}$ for which $\overline{\psi}|_{\mathfrak{g}\times T_xU_0}$ is linear for all $x\in U_0$, and that fulfils the three conditions from Example 4.6 can be written as $(\Phi^*\overline{\omega})|_{\mathfrak{g}\times TU_0}$ for some (even unique) invariant connection $\overline{\omega}$ on P.

   We consider the situation of the previous part, whereby we let

   $$X=\mathbb{R}^2 \qquad G=\mathrm{span}_{\mathbb{R}}(\vec{e}_1) \qquad W=\mathrm{span}_{\mathbb{R}}(\vec{e}_2) \qquad \text{and} \qquad P_0 = W\times\{e\}.$$

   Now, we are going to construct (U_0,τ_0,s_0) and $\overline{\psi}$ in such a way that the above statement is wrong:

   - First, we fix $0\neq\vec{s}\in\mathfrak{s}$ and define ω by (4.8) for $\psi\colon\mathfrak{g}\times TP_0\to\mathfrak{s}$ the map

     $$\psi_y(\lambda\cdot\vec{e}_1,\mu\cdot\vec{e}_2) := \mu\cdot f(y)\cdot\vec{s} \qquad \text{for} \qquad (\lambda\cdot\vec{e}_1,\mu\cdot\vec{e}_2)\in\mathfrak{g}\times T_{(y\cdot\vec{e}_2,e)}P_0$$

     with $f(0):=0$ and $f(y):=1/\sqrt[3]{y}$ if $y\neq 0$. Then, ω is easily seen to be smooth on $P' := Z\times S$ for $Z := \{(x,y)\in\mathbb{R}^2 \mid y\neq 0\}$, but it is not smooth at $((x,0),e)$ because

     $$\omega_{((x,y),e)}\big((\vec{0},\vec{e}_2),\vec{0}_{\mathfrak{s}}\big) = \psi_y\big(\vec{0},\vec{e}_2\big) = f(y)\cdot\vec{s} \qquad \forall\, y\in\mathbb{R}.$$

     Even more: there cannot exist any smooth invariant connection ω on P which coincides on P' with ω, just because $\lim_{y\to 0} f(y)\cdot\vec{s}$ does not exist.

[14] Pull back ω^ψ by Θ and restrict it to $\mathfrak{g}\times TP_0$.

[15] See also the discussions following Case 4.5.

- Now, we let $U_0 := \mathbb{R}$, $\tau_0\colon U_0 \to \mathbb{R}^2$, $t \mapsto \big(t, t^3\big)$ and $s_0\colon t \mapsto (\tau_0(t), e)$. Then, (U_0, τ_0, s_0) fulfils the conditions from [6], but we have[16]

$$\mathrm{im}[\mathrm{d}_0\tau_0] + \mathrm{im}\big[\mathrm{d}_e\varphi_{\tau_0(0)}\big] = \mathrm{span}_{\mathbb{R}}(\vec{e}_1) \neq T_0X = T_0\mathbb{R}^2 = \mathbb{R}^2.$$

As a consequence, $\overline{\psi}\colon \mathfrak{g} \times TU_0 \to \mathfrak{s}$ defined by $\overline{\psi}_t := (\Phi^*\omega)|_{\mathfrak{g}\times T_tU_0}$ is smooth, because for $t \neq 0$ and $r \in T_tU_0 = \mathbb{R}$ we have

$$\begin{aligned}\overline{\psi}_t(\lambda\vec{e}_1, r) &= (\Phi^*\omega)_{(e,s_0(t))}(\lambda\vec{e}_1, \mathrm{d}_t s_0(r)) = \big(\Phi^*\omega\big)\left(\lambda\vec{e}_1, r\cdot\vec{e}_1 + 3t^2r\cdot\vec{e}_2\right)\\ &= \omega_{((t,t^3),e)}\big((\lambda + r)\cdot\vec{e}_1 + 3t^2r\cdot\vec{e}_2, \vec{0}_{\mathfrak{s}}\big)\\ &= \psi_{t^3}\big((\lambda+r)\cdot\vec{e}_1, 3t^2r\cdot\vec{e}_2\big) = 3tr\cdot\vec{s}\end{aligned}$$

as well as $\overline{\psi}_0(\lambda\vec{e}_1, r) = 0$ if $t = 0$. For the first step, keep in mind that

$$(\Phi^*\omega)|_{\mathfrak{g}\times T_tU_0}(\vec{g}, r) = (\Phi^*\omega)(\vec{g}, \mathrm{d}_t s_0(r))$$

holds by Convention 3.1.2. Since ω fulfils the algebraic properties of an invariant connection, $\overline{\psi}$ fulfils the algebraical properties from Example 4.6.

- It remains to show that there is no smooth invariant connection $\overline{\omega}$ on P for which $\overline{\psi} = (\Phi^*\overline{\omega})|_{\mathfrak{g}\times TU_0}$ holds. This, however, follows from the first point as such an $\overline{\omega}$ necessarily had to coincide on P' with ω.

  In fact, let $U_0' := \mathbb{R}_{\neq 0}$, and $\tau_0'\colon U_0' \to Z$, $t \mapsto \big(t, t^3\big)$ as well as $s_0'\colon t \mapsto (\tau_0'(t), e)$ be defined as above. Then, (U_0', s_0') is a Θ-patch as we have removed the point $0 \in U_0$ for which transversality fails. Thus, (U_0', s_0') is a Φ-covering of P', so that

$$(\Phi^*\overline{\omega})|_{\mathfrak{g}\times TU_0'} = \overline{\psi}|_{\mathfrak{g}\times TU_0'} = (\Phi^*\omega)|_{\mathfrak{g}\times TU_0'}$$

implies $\overline{\omega} = \omega$ on P'.

5 Particular cases and applications

In the first part of this section, we will consider Φ-coverings of P arising from the induced action φ on the base manifold M of P. Then, we discuss the case where Φ acts via gauge transformations on P, which will lead us to a straightforward generalization of the description of connections by consistent families of local 1-forms on M. In the second part, we discuss the (almost) fibre transitive case, and deduce Wang's original theorem [8] from Theorem 4.9. Finally, we will consider the situation where P is trivial, and give examples in loop quantum gravity.

5.1 Φ-coverings and the induced action

Let (G, Φ) be a Lie group of automorphisms of the principal fibre bundle P. According to Lemma 3.4, for each $x \in M$ there is a φ-patch (with minimal dimension) M_x with $x \in M$. Consequently, we find an open neighbourhood $M_x' \subseteq M_x$ of x and a local section $s_x\colon U \to P$ with $M_x' \subseteq U$ for U an open neighbourhood of M. Let $I \subseteq M$ be a subset such that[17] each φ-orbit intersects at least one of the sets M_x for some $x \in I$. Then, it is immediate from Lemma 3.7.2 that $\{s_x(M_x')\}_{x\in I}$ is a Φ-covering of P. More generally, we have

[16] Thus, (U_0, s_0) cannot be a Θ-patch by the second part of Remark 3.6.3.

[17] It is always possible to choose $I = M$.

Corollary 5.1. *Let (P,π,M,S) be a principal fibre bundle and (G,Φ) a Lie group of automorphisms of P. Denote by $(M_\alpha,s_\alpha)_{\alpha\in I}$ a family consisting of a collection of φ-patches $\{M_\alpha\}_{\alpha\in I}$ and smooth sections*[18] *$s_\alpha\colon M_\alpha\to P$. Then, the sets $P_\alpha:=s_\alpha(M_\alpha)$ are Θ-patches. They provide a Φ-covering of P iff each φ-orbit intersects at least one patch M_α.*

Proof. This is immediate from Lemma 3.7.2. ■

We now consider the case where (G,Φ) is a Lie group of gauge transformations of P, i.e., $\varphi_g=\mathrm{id}_M$ for all $g\in G$. Here, we show that Theorem 4.9 can be seen as a generalization of the description of smooth connections by means of consistent families of local 1-forms on the base manifold M. For this, let $\{U_\alpha\}_{\alpha\in I}$ be an open covering of M and $\{s_\alpha\}_{\alpha\in I}$ a family of smooth sections $s_\alpha\colon U_\alpha\to P$. We define the open sets $U_{\alpha\beta}:=U_\alpha\cap U_\beta$ and consider the smooth maps $\delta_{\alpha\beta}\colon G\times U_{\alpha\beta}\to S$ determined by $s_\beta(x)=\Phi(g,s_\alpha(x))\cdot\delta_{\alpha\beta}(g,x)$, and for which $\delta_{\alpha\beta}(g,x)=\phi^{-1}_{s_\alpha(x)}(g)\cdot\delta_{\alpha\beta}(e,x)$ holds. Finally, we let $\mu_{\alpha\beta}(g,\vec{v}_x):=\mathrm{d}L_{\delta^{-1}_{\alpha\beta}(g,x)}\circ\mathrm{d}_x\delta_{\alpha\beta}(g,\cdot)(\vec{v}_x)$ for $\vec{v}_x\in T_xU_{\alpha\beta}$ and $g\in G$. Then, we have

Case 5.2 (Lie groups of gauge transformations)**.** *Let (G,Φ) be a Lie group of gauge transformations of the principal fibre bundle (P,π,M,S). Then, the Φ-invariant connections on P are in bijection with the families $\{\chi_\alpha\}_{\alpha\in I}$ of $\mathfrak{s}$-valued 1-forms $\chi_\alpha\colon U_\alpha\to\mathfrak{s}$ for which we have*

$$\chi_\beta(\vec{v}_x)=\big(\mathrm{Ad}_{\delta_{\alpha\beta}(g,x)}\circ\chi_\alpha\big)(\vec{v}_x)+\mu_{\alpha\beta}(g,\vec{v}_x)\qquad\forall\,\vec{v}_x\in T_xU_{\alpha\beta},\qquad\forall\,g\in G.\tag{5.1}$$

Proof. By Corollary 5.1 $\{s_\alpha(U_\alpha)\}_{\alpha\in I}$ is a Φ-covering of P. So, let $\{\psi_\alpha\}_{\alpha\in I}$ be a reduced connection w.r.t. this covering. We first show that condition $i)$ from Corollary 4.2 implies

$$\psi_\beta\big(\vec{g},\vec{0}_p\big)=\mathrm{d}_e\phi_p(\vec{g})\qquad\forall\,\vec{g}\in\mathfrak{g},\qquad\forall\,p\in s_\beta(U).$$

For this observe that condition $a)$ from Remark 4.4 means that for all $\beta\in I$, $p\in s_\beta(U_\beta)$, $\vec{w}_p\in T_ps_\beta(U_\beta)$ and $\vec{g}\in\mathfrak{g}$, $\vec{s}\in\mathfrak{s}$ we have

$$\mathrm{d}_e\Phi_p(\vec{g})+\vec{w}_p-\widetilde{s}(p)=0\quad\Longrightarrow\quad\psi_\beta(\vec{g},\vec{w}_p)-\vec{s}=0.$$

Now, $T_ps_\beta(U_\beta)$ is complementary to Tv_pP and $\mathrm{im}[\mathrm{d}_e\Phi_p]\subseteq\ker[\mathrm{d}_p\pi]$ so that $a)$ is the same as

$$a')\ \ \mathrm{d}_e\Phi_p(\vec{g})=\widetilde{s}(p)\quad\Longrightarrow\quad\psi_\beta\big(\vec{g},\vec{0}_p\big)=\vec{s}\qquad\text{for}\ \ \vec{g}\in\mathfrak{g},\ \vec{s}\in\mathfrak{s}\ \text{and all}\ p\in P_\beta.$$

But, since $G_x=G$ for all $x\in M$, this just means[19] $\psi_\beta\big(\vec{g},\vec{0}_p\big)=\mathrm{d}_e\phi_p(\vec{g})$ for all $\vec{g}\in\mathfrak{g}$ and already implies Condition $ii)$ from Corollary 4.2 as ϕ_p is a Lie group homomorphism. Consequently, we can ignore this condition in the following. Now, we have $p_\beta=q\cdot p_\alpha$ for $q\in Q$, $p_\alpha\in P_\alpha$, $p_\beta\in P_\beta$ iff $\pi(p_\alpha)=\pi(p_\beta)=x\in U_{\alpha\beta}$ and $q=\big(g,\delta^{-1}_{\alpha\beta}(g,x)\big)$. Consequently, the left hand side of condition $i)$ from Corollary 4.2 reads

$$\widetilde{g}(s_\beta(x))+\mathrm{d}_xs_\beta(\vec{v}_\beta)-\widetilde{s}(s_\beta(x))=\big(\mathrm{d}L_g\circ\mathrm{d}R_{\delta_{\alpha\beta}(g,x)}\circ\mathrm{d}_xs_\alpha\big)(\vec{v}_\alpha),$$

where $\vec{v}_\alpha,\vec{v}_\beta\in T_xM$ and $g\in G$. This is true for $\vec{v}_\alpha=\vec{v}_\beta=\vec{v}_x$, $\vec{g}=0$ and $\vec{s}=\mu_{\alpha\beta}(g,\vec{v}_x)$, which follows from

$$\begin{aligned}\mathrm{d}_xs_\beta(\vec{v}_\beta)&=\mathrm{d}_x\big[L_g\circ R_{\delta_{\alpha\beta}(g,\cdot)}\circ s_\alpha\big](\vec{v}_x)\\&=\mathrm{d}L_g\big[\mathrm{d}_{s_\alpha(x)}R\big(\mathrm{d}_x\delta_{\alpha\beta}(g,\cdot)(\vec{v}_x)\big)+\mathrm{d}R_{\delta_{\alpha\beta}(g,x)}(\mathrm{d}_xs_\alpha(\vec{v}_x))\big],\\\widetilde{s}(s_\beta(x))&=\tfrac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}L_g\circ R_{\delta_{\alpha\beta}(g,x)\cdot\exp(t\vec{s})}(s_\alpha(x))\\&=\mathrm{d}L_g\big[\mathrm{d}_{s_\alpha(x)}R\big(\mathrm{d}L_{\delta_{\alpha\beta}(g,x)}(\vec{s})\big)\big]=\mathrm{d}L_g\big[\mathrm{d}_{s_\alpha(x)}R\big(\mathrm{d}_x\delta_{\alpha\beta}(g,\cdot)(\vec{v}_x)\big)\big].\end{aligned}$$

[18] This is that $\pi\circ s_\alpha=\mathrm{id}_{M_\alpha}$.

[19] $\mathrm{d}_e\Phi_p(\vec{g})-\widetilde{s}(p)=0$ iff $(\vec{g},\vec{s})\in\mathfrak{q}_p$ iff $\vec{s}=\mathrm{d}_e\phi_p(\vec{g})$.

Consequently, by Corollary 4.2.i) and for

$$(\psi_\alpha \circ \mathrm{d}_x s_\alpha)(\vec{v}_x) := \psi_\alpha\big(\vec{0}_{\mathfrak{g}}, \mathrm{d}_x s_\alpha(\vec{v}_x)\big) \qquad \forall\, \vec{v}_x \in T_x U_{\alpha\beta}$$

we have

$$\psi_\beta\big(\vec{0}_{\mathfrak{g}}, \mathrm{d}_x s_\beta(\vec{v}_x)\big) = \big(\mathrm{Ad}_{\delta_{\alpha\beta}(g,x)} \circ \psi_\alpha \circ \mathrm{d}_x s_\alpha\big)(\vec{v}_x) + \mu_{\alpha\beta}(g, \vec{v}_x) \tag{5.2}$$

for all $g \in G$ and all $\vec{v}_x \in T_x U_{\alpha\beta}$. Due to part 2) in Remark 4.4 the condition i) from Corollary 4.2 now gives no further restrictions, so that for $\chi_\beta := \psi_\beta \circ \mathrm{d}s_\beta$ we have

$$\psi_\beta(\vec{g}, \mathrm{d}_x s_\beta(\vec{v}_x)) = \mathrm{d}_e \phi_{s_\beta(x)}(\vec{g}) + \chi_\beta(\vec{v}_x) \qquad \forall\, \vec{g} \in \mathfrak{g}, \qquad \forall\, \vec{v}_x \in T_x M, \qquad \forall\, x \in U_\beta.$$

Then, ψ_β is uniquely determined by χ_β for each $\beta \in I$, so that (5.2) yields the consistency condition (5.1) for the maps $\{\chi_\alpha\}_{\alpha \in I}$. ■

Example 5.3 (trivial action)**.** If G acts trivially, then for each $x \in U_{\alpha\beta}$ we have

$$\delta_{\alpha\beta}(g, x) = \phi^{-1}_{s_\alpha(x)}(g) \cdot \delta_{\alpha\beta}(e, x) = \delta_{\alpha\beta}(e, x).$$

Thus, $\delta_{\alpha\beta}$ is independent of $g \in G$, so that Case 5.2 just reproduces the description of smooth connections by means of consistent families of local 1-forms on the base manifold M.

5.2 (Almost) fibre transitivity

In this subsection we discuss the situation where M admits an element that is contained in the closure of each φ-orbit. For instance, this holds for all $x \in M$ if each φ-orbit is dense in M and, in particular, is true for fibre transitive actions.

Case 5.4 (almost fibre transitivity)**.** *Let $x \in M$ be contained in the closure of each φ-orbit and let $p \in F_x$. Then, each Θ-patch $P_0 \subseteq P$ with $p \in P_0$ is a Φ-covering of P. Hence, the Φ-invariant connections on P are in bijection with the smooth maps $\psi\colon \mathfrak{g} \times TP_0 \to \mathfrak{s}$ for which $\psi|_{\mathfrak{g} \times T_p P_0}$ is linear for all $p \in P_0$ and that fulfil the two conditions from Corollary 4.2.*

Proof. It suffices to show that $\pi(P_0)$ intersects each φ-orbit $[o]$. Since P_0 is a Θ-patch, there is an open neighbourhood $P' \subseteq P_0$ of p and a submanifold Q' of Q through (e_G, e_S) such that $\Theta|_{Q' \times P'}$ is a diffeomorphism to an open subset $U \subseteq P$. Then $\pi(U)$ is an open neighbourhood of $\pi(p)$ and by assumption we have $[o] \cap \pi(U) \neq \varnothing$ for each $[o] \in M/G$. Consequently, for $[o] \in M/G$ we find $\widetilde{p} \in U$ with $\pi(\widetilde{p}) \in [o]$. Let $\widetilde{p} = \Theta((g', s'), p')$ for $((g', s'), p') \in Q' \times P'$. Then

$$[o] \ni \pi(\widetilde{p}) = \pi\left(\Phi(g', p') \cdot s'\right) = \varphi(g', \pi(p')) \in [\pi(p')]$$

shows that $[o] = [\pi(p')]$ holds, hence $\pi(P_0) \cap [o] \neq \varnothing$. ■

The next example to Case 5.4 shows that evaluating the conditions i) and ii) from Corollary 4.2 at one single point can be sufficient to verify non-existence of invariant connections.

Example 5.5 (general linear group)**.**

1. Let $P := \mathrm{GL}(n, \mathbb{R})$ and $G = S = B \subseteq \mathrm{GL}(n, \mathbb{R})$ the subgroup of upper triangular matrices. Moreover, let $S_n \subseteq \mathrm{GL}(n, \mathbb{R})$ be the group of permutation matrices. Then, P is a principal fibre bundle with base manifold $M := P/S$, structure group S and projection

map $\pi\colon P \to M$, $p \mapsto [p]$. Moreover, G acts via automorphisms on P by $\Phi(g,p) := g \cdot p$, and we have the Bruhat decomposition

$$\mathrm{GL}(n,\mathbb{R}) = \bigsqcup_{w \in S_n} BwB.$$

Then, $M = \bigsqcup_{w\in S_n} G\cdot\pi(w)$, $G\cdot\pi(e) = \pi(e)$ and $\pi(e) \in \overline{G\cdot\pi(w)}$ for all $w \in S_n$. Now, $\mathrm{im}[\mathrm{d}_e\Theta_e] = \mathfrak{g}$ since $\mathrm{d}_e\Theta_e(\vec{g}) = \vec{g}$ for all $\vec{g} \in \mathfrak{g}$. Moreover, $\mathfrak{g} = \mathrm{span}_{\mathbb{R}}\{E_{ij}\,|\,1 \leq i \leq j \leq n\}$, so that $V := \mathrm{span}_{\mathbb{R}}\{E_{ij}\,|\,1 \leq j < i \leq n\}$ is an algebraic complement of $\mathfrak{g}$ in $T_eP = \mathfrak{gl}(n,\mathbb{R})$. By Lemma 3.4.2 we find a patch $H \subseteq P$ through e with $T_eH = V$, and due to Case 5.4 this is a Φ-covering.

2. A closer look at the point $e \in P$ shows that there cannot exist any Φ-invariant connection on $\mathrm{GL}(n,\mathbb{R})$. In fact, if $\psi\colon \mathfrak{g}\times TH \to \mathfrak{s}$ is a reduced connection w.r.t. H, for $\vec{w} := \vec{0}_e$ and $\vec{g} = \vec{s}$ we have

$$\widetilde{g}(e) + \vec{w} - \widetilde{s}(e) = \vec{g} + \vec{w} - \vec{s} = 0.$$

Thus, condition $i)$ from Corollary 4.2 gives $\psi\big(\vec{g},\vec{0}_e\big) - \vec{g} = 0$, hence $\psi\big(\vec{g},\vec{0}_e\big) = \vec{g}$ for all $\vec{g} \in \mathfrak{g}$. Now, $q\cdot e = e$ iff $q = (b,b)$ for some $b \in B$. Let

$$V \ni \vec{h} := E_{n1}, \qquad B \ni b := e + E_{1n}, \qquad \mathfrak{g} \ni \vec{g} := E_{11} - E_{1n} - E_{nn}.$$

Then, $\widetilde{g}(e) + \vec{h} = \vec{g} + \vec{h} = b\vec{h}b^{-1} = \mathrm{d}L_q\vec{h}$, so that condition $i)$ yields

$$\psi\big(\vec{g},\vec{h}\big) = \rho(q)\circ\psi\big(\vec{0}_{\mathfrak{g}},\vec{h}\big) = \mathrm{Ad}_b\circ\psi\big(\vec{0}_{\mathfrak{g}},\vec{h}\big),$$

hence $\vec{g} + [\mathrm{id} - \mathrm{Ad}_b]\circ\psi\big(\vec{0}_{\mathfrak{g}},\vec{h}\big) = 0$. But, $(\vec{g})_{11} = 1$ and

$$\Big(\psi\big(\vec{0}_{\mathfrak{g}},\vec{h}\big) - \mathrm{Ad}_b\circ\psi\big(\vec{0}_{\mathfrak{g}},\vec{h}\big)\Big)_{11} = \Big(\psi\big(\vec{0}_{\mathfrak{g}},\vec{h}\big)\Big)_{11} - \Big(\psi\big(\vec{0}_{\mathfrak{g}},\vec{h}\big)\Big)_{11} = 0,$$

so that ψ cannot exist.

Corollary 5.6. *If Φ is fibre transitive, then $\{p\}$ is a Φ-covering for all $p \in P$.*

Proof. It suffices to show that $\{\pi(p)\}$ is a φ-patch, since then $\{p\}$ is a Θ-patch by Corollary 5.1, and a Φ-covering by Case 5.4. This, however, is clear from Remark 3.3.1. In fact, if $x := \pi(p)$, then by general theory we know that M is diffeomorphic to G/G_x via $\vartheta\colon [g] \mapsto \varphi(g,x)$ and that for each $[g] \in G/G_x$ we find an open neighbourhood $U \subseteq G/G_x$ of $[g]$ and a smooth section $s\colon U \to G$. Then, surjectivity of $\mathrm{d}_e\varphi_x$ is clear from surjectivity of $\mathrm{d}_{[e]}\vartheta$ and

$$\mathrm{d}_e\varphi_x \circ \mathrm{d}_{[e]}s = \mathrm{d}_{[e]}(\varphi_x\circ s) = \mathrm{d}_{[e]}\varphi(s(\cdot),x) = \mathrm{d}_{[e]}\vartheta,$$

showing that $T_xM = \mathrm{d}_e\varphi_x(\mathfrak{g})$ holds. ∎

Let φ be transitive and $p \in P$. Then, $\{p\}$ is a Φ-covering by Corollary 5.6 and $T_p\{p\}$ is the zero vector space. Moreover, we have $p_\alpha = q\cdot p_\beta$ iff $p_\alpha = p_\beta = p$ and $q \in Q_p$. It follows that a reduced connection w.r.t. $\{p\}$ can be seen as a linear map $\psi\colon \mathfrak{g} \to \mathfrak{s}$ that fulfils the following two conditions:

- $\mathrm{d}_e\Theta_p(\vec{g},\vec{s}) = 0 \quad\Longrightarrow\quad \psi(\vec{g}) = \vec{s} \qquad$ for $\vec{g} \in \mathfrak{g}$, $\vec{s} \in \mathfrak{s}$,
- $\psi\big(\mathrm{Ad}_q(\vec{g})\big) = \rho(q)\circ\psi(\vec{g}) \qquad \forall\, q \in Q_p, \quad \forall\, \vec{g} \in \mathfrak{g}$.

Since $\ker[d_e\Theta_p] = \mathfrak{q}_p$, we have shown

Case 5.7 (Hsien-Chung Wang, [8]). *Let (G, Φ) be a fibre transitive Lie group of automorphisms of P. Then, for each $p \in P$ there is a bijection between the Φ-invariant connections on P and the linear maps $\psi\colon \mathfrak{g} \to \mathfrak{s}$ that fulfil*

a) $\psi(\vec{h}) = \mathrm{d}_e\phi_p(\vec{h}) \qquad \forall\, \vec{h} \in \mathfrak{g}_{\pi(p)}$,

b) $\psi \circ \mathrm{Ad}_h = \mathrm{Ad}_{\phi_p(h)} \circ \psi \qquad \forall\, h \in G_{\pi(p)}$.

This bijection is explicitly given by $\omega \mapsto \Phi_p^*\omega$.

Example 5.8.

1. **Homogeneous connections.** In the situation of Example 4.10 let $k = n$ and $X = \mathbb{R}^n$. Then, Φ is fibre transitive, and for $p = (0, e)$ we have $G_{\pi(p)} = \{e\}$ as well as $\mathfrak{g}_{\pi(p)} = \{0\}$. Thus, the reduced connections w.r.t. $\{p\}$ are just the linear maps $\psi\colon \mathbb{R}^n \to \mathfrak{s}$, and the corresponding homogeneous connections are given by

$$\omega^{\psi}{}_{(x,s)}(\vec{v}_x, \vec{\sigma}_s) = \mathrm{Ad}_{s^{-1}} \circ \psi(\vec{v}_x) + \mathrm{d}L_{s^{-1}}(\vec{\sigma}_s) \qquad \forall\, (\vec{v}_x, \vec{\sigma}_s) \in T_{(x,s)}P.$$

2. **Homogeneous isotropic connections.** Let $P = \mathbb{R}^3 \times \mathrm{SU}(2)$ and $\varrho\colon \mathrm{SU}(2) \to \mathrm{SO}(3)$ be the universal covering map. We consider the semi direct product $E := \mathbb{R}^3 \rtimes_{\varrho} \mathrm{SU}(2)$ whose multiplication is given by $(v, \sigma) \cdot_{\varrho} (v', \sigma') := (v + \varrho(\sigma)(v'), \sigma\sigma')$ for all $(v, \sigma), (v', \sigma) \in E$. Since E equals P as a set, we can define the action Φ of E on P just by $\cdot_{\varrho}$. Then, E is a Lie group which resembles the euclidean one, and it follows from Wang's theorem that the Φ-invariant connections are of the form (see, e.g., Appendix A.3 in [5])

$$\omega^c_{(x,s)}(\vec{v}_x, \vec{\sigma}_x) = c\, \mathrm{Ad}_{s^{-1}}[\mathfrak{z}(\vec{v}_x)] + s^{-1}\vec{\sigma}_s \qquad \forall\, (\vec{v}_x, \vec{\sigma}_s) \in T_{(x,s)}P.$$

Here, c runs over $\mathbb{R}$ and $\mathfrak{z}\colon \sum_{i=1}^{3} v^i \vec{e}_i \to \sum_{i=1}^{3} v^i \tau_i$ with matrices

$$\tau_1 := \begin{pmatrix} 0 & -\mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix}, \qquad \tau_2 := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \tau_3 := \begin{pmatrix} -\mathrm{i} & 0 \\ 0 & \mathrm{i} \end{pmatrix},$$

and $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ the standard basis in $\mathbb{R}^3$.

We close this section with a remark concerning the relations between sets of invariant connections that correspond to different lifts of the same Lie group action on the base manifold of a principal fibre bundle.

Remark 5.9. Let P be a principal fibre bundle and $\Phi, \Phi'\colon G \times P \to P$ two Lie groups of automorphisms with $\varphi = \varphi'$. Then, the respective sets of invariant connections can differ significantly. In fact, in the situation of the second part of Example 5.8 let $\Phi'((v, \sigma), (x, s)) := (v + \varrho(\sigma)(x), s)$. Then, $\varphi' = \varphi$ and Appendix B.1 shows that $\omega_0(\vec{v}_x, \vec{\sigma}_s) := s^{-1}\vec{\sigma}_s$ for $(\vec{v}_x, \vec{\sigma}_s) \in T_{(x,s)}P$ is the only Φ'-invariant connection on P.

5.3 Trivial bundles – applications to LQG

In this section, we will determine the set of spherically symmetric connections on $\mathbb{R}^3 \times \mathrm{SU}(2)$ to be used for the description of spherically symmetric gravitational systems (such as black holes) in the framework of loop quantum gravity. To this end, we reformulate Theorem 4.9 for trivial bundles.

The spherically symmetric connections on $P = \mathbb{R}^3 \times \mathrm{SU}(2)$ are such connections, invariant under the action $\Phi\colon \mathrm{SU}(2) \times P \to P$, $(\sigma, (x, s)) \mapsto (\sigma(x), \sigma s)$. Since Φ is not fibre transitive, we cannot use Case 5.7 for the necessary calculations. Moreover, it is not possible to apply the results from [6] (see Example 4.6) because the φ-stabilizer of $x = 0$ equals $\mathrm{SU}(2)$ whereas that of each $x \in \mathbb{R}^3\backslash\{0\}$ is given by the maximal torus $T_x := \{\exp(t\mathfrak{z}(x)\,|\,t \in \mathbb{R})\} \subseteq \mathrm{SU}(2)$. Of course, we could ignore the origin and consider the bundle $\mathbb{R}^3\backslash\{0\} \times \mathrm{SU}(2)$ together with the Φ-covering $\{\lambda \cdot \vec{e}_1 \mid \lambda \in \mathbb{R}_{>0}\}$. This, however, is a different situation because an invariant connection on $\mathbb{R}^3\backslash\{0\} \times \mathrm{SU}(2)$ is not necessarily extendible to an invariant connection on $\mathbb{R}^3 \times \mathrm{SU}(2)$ as the next example illustrates[20].

Example 5.10.

1. Let S be a Lie group and $P = \mathbb{R}^n \times S$. We consider the action $\Phi\colon \mathbb{R}_{>0} \times P \to P$, $(\lambda, (x, s)) \mapsto (\lambda x, s)$ and claim that the only Φ-invariant connection is given by

$$\omega_0(\vec{v}_x, \vec{\sigma}_s) := \mathrm{d}_s L_{s^{-1}}(\vec{\sigma}_s) \qquad \forall\, (\vec{v}_x, \vec{\sigma}_s) \in T_{(x,e)}P.$$

In fact, $P_\infty := \mathbb{R}^n \times \{e\}$ is a Φ-covering of P by Corollary 5.1, and it is straightforward to see (cf. Remark 4.4.3) that condition *i)* from Corollary 4.2 is equivalent to the conditions *a)* and *b)* from Remark 4.4. Let $\psi\colon \mathfrak{g} \times TP_\infty$ be a reduced connection w.r.t. P_∞ and define $\psi_x := \psi|_{\mathfrak{g}\times T_{(x,e)}}$.

Since the exponential map $\exp\colon \mathfrak{g} \to \mathbb{R}_{>0}$ is just given by $\mu \mapsto \mathrm{e}^\mu$ for $\mu \in \mathbb{R} = \mathfrak{g}$, we have $\widetilde{g}((x, e)) = \vec{g} \cdot x \in T_{(x,e)}P_\infty$ for $\vec{g} \in \mathfrak{g}$. Then, for $\vec{w} := -\vec{g} \cdot x \in T_{(x,e)}P_\infty$ from *a)* we obtain

$$\psi_x\big(\vec{g}, \vec{0}\big) = \psi_x\big(\vec{0}_{\mathfrak{g}}, \vec{g} \cdot x\big) \qquad \forall\, \vec{g} \in \mathfrak{g}, \qquad \forall\, x \in \mathbb{R}^n. \tag{5.3}$$

In particular, $\psi_0\big(\vec{g}, \vec{0}\big) = 0$, and since $Q_{(0,e)} = \mathbb{R}_{>0} \times \{e\}$, for $q = (\lambda, e)$ condition *b)* yields

$$\lambda\, \psi_0\big(\vec{0}_{\mathfrak{g}}, \vec{w}\big) = \psi_0\big(\vec{0}_{\mathfrak{g}}, \lambda\vec{w}\big) \overset{b)}{=} \psi_0\big(\vec{0}_{\mathfrak{g}}, \vec{w}\big) \qquad \forall\, \lambda > 0, \qquad \forall\, \vec{w} \in T_{(0,e)}P_\infty,$$

hence $\psi_0 = 0$. Analogously, for $x \neq 0$, $\vec{w} \in T_{(\lambda x,e)}P_\infty$, $\lambda > 0$ and $q = (\lambda, e)$, we obtain

$$\lambda\, \psi_{\lambda x}\big(\vec{0}_{\mathfrak{g}}, \vec{w}\big) = \psi_{\lambda x}\big(\vec{0}_{\mathfrak{g}}, \mathrm{d}L_q(\vec{w})\big) \overset{b)}{=} \rho(q) \circ \psi_x\big(\vec{0}_{\mathfrak{g}}, \vec{w}\big) = \psi_x\big(\vec{0}_{\mathfrak{g}}, \vec{w}\big),$$

i.e., $\psi_{\lambda x}\big(\vec{0}_{\mathfrak{g}}, \vec{w}\big) = \frac{1}{\lambda}\psi_x\big(\vec{0}_{\mathfrak{g}}, \vec{w}\big)$. Here, in the second step, we have used the canonical identification of the linear spaces $T_{(x,e)}P_\infty$ and $T_{(\lambda x,e)}P_\infty$. Using the same identification, from continuity (smoothness) of ψ and $\psi_0 = 0$ we obtain

$$0 = \lim_{\lambda\to 0} \psi_{\lambda x}\big(\vec{0}_{\mathfrak{g}}, \vec{w}\big) = \lim_{\lambda\to 0} \frac{1}{\lambda}\psi_x\big(\vec{0}_{\mathfrak{g}}, \vec{w}\big) \qquad \forall\, x \in \mathbb{R}^n, \qquad \forall\, \vec{w} \in T_{(x,e)}P_\infty$$

so that $\psi_x\big(\vec{0}_{\mathfrak{g}}, \cdot\big) = 0$ for all $x \in \mathbb{R}^n$, hence $\psi = 0$ by (5.3). Finally, it is straightforward to see that $(\Phi^*\omega_0)|_{\mathfrak{g}\times TP_\infty} = \psi = 0$ holds.

2. Let $P' = \mathbb{R}^n\backslash\{0\} \times S$ and Φ be defined as above. Then $K \times \{e\}$, for the unit-sphere $K := \{x \in \mathbb{R}^n \mid \|x\| = 1\}$, is a Φ-covering of P' with the properties from Example 4.6. Evaluating the corresponding conditions *i″)*, *ii″)*, *iii″)*, immediately shows that the set of Φ-invariant connections on P' is in bijection with the smooth maps $\psi\colon \mathbb{R} \times TK \to \mathfrak{s}$ for which $\psi|_{\mathbb{R}\times T_kK}$ is linear for all $k \in K$. The corresponding invariant connections are given by

$$\omega^{\psi}_{(x,s)}(\vec{v}_x, \vec{\sigma}_s) = \psi\left(\tfrac{1}{\|x\|}\mathrm{pr}_\parallel(\vec{v}_x), \mathrm{pr}_\perp(\vec{v}_x)\right) + s^{-1}\vec{\sigma}_s \qquad \forall\, (\vec{v}_x, \vec{\sigma}_s) \in T_{(x,s)}P'.$$

Here, $\mathrm{pr}_\parallel$ denotes the projection onto the axis defined by $x \in \mathbb{R}^n$, as well as $\mathrm{pr}_\perp$ the projection onto the corresponding orthogonal complement in $\mathbb{R}^n$.

[20] See also the remarks following Example 5.12, as well as the connection ω constructed in Example 4.10.2.

Also in the spherically symmetric case the φ-stabilizer of the origin has full dimension, and it turns out to be convenient (cf. Appendix B.2) to use the Φ-covering $\mathbb{R}^3\times\{e\}$ in this situation as well. Since the choice $P_\infty := M\times\{e\}$ is always reasonable (cf. Lemma 3.4.1) if there is a point in the base manifold M (of the trivial bundle $M\times S$) whose stabilizer is the whole group, we now adapt Theorem 4.9 to this situation. To this end, we identify T_xM with $T_{(x,e)}P_\infty$ for each $x\in M$ in the following.

Case 5.11 (trivial principal fibre bundles)**.** *Let (G,Φ) be a Lie group of automorphisms of the trivial principal fibre bundle $P=M\times S$. Then, the Φ-invariant connections are in bijection with the smooth maps $\psi\colon \mathfrak{g}\times TM\to\mathfrak{s}$ for which $\psi|_{\mathfrak{g}\times T_xM}$ is linear for all $x\in M$ and that fulfil the following properties.*

Let $\psi^\pm(\vec{g},\vec{v}_y,\vec{s}) := \psi(\vec{g},\vec{v}_y)\pm\vec{s}$ for $((\vec{g},\vec{s}),\vec{v}_y)\in\mathfrak{q}\times T_yM$. Then, for $q\in Q$, $x\in M$ with $q\cdot(x,e)=(y,e)\in M\times\{e\}$ and all $((\vec{g},\vec{s}),\vec{v}_x)\in\mathfrak{q}\times T_xM$ we have

i) $\widetilde{g}(x,e)+\vec{v}_x-\vec{s}=0 \quad\Longrightarrow\quad \psi^-(\vec{g},\vec{v}_x,\vec{s})=0,$

ii) $\psi^+(\mathrm{d}L_q\vec{v}_x)=\rho(q)\circ\psi\big(\vec{0}_{\mathfrak{g}},\vec{v}_x\big) \qquad \forall\,\vec{v}_x\in T_xM,$

iii) $\psi\big(\mathrm{Ad}_q(\vec{g}),\vec{0}_y\big)=\rho(q)\circ\psi\big(\vec{g},\vec{0}_x\big) \qquad \forall\,\vec{g}\in\mathfrak{g}.$

Proof. The elementary proof can be found in Appendix A. ■

Example 5.12 (spherically symmetric systems in loop quantum gravity)**.** Let $\varrho\colon\mathrm{SU}(2)\to\mathrm{SO}(3)$ be the universal covering map and $\sigma(x):=\varrho(\sigma)(x)$ for $x\in\mathbb{R}^3$. Moreover, let $\mathfrak{z}\colon\mathbb{R}^3\to\mathfrak{su}(2)$ be defined as in the second part of Example 5.8. We consider the action of $G=\mathrm{SU}(2)$ on $P=\mathbb{R}^3\times\mathrm{SU}(2)$ defined by $\Phi(\sigma,(x,s)):=(\varrho(\sigma)(x),\sigma s)$. It is shown in Appendix B.2 that the corresponding invariant connections are of the form

$$\begin{aligned}\omega^{abc}_{(x,s)}(\vec{v}_x,\vec{\sigma}_s) := \mathrm{Ad}_{s^{-1}}\big[&a(x)\mathfrak{z}(\vec{v}_x)+b(x)[\mathfrak{z}(x),\mathfrak{z}(\vec{v}_x)]\\ &+c(x)[\mathfrak{z}(x),[\mathfrak{z}(x),\mathfrak{z}(\vec{v}_x)]]\big]+s^{-1}\vec{\sigma}_s\end{aligned}\tag{5.4}$$

for $(\vec{v}_x,\vec{\sigma}_s)\in T_{(x,s)}P$ and with rotation invariant maps $a,b,c\colon\mathbb{R}^3\to\mathbb{R}$ for which the whole expression is a smooth connection.

We claim that the functions a, b, c can be assumed to be smooth as well. More precisely, we show that we can assume that

$$a(x)=f\big(\|x\|^2\big),\qquad b(x)=g\big(\|x\|^2\big),\qquad c(x)=h\big(\|x\|^2\big)$$

holds for smooth functions $f,g,h\colon(-\epsilon,\infty)\to\mathbb{R}$ with $\epsilon>0$. Then, each pullback of such a spherically symmetric connection by the global section $x\mapsto(x,e)$ can be written in the form

$$\widetilde{\omega}^{abc}_x(\vec{v}_x)=\widetilde{f}\big(\|x\|^2\big)\mathfrak{z}(\vec{v}_x)+\widetilde{g}\big(\|x\|^2\big)\mathfrak{z}(x\times\vec{v}_x)+\widetilde{h}\big(\|x\|^2\big)\mathfrak{z}\left(x\times(x\times\vec{v}_x)\right)$$

for smooth functions $\widetilde{f},\widetilde{g},\widetilde{h}\colon(-\epsilon,\infty)\to\mathbb{R}$ with $\epsilon>0$.

Proof of the claim. 1. Smoothness of ω^{abc} implies smoothness of the real functions

$$a_{\vec{n}}(\lambda):=a(\lambda\vec{n}),\qquad b_{\vec{n}}(\lambda):=\lambda b(\lambda\vec{n}),\qquad c_{\vec{n}}(\lambda):=\lambda^2c(\lambda\vec{n})\qquad\forall\,\lambda\in\mathbb{R}$$

for each $\vec{n}\in\mathbb{R}^3\backslash\{0\}$. In fact, $a_{\vec{n}}(\lambda)\cdot\mathfrak{z}(\vec{n})=\omega^{abc}_{(\lambda\vec{n},e)}(\vec{n})$ is smooth, so that smoothness of $b_{\vec{n}}$ and $c_{\vec{n}}$ is immediate from smoothness of $\lambda\mapsto\omega^{abc}_{(\lambda\vec{e}_1,e)}(\vec{e}_2)$.

2. Let $\vec{n}$ be fixed. Then, $a_{\vec{n}}$ is even so that $a_{\vec{n}}(\lambda)=f(\lambda^2)$ for a smooth function $f\colon(-\epsilon_1,\infty)\to\mathbb{R}$, see [10]. Moreover, $b_{\vec{n}}$ is smooth and odd, so that $b_{\vec{n}}(\lambda)=\lambda g\big(\lambda^2\big)$ for a smooth function

$g\colon (-\epsilon_2, \infty) \to \mathbb{R}$, again by [10]. Similarly, $c_{\vec{n}}(\lambda) = l(\lambda^2)$ for a smooth function $l\colon (-\epsilon_3, \infty) \to \mathbb{R}$. Since $\lambda \mapsto l(\lambda^2)$ is even and $l(0) = 0$, for $s \in \mathbb{N}_{>0}$ Taylor's formula yields

$$l\big(x^2\big) = a_1 x^2 + \cdots + a_s x^{2s} + x^{2(s+1)}\phi(x) = x^2\big(a_1 + \cdots + a_s x^{2s-2} + x^{2s}\phi(x)\big) = x^2 L(x)$$

with remainder term $\phi(x) := \frac{1}{(2s+1)!}\frac{1}{x^{2s+2}}\int_0^x (x-t)\, l^{(2s+2)}(t)\,\mathrm{d}t$ for $x \neq 0$ and $\phi(0) := l^{(2s+2)}(0)$. Now, ϕ is continuous by Theorem 1 in [9], so that L is continuous as well. But $x \mapsto x^2 L(x)$ is smooth, so that Corollary 1 in [9] shows that L is smooth as well. Now, L is even, hence $L(x) = h(x^2)$ for some smooth function $h\colon (-\epsilon_4, \infty) \to \mathbb{R}$. Then, $c_{\vec{n}}(\lambda) = l\big(\lambda^2\big) = \lambda^2 h\big(\lambda^2\big)$, and for $x \neq 0$ we get

$$\begin{aligned} b(x) &= \|x\|\, b\left(\|x\|\frac{x}{\|x\|}\right)\frac{1}{\|x\|} = b_{\frac{x}{\|x\|}}(\|x\|)\frac{1}{\|x\|} = g\left(\|x\|^2\right), \\ c(x) &= \|x\|^2 c\left(\|x\|\frac{x}{\|x\|}\right)\frac{1}{\|x\|^2} = c_{\frac{x}{\|x\|}}\left(\|x\|\right)\frac{1}{\|x\|^2} = h\left(\|x\|^2\right). \end{aligned}$$

Moreover, for $x = 0$ we have

$$\begin{aligned} b(x)[\mathfrak{z}(x), \mathfrak{z}(\vec{v}_x)] &= 0 = g\big(\|x\|^2\big)[\mathfrak{z}(x), \mathfrak{z}(\vec{v}_x)], \\ c(x)[\mathfrak{z}(x), [\mathfrak{z}(x), \mathfrak{z}(\vec{v}_x)]] &= 0 = h(x)[\mathfrak{z}(x), [\mathfrak{z}(x), \mathfrak{z}(\vec{v}_x)]] \end{aligned}$$

so that we can assume $a(x) = f(\|x\|^2)$, $b(x) = g(\|x\|^2)$ and $c(x) = h(\|x\|^2)$ for the smooth functions $f, g, h\colon (-\min(\epsilon_1, \ldots, \epsilon_4), \infty) \to \mathbb{R}$. ■

In particular, there are spherically symmetric connections on $\mathbb{R}^3\backslash\{0\}\times \mathrm{SU}(2)$ which cannot be extended to those on P. For instance, if $b = c = 0$ and $a(x) := 1/\|x\|$ for $x \in \mathbb{R}^3\backslash\{0\}$, then ω^{abc} cannot be extended smoothly to an invariant connection on $\mathbb{R}^3 \times \mathrm{SU}(2)$ since elsewise $a_{\vec{n}}$ could be extended to a continuous (smooth) function on $\mathbb{R}$.

6 Conclusions

We conclude with a short review of the particular cases that follow from Theorem 4.9. For this let (G, Φ) be a Lie group of automorphisms of the principal fibre bundle (P, π, M, S) and φ the induced action on M.

- If $P = M \times S$ is trivial, then $M \times \{e\}$ is a Φ-covering of P. As we have demonstrated in the spherically symmetric and scale invariant case (cf. Examples 5.10 and 5.12), this choice can be useful for calculations if there is a point in M whose φ-stabilizer is the whole group G.
- If there is an element $x \in M$ which is contained in the closure of each φ-orbit, each Θ-patch which contains some $p \in \pi^{-1}(x)$ is a Φ-covering of P, see Example 5.5. If φ acts transitively on M, for each $p \in P$ the zero-dimensional submanifold $\{p\}$ is a Φ-covering of P; giving back Wang's original theorem, see Case 5.7 and Example 5.8.
- Let Φ act via gauge transformations on P. In this case each open covering $\{U_\alpha\}_{\alpha\in I}$ of M together with smooth sections $s_\alpha\colon U_\alpha \to P$ provides the Φ-covering $\{s_\alpha(U_\alpha)\}_{\alpha\in I}$ of P. If G acts trivially, this specializes to the usual description of smooth connections by means of consistent families of local 1-forms on the base manifold M.
- If P_0 is a Θ-patch such that $\pi(P_0)$ intersects each φ-orbit in a unique point, it is a Φ-covering. If in addition the stabilizer Q_p does not depend on $p \in P_0$, we get back the characterization from [6], see Example 4.6.

- Assume there is a collection of φ-orbits forming an open subset $U \subseteq M$. Then, $O := \pi^{-1}(U)$ is a principal fibre bundle and each Φ-invariant connection on P restricts to a Φ-invariant connection on O. Conversely, if U is in addition dense in M, one can ask the question whether a Φ-invariant connection on O extends to a Φ-invariant connection on P. Since such an extension is necessarily unique (continuity), φ-orbits not contained in U can be seen as sources of obstructions for the extendability of invariant connections on O to P. Indeed, as the examples in Subsection 5.3 show, smoothness of these extension can give crucial restrictions. Moreover, by Example 5.5, taking one additional orbit into account can shrink the number of invariant connections to zero. Of particular interest, in this context, is the case where G is compact, as then the orbits of principal type always form a dense and open subset of M on which the situation of [6] always holds locally [7]. This gives rise to a canonical Φ-covering O consisting of convenient patches. Thus, using the present characterization theorem, there is a realistic chance to get some general classification results in the compact case[21].

As Corollary 5.1 shows, in the general situation one can always construct Φ-coverings of P from families of φ-patches in M. In particular, the first three cases arise in this way.

Appendix

A A technical proof

Proof of Case 5.11. The only patch is $M \times \{e\}$, so that a reduced connection is a smooth map $\psi\colon \mathfrak{g} \times TM \to \mathfrak{s}$ with the claimed linearity property and that fulfils the two conditions from Corollary 4.2. Obviously, *ii*) and iii) are equivalent. Moreover, i) follows from *i*) for $p_\alpha = p_\beta = (x, e)$, $q = (e, e)$, $\vec{w}_{p_\beta} = \vec{v}_x$ and $\vec{w}_{p_\alpha} = \vec{0}_{(x,e)}$, see also *a*) in Remark 4.4. Now, to obtain ii), let $\vec{v}_x \in T_xM$, $q \in Q$ and $q \cdot (x, e) = (y, e)$. Then, $\mathrm{d}L_q\vec{v}_x = (\vec{v}_y, -\vec{s})$ for elements $\vec{v}_y \in T_yM$ and $\vec{s} \in \mathfrak{s}$ so that

$$\psi^+(\mathrm{d}L_q\vec{v}_x) = \psi^+(\vec{v}_y, -\vec{s}) = \psi\big(\vec{0}_\mathfrak{g}, \vec{v}_y\big) - \vec{s} \overset{i)}{=} \rho(q) \circ \psi\big(\vec{0}_\mathfrak{g}, \vec{v}_x\big).$$

It remains to show that i) and ii) imply *i*). To this end, let $(y, e) = q \cdot (x, e)$ for $x, y \in M$ and $q \in Q$. Then *i*) reads

$$\widetilde{g}(y, e) + \vec{v}_y - \vec{s} = \mathrm{d}L_q\vec{v}_x \quad\Longrightarrow\quad \psi^-(\vec{g}, \vec{v}_y, \vec{s}) = \rho(q) \circ \psi\big(\vec{0}_\mathfrak{g}, \vec{v}_x\big),$$

where $\vec{v}_x \in T_xM$, $\vec{v}_y \in T_yM$, $\vec{s} \in \mathfrak{s}$ and $\vec{g} \in \mathfrak{g}$. Let $\mathrm{d}L_q\vec{v}_x = (\vec{v}_y, -\vec{s})$ be as above. If ii) is true, then it is clear from

$$\psi^-(\vec{v}_y, \vec{s}) = \psi^+(\mathrm{d}L_q\vec{v}_x) \overset{\text{ii)}}{=} \rho(q) \circ \psi\big(\vec{0}_\mathfrak{g}, \vec{v}_x\big)$$

that *i*) is true for $\big((\vec{0}_\mathfrak{g}, \vec{s}), \vec{v}_y\big)$, i.e.,

$$\vec{0}_\mathfrak{g} + \vec{v}_y - \vec{s} = \mathrm{d}L_q\vec{v}_x \quad\Longrightarrow\quad \psi\big(\vec{0}_\mathfrak{g}, \vec{v}_y\big) - \vec{s} = \rho(q) \circ \psi\big(\vec{0}_\mathfrak{g}, \vec{v}_x\big).$$

Due to i) and the linearity properties of ψ, the condition *i*) then also holds for each other element $((\vec{g}', \vec{s}'), \vec{v}'_y) \in \mathfrak{q} \times T_yM$ with $\widetilde{g}'(y, e) + \vec{v}'_y - \vec{s}' = \mathrm{d}L_q\vec{v}_x$. ■

[21] To be used, e.g., to extend the framework of the foundational LQG reduction paper [2].

B Technical calculations

Let $P = \mathbb{R}^3 \times \mathrm{SU}(2)$, $\varrho\colon \mathrm{SU}(2) \to \mathrm{SO}(3)$ the universal covering map, $E = \mathbb{R}^3 \rtimes_\varrho \mathrm{SU}(2)$ and $\mathfrak{z}\colon \mathbb{R}^3 \to \mathfrak{su}(2)$ be defined as in the second part of Example 5.8. Then, $\varrho(\sigma) = \mathfrak{z}^{-1} \circ \mathrm{Ad}_\sigma \circ \mathfrak{z}$ and each $\sigma \in \mathrm{SU}(2)$ can be written as

$$\sigma = \cos(\alpha/2)\mathbb{1} + \sin(\alpha/2)\mathfrak{z}(\vec{n}) = \exp\left(\alpha/2 \cdot \mathfrak{z}(\vec{n})\right)$$

for some $|\vec{n}| = 1$ and $\alpha \in [0, 2\pi]$. In this case $\varrho(\sigma)$ rotates a point x by the angle α w.r.t. the axis $\vec{n}$. For simplicity, if $\sigma \in \mathrm{SU}(2)$ and $x \in \mathbb{R}^3$, we write $\sigma(x)$ instead of $\varrho(\sigma)(x)$ in the following.

B.1 A result used in the end of Section 5

We consider the fibre transitive action $\Phi'\colon E \times P \to P$ defined by $\Phi'((v,\sigma),(x,s)) := (v+\sigma(x), s)$ and claim that the connection

$$\omega_0(\vec{v}_x, \vec{\sigma}_s) = s^{-1}\vec{\sigma}_s \qquad \forall\, (\vec{v}_x, \vec{\sigma}_s) \in T_{(x,s)}P$$

is the only Φ'-invariant one. For this, observe that the stabilizer of $x = 0$ w.r.t. φ' is given by $\mathrm{SU}(2)$ and $\phi'_{(0,e)}(\sigma) = e$ for all $\sigma \in \mathrm{SU}(2)$. We apply Wang's theorem to $p = (0, e)$. Then condition a) yields $\psi(\vec{s}) = 0$ for all $\vec{s} \in \mathfrak{su}(2)$, and b) now reads $\psi \circ \mathrm{Ad}_\sigma = \psi$ for all $\sigma \in \mathrm{SU}(2)$. Consequently, for $\vec{v} \in \mathbb{R}^3 \subseteq \mathfrak{e} = \mathbb{R}^3 \times \mathfrak{su}(2)$ we obtain

$$0 = \tfrac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}\, \psi(\vec{v}) = \tfrac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}\, \psi \circ \mathrm{Ad}_{\exp(t\vec{s})}(\vec{v}) = \psi\left(\tfrac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}\, \varrho(\exp(t\vec{s}))(\vec{v})\right) = \psi \circ \mathfrak{z}^{-1}([\vec{s}, \mathfrak{z}(\vec{v})])$$

for all $\vec{s} \in \mathfrak{su}(2)$, just by linearity of ψ. This gives

$$0 = \psi\big(\mathfrak{z}^{-1}([\tau_i, \mathfrak{z}(\vec{e}_j)])\big) = \psi\big(\mathfrak{z}^{-1}([\tau_i, \tau_j])\big) = 2\epsilon_{ijk}\psi(\vec{e}_k),$$

hence $\psi = 0 = \Phi'^*_p\omega_0$.

B.2 Spherically symmetric connections

We consider the action Φ of $\mathrm{SU}(2)$ on P defined by $\Phi(\sigma,(x,s)) := (\sigma(x), \sigma s)$, and show that the corresponding invariant connections are given by (see (5.4) in Example 5.12)

$$\omega^{abc}_{(x,s)}(\vec{v}_x, \vec{\sigma}_s) := \mathrm{Ad}_{s^{-1}}\left[a(x)\mathfrak{z}(\vec{v}_x) + b(x)[\mathfrak{z}(x), \mathfrak{z}(\vec{v}_x)] + c(x)[\mathfrak{z}(x), [\mathfrak{z}(x), \mathfrak{z}(\vec{v}_x)]]\right] + s^{-1}\vec{\sigma}_s$$

with rotation invariant maps $a, b, c\colon \mathbb{R}^3 \to \mathbb{R}$ for which the whole expression is a smooth connection. Now, a straightforward calculation shows that each ω^{abc} is Φ-invariant, so that it remains to verify that each Φ-invariant connection is of the upper form. For this, we reduce the connections ω^{abc} w.r.t. $P_\infty = \mathbb{R}^3 \times \{e\}$ and show that each map ψ as in Case 5.11 can be obtained in this way. To this end, let $\vec{g} \in \mathfrak{g}$, $p = (x, e) \in P_\infty$ and $\gamma_x\colon (-\epsilon, \epsilon) \to M$ be a smooth curve with $\dot{\gamma}_x(0) = \vec{v}_x \in T_xM \subseteq T_pP_\infty$. Then,

$$\begin{aligned} \mathrm{d}_{(e,p)}\Phi(\vec{g}, \vec{v}_x) &= \left(\tfrac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}\mathfrak{z}^{-1}\left(\exp(t\vec{g})\mathfrak{z}(\gamma_x(t))\exp(t\vec{g})^{-1}\right), \exp(t\vec{g})\right) \\ &= \left(\mathfrak{z}^{-1}\left([\vec{g}, \mathfrak{z}(x)]\right) + \vec{v}_x, \vec{g}\right). \end{aligned} \tag{B.1}$$

This equals $\vec{s}$ iff $\vec{g} = \vec{s}$ and $\vec{v}_x = \mathfrak{z}^{-1}([\mathfrak{z}(x), \vec{g}])$. Consequently, for the reduced connection ψ^{abc} which corresponds to ω^{abc} we obtain

$$\begin{aligned} \psi^{abc}(\vec{g}, \vec{v}_x) &= \left(\Phi^*\omega^{abc}\right)_{(e,p)}(\vec{g}, \vec{v}_x) = \omega^{abc}_p\left(\mathfrak{z}^{-1}\left([\vec{g}, \mathfrak{z}(x)] + \mathfrak{z}(\vec{v}_x)\right), \vec{g}\right) \\ &= a(x)\big[[\vec{g}, \mathfrak{z}(x)] + \mathfrak{z}(\vec{v}_x)\big] + b(x)\big[[\mathfrak{z}(x), [\vec{g}, \mathfrak{z}(x)]] + [\mathfrak{z}(x), \mathfrak{z}(\vec{v}_x)]\big] \\ &\quad + c(x)\big[[\mathfrak{z}(x), [\mathfrak{z}(x), [\vec{g}, \mathfrak{z}(x)]]] + [\mathfrak{z}(x), [\mathfrak{z}(x), \mathfrak{z}(\vec{v}_x)]]\big] + \vec{g}. \end{aligned}$$

Now, assume that ψ is as in Case 5.11. Then for $q \in Q$ and $p \in P_\infty$ we have $q \cdot p \in P_\infty$ iff $q = (\sigma, \sigma)$ for some $\sigma \in \mathrm{SU}(2)$ and $p = (x, e)$ for some $x \in M$. Consequently, $q \cdot p = (\sigma(x), e)$ as well as $\mathrm{d}L_q(\vec{v}_x) = \sigma(\vec{v}_x)$ for all $\vec{v}_x \in T_xM$ so that ii) gives

$$\psi\big(\vec{0}_{\mathfrak{g}}, \sigma(\vec{v}_x)\big) = \psi^+(\mathrm{d}L_q(\vec{v}_x)) = \mathrm{Ad}_\sigma \circ \psi\big(\vec{0}_{\mathfrak{g}}, \vec{v}_x\big),$$

hence

$$\psi\big(\vec{0}_{\mathfrak{g}}, \vec{v}_x\big) = \mathrm{Ad}_{\sigma^{-1}} \circ \psi\big(\vec{0}_{\mathfrak{g}}, \sigma(\vec{v}_x)\big) \qquad \forall\, \vec{v}_x \in T_xM. \tag{B.2}$$

If $x \neq 0$, then for $\sigma_t := \exp(t\mathfrak{z}(x))$ we have $\sigma_t(x) = x$ and $\sigma_t(\vec{v}_x) \in T_xM$ for all $t \in \mathbb{R}$. Then, linearity of $\psi_x := \psi|_{\mathfrak{g}\times T_{(x,e)}P_\infty}$ yields

$$\begin{aligned} 0 &= \tfrac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}\psi\big(\vec{0}_{\mathfrak{g}}, \vec{v}_x\big) \overset{(B.2)}{=} \tfrac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}\mathrm{Ad}_{\sigma_t^{-1}} \circ \psi\big(\vec{0}_{\mathfrak{g}}, \sigma_t(\vec{v}_x)\big) \\ &= \tfrac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}\sigma_t^{-1}\big(\psi_x \circ \mathfrak{z}^{-1}\big)\left(\sigma_t\, \mathfrak{z}(\vec{v}_x)\, \sigma_t^{-1}\right)\sigma_t \\ &\overset{\text{lin.}}{=} -\mathfrak{z}(x)\psi_x\big(\vec{0}_{\mathfrak{g}}, \vec{v}_x\big) + \big(\psi_x \circ \mathfrak{z}^{-1}\big)\left[\mathfrak{z}(x)\mathfrak{z}(\vec{v}_x) - \mathfrak{z}(\vec{v}_x)\mathfrak{z}(x)\right] + \psi_x\big(\vec{0}_{\mathfrak{g}}, \vec{v}_x\big)\mathfrak{z}(x), \end{aligned}$$

hence $\big[\mathfrak{z}(x), \psi\big(\vec{0}_{\mathfrak{g}}, \vec{v}_x\big)\big] = \big(\psi \circ \mathfrak{z}^{-1}\big)\left([\mathfrak{z}(x), \mathfrak{z}(\vec{v}_x)]\right)$. For $x = \lambda\vec{e}_1 \neq 0$ and $\kappa_j := \psi\big(\vec{0}_{\mathfrak{g}}, \vec{v}_x\big)$ with $\vec{v}_x = \vec{e}_j$ this reads

$$[\tau_1, \kappa_j] = \big(\psi_x \circ \mathfrak{z}^{-1}\big)([\tau_1, \tau_j]) = \big(\psi_x \circ \mathfrak{z}^{-1}\big)(2\epsilon_{1jk}\tau_k) = 2\epsilon_{1jk}\psi_x\big(\vec{0}_{\mathfrak{g}}, \vec{e}_k\big) = 2\epsilon_{1jk}\kappa_k.$$

From these relations, it follows that

$$\kappa_1 = r(\lambda)\tau_1, \qquad \kappa_2 = s(\lambda)\tau_2 + t(\lambda)\tau_3, \qquad \kappa_3 = s(\lambda)\tau_3 - t(\lambda)\tau_2$$

for real constants $r(\lambda), s(\lambda), t(\lambda)$ depending on $\lambda \in \mathbb{R}\backslash\{0\}$. Then, for $x = \lambda\vec{e}_1$ and

$$a(\lambda\vec{e}_1) := r(\lambda), \qquad b(\lambda\vec{e}_1) := \frac{t(\lambda)}{2\lambda}, \qquad c(\lambda\vec{e}_1) := \frac{r(\lambda) - s(\lambda)}{4\lambda^2}$$

linearity of ψ_x yields

$$\psi\big(\vec{0}_{\mathfrak{g}}, \vec{v}_x\big) = a(x)\mathfrak{z}(\vec{v}_x) + b(x)[\mathfrak{z}(x), \mathfrak{z}(\vec{v}_x)] + c(x)[\mathfrak{z}(x), [\mathfrak{z}(x), \mathfrak{z}(\vec{v}_x)]].$$

Now, if $x \neq 0$ is arbitrary, then $x = \sigma(\lambda\vec{e}_1)$ for some $\sigma \in \mathrm{SU}(2)$ and $\lambda > 0$. So, $(\sigma, \sigma) \cdot (\lambda\vec{e}_1, e) = (x, e)$ and if we consider $\sigma^{-1}(\vec{v}_x)$ as an element of $T_{(\lambda\vec{e}_1,e)}P_\infty$, then ii) gives

$$\begin{aligned} \psi\big(\vec{0}_{\mathfrak{g}}, \vec{v}_x\big) &= \psi^+(\vec{v}_x) = \psi^+\big(\mathrm{d}L_{(\sigma,\sigma)}\big(\sigma^{-1}(\vec{v}_x)\big)\big) \\ &\overset{\text{ii)}}{=} \mathrm{Ad}_\sigma \circ \psi^+\big(\sigma^{-1}(\vec{v}_x)\big) = \mathrm{Ad}_\sigma \circ \psi\big(\vec{0}_{\mathfrak{g}}, \sigma^{-1}(\vec{v}_x)\big) \\ &= a(\lambda\vec{e}_1)\mathfrak{z}(\vec{v}_x) + b(\lambda\vec{e}_1)\left[\mathfrak{z}(x), \mathfrak{z}(\vec{v}_x)\right] + c(\lambda\vec{e}_1)[\mathfrak{z}(x), [\mathfrak{z}(x), \mathfrak{z}(\vec{v}_x)]]. \end{aligned}$$

For $x = 0$ we have $\sigma(x) = x$ for all $\sigma \in \mathrm{SU}(2)$, and analogous to the case $x \neq 0$, but now for $\sigma_t := \exp(t\vec{g})$ with $\vec{g} \in \mathfrak{g}$, we obtain from (B.2) that

$$\big[\vec{g}, \psi_0\big(\vec{0}_{\mathfrak{g}}, \vec{v}_0\big)\big] = \big(\psi_0 \circ \mathfrak{z}^{-1}\big)\left([\vec{g}, \mathfrak{z}(\vec{v}_0)]\right) \qquad \forall\, \vec{g} \in \mathfrak{su}(2), \qquad \forall\, \vec{v}_0 \in T_0M.$$

This gives $\big[\tau_i, \psi\big(\vec{0}_{\mathfrak{g}}, \vec{e}_j\big)\big] = 2\epsilon_{ijk}\psi\big(\vec{0}_{\mathfrak{g}}, \vec{e}_k\big)$ and forces $\psi(\vec{v}_0) = a(0)\mathfrak{z}(\vec{v}_0)$ for all $\vec{v}_0 \in T_{(0,e)}P_\infty$ whereby $a(0) \in \mathbb{R}$ is some constant. Together, this shows

$$\psi\big(\vec{0}_{\mathfrak{g}}, \vec{v}_x\big) = a(x)\mathfrak{z}(\vec{v}_x) + b(x)[\mathfrak{z}(x), \mathfrak{z}(\vec{v}_x)] + c(x)[\mathfrak{z}(x), [\mathfrak{z}(x), \mathfrak{z}(\vec{v}_x)]]$$

with functions a, b, c that depend on $\|x\|$ in such a way that the whole expression is smooth. Finally, to determine $\psi\big(\vec{g}, \vec{0}_x\big)$ for $\vec{g} \in \mathfrak{su}(2) = \mathfrak{g}$, we consider $\mathfrak{z}^{-1}([\mathfrak{z}(x), \vec{g}])$ as an element of $T_{(x,e)}P_\infty$. Then by (B.1) we obtain from i) that $\psi\big(\vec{g}, \mathfrak{z}^{-1}([\mathfrak{z}(x), \vec{g}])\big) - \vec{g} = 0$, hence

$$\begin{aligned} \psi\big(\vec{g}, \vec{0}_x\big) &= \vec{g} - \psi\big(\vec{0}_{\mathfrak{g}}, \mathfrak{z}^{-1}([\mathfrak{z}(x), \vec{g}])\big) \\ &= \vec{g} - a(x)[\mathfrak{z}(x), \vec{g}] - b(x)[\mathfrak{z}(x), [\mathfrak{z}(x), \vec{g}]] - c(x)[\mathfrak{z}(x), [\mathfrak{z}(x), [\mathfrak{z}(x), \vec{g}]]] \\ &= a(x)[\vec{g}, \mathfrak{z}(x)] + b(x)[\mathfrak{z}(x), [\vec{g}, \mathfrak{z}(x)]] + c(x)[\mathfrak{z}(x), [\mathfrak{z}(x), [\mathfrak{z}(x), \vec{g}]]] + \vec{g}. \end{aligned}$$

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References

[1] Ashtekar A., Bojowald M., Lewandowski J., Mathematical structure of loop quantum cosmology, *Adv. Theor. Math. Phys.* **7** (2003), 233–268, gr-qc/0304074.

[2] Bojowald M., Kastrup H.A., Symmetry reduction for quantized diffeomorphism-invariant theories of connections, *Classical Quantum Gravity* **17** (2000), 3009–3043, hep-th/9907042.

[3] Duistermaat J.J., Kolk J.A.C., Lie groups, *Universitext*, Springer-Verlag, Berlin, 2000.

[4] Fleischhack Ch., Loop quantization and symmetry: configuration spaces, arXiv:1010.0449.

[5] Hanusch M., Invariant connections in loop quantum gravity, arXiv:1307.5303.

[6] Harnad J., Shnider S., Vinet L., Group actions on principal bundles and invariance conditions for gauge fields, *J. Math. Phys.* **21** (1980), 2719–2724.

[7] Rudolph G., Schmidt M., Differential geometry and mathematical physics. Part I. Manifolds, Lie groups and Hamiltonian systems, *Theoretical and Mathematical Physics*, Springer, Dordrecht, 2013.

[8] Wang H.C., On invariant connections over a principal fibre bundle, *Nagoya Math. J.* **13** (1958), 1–19.

[9] Whitney H., Differentiability of the remainder term in Taylor's formula, *Duke Math. J.* **10** (1943), 153–158.

[10] Whitney H., Differentiable even functions, *Duke Math. J.* **10** (1943), 159–160.