Twistors and Bi-Hermitian Surfaces of Non-Kähler Type[⋆]

Akira FUJIKI [†] *and Massimiliano PONTECORVO* [‡]

[†] *Research Institute for Mathematical Sciences, Kyoto University, Japan*
  E-mail: *fujiki@math.sci.osaka-u.ac.jp*

[‡] *Dipartimento di Matematica e Fisica, Università Roma Tre., Italy*
  E-mail: *max@mat.uniroma3.it*

Abstract. The aim of this work is to give a twistor presentation of recent results about bi-Hermitian metrics on compact complex surfaces with odd first Betti number.

Key words: non-Kähler surfaces; bi-Hermitian metrics; twistor space

2010 Mathematics Subject Classification: 53C15; 53C28

1 Introduction

We treat in this work bi-Hermitian surfaces, by which we mean orientable Riemannian conformal four-manifolds $(M, [g])$ with a pair of $[g]$-orthogonal complex structures J_+ and J_- which induce the *same* orientation and are linearly independent, i.e., there is some $p \in M$ where $J_+(p) \neq \pm J_-(p)$. The original motivation is given by a general interest for Riemannian manifolds admitting more than one orthogonal complex structure [22, 38, 39, 40].

The case when $J_\pm$ are opposite oriented is also interesting and gives rise to a holomorphic splitting of the tangent bundle; see [6] for a complete account.

In what follows it will be important to look at possible relations between the two Lee forms (denoted by $\beta_\pm$) of the Hermitian metrics $(g, J_\pm)$. For a general Hermitian surface (M, g, J) with fundamental $(1,1)$-form $\omega(\cdot, \cdot) := g(\cdot, J\cdot)$ the Lee form is the 1-form uniquely defined by the relation $d\omega = \beta \wedge \omega$. Recall that we are in real dimension four. The 1-form β captures most of the conformal features of the Hermitian metric: (g, J) is Kähler if and only if $\beta = 0$, conformally Kähler if and only if β is exact and is locally conformally Kähler, abbreviated by l.c.K., if and only if β is closed.

We will always assume that M is compact and connected; let us recall at this point an important result of Gauduchon [20] which says in particular that on any compact conformal Hermitian surface $(M, [g], J)$ there always is a unique metric (up to homothety) in the conformal class $[g]$ for which the Lee form is co-closed; this metric is usually called the *Gauduchon metric* of $(M, [g], J)$. In particular one can take β harmonic if $(M, [g], J)$ is compact l.c.K.

Our main interest is to study the complex structure of the two surfaces $S_+ := (M, J_+)$ and $S_- := (M, J_-)$ under the condition that the first Betti number $b_1(M)$ is *odd*. In other words we will be concerned with non-Kähler surfaces and it is therefore important to find complex curves on them. To this end let

$$T := \{ p \in M \mid J_+(p) = \pm J_-(p) \}$$

---

denote the set of points where the two complex structures agree up to sign; it plays an important role in understanding the complex geometry of the surfaces because it turns out to always be the zero-set of a holomorphic section of a line bundle:

Proposition 1.1 ([5, 38]). *The (possibly empty) set T is the support of a common effective divisor $\mathbf{T} \geq 0$ in both surfaces $S_{\pm}$.*

Before discussing how T can be described as a complex curve inside the twistor space of $(M, [g])$ and compute the first Chern class $c_1(\mathbf{T})$ using the twistor picture, we have the following

Remark 1.2. From the differential geometric viewpoint, T is the closure of the union of all smooth surfaces in M which are simultaneously $J_{\pm}$-holomorphic.

To see this, let $C \subset (M, [g], J_{\pm})$ be a smooth surface (real 2-dimensional) which is both a J_+-holomorphic and J_--holomorphic curve and consider the tangent space T_pC at any point $p \in C$. Let V denote this real 2-dimensional subspace of T_pM. Because $J_{\pm}$ are assumed to be g-compatible the orthogonal complement W is also a $J_{\pm}$-complex subspace of T_pM; in each 2-dimensional subspace V and W the two complex structures $J_{\pm}$ are rotation by $90°$ and induce the same orientation on T_pM if and only if they are linearly dependent a p.

The complex curve T turns out in general to be non-smooth with several irreducible components meeting transversally, the smooth open set of each component is a $J_{\pm}$ holomorphic curve C as above and T is the closure of the union of all such C.

This observation is relevant to the blow-up construction of Cavalcanti–Gualtieri [12] because in their case the exceptional divisor E does not belong to the anti-canonical divisor $-K$, which is supported precisely on T. This shows that E is a J_+-holomorphic curve which is not J_--holomorphic.

In order to geometrically describe how T turns out to be a complex curve with a natural structure of a divisor in each surface $(M, J_{\pm})$ we now present a *twistor approach* to bi-Hermitian metrics in four-dimension – following [38] – which produced new examples with b_1 odd [19] as we shall discuss later.

An orthogonal almost complex structure is the same as a smooth section $J : M \to Z$ of the twistor space Z which is the fiber bundle of all linear complex structures at T_pM compatible with the metric and orientation; in four-dimension, the fiber at $p \in M$ is the homogeneous space $\mathrm{SO}(4)/\mathrm{U}(2) \cong \mathbb{CP}_1$. It is known that Z is an almost complex 6-manifold – let $\mathbb{J}$ denote its almost complex structure – which only depends on the fixed conformal structure $[g]$ and orientation of M. The integrability of J is equivalent to the fact that $J : (M, J) \to (Z, \mathbb{J})$ is an almost holomorphic map and that its image $J(M) =: S$ is an almost complex submanifold of $(Z, \mathbb{J})$ which is therefore (tautologically) biholomorphic to the original complex surface: $S \cong (M, J)$, see for example [1]. Notice that we make no assumption on the integrability of the twistor almost complex structure $\mathbb{J}$.

The twistor space Z is also equipped with a natural *real structure*, namely the anti-holomorphic involution $\sigma : Z \to Z$ that sends $J \mapsto -J$; objects that are σ-invariant are then called 'real'. For example, setting $X := S \amalg \sigma(S)$ we get a 'real' submanifold of Z and as M is compact we can consider its Poincaré dual $X^* \in H^2_\sigma(Z, \mathbb{Z})$ which is a 'real' 2-cohomology class, i.e. σ-invariant. By Leray–Hirsch theorem this space is 1-dimensional generated, over $\mathbb{Q}$, by the first Chern class $c_1(Z)$ so that $X^* = q \cdot c_1(Z)$ for some $q \in \mathbb{Q}$.

Each fiber $L \cong \mathrm{SO}(4)/\mathrm{U}(2) \cong \mathbb{CP}_1$ is also σ-invariant, usually called a 'real twistor line', and turns out to have normal bundle $\nu_{L/Z} \cong \mathcal{O}_{\mathbb{CP}_1}(1) \oplus \mathcal{O}_{\mathbb{CP}_1}(1)$. Therefore, $c_1(Z)|_L = 4$ and because for each L the two almost complex submanifolds X and L intersect in exactly two points we conclude that $q = \frac{1}{2}$.

This can be used to compute the first Cherm class of a Hermitian conformal surface $(M, [g], J)$ from its twistor space Z by adjunction formula to the almost complex submanifold $S \subset Z$. Using

that $S \cong (M, J)$ and $X = S \amalg \sigma(S)$ we get from adjunction that

$$c_1(M, J) = [c_1(Z) - X^*]|_S = \frac{1}{2} c_1(Z)|_S. \tag{1}$$

In our case we have a bi-Hermitian structure $([g], J_\pm)$ from which we get two complex surfaces in the twistor space Z. We will use the following notation $S_\pm := J_\pm(M)$ and similarly $X_\pm := S_\pm \amalg \sigma(S_\pm)$. Notice again that there is a tautological biholomorphism $(M, J_\pm) \cong S_\pm$.

In this notation the set of points T can be identified with either of the following two subsets in the twistor space: $X_+ \cap S_-$ or $S_+ \cap X_- \subset Z$; this exhibits T as an almost complex subvariety of either S_+ and S_-; T is therefore a complex curve in each of the two smooth surfaces, in particular closed in the analytic Zariski topology and nowhere dense [38, Proposition 1.3].

Proposition 1.3. *The complex curve T has a natural structure of divisor $\mathbf{T}$ in both surfaces $S_\pm$ given by the property*

$$c_1(\mathbf{T}) = c_1(S_\pm).$$

In other words, for each compact complex surface $(S_\pm)$ there is a holomorphic line bundle $F_\pm$ such that

(i) $c_1(F_\pm) = 0$, *and*

(ii) $\mathbf{T} = F_\pm - K_\pm$ *where $K_\pm$ is the canonical line bundle of the surface $S_\pm$.*

Proof. In the twistor space Z we have that $\mathbf{T} \cong X_- \cap S_+ \cong X_+ \cap S_-$ therefore $c_1(\mathbf{T}) = \frac{1}{2} c_1(Z)|_{S_+} = \frac{1}{2} c_1(Z)|_{S_-}$ and the conclusion follows from (1). ∎

Remark 1.4. In general, the complex curve T has several irreducible components and it gets the structure of effective divisor $\mathbf{T}$, with multiplicities, by taking the (only) linear combination such that $c_1(\mathbf{T}) = c_1(J_-) = c_1(J_+)$. As usual, in what follows $\mathbf{T}$ will denote the divisor as well as the holomorphic line bundle.

Although the two complex structures J_+ and J_- are in general not biholomorphic, it turns out that they share common properties. For this reason we use the notation $J_\pm$ to denote either of them.

As shown in [2], the flat line bundles F_+ and F_- are also isomorphic because they come from the same representation of $\pi_1(M)$. In fact, it is important to recall that as flat line bundles on M they correspond to the 1-form $-\frac{1}{2}(\beta_+ + \beta_-)$ [2] the opposite average of the two Lee forms of $(g, J_\pm)$); by [5, p. 420] this turns out to be always a closed 1-form on a compact M. We will then denote by F the line bundle of zero Chern class $F_\pm$ and we will be looking at the fundamental equation

$$\mathbf{T} = F - K, \tag{2}$$

which on each of the two surfaces $(M, J_\pm)$ relates the divisor $\mathbf{T}$ to the canonical line bundle K.

We will use the fundamental equation (2) above to understand the complex structures of a bi-Hermitian surface as well as its Riemannian properties. Denote, as usual, by $\beta_\pm$ the Lee forms of the bi-Hermitian surface $(M, [g], J_\pm)$; we next want to consider the following conformally invariant conditions:

1) $\beta_+ - \beta_- = 0$;

2) $\beta_+ + \beta_- = df$;

3) $\mathbf{T} = 0$.

The first equality is the *hyper-Hermitian* condition meaning that J_+ and J_- span an S^2-worth of complex structures on M like in the hyper-Kähler case. In this situation it was shown by Boyer [10] that $(M, J_\pm)$ must be a Hopf surface, when $b_1(M)$ is assumed to be odd. In this article we will not be concerned with the hyper-Hermitian case and consider a bi-Hermitian surface to have exactly two complex structures (up to sign).

The second condition is equivalent to say that J_+ and J_- have the same Gauduchon metric in the conformal class $[g]$ and that the sum of Lee forms vanishes $\beta_+ + \beta_- = 0$, for this metric.

Equivalently, the flat line bundle F is holomorphically trivial or in other words: $\mathbf{T} = -K$. Indeed, by Gualtieri [25] generalized Kähler structures are in bijective correspondence with bi-Hermitian structures satisfying $\mathbf{T} = -K$.

Finally, the third condition says that J_+ and J_- are different (up to sign) at *every* point; these are called *strongly bi-Hermitian* metrics. The equation $\mathbf{T} = 0$ implies $c_1(S_\pm) = 0$ in $H^2(M, \mathbb{Z})$ which, as $b_1(M)$ is odd and $\mathrm{Kod}(S_\pm) \leq 0$, implies that $S_\pm$ is a Kodaira, a Hopf or a Bombieri–Inoue surface. However it is shown in [4] that only the second possibility can actually occur. We will explain this point in the next section.

From now on we will indicate by S any of the two surfaces $S_\pm$. The fundamental equation (2) says that $\mathbf{T}$ is a numerically anti-canonical divisor in S and it turns out that T can have at most two connected components. Setting $t := b_0(T)$ we obviously have that $t = 0$ corresponds to the strongly bi-Hermitian situation, while $t = 2$ is equivalent to the equation $\mathbf{T} = -K$; this has been observed in [3] and will be explained in the last section. In Section 3 we construct a twistor example with $t = 1$ and illustrate a very interesting, more general result by [3].

We conclude this introduction with a short outline of the rest of the paper.

In Section 2 we present some preliminary results from [2, 4, 5, 38] and a technical lemma. We then explain a result of Apostolov and Dloussky, which asserts that the minimal model of a bi-Hermitian S with $b_1(S)$ odd can only be a Hopf surface or a Kato surface.

In Section 3 we discuss the case of Hopf surfaces which admit bi-Hermitian metrics of all kinds, i.e. with all possible values of $t = 0, 1, 2$. The first two cases are due to Apostolov–Dloussky and Apostolov–Bailey–Dloussky.

Section 4 is devoted to study Kato surfaces. We divide them into two main types: Kato surfaces with branches and without. The second type is best known and we have a description of all possible anti-canonical divisors on them. This gives important information concerning existence of bi-Hermitian metrics. For example: there is no bi-Hermitian metric whatsoever if the minimal model is a general Enoki or half Inoue surface. On the positive side, the first bi-Hermitian metrics on Kato surfaces were constructed explicitly by LeBrun [32] on parabolic Inoue surfaces; we present in this section a brief outline of a twistor construction due to Fujiki–Pontecorvo of bi-Hermitian metrics on hyperbolic and parabolic Inoue surfaces [19]. Because these metrics are anti-self-dual they satisfy $t = 2$ [38].

Finally, we show that the situation is fairly satisfactory for blown-up hyperbolic Inoue surfaces because they do admit bi-Hermitian metrics whenever they can; furthermore $t = 2$ always in this case. The situation for *intermediate* Kato surfaces, i.e. with branches, is still open: there are no examples of bi-Hermitian structures, as far as we know.

2   Preliminary results

The main tool for our study of bi-Hermitian surfaces of non-Kähler type is the fundamental equation (2); we start this section with the following useful result of Apostolov about the fundamental line bundle F:

Proposition 2.1 ([2, Lemma 1], [4, Proposition 2])**.** *Let $(M, [g], J_\pm)$ be any compact bi-Hermitian surface. Then the flat line bundle F comes from a real representation of the fundamental*

group $\pi_1(M)$. Furthermore, with respect to any J_+-standard metric, we have

$$\deg(F) = -\frac{1}{8\pi} \int_M \|\beta_+ + \beta_-\|^2.$$

In particular, the degree of F is non-positive and $\deg(F) = 0$ if and only if F is holomorphically trivial if and only if the sum of Lee forms $\beta_+ + \beta_-$ is an exact 1-form for (one and hence) any metric in the conformal class.

Remark 2.2. Notice that in general, the degree of a holomorphic line bundle L computes the volume (with sign) of a virtual meromorphic section with respect to a fixed Gauduchon metric, i.e. with $\partial\bar{\partial}$-closed fundamental $(1,1)$-form, otherwise the degree would not be well defined.

In the special case $c_1(L) = 0$, the *sign* of $\deg(L)$ is independent of the chosen Gauduchon metric by a general result [34] and therefore only depends on the representation of the fundamental group $\pi_1(M)$.

In order to deal with non-minimal surfaces we now introduce the following notation: we let S_0 denote the minimal model of any of $S := (M, J_\pm)$ with blow-down map $\mathbf{b} : S \to S_0$. Then, F_0 will denote the unique real flat line bundle on S_0 such that $F = \mathbf{b}^*(F_0)$, because $\pi_1(S) = \pi_1(S_0)$; while $\mathbf{b}_* : \mathrm{Div}(S) \to \mathrm{Div}(S_0)$ will be the natural projection and K_0 the canonical bundle on S_0. The following simple observation, see also [13], will be repeatedly used:

Lemma 2.3. *Let $(M, [g], J_\pm)$ be a bi-Hermitian surface and let S_0 denote the minimal model of any of $S := (M, J_\pm)$. Using the notations introduced above, let $\mathbf{T}_0 := \mathbf{b}_*(\mathbf{T})$. Then the following hold:*

   i) $\mathbf{T}_0 = F_0 - K_0$ *is an effective divisor on S_0;*

   ii) S *is obtained by blowing up S_0 at points lying on $\mathbf{T}_0$, in particular $t = b_0(\mathbf{T}) = b_0(\mathbf{T}_0)$;*

   iii) $F_0 = 0$ *if and only if $F = 0$, otherwise F_0 has negative degree on S_0.*

Proof. This is a standard argument [2, 13, 19, 38]. For simplicity we prove it for a one point blow-up S of S_0 with exceptional divisor E. In this case the adjunction formula reads $-K = -\mathbf{b}^*K_0 - E$ [24, p. 187]. We can add F to both sides of the equation to get $F - K = F - \mathbf{b}^*K_0 - E$; taking $\mathbf{b}_*$ gives the first statement i).

Next we compare the total transform $\mathbf{b}^*\mathbf{T}_0$ with the proper transform $\tilde{\mathbf{T}}_0$. By adjunction formula again, $\mathbf{b}^*\mathbf{T}_0 = \mathbf{b}^*F_0 - \mathbf{b}^*K_0 = F - K + E$; on the other hand, we always have $\mathbf{b}^*\mathbf{T}_0 = \tilde{\mathbf{T}}_0 + mE$ where m is the multiplicity of the blown-up point p along T_0; we conclude that $\mathbf{T} = \tilde{\mathbf{T}}_0 + (m-1)E$. This shows that $\mathbf{T}$ is effective only when $m \geq 1$, i.e. $p \in \mathbf{T}_0$, and therefore they have the same number of connected components.

It remains to prove part iii) of the statement. First of all, it is clear that F_0 is torsion if and only if F is torsion which however implies $F = 0$ and therefore $F_0 = 0$.

On the other hand, since S admits a bi-Hermitian metric, it will be shown below and independently of this lemma that S and therefore S_0 are class-VII surfaces. In this case the degree map $H^1(\mathbb{R}^+)(\cong \mathbb{R}^+) \to \mathbb{R}$ is isomorphic, modulo torsion, [34, Proposition 3.1.13] so that for both F and F_0 they are trivial if and only if their degree is zero. We are left to show $F \neq 0$ implies $\deg(F_0) < 0$.

For this purpose, let ω be the Kähler form of a Gauduchon metric on the minimal model S_0. As is well-known [24, p. 186] there exist small neighborhoods $U \subset V$ of E in S such that the line bundle $[E]$ admits a Hermitian metric whose Chern form ρ has support in V, is semi-positive in U and is positive definite when restricted to each tangent space of E. This immediately implies that $\tilde{\omega}_\epsilon := b^*\omega + \epsilon\rho$ is everywhere positive definite for any small constant ϵ and therefore is the Kähler form of a Gauduchon metric on S with respect to which we can compute degrees

$$\deg(F) = \int_S c_1(F) \wedge \tilde{\omega}_\epsilon = \int_S c_1(F) \wedge \mathbf{b}^*\omega + \epsilon \int_S c_1(F) \wedge \rho = \deg(F_0) + \epsilon \int_S c_1(F) \wedge \rho.$$

Therefore, if by contradiction $\deg(F_0) > 0$ we can find ϵ small such that $\deg(F) > 0$ which is absurd. ∎

Notice that, the divisor $\mathbf{T}_0$ on the minimal model may not, in general, come from a bi-Hermitian metric on S_0. It will also be shown that the number of its connected components can only be 0, 1 or 2 and that all these possibilities actually occur.

We now present the following important result; the first alternative was proved in [2, 4] and the second one in [13].

Proposition 2.4. *Let* $(M, [g], J_{\pm})$ *be a bi-Hermitian surface with* $b_1(M) = odd$ *and let* S_0 *denote the minimal model of any of* $S := (M, J_{\pm})$. *Then there are two possibilities:*

  i) $b_2(S_0) = 0$ *and* S_0 *is a Hopf surface, or else*

  ii) $b_2(S_0) > 0$ *and* S_0 *is a Kato surface.*

In particular, M *is diffeomorphic to* $(S^1 \times S^3) \# m\overline{\mathbb{CP}}_2$ *with* $m = b_2(M)$; *or is a finite quotient of* $S^1 \times S^3$.

Proof. We start by showing that the Kodaira dimension must be negative [2]. Suppose not, from the fundamental equation (2) we have $\deg(K) = \deg(F) - \deg(\mathbf{T}) \leq 0$ because $\deg(F) \leq 0$ and $\mathbf{T} \geq 0$. Therefore it is enough to prove that the degree of K cannot vanish. In fact $\deg(K) = 0$ is equivalent to say that $\deg(F) = 0 = \mathbf{T}$ and therefore equivalent to $K = 0$. By Kodaira classification it is now enough to exclude that a Kodaira surface can admit bi-Hermitian metrics.

For this purpose we present a slightly different argument from Apostolov's original proof. Let ω_+ denote the fundamental $(1,1)$-form of any Hermitian structure (M, g, J_+) on a Kodaira surface (M, J_+) and let $\Omega = \gamma_+ - i\delta_+$ be a holomorphic section of the trivial canonical bundle with the property that $\gamma_+ \wedge \gamma_+ = \delta_+ \wedge \delta_+ = \omega_+ \wedge \omega_+ = \text{vol}(g)$.

Then $\bar{\partial}\Omega = 0$ implies that the real and imaginary parts γ_+ and δ_+ are self-dual closed symplectic forms since $\partial\gamma_+ = \partial\delta_+ = 0 \in \Lambda^{3,0} = 0$ and $d = \partial + \bar{\partial}$ by the integrability of J_+.

Therefore, whenever (M, J_+) is a Kodaira surface, both (M, δ_+) and (M, γ_+) provide Thurston examples of a compact symplectic manifold which is not Kählerian [45]. Recall that the topological invariants are: $b_1(M) = 3$ while the Euler characteristic and the signature both vanish.

The Kodaira complex structure is not tamed by the Thurston symplectic structures and it was noticed by Salamon [39] that the relation among them is that they define an almost hyper-Hermitian structure $\{J_+, I_+, K_+\}$ containing one complex and two symplectic structures: in fact I_+ and K_+ are tamed by γ_+ and δ_+, respectively.

Assume now by contradiction that there is another g-orthogonal complex structure J_-. With the same procedure as above we produce two more self-dual symplectic forms γ_- and δ_- but because $b_+(M) = 2$ we would get that they are linear combination with *constant* coefficients of γ_+ and δ_+; this however would imply that the *angle function* between J_+ and J_- is also constant, forcing J_+ and J_- to span a hyper-Hermitian structure on M [1, Proposition 2.5]; this is however impossible because such structures can only live on Hopf surfaces, by a result of Boyer [10].

We have shown so far that $\deg(K) < 0$ and this certainly implies $\text{Kod}(S) = -\infty$ – i.e. S belongs to class VII in Kodaira classification of complex surfaces. However, $\deg(K) < 0$ also implies by Lemma 2.3 that $\deg(K_0) < 0$ on the minimal model S_0. We conclude that S_0 cannot be a Bombieri–Inoue [9, 27] because Teleman [42] shows that these surfaces have canonical bundle of positive degree. Therefore $b_2(S_0) = 0$ implies S_0 is a Hopf surface by a theorem of Bogomolov [8], later clarified by [33, 41].

It remains to discuss the case $b_2(S) > 0$; by the main result of [13] is enough to show that $\mathbf{T}_0$ is a non-trivial divisor on S_0. In fact, if by contradiction $\mathbf{T}_0 = 0$ we will have $c_1(K_0) = c_1(F_0) = 0$

so that $c_1^2(S_0) = 0$ but this equation implies $b_2(S_0) = 0$ on class VII surfaces because $b_1 = 1$ and $b_2^+ = 0$. ∎

3 Hopf surfaces

A Hopf surface is a compact complex surface whose universal cover is $\mathbb{C}^2 \setminus \{0\}$. The aim of the section is to show that they admit a surprising abundance of bi-Hermitian metrics, with all possible values of $t = 0, 1, 2$.

It was shown in [38] that any conformally-flat metric $[g]$ on a Hopf surface M admits two ortogonal complex structures J_+ and J_-. Therefore $(M, [g], J_\pm)$ becomes a bi-Hermitian surface with anti-self-dual metric. In particular $\mathbf{T} = -K$ consists of two disjoint smooth elliptic curves of multiplicity 1 so that $t = 2$.

For some special conformally flat Hopf surface, J_+ belongs to a hyper-Hermitian structure $\{I_+, J_+, K_+\}$ and in some of these cases the same holds for J_- – i.e. some very special Hopf surfaces have two hyper-Hermitian structures. The divisor $\mathbf{T}$ is zero in this case.

More in general, for bi-Hermitian metrics with $t = 0$ there is a complete result of Apostolov–Dloussky which says that such metrics exist if and only if any of $(M, J_\pm)$ is a Hopf surface whose canonical bundle comes from a *real* representation of the fundamental group [4]. For sometime these were the only examples of bi-Hermitian metrics with $F \neq 0$.

More recently however, the first examples of bi-Hermitian metrics with $t = 1$ have been constructed in [3], again on Hopf surfaces.

The techniques are very interesting and general: the aim is to construct a bi-Hermitian metric with a *connected* divisor $\mathbf{T} = F - K$. The ingredient for doing it is a l.c.K. metric which we can think of as a twisted Kähler metric with values in a flat line L (the degree of which will automatically be positive).

Assuming that it is possible to take $L = -F$, one can contract a holomorphic section of $\mathbf{T}$ with the l.c.K. $(1, 1)$-form to obtain a tensor field which is in fact an infinitesimal deformation of the complex structure, trivial along $\mathbf{T}$. It is shown in [3] that this defines a true deformation and the deformed complex structure as well as the original one are both orthogonal with respect to a Riemannian conformal metric. This circle of ideas was used in [23] to show that a surface of Kähler type is bi-Hermitian if and only if admits holomorphic anti-canonical sections.

We can now give a twistor proof of the existence of bi-Hermitian metrics with $t = 1$, for some very special Hopf surfaces.

Proposition 3.1. *Let S be a Hopf surface which is an elliptic fiber bundle over $\mathbb{CP}_1$. Then S admits a bi-Hermitian metric with $t = 1$.*

Proof. The usual Vaisman metric on S is l.c.K., conformally flat and hyper-Hermitian because $S \to \mathbb{CP}_1$ is a smooth bundle [38]. In order to find the twisting l.c.K. line bundle L we can use its twistor space Z as follows. The hyper-Hermitian structure naturally defines a holomorphic projection $Z \to \mathbb{CP}_1$ so that the normal bundle of the image X of the l.c.K. metric in Z is trivial. On the other hand – because the twistor fibration induces an isomorphism of fundamental groups and therefore of flat line bundles – the twisting line bundle L can also be read–off from the twistor space. In fact, by the main result of [37] $X = -\frac{1}{2}K_Z - L$, therefore the triviality of the normal bundle $X|_X$ implies $2L = -(K_Z)|_S$ on the Hopf surface. By adjunction formula we then get $2L = -K_S$. Now, if E denotes any fiber of the elliptic fibration $S \to \mathbb{CP}_1$ we have $-K_S = 2E$ so that $L = E$. The machinery of [3] therefore yields a bi-Hermitian metric with $\mathbf{T} = F - K = -L - K = -E + 2E = E$, in particular $t = 1$. ∎

The above is just the easiest example of the following general result which uses the fact that every Hopf surface is l.c.K. [21].

Theorem 3.2 ([3])**.** *Hopf surfaces admits bi-Hermitian metrics with $t = 1$.* **T** *is supported on an elliptic curve which can have arbitrary* (*positive*) *multiplicity.*

Furthermore, bi-Hermitian metrics on blown-up Hopf surfaces were constructed by LeBrun, using a hyperbolic ansatz for anti-self-dual metrics with circle symmetry [32]; by Kim–Pontecorvo [31] using twistor methods and more recently by a general construction of Cavalcanti–Gualtieri [12].

In [19] anti-self-dual bi-Hermitian structures with $\mathbf{T} = -K$ are constructed on any blown-up Hopf surfaces, blown-up hyperbolic Inoue surfaces and blown-up parabolic Inoue surfaces S_t which are obtained via a small deformation $(S_t, -K_t)$ of 'anti-canonical pairs' $(S, -K)$, where S is any hyperbolic Inoue surface.

4   Kato surfaces

A minimal non-Kähler surface of Kodaira dimension $-\infty$ and positive second Betti number is said to belong to class VII$_0^+$. All known examples are so called Kato surfaces which were introduced in [30] and by defintion are compact complex surfaces S admitting a global spherical shell: there is a holomorphic embedding $\phi \colon U \to S$, where $U \subset \mathbb{C}^2 \setminus \{0\}$ is a neighborhood of the unit sphere S^3, such that $S \setminus \phi(U)$ is connected.

The following statement summarizes some of the main results about Kato surfaces

Theorem 4.1 ([13, 15, 30])**.** *For a surface S in class VII_0^+ the following conditions are equivalent and they imply that S is diffeomorphic to $(S^1 \times S^3) \# n\overline{\mathbb{CP}}_2$.*

1. *S is a Kato surface.*

2. *S contains $b_2(S)$ rational curves.*

3. *S admits a divisor $D = G - mK$ with $c_1(G) = 0$ and $m \in \mathbb{Z}^+$.*

A divisor of the form $D = G - mK$ with $c_1(G) = 0$ is called a NAC (numerically anti-canonical) divisor in the terminology of Dloussky [13] because its Chern class is a multiple of the anti-canonical class. It is known that D is automatically effective on $S \in \text{VII}_0^+$. The smallest $m \geq 1$ for which a NAC divisor exists is called the *index* of the Kato surface S.

There are recent important results about Kato surfaces going in different directions; concerning their Hermitian geometry Brunella proved the following strong result

Theorem 4.2 ([11])**.** *Every Kato surface admits l.c.K. metrics.*

But Kato surfaces are important mainly because they are the only known examples of surfaces in class-VII$_+^0$. A strong conjecture of Nakamura [36, Conjecture 5.5] says that there should be no other examples in this class. We only point out here that there is recent important progress in this direction by A. Teleman [43, 44].

Because on any bi-Hermitian surface **T** is a NAC divisor with $m = 1$ we are led to study these Kato surfaces. For doing this we divide them into classes according to the intersection properties of the b_2 rational curves.

First of all, each Kato surface has a cycle C of rational curves; we then consider two broad classes: Kato surfaces with no branch – also called *extreme* – that is, every rational curve belongs to some cycle C; and surfaces with branches – also called *intermediate* – in this case the maximal curve consists of the union of one cycle with chains of rational curves intersecting it transversally at a single point [35].

We will only be concerned about unbranched Kato surfaces which we are now going to describe in more detail. These surfaces can be divided into two subclasses; for simplicity we consider

their characterizations due to Nakamura [35] in terms of the configuration of complex curves on them, rather than their original definitions due to Inoue [28] and Enoki [17].

We start by considering the class of Enoki surfaces, containing parabolic Inoue surfaces. The characterizing property of these surfaces is that the unique cycle C has self-intersection zero: $C^2 = 0$, which (as $b_2 \neq 0$) is the same as saying that every irreducible component of C has self-intersection number -2. Because the intersection form is negative definite on any surface of class VII$^+$, it follows that $C = 0$ in $H^2(M, \mathbb{Z})$. We can now state the following important result of Enoki

Theorem 4.3 ([17]). *A surface $S \in \text{VII}_0^+$ has a divisor D of zero self-intersection if and only if S is an Enoki surface and $D = mC$ is a multiple of the cycle of rational curves in S.*

Enoki surfaces are exceptional compactifications of affine line budles in the sense that the complement $S \setminus C$ is always an affine line bundle over an elliptic curve. In the special case that this bundle has a section the Enoki surface S is special because it contains an elliptic curve E; such special Enoki surfaces are called *parabolic Inoue* surfaces, it turns out that they are the only surfaces in class VII$_0^+$ containing one (and in fact only one) elliptic curve E; because the anti-canonical bundle $-K = E + C$ it follows that $E^2 = -b_2(S)$. We will use the following terminology: a *general Enoki* surface is an Enoki surface which is not parabolic Inoue.

The other class of Kato surfaces without branches is that of *hyperbolic Inoue* surfaces, these are the only Kato surfaces with two cycles of rational curves and are also called even Hirzebruch–Inoue surfaces. In the special case when these cycles are isomorphic – in particular their components have the same intersection numbers in the same cyclic order – the hyperbolic Inoue surface has a fix-point-free involution whose quotient is a Kato surface called *half Inoue* surface, or also odd Hirzebruch–Inoue surface. Finally, every unbranched Kato surface falls into one of these classes.

We can now start reporting about bi-Hermitian metrics on them, most of the results below can also be found in [3, Appendix A].

The first examples are due to LeBrun [32] who used his hyperbolic ansatz to produce anti-self-dual Hermitian metrics with S^1-action on parabolic Inoue surfaces. It was noticed later on that these metrics are actually bi-Hermitian with $t = 2$ [38].

Let us notice here that the hyperbolic ansatz can only work in this case because by [38] the isometric action is automatically holomorphic but by a result of Hausen [26] parabolic Inoue surfaces are the only Kato surfaces which can admit holomorphic S^1-action.

The other known examples are again anti-self-dual and came from the twistor construction of Fujiki–Pontecorvo [19] which we are now going to briefly describe.

We started from a Joyce twistor space, that is the twistor space Z of a $(S^1 \times S^1)$-invariant self-dual metric on the connected sum $m\mathbb{CP}_2$ of complex projective planes [29]. These twistor spaces were studied by Fujiki [18] who showed that there are $(m + 2)$ invariant twistor lines and each of them is the transverse intersection of a pair of *elemetary* divisors S_+, S_- – this merely means that their intersection number with a twistor line equals 1.

As explained in the introduction a bi-Hermitian metric produces two pairs of *disjoint* complex hypersurfaces in the twistor space Z which turns out to be a complex 3-manifold precisely when the metric is anti-self-dual [7]. In order to obtain these desired configuration of hypersurfaces we therefore blow up two invariant twistor lines and obtain a smooth complex 3-fold $\tilde{Z}$ with two pairs of disjoint hypersurfaces. Even though $\tilde{Z}$ is not a twistor space anymore, this is the starting point of a method introduced in [16] in order to construct the twistor space of a connected sum.

Following this method and its relative version [31] we now carefully identify the resulting two exceptional divisors in $\tilde{Z}$ and consider the resulting quotient space $\hat{Z}$ which is a 'singular twistor space' containing a Cartier divisor $\hat{S}$ consisting of two pairs of disjoint singular hypersurfaces. $\hat{S}$ is in fact an anti-canonical divisor in $\hat{Z}$.

The general theory of [16] and [31] then predicts that if there exist smooth and 'real' deformations (Z_t, S_t) of the singular pair $(\hat{Z}, \hat{S})$ then Z_t is a smooth twistor space containing an effective anti-canonical divisor S_t having 4 irreducible components, pairwise disjoint, and each component meets every twistor line in exactly one point. In other words any smooth deformation will yield exactly the twistor data of a bi-Hermitian anti-self-dual structure on the connected sum. The existence of a real structure on the deformation is a consequence of the general theory.

Because we started by identifying two exceptional divisors in the same $\tilde{Z}$ the resulting 4-manifold is actually a self-connected sum, diffeomorphic to $(S^1 \times S^3) \# m\overline{\mathbb{CP}}_2$ which is the diffeomorphism type of a Kato surface, as desired.

The deformation theory of the constructed singular pair $(\hat{Z}, \hat{S})$ is governed by the local Ext sheaves and global Ext groups of $\Omega_{\hat{Z}}(\log \hat{S})$ which is the sheaf of germs of holomorphic 1-form having at worst logarithmic poles along $\hat{S}$. We then showed existence of smooth deformations of the pair by proving the vanishing of both $\mathrm{Ext}^2(\Omega_{\hat{Z}}(\log \hat{S}))$ and $H^2(\Theta_{\hat{Z}}(-\log \hat{S}))$, this last sheaf being holomorphic vector fields on $\hat{Z}$ which are tangent to $\hat{S}$, along $\hat{\hat{S}}$.

In this construction the various choices of original $(S^1 \times S^1)$-action on $m\mathbb{CP}_2$ and of pairs of invariant twistor lines we started with allow us to obtain every hyperbolic Inoue surface as an irreducibile component of S_t. We can finally state our main result

Theorem 4.4 ([19])**.** *Every properly blown up hyperbolic Inoue surface admits families of bi-Hermitian anti-self-dual structures. The same result holds for some parabolic Inoue surfaces. In particular, $t = 2$ for all these metrics.*

Remark 4.5. Properly blown up means that we are only allowed to blow up points which are nodes of (the anti-canonical cycle) $-K$.

Furthermore, a variation of the construction gives existence of families of anti-self-dual Hermitian metrics on every properly blown up half Inoue surface. These surfaces however cannot admit bi-Hermitian metrics as will soon be clear.

The above were the first examples of l.c.K. metrics without symmetries on Kato surfaces.

Now that we have an existence result we can describe the situation for bi-Hermitian metrics on unbranched Kato surfaces; this is made possible by the fact that all their NAC divisors can be described. A useful consequences of Enoki theorem is that the rational curves on a Kato surface form a bases of $H^2(S, \mathbb{Q})$ unless S is Enoki. In particular a NAC divisor $D = G - K$ is unique or else the surface is Enoki.

As usual, the statements below hold for any one of the two complex surfaces $S = (M, J_\pm)$.

Proposition 4.6. *There are no bi-Hermitian metrics at all if the minimal model of S is a general Enoki or half Inoue surface.*

Proof. By Lemma 2.3 it is enough to show that the minimal model contains no NAC divisors of index 1: $D = G - K$. We start by considering a general Enoki surface, it is known in this case that $-K = C + \hat{E}$ where C is the unique cycle and $\hat{E}$ is a line bundle without meromorphic sections. The second cohomology $H^2(S, \mathbb{Z})$ is spanned by the Chern class $c_1(\hat{E})$ and the irreducible components of the cycle C which however are subject to the relation $C = 0 \in H^2(S, \mathbb{Z})$. If there was a NAC divisor $D = G - K$ on a general Enoki surface we would have $D - C = G + \hat{E}$. But this equation is impossible in cohomology because C is the maximal curve so that $c_1(D - C)$ is spanned by the irreducible components of the cycle while $c_1(G + \hat{E}) = c_1(\hat{E})$ which is linearly independent from the components of C. This argument shows that there is no NAC divisor on a general Enoki surface.

Suppose now that the minimal model is a half Inoue surface. In this case $-K = C + L$ where C is the unique cycle and L is a non-trivial line bundle of order 2. Then the unique NAC divisor is C which however does not satisfy the fundamental equation (2) $C = F - K$ because

the degree of F should vanish if and only if F is the trivial line bundle, while the degree of L vanishes – since $2L = \mathcal{O}_S$ – even though L is non-trivial.   ∎

We can use this techniques to show the following result stated in the introduction

Proposition 4.7. *Let M be a compact four manifold with odd first Betti number. A bi-Hermitian surface S satisfies $\mathbf{T} = -K$ if and only if T is disconnected, if and only if $t = 2$. Therefore its minimal model S_0 is either* (*a finite quotient of*) *a diagonal Hopf surface; or a parabolic Inoue surface or a hyperbolic Inoue surface.*

Proof. One direction was proved in [5]. Therefore, suppose that T is disconnected we need to prove $\mathbf{T} = -K$. By Lemma 2.3 the minimal model S_0 has a disconnected NAC divisor $\mathbf{T}_0 = F_0 - K_0$ with $\deg(F_0) \leq 0$ and is enough to show $\deg(F_0) = 0$.

First of all, S_0 cannot be a Kato surface with branches because it is shown in [14] that any NAC divisor is supported on the maximal curve which is connected. Therefore S_0 is either a Hopf or an unbranched Kato surface with a disconnected NAC divisor $\mathbf{T}_0 = F_0 - K_0$ and of course $t = 2$ otherwise S_0 is an elliptic Hopf surface with $\mathrm{vol}(\mathbf{T}_0)$ bigger than $\mathrm{vol}(-K_0)$, which is impossible. Suppose S_0 is a Hopf surface, if it has a disconnected divisor it must be a diagonal one in which case $-K_0 = E_1 + E_2$ is a union of two elliptic curves with multiplicity one. Because $\mathbf{T}_0$ is an effective and disconnected divisor $\mathbf{T}_0 = aE_1 + bE_2$ with $a, b \geq 1$. The conclusion is that $F_0 = (a-1)E_1 + (b-1)E_2$ has positive degree unless $a = b = 1$ so that $F_0 = 0$, as wanted.

We can now assume that S_0 is an unbranched Kato surface with effective and disconnected NAC divisor $T_0 = F_0 - K_0$. Now, a general Enoki has no NAC divisors, the only NAC divisor of a half Inoue surface is the cycle C which is connected and so we are left with parabolic and hyperbolic Inoue surfaces which both have effective $-K_0$; but in this case $F_0 = T_0 + K_0$ is a divisor and satisfies the hypothesis of Enoki theorem so that S_0 is a parabolic Inoue surface with $F_0 = nC$ leading to $T_0 = nC + C + E = (n+1)C + E$. But $n < 0$ because F_0 has negative degree therefore T_0 effective implies $n = -1$ in which case $T_0 = E$ is connected.   ∎

We now come back to existence of bi-Hermitian metrics and consider hyperbolic Inoue surfaces; in this case the situation is fairly clear (and best possible):

Proposition 4.8. *Let S_0 be an arbitrary hyperbolic Inoue surface and suppose S is a blow up of S_0. Then every bi-Hermitian metric on S satisfies $t = 2$. Furthermore, such metrics exist if and only if S is obtained by blowing up points on the anti-canonical divisor of S_0.*

Proof. The anti-canonical divisor of S_0 is effective: in fact $-K = C_1 + C_2$ is the maximal curve consisting of the union of two cycles. This is the only NAC divisor with $m = 1$ by uniqueness and using Lemma 2.3 this proves the first part of the statement.

To prove the second part, recall that anti-self-dual bi-Hermitian metrics were constructed in [19] on every properly blown up hyperbolic Inoue surface. Therefore using the result of [12] – which says that generalizad Kähler metrics persist by blowing up *smooth* points of $-K$ – we get such metrics on any S obtained by blowing up points on the anti-canonical divisor of its minimal model.

It is important to notice that we get all possible blow ups because the anticanonical divisor of a hyperbolic Inoue surface is reduced; therefore the exceptional curve obtained by blowing up a smooth point will not belong to the anticanonical divisor of the blown up surface.   ∎

A similar but weaker existence result holds for bi-Hermitian metrics with $T = -K$ on blown up parabolic Inoue surfaces, with an additional reality condition. It remains to be seen whether there are bi-Hermitian metrics with $t = 1$ on parabolic Inoue surfaces and on intermediate (i.e. branched) Kato surfaces.

Acknowledgements

References

[1] Alekseevsky D.V., Marchiafava S., Pontecorvo M., Compatible complex structures on almost quaternionic manifolds, *Trans. Amer. Math. Soc.* **351** (1999), 997–1014.

[2] Apostolov V., Bihermitian surfaces with odd first Betti number, *Math. Z.* **238** (2001), 555–568.

[3] Apostolov V., Bailey M., Dloussky G., From locally conformally Kähler to bi-Hermitian structures on non-Kähler complex surfaces, arXiv:1307.3660.

[4] Apostolov V., Dloussky G., Bihermitian metrics on Hopf surfaces, *Math. Res. Lett.* **15** (2008), 827–839, arXiv:0710.2266.

[5] Apostolov V., Gauduchon P., Grantcharov G., Bi-Hermitian structures on complex surfaces, *Proc. London Math. Soc.* **79** (1999), 414–428.

[6] Apostolov V., Gualtieri M., Generalized Kähler manifolds, commuting complex structures, and split tangent bundles, *Comm. Math. Phys.* **271** (2007), 561–575.

[7] Atiyah M.F., Hitchin N.J., Singer I.M., Self-duality in four-dimensional Riemannian geometry, *Proc. Roy. Soc. London Ser. A* **362** (1978), 425–461.

[8] Bogomolov F.A., Classification of surfaces of class VII_0 with $b_2 = 0$, *Math. USSR-Izv.* **10** (1976), 255–269.

[9] Bombieri E., Letter to Kodaira, 1973.

[10] Boyer C.P., A note on hyper-Hermitian four-manifolds, *Proc. Amer. Math. Soc.* **102** (1988), 157–164.

[11] Brunella M., Locally conformally Kähler metrics on Kato surfaces, *Nagoya Math. J.* **202** (2011), 77–81, arXiv:1001.0530.

[12] Cavalcanti G.R., Gualtieri M., Blowing up generalized Kähler 4-manifolds, *Bull. Braz. Math. Soc. (N.S.)* **42** (2011), 537–557, arXiv:1106.1481.

[13] Dloussky G., On surfaces of class VII_0^+ with numerically anticanonical divisor, *Amer. J. Math.* **128** (2006), 639–670, math.CV/0406387.

[14] Dloussky G., Oeljeklaus K., Vector fields and foliations on compact surfaces of class VII_0, *Ann. Inst. Fourier (Grenoble)* **49** (1999), 1503–1545.

[15] Dloussky G., Oeljeklaus K., Toma M., Class VII_0 surfaces with b_2 curves, *Tohoku Math. J.* **55** (2003), 283–309, math.CV/0201010.

[16] Donaldson S., Friedman R., Connected sums of self-dual manifolds and deformations of singular spaces, *Nonlinearity* **2** (1989), 197–239.

[17] Enoki I., Surfaces of class VII_0 with curves, *Tôhoku Math. J.* **33** (1981), 453–492.

[18] Fujiki A., Compact self-dual manifolds with torus actions, *J. Differential Geom.* **55** (2000), 229–324.

[19] Fujiki A., Pontecorvo M., Anti-self-dual bihermitian structures on Inoue surfaces, *J. Differential Geom.* **85** (2010), 15–71, arXiv:0903.1320.

[20] Gauduchon P., La 1-forme de torsion d'une variété hermitienne compacte, *Math. Ann.* **267** (1984), 495–518.

[21] Gauduchon P., Ornea L., Locally conformally Kähler metrics on Hopf surfaces, *Ann. Inst. Fourier (Grenoble)* **48** (1998), 1107–1127.

[22] Gentili G., Salamon S., Stoppato C., Twistor transforms of quaternionic functions and orthogonal complex structures, arXiv:1205.3513.

[23] Goto R., Unobstructed K-deformations of generalized complex structures and bi-Hermitian structures, *Adv. Math.* **231** (2012), 1041–1067, arXiv:0911.2958.

[24] Griffiths P., Harris J., Principles of algebraic geometry, *Pure and Applied Mathematics*, John Wiley & Sons, New York, 1978.

[25] Gualtieri M., Generalized Kähler geometry, arXiv:1007.3485.

[26] Hausen J., Zur Klassifikation glatter kompakter $\mathbb{C}^*$-Flächen, *Math. Ann.* **301** (1995), 763–769.

[27] Inoue M., On surfaces of class VII$_0$, *Invent. Math.* **24** (1974), 269–310.

[28] Inoue M., New surfaces with no meromorphic functions. II, in Complex Analysis and Algebraic Geometry, Iwanami Shoten, Tokyo, 1977, 91–106.

[29] Joyce D.D., Explicit construction of self-dual 4-manifolds, *Duke Math. J.* **77** (1995), 519–552.

[30] Kato M., Compact complex manifolds containing "global" spherical shells. I, in Proceedings of the International Symposium on Algebraic Geometry (Kyoto Univ., Kyoto, 1977), Kinokuniya Book Store, Tokyo, 1978, 45–84.

[31] Kim J., Pontecorvo M., A new method of constructing scalar-flat Kähler surfaces, *J. Differential Geom.* **41** (1995), 449–477.

[32] LeBrun C., Anti-self-dual Hermitian metrics on blown-up Hopf surfaces, *Math. Ann.* **289** (1991), 383–392.

[33] Li J., Yau S.-T., Zheng F., On projectively flat Hermitian manifolds, *Comm. Anal. Geom.* **2** (1994), 103–109.

[34] Lübke M., Teleman A., The Kobayashi–Hitchin correspondence, World Scientific Publishing Co., Inc., River Edge, NJ, 1995.

[35] Nakamura I., On surfaces of class VII$_0$ with curves, *Invent. Math.* **78** (1984), 393–443.

[36] Nakamura I., On surfaces of class VII$_0$ with curves. II, *Tohoku Math. J.* **42** (1990), 475–516.

[37] Pontecorvo M., On twistor spaces of anti-self-dual Hermitian surfaces, *Trans. Amer. Math. Soc.* **331** (1992), 653–661.

[38] Pontecorvo M., Complex structures on Riemannian four-manifolds, *Math. Ann.* **309** (1997), 159–177.

[39] Salamon S., Special structures on four-manifolds, *Riv. Mat. Univ. Parma (4)* **17** (1991), 109–123.

[40] Salamon S., Viaclovsky J., Orthogonal complex structures on domains in $\mathbb{R}^4$, *Math. Ann.* **343** (2009), 853–899, arXiv:0704.3422.

[41] Teleman A., Projectively flat surfaces and Bogomolov's theorem on class VII$_0$ surfaces, *Internat. J. Math.* **5** (1994), 253–264.

[42] Teleman A., The pseudo-effective cone of a non-Kählerian surface and applications, *Math. Ann.* **335** (2006), 965–989, arXiv:0704.2948.

[43] Teleman A., Instantons and curves on class VII surfaces, *Ann. of Math.* **172** (2010), 1749–1804, arXiv:0704.2634.

[44] Teleman A., A variation formula for the determinant line bundle. Compact subspaces of moduli spaces of stable bundles over class VII surfaces, arXiv:1309.0350.

[45] Thurston W.P., Some simple examples of symplectic manifolds, *Proc. Amer. Math. Soc.* **55** (1976), 467–468.