Wong's Equations and Charged Relativistic Particles in Non-Commutative Space[⋆]

Herbert BALASIN [†], *Daniel N. BLASCHKE* [‡], *François GIERES* [§] *and Manfred SCHWEDA* [†]

[†] *Institute for Theoretical Physics, Vienna University of Technology,*
  Wiedner Hauptstraße 8-10, A-1040 Vienna, Austria
  E-mail: *hbalasin@tph.tuwien.ac.at, mschweda@tph.tuwien.ac.at*

[‡] *Los Alamos National Laboratory, Theory Division, Los Alamos, NM, 87545, USA*
  E-mail: *dblaschke@lanl.gov*

[§] *Université de Lyon, Université Claude Bernard Lyon 1 and CNRS/IN2P3,*
  Institut de Physique Nucléaire, Bat. P. Dirac,
  4 rue Enrico Fermi, F-69622-Villeurbanne, France
  E-mail: *gieres@ipnl.in2p3.fr*

Abstract. In analogy to Wong's equations describing the motion of a charged relativistic point particle in the presence of an external Yang–Mills field, we discuss the motion of such a particle in non-commutative space subject to an external $U_\star(1)$ gauge field. We conclude that the latter equations are only consistent in the case of a constant field strength. This formulation, which is based on an action written in Moyal space, provides a coarser level of description than full QED on non-commutative space. The results are compared with those obtained from the different Hamiltonian approaches. Furthermore, a continuum version for Wong's equations and for the motion of a particle in non-commutative space is derived.

Key words: non-commutative geometry; gauge field theories; Lagrangian and Hamiltonian formalism; symmetries and conservation laws

2010 Mathematics Subject Classification: 81T13; 81T75; 70S05

1 Introduction

Over the last twenty years, non-commuting spatial coordinates have appeared in various contexts in the framework of quantum gravity and superstring theories. This fact contributed to the motivation for studying classical and quantum theories with a finite or infinite number of degrees of freedom on non-commutative spaces. Different mathematical approaches have been pursued and various physical applications have been explored, e.g. see references [8, 11, 24, 35, 39] for some partial reviews. Beyond the applications, classical and quantum mechanics on non-commutative space are of interest as toy models for field theories which are more difficult to handle, in particular in the case of interactions with gauge fields. In this respect, we recall a similar situation concerning the coupling of matter to Yang–Mills fields on ordinary space: A coarser level of description for the latter theories has been proposed by Wong [45] who considered the motion of charged point particles in an external gauge field [6, 7, 9, 15, 16, 26, 27, 44]. The latter equations allow for various physical applications, e.g. to the dynamics of quarks and their interaction with gluons [26, 27]. Somewhat similar equations, known as Mathisson–Papapetrou–Dixon equations [13, 30, 33] appear in general relativity for a spinning particle in

---

curved space. In this spirit, we will consider in the present work the dynamics of a relativistic "point" particle in non-commutative space subject to an external $U_\star(1)$ gauge field (thereby implementing a suggestion made in an earlier work [38], see also [20] for some related studies based on a first order expansion in the non-commutativity parameter).

More precisely, we are motivated by the motion in Moyal space, the latter space having been widely discussed as the arena for field theory in non-commutative spaces. By proceeding along the lines of Wong's equations (which are discussed in Section 2), we derive a set of equations for the dynamics of the particle in Moyal space in Section 3. For the description of the coupling of a particle to a gauge field, the relativistic setting is the most natural one, but our discussions (of Wong's equations in commutative space or of a particle in non-commutative space) could equally well be done within the non-relativistic setting. The particular case of a constant field strength is the most tractable one (and the only consistent one for Wong's equations in non-commutative space) and will be considered in some detail in Section 4. Subsequently in Section 5, we briefly recall the different Hamiltonian approaches which have previously been pursued for the formulation of classical mechanics in non-commutative space and we compare the resulting equations governing the dynamics of particles coupled to an electromagnetic field. In the appendix we present a continuum formulation of Wong's equations on a generic space-time manifold, a formulation which readily generalizes to Moyal space.

For the motion of a relativistic point particle in four-dimensional (commutative or non-commutative) space-time, the following notations will be used. The metric tensor is given by

$$(\eta_{\mu\nu})_{\mu,\nu\in\{0,1,2,3\}} = \mathrm{diag}(1,-1,-1,-1)$$

and we choose the natural system of units ($c \equiv 1 \equiv \hbar$). The proper time τ for the particle is defined (up to an additive constant) by

$$d\tau^2 = ds^2, \qquad \text{with} \qquad ds^2 \equiv dx^\mu dx_\mu = (dt)^2 - (d\vec{x})^2, \tag{1.1}$$

and for the massive particle we have $ds^2 > 0$. From (1.1) it follows that $\dot{x}^2 = 1$ where $\dot{x}^2 \equiv \dot{x}^\mu \dot{x}_\mu$ and $\dot{x}^\mu \equiv dx^\mu/d\tau$.

2   Reminder on particles in commutative space

Abelian gauge field in flat space. We consider the interaction of a charged massive relativistic particle with an external electromagnetic field given by the $U(1)$ gauge potential (A^μ). The motion of this particle along its space-time trajectory $\tau \mapsto x^\mu(\tau)$ is described by the action[1]

$$S[x] = -m \int ds - q \int dx^\mu A_\mu(x(\tau)) = -m \int d\tau \sqrt{\dot{x}^2} - q \int d\tau \dot{x}^\mu A_\mu(x(\tau)). \tag{2.1}$$

Here, q denotes the conserved electric charge of the particle associated with the conserved current density (j^μ):

$$j^\mu(y) = q \int d\tau \dot{x}^\mu(\tau)\delta^4(y - x(\tau)), \qquad \partial_\mu j^\mu = 0, \qquad \int d^3y j^0(y) = q. \tag{2.2}$$

We note that the interaction term in the functional (2.1) may be rewritten in terms of the above current according to

$$q \int d\tau \dot{x}^\mu A_\mu(x(\tau)) = \int d^4y j^\mu(y) A_\mu(y). \tag{2.3}$$

---

[1]To be more precise, in the integral (2.1) the variable τ is viewed as a purely mathematical parameter which is only identified with proper time after deriving the equations of motion from the action. Thus, the relation $\dot{x}^2 = 1$ is only to be used at the latter stage.

This expression is invariant under infinitesimal gauge transformations, i.e. $\delta_\lambda A_\mu = \partial_\mu \lambda$, thanks to the conservation of the current:

$$\delta_\lambda \int d^4 y\, j^\mu A_\mu = \int d^4 y\, j^\mu \partial_\mu \lambda = -\int d^4 y\, (\partial_\mu j^\mu)\lambda = 0.$$

We note that the parameter q which describes the coupling of the particle to the gauge field might in principle depend on the world line parameter τ: if this assumption is made, $q(\tau)$ appears under the τ-integral in (2.2), the current j^μ is no longer conserved and the coupling (2.3) is no longer gauge invariant. Thus, q is necessarily constant along the path.

Variation of the action (2.1) with respect to x^μ and substitution of the relation $\dot{x}^2 = 1$ leads to the familiar equation of motion

$$m\ddot{x}^\mu = qF^{\mu\nu}\dot{x}_\nu, \qquad \text{with} \qquad F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu.$$

As is well known, the free particle Lagrangian $L_{\text{free}} = -m\sqrt{\dot{x}^2}$ which represents the first term of the action (2.1) and which is non-linear in $\dot{x}^2$ can be replaced by $\tilde{L}_{\text{free}} = \frac{m}{2}\dot{x}^2$ which is linear in $\dot{x}^2$ since both Lagrangians yield the same equation of motion. Indeed, we will consider the latter Lagrangian in equation (2.12) and in Section 5 where we go over to the Hamiltonian formulation.

The aim of this work is to generalize this setting to a non-commutative space-time. Since a gauge field on non-commutative space entails a non-Abelian structure for the field strength tensor $F_{\mu\nu}$ due to the star product, it is worthwhile to understand first the coupling of a spinless particle to a non-Abelian gauge field on commutative space-time.

Non-Abelian gauge field in flat space. We consider a compact Lie group G (e.g. $G = \text{SU}(N)$ for concreteness) with generators T^a satisfying

$$[T^a, T^b] = \mathrm{i}f^{abc}T^c \qquad \text{and} \qquad \text{Tr}\left(T^a T^b\right) = \delta^{ab}.$$

Just as for the Abelian gauge field, the source $j_\mu^a(x)$ of the non-Abelian gauge field $A_\mu(x) \equiv A_\mu^a(x)T^a$ (e.g. a field theoretic expression like $\bar{\psi}\gamma_\mu \hat{T}^a \psi$ involving a multiplet ψ of spinor fields) is considered to be given by the current density $j_\mu^a(x)$ of a relativistic point particle [6, 7, 15, 16, 26, 27, 44, 45]. Instead of an electric charge, the particle moving in an external Yang–Mills field is thus assumed to carry a color-charge or isotopic spin $\vec{q} \equiv (q^a)_{a=1,\dots,\dim G}$ which transforms under the adjoint representation of the structure group G. Henceforth, one considers the Lie algebra-valued variable $q(\tau) \equiv q^a(\tau)T^a$ which is assumed to be τ-dependent. The particle is then described in terms of its space-time coordinates $x^\mu(\tau)$ and its isotopic spin $q(\tau)$, i.e. it is referred to with respect to geometric space and to internal space. For the moment, we assume $q(\tau)$ to represent a given non-dynamical (auxiliary) variable and we will comment on a different point of view below. Its coupling to an external non-Abelian gauge field (A_μ) is now described by the action

$$\begin{aligned}
S[x] &= -m\int ds - \int d\tau\, \dot{x}^\mu \, \text{Tr}\{q(\tau)A_\mu(x(\tau))\} \\
&= -m\int d\tau\sqrt{\dot{x}^2} - \int d\tau\, \dot{x}^\mu q^a(\tau)A_\mu^a(x(\tau)).
\end{aligned} \tag{2.4}$$

The current density (2.2) presently generalizes to a Lie algebra-valued expression $j_\mu \equiv j_\mu^a T^a$ given by

$$j^\mu(y) = \int d\tau\, \dot{x}^\mu(\tau)q(\tau)\delta^4(y - x(\tau)), \tag{2.5}$$

and thereby the interaction term in the functional (2.4) can be rewritten as

$$-\int d^4 y\, \text{Tr}(j^\mu A_\mu). \tag{2.6}$$

For an infinitesimal gauge transformation with Lie algebra-valued parameter λ, i.e. $\delta_\lambda A_\mu = D_\mu \lambda \equiv \partial_\mu \lambda - \mathrm{i}g[A_\mu, \lambda]$, we have

$$\delta_\lambda \int d^4 y \, \mathrm{Tr}\{j^\mu A_\mu\} = \int d^4 y \, \mathrm{Tr}\{j^\mu D_\mu \lambda\} = -\int d^4 y \, \mathrm{Tr}\{(D_\mu j^\mu)\lambda\}. \tag{2.7}$$

Thus, gauge invariance of the action (2.4) requires the current to be *covariantly conserved,* i.e. $D_\mu j^\mu = 0$. From (2.5) we can deduce by a short calculation that

$$(D_\mu j^\mu)^a(y) = \int d\tau \frac{Dq^a}{d\tau} \delta^4(y - x(\tau)), \qquad \text{with} \qquad \frac{Dq^a}{d\tau} \equiv \frac{dq^a}{d\tau} - \mathrm{i}g\dot{x}^\mu [A_\mu(x(\tau)), q]^a, \tag{2.8}$$

hence j^μ is covariantly conserved if the charge q is *covariantly constant* along the world line: $Dq^a/d\tau = 0$ (subsidiary condition).

Variation of the action with respect to x^μ (and use of $\dot{x}^2 = 1$) yields the equations of motion

$$m\ddot{x}^\mu = \mathrm{Tr}(qF^{\mu\nu})\dot{x}_\nu, \qquad \text{where} \qquad \frac{Dq^a}{d\tau} = 0, \tag{2.9}$$

and $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu - \mathrm{i}g[A_\mu, A_\nu]$. By construction, these equations are invariant under infinitesimal gauge transformations parametrized by $\lambda(x)$ since the charge q is a non-dynamical (auxiliary) variable transforming as $\delta_\lambda q(\tau) \equiv -\mathrm{i}g[q(\tau), \lambda(x(\tau))] = 0$ for all τ, hence

$$\delta_\lambda \big(\mathrm{Tr}(qF_{\mu\nu})\dot{x}^\nu \big) = -\mathrm{i}g\,\mathrm{Tr}(q[F_{\mu\nu}, \lambda])\dot{x}^\nu = \mathrm{i}g\,\mathrm{Tr}([q, \lambda]F_{\mu\nu})\dot{x}^\nu = 0,$$

$$\delta_\lambda \left(\frac{Dq}{d\tau} \right) = -\mathrm{i}g\dot{x}^\mu[\delta_\lambda A_\mu, q] = -\mathrm{i}g\left[\frac{D\lambda}{d\tau}, q \right] = -\mathrm{i}g\frac{D}{d\tau}[\lambda, q] = 0.$$

Equations (2.9), which represent the Lorentz–Yang–Mills force equation for a relativistic particle in an external Yang–Mills field, are known as *Wong's equations* [45]. More specifically, the relation $Dq^a/d\tau = 0$ may be viewed as charge transport equation and it is the geometrically natural generalization (to the charge vector (q^a)) of the constancy of the charge q in electrodynamics.

A few remarks are in order here. We should point out that the equation of motion for x^μ had already been obtained earlier in curved space by Kerner [25], but the charge transport equation was not established in that setting. We refer to the works [6, 7, 26, 27] for a treatment of dynamics involving the Lagrangian $-\frac{1}{4}\int d^3 x \, \mathrm{Tr}(F^{\mu\nu}F_{\mu\nu})$ of the gauge field: the latter approach yields the covariant conservation law $D_\mu j^\mu = 0$ as a consequence of the equation of motion $D_\nu F^{\nu\mu} = j^\mu$ and the relation $[D_\mu, D_\nu]F^{\nu\mu} = 0$. For a general discussion of the issue of gauge invariance for the coupling of gauge fields to non-dynamical external sources, we refer to [34]. Equations (2.9) and their classical solutions have been investigated in the literature and applied for instance to the study of the quark gluon plasma [26]. The particular case of a constant field strength $F_{\mu\nu}$ ("uniform fields") exhibits interesting features [9] which will be commented upon in Section 4.

If one regards (q^a) as a dynamical variable which satisfies the equation of motion $Dq^a/d\tau = 0$ and which transforms under gauge variations with the adjoint representation, i.e.

$$\delta_\lambda q(\tau) = -\mathrm{i}g[q(\tau), \lambda(x(\tau))], \tag{2.10}$$

then equations (2.9) are obviously gauge invariant since $F_{\mu\nu}$ also transforms with the adjoint representation[2]. However, we emphasize that we started from the action (2.4) to obtain the equation of motion of x^μ, the one of q following from the requirement of gauge invariance of the

---

[2]If j^μ is assumed to transform covariantly, then gauge invariance of the action (2.6) implies $\partial_\mu j^\mu = 0$ as has already been noticed in reference [36].

initial action[3]. An action which yields the equations (2.9) as equations of motion of both x^μ and q has been constructed in the non-relativistic setting in reference [28]. The relativistic generalization of this approach proceeds as follows. (For simplicity, we put the coupling constant g equal to one.) One introduces a Lie algebra-valued variable $\Lambda(\tau) \equiv \Lambda^a(\tau) T^a$ where the functions $\Lambda^a(\tau)$ are Grassmann odd, the charge q being defined as an expression which is bilinear in Λ (i.e. a description which is familiar for the spin):

$$q \equiv -\frac{1}{2}[\Lambda, \Lambda]_+, \qquad \text{i.e.} \qquad q^a = -\frac{\mathrm{i}}{2} f^{abc} \Lambda^b \Lambda^c. \tag{2.11}$$

The Lagrangian

$$L(x, \dot{x}, \Lambda, \dot{\Lambda}) \equiv \frac{m}{2}\dot{x}^2 + \frac{\mathrm{i}}{2} \operatorname{Tr}\left(\Lambda \frac{D\Lambda}{d\tau}\right) = \frac{m}{2}\dot{x}^2 + \frac{\mathrm{i}}{2}\operatorname{Tr}(\Lambda\dot{\Lambda}) + \dot{x}^\mu \operatorname{Tr}(qA_\mu), \tag{2.12}$$

is invariant under gauge transformations for which $\delta_\lambda A_\mu = D_\mu \lambda$ and

$$\delta_\lambda \Lambda(\tau) = -\mathrm{i}[\Lambda(\tau), \lambda(x(\tau))],$$

which implies the transformation law (2.10). Moreover, the Lagrangian leads to the following equations of motion for x^μ and Λ^a:

$$m\ddot{x}_\mu = \dot{x}^\nu \operatorname{Tr}\{q(\partial_\mu A_\nu - \partial_\nu A_\mu)\} + \operatorname{Tr}(\dot{q}A_\mu), \qquad 0 = \frac{D\Lambda^a}{d\tau}. \tag{2.13}$$

From (2.11) it follows that $\frac{Dq}{d\tau} = -\left[\Lambda, \frac{D\Lambda}{d\tau}\right]_+$, hence (2.13) implies that $\frac{Dq^a}{d\tau} = 0$, i.e. the charge q is covariantly conserved. Substitution of the latter result into the first of equations (2.13) yields the equation of motion $m\ddot{x}^\mu = \operatorname{Tr}(qF^{\mu\nu})\dot{x}_\nu$. We note that the Hamiltonian associated to the Lagrangian (2.12) reads

$$H = \frac{1}{2m}[p^\mu - \operatorname{Tr}(qA^\mu)][p_\mu - \operatorname{Tr}(qA_\mu)],$$

or $H = \frac{m}{2}\dot{x}^2$ if expressed in terms of the velocity. The Poisson brackets

$$\{x^\mu, p_\nu\} = \delta^\mu_\nu, \qquad \{\Lambda_a, \Lambda_b\} = -\mathrm{i}\delta_{ab},$$

which imply the non-Abelian algebra of charges

$$\{q_a, q_b\} = f_{abc}q_c,$$

again allow us to recover all previous equations of motion from the evolution equation of dynamical variables F, i.e. from $\dot{F} = \{F, H\}$. In terms of the kinematical momentum $\Pi_\mu \equiv p_\mu - \operatorname{Tr}(qA_\mu)$, the Hamiltonian reads $H = \frac{1}{2m}\Pi^2$ and the Poisson brackets take the form

$$\{x^\mu, \Pi_\nu\} = \delta^\mu_\nu, \qquad \{\Pi_\mu, \Pi_\nu\} = \operatorname{Tr}(qF_{\mu\nu}), \qquad \{q_a, q_b\} = f_{abc}q_c.$$

The dynamical variable Λ which allowed for the Lagrangian formulation is well hidden in the latter equations.

Continuum formulation of the dynamics. The field strength $F_{\mu\nu}$ manifests itself physically by the force field, i.e. by an exchange of energy and momentum between the charge carrier

---

[3]In fact, if the charge q is treated as a dynamical variable in the action (2.4), then it amounts to a Lagrange multiplier leading to the equation of motion $\dot{x}^\mu A_\mu^a(x(\tau)) = 0$ which is not gauge invariant. We thank the anonymous referee for drawing our attention to this point.

and the field [40]. In the framework of field theory, the physical entities are described by local fields, i.e. one has a continuum formulation. In order to obtain such a formulation for the particle's equations of motion (2.9), we have to integrate these relations over the variable τ with a delta function concentrated on the particle's trajectory. The resulting expressions then involve the current density $j^{a\mu}(y)$ defined in equation (2.5) as well as the energy-momentum tensor (density) of the point particle which is given by (see Appendix A)

$$T^{\mu\nu}(y) = \int d\tau m \dot{x}^\mu \dot{x}^\nu \delta^4(y - x(\tau)). \tag{2.14}$$

More explicitly, by using

$$\dot{x}^\nu \partial_\nu^y \delta^4(y - x(\tau)) = -\dot{x}^\nu \partial_\nu^x \delta^4(y - x(\tau)) = -\frac{d}{d\tau} \delta^4(y - x(\tau)), \tag{2.15}$$

we have

$$\partial_\nu^y T^{\nu\mu}(y) = \int d\tau m \dot{x}^\mu \dot{x}^\nu \partial_\nu^y \delta^4(y - x(\tau)) = -\int d\tau m \dot{x}^\mu \frac{d}{d\tau} \delta^4(y - x(\tau))$$

$$= \int d\tau m \ddot{x}^\mu \delta^4(y - x(\tau)),$$

and substitution of the particle's equation of motion $m\ddot{x}^\mu = \mathrm{Tr}(qF^{\mu\nu})\dot{x}_\nu$ then yields

$$\partial_\nu^y T^{\nu\mu}(y) = \int d\tau \dot{x}_\nu q^a F_a^{\mu\nu}(x(\tau)) \delta^4(y - x(\tau)) = F_a^{\mu\nu}(y) \int d\tau \dot{x}_\nu q^a \delta^4(y - x(\tau))$$

$$= F_a^{\mu\nu}(y) j_\nu^a(y).$$

The *continuum version of Wong's equations* (2.9) thus reads

$$\partial_\nu T^{\nu\mu} = \mathrm{Tr}(F^{\mu\nu} j_\nu), \qquad \text{where} \qquad D_\mu j^\mu = 0. \tag{2.16}$$

These equations describe the exchange of energy and momentum between the field $F^{\mu\nu}$ and the current j^μ (i.e. the matter). They admit an obvious generalization to Moyal space, see equation (3.8) below. In Appendix A, we show that they also admit a natural extension to curved space (endowed with a metric tensor $(g_{\mu\nu}(x))$). Moreover, we will prove there that they have to hold for arbitrary dynamical matter fields ϕ whose dynamics is described by a generic action $S[\phi; g_{\mu\nu}, A_\mu^a]$ which is invariant under both gauge transformations and general coordinate transformations ($g_{\mu\nu}$ and A_μ^a representing fixed external fields). We note that the expectation values of equations (2.16) viewed as operatorial relations in quantum field theory imply Wong's classical equations of motion for sufficiently localized, quantum "wave-packet" states [9].

Curved space. Finally, we also point out that equations which are somewhat similar to Wong's equations appear in general relativity for a spinning particle in curved space, for which case the contraction of the Riemann tensor $R^\alpha{}_{\beta\mu\nu}$ with the spin tensor $S^{\mu\nu}$ plays a role which is similar to the field strength $F^{\mu\nu}$ in Yang–Mills theories. The explicit form of these equations of motion, which are known as the *Mathisson–Papapetrou–Dixon equations* [13, 30, 33], is given by

$$\frac{\nabla}{d\tau}(mu^\alpha) + \frac{1}{2} S^{\mu\nu} u^\sigma R^\alpha{}_{\sigma\mu\nu} = 0, \qquad \frac{\nabla S^{\alpha\beta}}{d\tau} = 0,$$

where $\frac{\nabla}{d\tau}$ denotes the covariant derivative along the trajectory and (u^μ) is the particle's four-velocity.

3 Lagrangian approach to particles in NC space

Moyal space and distributions. We consider four dimensional Moyal space, i.e. we assume that the space-time coordinates fulfill a Heisenberg-type algebra (for a review see [8, 35, 39] and references therein). Thus, the star product of functions is defined by

$$(f \star g)(x) \equiv \left(e^{\frac{i}{2} \theta^{\mu\nu} \partial_\mu^x \partial_\nu^y} f(x) g(y) \right) \big|_{x=y},$$

where the parameters $\theta^{\mu\nu} = -\theta^{\nu\mu}$ are constant, and their star commutator is defined by $[f \stackrel{\star}{,} g] \equiv f \star g - g \star f$, which implies that $[x^\mu \stackrel{\star}{,} x^\nu] = i\theta^{\mu\nu}$. In the sequel we will repeatedly use the following fundamental properties of the star product:

$$\int d^4x \, f \star g = \int d^4x \, f \cdot g, \qquad \int d^4x \, f \star g \star h = \int d^4x \, h \star f \star g. \tag{3.1}$$

An important point to note is that in Moyal space, the integral $\int d^4x$ plays the role of a trace[4], and hence equations of motion must always be derived from the action rather than the Lagrangian. For a detailed discussion of the algebras of functions and of distributions on Moyal space in the context of non-commutative spaces and of quantum mechanics in phase space, we refer to [43] and [21, 42], respectively. Here, we only note that the star product of the delta distribution δ_y (with support in y) with a function ψ may be defined by application to a test function φ:

$$\langle \delta_y \star \psi, \varphi \rangle \equiv \int d^4x (\delta_y \star \psi)(x)\varphi(x) = \int d^4x (\delta_y \star \psi \star \varphi)(x) = \int d^4x \delta_y(x)(\psi \star \varphi)(x)$$
$$= (\psi \star \varphi)(y).$$

Hence, the action of the distribution $\delta_y \star \psi$ on the test function φ is equal to the action of the distribution δ_y on the test function $\psi \star \varphi$. Similarly, we find

$$\langle \psi \star \delta_y, \varphi \rangle \equiv \int d^4x (\psi \star \delta_y)(x)\varphi(x) = \int d^4x (\psi \star \delta_y \star \varphi)(x) = \int d^4x (\delta_y \star \varphi \star \psi)(x)$$
$$= (\varphi \star \psi)(y).$$

The following considerations hold for an arbitrary antisymmetric matrix $(\theta^{\mu\nu})$, but for the physical applications it is preferable to assume that $\theta^{\mu 0} = 0$, i.e. assume the time to be commuting with the spatial coordinates. This choice is motivated by the fact that the parameters θ^{ij} have close analogies with a constant magnetic field both from the algebraic and dynamical points of view [11], and by the fact that a non-commuting time leads to problems with time-ordering in quantum field theory [5].

Charged particle in Moyal space. Since a $U_\star(1)$ gauge field (A^μ) on Moyal space entails a non-Abelian structure for the field strength tensor $(F_{\mu\nu})$ due to the star product [8, 35, 39],

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu \stackrel{\star}{,} A_\nu],$$

one expects that the treatment of a source for this gauge field as given by a "point" particle should allow for a description of such particles on non-commutative space which is quite similar to the one found by Wong in the case of Yang–Mills theory. If the matter content in field theory is given by a spinor field ψ, then the interaction term with the gauge field reads

$$\int d^4y \, J^\mu \star A_\mu, \qquad \text{with} \qquad J^\mu \equiv g\bar{\psi}\gamma^\mu \star \psi,$$

---

[4]This is best seen by employing the Weyl map from operators to functions, and is also the reason for the cyclicity property (3.1). Furthermore, when considering gauge fields only the action is gauge invariant, not the Lagrangian.

i.e. it involves two star products. By virtue of the properties (3.1) one of these star products can be dropped under the integral, but not both of them. If we consider the particle limit (i.e. J^μ representing the current density of the particle), it is judicious to maintain the star product between J^μ and A_μ so as not to hide the non-commutative nature of the underlying space (over which we integrate) and to allow for the use of the cyclic invariance property (3.1) later on in our derivation. In fact, the pairing $\langle J, A\rangle \equiv \int d^4y J^\mu \star A_\mu$ represents the analogue of the pairing $\langle j, A\rangle \equiv \int d^4y \operatorname{Tr}(j^\mu A_\mu)$ in Yang–Mills theory on commutative space. Accordingly, we will require the action for the particle to be invariant under non-commutative gauge transformations defined at the infinitesimal level by $\delta_\lambda A_\mu = D_\mu \lambda \equiv \partial_\mu \lambda - ig[A_\mu \overset{\star}{,} \lambda]$ where the parameter λ is an arbitrary function. As was done for Wong's equations, we assume that the charge q of the relativistic particle in non-commutative space depends on the parameter τ parametrizing the particle's world line and that it represents a given non-dynamical variable. The coupling of this particle to an external $U_\star(1)$ gauge field (A^μ) can be described by the action[5]

$$S[x] \equiv S_{\text{free}}[x] - S_{\text{int}}[x] \equiv -m \int d\tau \sqrt{\dot{x}^2} - \int d^4y J^\mu A_\mu, \tag{3.2}$$

where

$$J^\mu(y) \equiv \int d\tau q(\tau)\dot{x}^\mu(\tau)\delta^4(y - x(\tau)). \tag{3.3}$$

Keeping in mind the formulation of field theory on non-commutative space [8, 35, 39], we will argue directly with the action rather than the Lagrangian function and require this action to be invariant under non-commutative gauge transformations. For such an infinitesimal transformation we have

$$\delta_\lambda \int d^4y J^\mu A_\mu = \int d^4y J^\mu D_\mu \lambda = -\int d^4y (D_\mu J^\mu) \star \lambda.$$

Hence, invariance of the action (3.2) under non-commutative gauge transformations requires the current to be *covariantly conserved*, i.e. $D_\mu J^\mu = 0$. By virtue of equation (3.3) we now infer that

$$0 \overset{!}{=} (D_\mu J^\mu)(y) = \int d\tau q\dot{x}^\mu D^y_\mu \delta^4(y - x(\tau))$$

$$= \int d\tau q\dot{x}^\mu \left\{ \partial^y_\mu \delta^4(y - x(\tau)) - ig[A_\mu(y) \overset{\star}{,} \delta^4(y - x(\tau))] \right\}. \tag{3.4}$$

Equation (2.15) entails that the first term in the last line can be rewritten as

$$\int d\tau q\dot{x}^\mu \partial^y_\mu \delta^4(y - x(\tau)) = -\int d\tau q \frac{d}{d\tau}\delta^4(y - x(\tau)) = \int d\tau \frac{dq}{d\tau}\delta^4(y - x(\tau)).$$

From condition (3.4) it thus follows that the charge q has to be *covariantly conserved* along the world line in the sense that

$$0 = \int d\tau \frac{Dq}{d\tau}\delta^4(y - x(\tau)) \equiv \int d\tau \left\{ \frac{dq}{d\tau}\delta^4(y - x(\tau)) - igq\dot{x}^\mu[A_\mu(y) \overset{\star}{,} \delta^4(y - x(\tau))] \right\}. \tag{3.5}$$

For later reference, we note that this relation yields the following equality after star multiplication with $A_\nu(y)\delta y^\nu$ and integration over y:

$$\int d^4y \int d\tau \delta x^\nu(\tau) \frac{dq}{d\tau}\delta^4(y - x(\tau)) \star A_\nu(y)$$

---

[5]We note that the coupling of $U_\star(1)$ gauge fields to external currents has also been addressed in the recent work [4].

$$= \mathrm{i}g \int d^4y \int d\tau \delta x^\nu(\tau) q\dot{x}^\mu [A_\mu(y) \stackrel{\star}{,} \delta^4(y - x(\tau))] \star A_\nu(y)$$

$$= -\mathrm{i}g \int d^4y \int d\tau \delta x^\nu(\tau) q\dot{x}^\mu \delta^4(y - x(\tau)) \star [A_\mu(y) \stackrel{\star}{,} A_\nu(y)]. \tag{3.6}$$

In order to derive the equation of motion for the particle determined by the action (3.2), we vary the latter with respect to x^μ. The variation of S_{free} being the same as in commutative space, we only work out the variation of the interaction part S_{int}, all star products being viewed as functions of the variable y:

$$\delta S_{\mathrm{int}} = \delta \int d^4y \int d\tau \big\{ \dot{x}^\mu q\delta^4(y - x(\tau)) \star A_\mu(y) \big\}$$

$$= \int d^4y \int d\tau \big\{ \delta\dot{x}^\mu q\delta^4(y - x(\tau)) \star A_\mu(y) + \dot{x}^\mu q\delta \big[\delta^4(y - x(\tau))\big] \star A_\mu(y) \big\}$$

$$= \int d^4y \int d\tau \left\{ \frac{d(\delta x^\mu)}{d\tau} q\delta^4(y - x(\tau)) \star A_\mu(y) + \dot{x}^\mu q(\delta x^\nu)\partial^x_\nu \delta^4(y - x(\tau)) \star A_\mu(y) \right\}$$

$$= \int d^4y \int d\tau(\delta x^\nu) \left\{ -\frac{d}{d\tau} \big[q\delta^4(y - x(\tau))\big] \star A_\nu(y) + \dot{x}^\mu q\delta^4(y - x) \star \partial^y_\nu A_\mu(y) \right\}.$$

By virtue of the product rule, the first term in the last line yields two terms, one involving $\frac{dq}{d\tau}$ which can be rewritten using relation (3.6), and one involving $\frac{d}{d\tau}\delta^4(y - x(\tau))$ which can be rewritten using (2.15):

$$- \int d^4y \int d\tau(\delta x^\nu) q\frac{d}{d\tau}\delta^4(y - x(\tau)) \star A_\nu(y)$$

$$= \int d^4y \int d\tau(\delta x^\nu) q\dot{x}^\mu \big[\partial^y_\mu \delta^4(y - x(\tau))\big] \star A_\nu(y)$$

$$= - \int d^4y \int d\tau(\delta x^\nu) q\dot{x}^\mu \delta^4(y - x(\tau)) \star \partial^y_\mu A_\nu(y).$$

Hence, we arrive at

$$\delta S_{\mathrm{int}} = \int d^4y \int d\tau(\delta x^\nu)\dot{x}^\mu q\delta^4(y - x(\tau)) \star \big\{ \partial^y_\nu A_\mu(y) - \partial^y_\mu A_\nu(y) + \mathrm{i}g[A_\mu(y) \stackrel{\star}{,} A_\nu(y)] \big\}$$

$$= \int d^4y \int d\tau(\delta x^\nu)\dot{x}^\mu q\delta^4(y - x(\tau)) \star F_{\nu\mu}(y) = \int d\tau(\delta x^\nu)\dot{x}^\mu q F_{\nu\mu}(x(\tau)).$$

Note that it is the constraint equation (3.6) following from (3.5) that yields the terms which are quadratic in the gauge field. This is very much the same mechanism as in the commutative space calculation which leads to Wong's equations, cf. (2.4)–(2.9).

In conclusion, we obtain the following *equation of motion for the charged relativistic particle in non-commutative space*:

$$m\ddot{x}^\mu = qF^{\mu\nu}\dot{x}_\nu, \qquad \text{with} \qquad F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu - \mathrm{i}g[A_\mu \stackrel{\star}{,} A_\nu]. \tag{3.7}$$

We note that the antisymmetry of $F_{\mu\nu}$ with respect to its indices implies

$$0 = \ddot{x}^\mu \dot{x}_\mu = \frac{1}{2}\frac{d\dot{x}^2}{d\tau},$$

which is consistent with $\dot{x}^2 = 1$. Consistency of the equation of motion (3.7) requires its gauge invariance, i.e. the gauge invariance of its right-hand-side. Since

$$\delta_\lambda(qF^{\mu\nu}\dot{x}_\nu) = q(\delta_\lambda F^{\mu\nu})\dot{x}_\nu = -\mathrm{i}gq[F^{\mu\nu} \stackrel{\star}{,} \lambda]\dot{x}_\nu = gq\theta^{\rho\sigma}(\partial_\rho F^{\mu\nu})(\partial_\sigma \lambda)\dot{x}_\nu + \mathcal{O}(\theta^2),$$

the gauge invariance only holds for constant field strengths by contrast to the case of Wong's equation for Yang–Mills fields in commutative space. This difference can be traced back to the fact that the analogue of the trace in Yang–Mills theory is given in Moyal space by the integral $\int d^4x$: since such an integral does not occur in the differential equation (3.7), the gauge invariance is only realized for constant $F_{\mu\nu}$. We will further discuss the latter fields and the comparison with Yang–Mills theory in Section 4 where we will also consider the subsidiary condition for the charge q. Here we only note the following points in this respect. The equation of motion (3.7) taken for itself is consistent for any constant values of q and $F_{\mu\nu}$. Furthermore in non-commutative space the auxiliary variable q cannot be rendered dynamical with a non-trivial gauge transformation law (2.10) for $q(\tau)$ since the star commutator of $q(\tau)$ with a gauge transformation parameter $\lambda(x(\tau))$ vanishes.

In summary, the coupling of a relativistic particle to a gauge field (A^μ) is described in general by the Lagrangian

$$L(x,\dot{x}) = -m\sqrt{\dot{x}^2} - qA_\mu\dot{x}^\mu, \qquad \text{or} \qquad L(x,\dot{x}) = \frac{m}{2}\dot{x}^2 + qA_\mu\dot{x}^\mu.$$

The non-commutativity of space-time can be implemented in the Lagrangian framework by rewriting the interaction term of the action as an integral $\int d^4y(J^\mu \star A_\mu)(y)$ where the current J^μ is defined by (3.3) and by requiring this action to be invariant under non-commutative gauge transformations. The resulting equation of motion (3.7) is only gauge invariant for constant field strengths.

Continuum formulation of the dynamics. By following the same lines of arguments as for the Lorentz–Yang–Mills force equation (see equations (2.14)–(2.16)), we can obtain a continuum version of the equation of motion (3.7) by multiplying this equation with $\delta^4(y-x(\tau))\varphi(y)$, where $\varphi(y)$ is a suitable test function, and integrating over τ and over y. More explicitly, by starting from the energy-momentum tensor (2.14) for the point particle and using relation (2.15), we get

$$\int d^4y(\partial_\nu T^{\nu\mu})(y)\varphi(y) = \int d^4y \int d\tau m\dot{x}^\mu\dot{x}^\nu\partial_\nu^y\delta^4(y-x(\tau))\varphi(y)$$

$$= -\int d^4y \int d\tau m\dot{x}^\mu\frac{d}{d\tau}\delta^4(y-x(\tau))\varphi(y) = \int d^4y \int d\tau m\ddot{x}^\mu\delta^4(y-x(\tau))\varphi(y).$$

Substitution of equation (3.7) then yields the expression

$$\int d^4y \int d\tau qF^{\mu\nu}(x(\tau))\dot{x}_\nu\delta^4(y-x(\tau))\varphi(y) = \int d^4y \int d\tau qF^{\mu\nu}(y)\dot{x}_\nu\delta^4(y-x(\tau))\varphi(y)$$

$$= \int d^4y(F^{\mu\nu}J_\nu)(y)\varphi(y),$$

where we considered the current density (3.3) in the last line. Thus, we have the result

$$\partial_\nu T^{\nu\mu} = F^{\mu\nu}J_\nu, \qquad \text{where} \qquad D_\mu J^\mu = 0, \tag{3.8}$$

and these relations are completely analogous to the continuum equations (2.16) which correspond to Wong's equations (apart from the fact that the invariance of (3.8) under non-commutative gauge transformations requires the field strength $F^{\mu\nu}$ to be constant).

4   Case of a constant field strength

We will now discuss the dynamics of charged particles coupled to a constant field strength on the basis of the results obtained in Sections 2 and 3. Indeed this case represents a mathematically tractable and physically interesting application of the general formalism.

We successively discuss the case of non-Abelian Yang–Mills theory on Minkowski space and the case of a $U_\star(1)$ gauge field on Moyal space while emphasizing the differences that exist for constant field strengths.

4.1   Wong's equations in commutative space

The case of a "uniform field strength" in Yang–Mills theory has been addressed some time ago by the authors of reference [9], see also [19] for related points. Since the Yang–Mills field strength $F_{\mu\nu}(x) \equiv F_{\mu\nu}^a(x)T^a$ is not gauge invariant, but transforms under finite gauge transformations as $F_{\mu\nu}' = U^{-1}F_{\mu\nu}U$ (with $U(x) \in G =$ structure group), one has to specify first what is meant by a constant field. The field $F_{\mu\nu}$ is said to be *uniform* if the gauge field $A_\mu(x) \equiv A_\mu^a(x)T^a$ at a point x can be related by a gauge transformation to the gauge field $A_\mu(y)$ at any other point y. More precisely, for a space-time translation parametrized by $a \in \mathbb{R}^4$, there exists a gauge transformation $x \mapsto U(x;a) \in G$ such that

$$A_\mu(x + a) = U^{-1}(x;a)A_\mu(x)U(x;a) + \mathrm{i}U^{-1}(x;a)\partial_\mu U(x;a),$$

and thereby

$$F_{\mu\nu}(x + a) = U^{-1}(x;a)F_{\mu\nu}(x)U(x;a).$$

In this case a gauge may be chosen in which all components of $F_{\mu\nu}$ are constant because $F_{\mu\nu}$ at the point x can be made equal to its value at some arbitrary point y by transforming it by an appropriate gauge group element.

Since the field strength $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu + \mathrm{i}[A_\mu, A_\nu]$ contains two terms, namely $\partial_\mu A_\nu - \partial_\nu A_\mu$ which has the Abelian form, and $[A_\mu, A_\nu]$ which does not involve derivatives, a constant non-zero field strength $F_{\mu\nu}$ can be obtained either from a linear gauge potential (i.e. an Abelian-like gauge field),

$$A_\mu = -\frac{1}{2}F_{\mu\nu}x^\nu, \qquad \partial_\mu A_\nu - \partial_\nu A_\mu = F_{\mu\nu} = \text{const}, \qquad [A_\mu, A_\nu] = 0,$$

or from a constant non-Abelian-like gauge potential,

$$A_\mu = \text{const}, \qquad \partial_\mu A_\nu = 0, \qquad \mathrm{i}[A_\mu, A_\nu] = F_{\mu\nu} = \text{const}.$$

In fact [9], these two types of potentials exhaust all possibilities for a constant field strength. It has been shown for the structure group SU(2) that the two types of gauge potentials leading to a same constant field strength are gauge inequivalent and result in physically different behavior when matter interacts with them, e.g. the solutions of Wong's equations have completely different properties in both cases. An explicit example for $G = \text{SU}(2)$ is given by a constant magnetic field in z-direction [19]: let σ_k (with $k = 1, 2, 3$) denote the Pauli matrices and suppose

$$F_{0i} = 0, \qquad F_{ij} = \varepsilon_{ijk}B_k, \qquad \text{with} \qquad (B_k)_{k=1,2,3} = (0, 0, 2\sigma_3).$$

This constant field strength derives from the linear Abelian-like potential

$$\vec{A} \equiv (A_k)_{k=1,2,3} = -\frac{1}{2}\vec{x} \wedge \vec{B} = (-y, x, 0)\sigma_3,$$

or from a constant non-Abelian-like potential $\vec{A} = (-\sigma_2, \sigma_1, 0)$.

4.2   Wong's equations in non-commutative space

The field strength associated to a $U_\star(1)$ gauge field (A_μ) on Moyal space reads

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu - \mathrm{i}g[A_\mu \overset{\star}{,} A_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu + g\theta^{\rho\sigma}(\partial_\rho A_\mu)(\partial_\sigma A_\nu) + \mathcal{O}(\theta^2).$$

Due to the derivatives appearing in the star commutator, the Abelian-like term $\partial_\mu A_\nu - \partial_\nu A_\mu$ and the non-Abelian-like term $-\mathrm{i}g[A_\mu \overset{\star}{,} A_\nu]$ cannot vanish independently of each other: the

non-commutative field strength $F_{\mu\nu}$ can only be constant for a linear Abelian-like potential. More precisely, for

$$A_\mu = -\frac{1}{2}\bar{B}_{\mu\nu}x^\nu, \tag{4.1}$$

where the coefficients $\bar{B}_{\mu\nu} \equiv -\bar{B}_{\nu\mu}$ are constant, we obtain

$$F_{\mu\nu} = \bar{B}_{\mu\nu} - \frac{g}{4}\bar{B}_{\mu\rho}\theta^{\rho\sigma}\bar{B}_{\sigma\nu}. \tag{4.2}$$

This field strength is constant, but dependent on the non-commutativity parameters $\theta^{\mu\nu}$. If we interpret $\bar{B}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ as the physical field strength, then (4.2) means that the non-commutativity parameters $\theta^{\mu\nu}$ modify in general the trajectories of the particle as compared to its motion in commutative space[6].

Since the field strength transforms under a finite gauge transformation $U(x) \in U_\star(1)$ as $F_{\mu\nu} = U^{-1}\star F_{\mu\nu}\star U$, a constant field $F_{\mu\nu}$ is gauge invariant. Thus, for constant field strengths the situation is quite different for $U_\star(1)$ gauge fields and for non-Abelian gauge fields on Minkowski space despite the fact that we encounter the same structure for the gauge transformations, the field strength and the action functional in both cases.

Let us again come back to the expression (4.1) for the gauge field. Substitution of this expression into the subsidiary condition (3.5) yields

$$
\begin{aligned}
0 &= \int d\tau \big\{ \dot{q}\delta^4(y-x(\tau)) - \mathrm{i}gq\dot{x}^\mu[A_\mu(y) \overset{\star}{,} \delta^4(y-x(\tau))] \big\} \\
&= \int d\tau \big\{ \dot{q}\delta^4(y-x(\tau)) - \frac{g}{2}q\dot{x}^\mu\bar{B}_{\mu\rho}\theta^{\rho\sigma}\partial_\sigma^y\delta^4(y-x(\tau)) \big\}.
\end{aligned} \tag{4.3}
$$

If the matrix $(\bar{B}_{\mu\nu})$ is the inverse of the matrix $(\theta^{\mu\nu})$, i.e. $\bar{B}_{\mu\rho}\theta^{\rho\sigma} = \delta_\mu^\sigma$, then $F_{\mu\nu} = (1 - \frac{g}{4})\bar{B}_{\mu\nu}$ and, by virtue of (2.15) and an integration by parts, condition (4.3) takes the form

$$0 = \int d\tau \dot{q}\delta^4(y-x(\tau))\left\{1 - \frac{g}{2}\right\}.$$

The latter relation is obviously satisfied for a constant q. In this case, the equation of motion (3.7) for the particle in non-commutative space, i.e. $m\ddot{x}^\mu = qF^{\mu\nu}\dot{x}_\nu$, has the same form as the one of an electrically charged particle in ordinary space. This result is analogous to the one obtained for a constant magnetic field in x^3-direction within the Hamiltonian approaches, see equations (5.10) and (5.13) below.

5   Hamiltonian approaches to particles in NC space

To start with, we briefly review the Hamiltonian approaches in commutative space before considering the generalization to the non-commutative setting. In the latter setting, we will notice that various approaches yield different results since several expressions which coincide in commutative space no longer agree.

---

[6]We note that the latter dependence on the non-commutativity parameters can be eliminated mathematically if one assumes that $\bar{B}_{\mu\nu}$ depends in a specific way on the parameters $\theta^{\mu\nu}$ and some θ-independent constants $B_{\mu\nu}$. To illustrate this point [11], we assume that the only non-vanishing components of $\bar{B}_{\mu\nu}$ and $\theta^{\mu\nu}$ are as follows: $\bar{B}_{12} = -\bar{B}_{21} \equiv \bar{B}, \theta^{12} = -\theta^{21} \equiv \theta$, i.e. $F_{12} = \bar{B}(1 + \frac{g}{4}\theta\bar{B}) = -F_{21}$. If $\bar{B}$ depends on θ and on a θ-independent constant B according to

$$\bar{B} = \bar{B}(B;\theta) \equiv \frac{2}{g\theta}\left(\sqrt{1 + g\theta B} - 1\right) = B\left(1 - \frac{g}{4}\theta B\right) + \mathcal{O}(\theta^2),$$

then relation (4.2) implies that the non-commutative field strength F_{12} is a θ-independent constant: $F_{12} = B$.

5.1   Reminder on the Poisson bracket approach

The Hamiltonian formulation of relativistic (as well as non-relativistic) mechanics is based on two inputs (e.g. see reference [29]): a Hamiltonian function and a Poisson structure (or equivalently a symplectic structure). If one starts from the Lagrangian formulation, the Hamiltonian function is obtained from the Lagrange function by a Legendre transformation. E.g. the Lagrangian $L(x, \dot{x}) = \frac{m}{2}\dot{x}^2 + eA_\mu\dot{x}^\mu$ (involving the constant charge e) yields the *Hamiltonian*

$$H(x, p) = \frac{1}{2m}(p - eA)^2 = \frac{1}{2m}(p^\mu - eA^\mu)(p_\mu - eA_\mu). \tag{5.1}$$

The trajectories in phase space are parametrized by $\tau \mapsto (x(\tau), p(\tau))$ where τ denotes a real variable to be identified with proper time after the equations of motion have been derived. The *Poisson brackets* $\{\cdot, \cdot\}$ of the phase space variables x^μ, p^μ are chosen in such a way that the evolution equation $\dot{F} = \{F, H\}$ (where $\dot{F} \equiv dF/d\tau$) yields the Lagrangian equation of motion for x^μ, though written as a system of first order differential equations. For instance, if we consider the usual form of the Poisson brackets, i.e. the *canonical Poisson brackets*

$$\{x^\mu, x^\nu\} = 0, \qquad \{p^\mu, p^\nu\} = 0, \qquad \{x^\mu, p^\nu\} = \eta^{\mu\nu}, \tag{5.2}$$

then substitution of $F = x^\mu$ and $F = p^\mu$ into $\dot{F} = \{F, H\}$ (with H given by (5.1)) yields the system of equations

$$m\dot{x}^\mu = p^\mu - eA^\mu, \qquad m\dot{p}^\mu = e(p_\nu - eA_\nu)\partial^\mu A^\nu,$$

from which we conclude that

$$m\ddot{x}^\mu = \dot{p}^\mu - e\dot{A}^\mu = \underbrace{\frac{e}{m}(p_\nu - eA_\nu)}_{=e\dot{x}_\nu}\partial^\mu A^\nu - e\dot{x}^\nu\partial_\nu A^\mu = e\dot{x}_\nu\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) \equiv ef^{\mu\nu}\dot{x}_\nu.$$

This equation coincides with the Euler–Lagrange equation for x^μ following from the Lagrangian $L(x, \dot{x}) = \frac{m}{2}\dot{x}^2 + eA_\mu\dot{x}^\mu$.

Concerning the gauge invariance, we emphasize a result [40] which does not seem to be very well known. The Hamiltonian (5.1) is *not* invariant under a gauge transformation $A_\mu \to A_\mu + \partial_\mu\lambda$ which is quite intriguing. However, it is invariant if this transformation is combined with the phase space transformation $(x^\mu, p^\mu) \to (x^\mu, p^\mu + e\partial^\mu\lambda)$, the latter being a canonical transformation since it preserves the fundamental Poisson brackets (5.2). Indeed, under this combined transformation the kinematical momentum $\Pi^\mu \equiv p^\mu - eA^\mu$ (which coincides with $m\dot{x}^\mu$) is invariant.

We note that the Hamiltonian (5.1) can be rewritten in terms of the variable $\Pi_\mu \equiv p_\mu - eA_\mu$ as $H \equiv \frac{1}{2m}\Pi^2$. Thereby H has the form of a free particle Hamiltonian, but the Poisson brackets are now modified: from $\Pi_\mu = p_\mu - eA_\mu$ and (5.2) it follows that

$$\{x^\mu, x^\nu\} = 0, \qquad \{\Pi^\mu, \Pi^\nu\} = ef^{\mu\nu}(x), \qquad \{x^\mu, \Pi^\nu\} = \eta^{\mu\nu}, \tag{5.3}$$

with $f^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu$. Since the electromagnetic field strength $f^{\mu\nu}$ is gauge invariant, the latter invariance is manifestly realized in this formulation.

In summary, the coupling of a charged particle to an electromagnetic field can either be described by the canonical Poisson brackets (5.2) and the minimally coupled Hamiltonian (5.1) or by introducing the field strength into the Poisson brackets (as a non-commutativity of the momenta) and considering a Hamiltonian which has the form of a free particle Hamiltonian.

5.2   Reminder on the symplectic form approach

If we gather all phase space variables into a vector $\vec{\xi} \equiv (\xi^I) \equiv (x^0, \ldots, x^3, p^0, \ldots, p^3)$, the fundamental Poisson brackets (5.2) read

$$\left\{\xi^I, \xi^J\right\} = \Omega^{IJ}, \qquad \text{with} \qquad \left(\Omega^{IJ}\right) \equiv \begin{bmatrix} 0 & \eta^{\mu\nu} \\ -\eta^{\mu\nu} & 0 \end{bmatrix}.$$

The inverse of the Poisson matrix $\Omega \equiv (\Omega^{IJ})$ is the matrix with entries $\omega_{IJ} \equiv (\Omega^{-1})_{IJ}$ which defines the *symplectic* 2-*form*

$$\omega \equiv \frac{1}{2}\sum_{I,J} \omega_{IJ} d\xi^I \wedge d\xi^J = dp^\mu \wedge dx_\mu, \tag{5.4}$$

e.g. see reference [29] for mathematical details. The Hamiltonian equations of motion can be written as

$$\dot{\xi}^I = \left\{\xi^I, H\right\} = \Omega^{KJ}\partial_K \xi^I \partial_J H, \qquad \text{i.e.} \qquad \dot{\xi}^I = \Omega^{IJ}\partial_J H,$$

or equivalently as $\omega_{IJ}\dot{\xi}^J = \partial_I H$.

    In terms of the phase space variables (x^μ, Π^μ) appearing in the non-canonical Poisson brackets (5.3), the symplectic 2-form (5.4) reads

$$\omega = d\Pi^\mu \wedge dx_\mu + \frac{1}{2}e f_{\mu\nu} dx^\mu \wedge dx^\nu.$$

This formulation of the electromagnetic interaction based on the symplectic 2-form and the evolution equation $\omega_{IJ}\dot{\xi}^J = \partial_I H$ goes back to the seminal work of Souriau [37].

5.3   Standard (Poisson bracket) approach to NC space-time

In order to introduce a non-commutativity for the configuration space, one generally starts from a function H on phase space to which one refers as the Hamiltonian without any reference to a Lagrangian, e.g. we can consider the function H given in equation (5.1). The non-commutativity of space-time is then implemented by virtue of the Poisson brackets

$$\{x^\mu, x^\nu\} = \theta^{\mu\nu}, \qquad \{p^\mu, p^\nu\} = 0, \qquad \{x^\mu, p^\nu\} = \eta^{\mu\nu}, \tag{5.5}$$

where $\theta^{\mu\nu} = -\theta^{\nu\mu}$ is again assumed to be constant. (For an overview of the description of non-relativistic charged particles in non-commutative space we refer to [3, 11, 24], the pioneering work being [17, 18], see also [1, 2, 31, 32] for some subsequent early work. We also mention that dynamical systems in non-commutative space can be constructed by applying Dirac's treatment of constrained Hamiltonian systems to an appropriate action functional, see [12] and references therein.)

    As in the commutative setting, we gather all phase space variables into a vector $\vec{\xi} \equiv (\xi^I) \equiv (x^0, \ldots, x^3, p^0, \ldots, p^3)$, the fundamental Poisson brackets (5.5) being now given by

$$\left\{\xi^I, \xi^J\right\} = \Omega^{IJ}, \qquad \text{with} \qquad \left(\Omega^{IJ}\right) \equiv \begin{bmatrix} \theta^{\mu\nu} & \eta^{\mu\nu} \\ -\eta^{\mu\nu} & 0 \end{bmatrix}.$$

Quite generally, the Poisson bracket of two arbitrary functions F, G on phase space reads

$$\{F, G\} \equiv \sum_{I,J} \Omega^{IJ}\partial_I F \partial_J G = \theta^{\mu\nu}\frac{\partial F}{\partial x^\mu}\frac{\partial G}{\partial x^\nu} + \frac{\partial F}{\partial x^\mu}\frac{\partial G}{\partial p_\mu} - \frac{\partial F}{\partial p^\mu}\frac{\partial G}{\partial x_\mu}.$$

Substitution of $F = x^\mu$ and $F = p^\mu$ into $\dot{F} = \{F, H\}$ (with H given by (5.1)) yields the system of equations

$$m\dot{x}^\mu = (p_\nu - eA_\nu)(\eta^{\mu\nu} - e\theta^{\mu\rho}\partial_\rho A^\nu),$$
$$m\dot{p}^\mu = e(p_\nu - eA_\nu)\partial^\mu A^\nu. \tag{5.6}$$

In the present case, the phase space transformation $(x^\mu, p^\mu) \to (x^\mu, p^\mu + e\partial^\mu\lambda)$ does not represent a canonical transformation since it does not preserve the Poisson brackets (5.5) if $\theta^{\mu\nu} \neq 0$. Hence, the resulting Hamiltonian equations of motion (5.6) are not gauge invariant, as has already been pointed out in reference [11] by considering different gauges.

In the next two subsections, we recall how this problem can be overcome for the particular case of a constant field strength as well as more generally, and we compare with the results obtained in Section 3 from the action involving star products. Here, we only note that a non-Abelian structure of the field strength is hidden in equation (5.6). To illustrate this point, we consider the particular case where the only non-vanishing components of $\theta^{\mu\nu}$ are $\theta^{ij} = \varepsilon^{ij}\theta$ (with $i, j \in \{1, 2\}$ and $\varepsilon^{12} \equiv -\varepsilon^{21} \equiv 1$) and where the only non-vanishing components of A^μ are $A^i(x^1, x^2)$ (with $i \in \{1, 2\}$). For this situation which describes a time-independent magnetic field perpendicular to the $x^1 x^2$-plane, the first of equations (5.6) yields

$$m\dot{x}_i = (p_k - eA_k)(\delta_{ik} - e\theta\varepsilon_{ij}\partial_j A_k),$$

and implies

$$m\frac{d}{dt}(x_i + e\theta\varepsilon_{ij}A_j) = (1 + e\mathcal{F}_{12})(p_i - eA_i) \tag{5.7}$$

with

$$\mathcal{F}_{12} \equiv \partial_1 A_2 - \partial_2 A_1 + e\{A_1, A_2\} = \partial_1 A_2 - \partial_2 A_1 + e\theta^{\rho\sigma}(\partial_\rho A_1)(\partial_\sigma A_2). \tag{5.8}$$

Thus, we find a non-Abelian structure for the generalized field strength, but in the present approach the field $\mathcal{F}_{\mu\nu}$ is only linear in the non-commutativity parameters in contrast to the field $F_{\mu\nu}$ in (3.7) which involves the star commutator

$$-i[A_\mu \overset{\star}{,} A_\nu] = \theta^{\rho\sigma}(\partial_\rho A_\mu)(\partial_\sigma A_\nu) + \mathcal{O}(\theta^2).$$

If the gauge potential is linear in x, the field strengths $\mathcal{F}_{\mu\nu}$ and $F_{\mu\nu}$ as defined by equations (5.8) and (3.7), respectively, coincide with each other (if one identifies the coupling constant g with e).

To conclude, we note that (5.7) can be solved for $p_i - eA_i$ in terms of $m\dot{x}_i$: the system of first order differential equations (5.6) can then be written as a second order equation for x^μ, but the resulting equations of motion are not gauge invariant and they cannot be derived from a Lagrangian [3].

5.4   Standard approach to NC space-time continued

The reasoning presented concerning the brackets (5.3) suggests to consider a Hamiltonian which has a free form and to introduce a field strength $B^{\mu\nu}(x)$ as a non-commutativity of the momenta, i.e. consider phase space variables (x^μ, p^μ) satisfying the non-canonical Poisson algebra

$$\{x^\mu, x^\nu\} = \theta^{\mu\nu}, \qquad \{p^\mu, p^\nu\} = eB^{\mu\nu}, \qquad \{x^\mu, p^\nu\} = \eta^{\mu\nu}, \tag{5.9}$$

with $\theta^{\mu\nu}$ constant. As pointed out in reference [22], the Jacobi identities for the algebra (5.9) are only satisfied if *the field strength is constant:*

$$\{x^\mu, \{p^\nu, p^\lambda\}\} + \{\text{cyclic permutations of } \mu, \nu, \lambda\} = e\theta^{\mu\rho}\partial_\rho B^{\nu\lambda}.$$

Thus, the dynamics of a charged particle coupled to a *constant field* $B^{\mu\nu}$ on non-commutative space-time can be described in terms of phase space variables (x^μ, p^μ) satisfying the non-canonical Poisson algebra (5.9), the Hamiltonian being given by $H(p) = \frac{1}{2m}p^2 = \frac{1}{2m}p^\mu p_\mu$. The Hamiltonian equations of motion

$$m\dot{x}^\mu = p^\mu, \qquad m\dot{p}^\mu = eB^{\mu\nu}p_\nu,$$

then imply the second order equation

$$m\ddot{x}^\mu = eB^{\mu\nu}\dot{x}_\nu. \tag{5.10}$$

This equation of motion for x^μ coincides with the one that one encounters for $\theta^{\mu\nu} = 0$ since the Hamiltonian only depends on p and not on the coordinates x^μ whose Poisson brackets do not vanish. However the non-commutativity parameters $\theta^{\mu\nu}$ appear in quantities like the volume form on phase space which is the 4-fold exterior product of the symplectic form with itself,

$$dV \equiv \frac{1}{4!}\omega^4 = \frac{1}{\sqrt{\det\Omega}}d\xi^1 \cdots d\xi^8,$$

where Ω denotes the Poisson matrix and where we suppressed the exterior product symbol.

5.5   "Exotic" (symplectic form) approach to NC space-time

The Hamiltonian approach to mechanics on non-commutative space based on the simple form (5.9) of the Poisson algebra (in which the Poisson bracket $\{x^\mu, p^\nu\}$ has the canonical form) has been nicknamed the *standard approach*. As we just recalled, it does not allow for the inclusion of a non-constant field strength. By contrast, the so-called *exotic approach* [18, 24] which is based on a simple form of the symplectic 2-form allows us to describe generic field strengths $B_{\mu\nu}(\vec{x})$. In this setting, the constant non-commutativity parameters $\theta^{\mu\nu}$ are introduced into the symplectic 2-form[7] defined on the phase space parametrized by (x^μ, p^μ):

$$\omega = dp^\mu \wedge dx_\mu + \frac{1}{2}eB_{\mu\nu}dx^\mu \wedge dx^\nu + \frac{1}{2}\theta_{\mu\nu}dp^\mu \wedge dp^\nu.$$

The Poisson matrix is obtained by the inversion of the symplectic matrix (e.g. see reference [41] for the case of a space-time of arbitrary dimension), and therefore it has a more complicated form than the one corresponding to (5.9). By way of illustration, we recall the result that one obtains for the simplest instance [18] where one has only two spatial coordinates, i.e. $\vec{x} \equiv (x_1, x_2)$:

$$\{x_1, x_2\} = \kappa^{-1}\theta, \qquad \{p_1, p_2\} = \kappa^{-1}eB, \qquad \{x_i, p_j\} = \kappa^{-1}\delta_{ij},$$

where $\kappa(\vec{x}) \equiv 1 - e\theta B(\vec{x})$ with $\theta_{12} \equiv \theta$ and $B_{12} \equiv B$. None of the brackets now has a canonical form. The equations of motion following from the Hamiltonian $H(\vec{x}, \vec{p}) \equiv \frac{1}{2m}\vec{p}^2 + eV(\vec{x})$ read

$$\dot{p}_i = eE_i + eB\varepsilon_{ij}\dot{x}_j, \qquad \text{with} \qquad E_i \equiv -\partial_i V, \qquad i \in \{1, 2\}$$
$$m^*\dot{x}_i = p_i - em\theta\varepsilon_{ij}E_j, \qquad \text{with} \qquad m^* \equiv \kappa m, \qquad \kappa(\vec{x}) = 1 - e\theta B(\vec{x}), \tag{5.11}$$

where ε_{ij} denotes the components of the constant antisymmetric tensor normalized by $\varepsilon_{12} = 1$. The parameter $m^*(\vec{x}) \equiv m\kappa(\vec{x})$ may be viewed as an effective mass depending on the position of the particle. Various physical applications of this system of evolution equations have been found in recent years, see [24] and references therein. For $V \equiv 0$, we have $p_i = m^*\dot{x}_i = m\kappa\dot{x}_i$, hence

$$\dot{p}_i = m\dot{\kappa}\dot{x}_i + m\kappa\ddot{x}_i, \qquad \text{with} \qquad \dot{\kappa} = -e\theta\dot{B} = -e\theta\dot{x}_j\partial_j B.$$

---

[7]One may as well consider $\vec{p}$-dependent parameters $\theta^{\mu\nu}$.

Substitution of this expression into the first of equations (5.11) yields a second order differential equation for x_i:

$$m^*(\vec{x})\ddot{x}_i = e\varepsilon_{ij}\dot{x}_j B^*, \qquad \text{with} \qquad B^* \equiv B + \frac{1}{2}m\theta\varepsilon_{ij}\dot{x}_i(\partial_j B). \tag{5.12}$$

This equation, which looks somewhat exotic, includes a θ-dependent term depending on the derivative of the field strength and it involves an $\vec{x}$-dependent mass, i.e. there is a dependence of parameters on the localization of the particle in the space in which it evolves.

The expressions in (5.12) simplify greatly in the case of a *constant magnetic field:* equation (5.12) then reduces to

$$m\ddot{x}_i = e\frac{B}{\kappa}\varepsilon_{ij}\dot{x}_j, \qquad \text{with} \qquad \kappa = 1 - e\theta B = \text{const}. \tag{5.13}$$

As was pointed out earlier [23], this equation of motion coincides with the "standard approach" equation (5.10) after a rescaling of time $t \to \kappa t$. We note that the value $\check{B} \equiv \frac{B}{1-e\theta B}$ coincides with the one obtained for a constant magnetic field in two dimensions from the Seiberg–Witten map in non-commutative gauge field theory [11], but it differs from the constant non-commutative field strength

$$F_{12} \equiv \partial_1 A_2 - \partial_2 A_1 - \mathrm{i}e[A_1 \overset{\star}{,} A_2] = \partial_1 A_2 - \partial_2 A_1 + e\{A_1, A_2\},$$

e.g. in the symmetric gauge $(A_1, A_2) = (-\frac{B}{2}x_2, \frac{B}{2}x_1)$, where one finds $F_{12} = B(1 + \frac{e\theta B}{4\kappa})$.

In conclusion, different Hamiltonian formulations for a charged "point" particle in a non-commutative space lead to different results. However, for the special case of a constant magnetic field strength we have seen in the previous two subsections that the different Hamiltonian formulations lead to the same results (or to results that are related to each other by a redefinition of the magnetic field). So does the Lagrangian formulation of Section 3 as we have shown in Section 4.

6   Concluding remarks

Just as there exist different approaches to the formulation of gauge field theories on non-commutative spaces (e.g. the star product approach [39], the approach of spectral triples [10], of matrix models [14], . . .), there appear to exist different approaches to the dynamics of relativistic or non-relativistic particles in non-commutative space which are subject to a background gauge field. It is plausible that these approaches yield essentially the same results in the particular case of a constant magnetic field, i.e. a field strength which does not depend on the non-commuting coordinates. The "exotic" (symplectic form) approach to non-commutative space-time can be viewed as an extension of all other approaches to the case of a generic field strength.

A   Continuum formulation on a generic manifold

In this appendix, we show that Wong's equations, as formulated on a generic space-time manifold, admit a simple continuum version. Moreover, we will prove that the latter formulation has to hold for arbitrary dynamical matter fields ϕ whose dynamics is described by a generic action $S[\phi; g_{\mu\nu}, A_\mu^a]$ which is invariant under both gauge transformations and general coordinate transformations ($g_{\mu\nu}$ and A_μ^a representing fixed external fields). These arguments generalize to Moyal space in the particular case of a constant field strength.

Let M be a four dimensional space-time manifold endowed with a fixed metric tensor $(g_{\mu\nu})$ of signature $(+, -, -, -)$. We denote the covariant derivative of a tensor field with respect

to the Levi-Civita-connection by ∇_μ (e.g. $\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\rho} V^\rho$ where the coefficients $\Gamma^\nu_{\mu\rho}$ are the Christoffel symbols) and the gauge covariant derivative as before by D_μ (e.g. $\delta_\lambda A^a_\mu = D_\mu \lambda^a \equiv \partial_\mu \lambda^a - \mathrm{i} g [A_\mu, \lambda]^a$ for the infinitesimal gauge transformation of the Yang–Mills gauge field (A^a_μ)). Since we used the notation $\frac{Dq^a}{d\tau} \equiv \dot{x}^\mu D_\mu q^a$ in the main body of the text, we will write $\frac{\nabla V^\mu}{d\tau} \equiv \dot{x}^\nu \nabla_\nu V^\mu$ for the derivative of the vector field $V^\mu(x(\tau))$ along the trajectory $\tau \mapsto x(\tau)$.

Lorentz-force and its non-Abelian generalization. The Lorentz-force equation on the space-time manifold M reads

$$m\frac{\nabla u^\mu}{d\tau} = e F^\mu{}_\nu u^\nu, \tag{A.1}$$

where $u^\mu \equiv \dot{x}^\mu$ denotes the 4-velocity of the particle of constant charge $q \equiv e$ and where $F_{\mu\nu}$ represents a given electromagnetic field strength. This equation of motion follows from the point particle action

$$S[x] = \frac{m}{2}\int d\tau\, g_{\mu\nu}(x(\tau))\dot{x}^\mu \dot{x}^\nu + e \int d\tau\, A_\mu(x(\tau))\dot{x}^\mu$$

upon variation with respect to x^μ.

The natural generalization of (A.1) to non-Abelian Yang–Mills theory is given by *Wong's equations* as written *on the space-time manifold M*:

$$m\frac{\nabla^2 x^\mu}{d\tau^2} = q^a F^{a\mu}{}_\nu \dot{x}^\nu, \qquad \text{where} \qquad \frac{Dq^a}{d\tau} = 0. \tag{A.2}$$

Here, the covariant constancy of the charge-vector (q^a) represents the geometrically natural generalization of the ordinary constancy of the charge e appearing in the Abelian gauge theory. The equation of motion of x^μ follows from the action functional

$$S_{\mathrm{W}}[x] = \frac{m}{2}\int d\tau\, g_{\mu\nu}(x(\tau))\dot{x}^\mu \dot{x}^\nu + \int d\tau\, q^a A^a_\mu(x(\tau))\dot{x}^\mu. \tag{A.3}$$

Continuum formulation. The components $T^{\mu\nu}$ of the energy-momentum tensor (density) and the components of the current density may be defined as functional derivatives of the action,

$$T^{\mu\nu}(x) \equiv 2\frac{\delta S_{\mathrm{W}}}{\delta g_{\mu\nu}(x)}, \qquad j^\mu_a(x) \equiv \frac{\delta S_{\mathrm{W}}}{\delta A^a_\mu(x)}$$

so that expression (A.3) implies

$$T^{\mu\nu}(y) = \int d\tau\, \delta^4(y - x(\tau)) m \dot{x}^\mu(\tau)\dot{x}^\nu(\tau),$$

$$j^{a\mu}(y) = \int d\tau\, \delta^4(y - x(\tau)) q^a(\tau)\dot{x}^\mu(\tau).$$

We note that the energy-momentum 4-vector is then given by $P^\mu = \int_{\mathbb{R}^3} d^3 x\, T^{\mu 0}$ which yields the standard expressions:

$$P^0 = \int_{\mathbb{R}^3} d^3 x\, T^{00} = m\dot{x}^0 = m\frac{dt}{d\tau} = \frac{m}{\sqrt{1 - \vec{v}^2}}, \qquad P^i = m\dot{x}^i = \frac{mv^i}{\sqrt{1 - \vec{v}^2}}.$$

The 4-divergence of the energy-momentum tensor can be evaluated by substituting the equation of motion $m\frac{\nabla^2 x^\mu}{d\tau^2} = q^a F^{a\mu}{}_\nu \dot{x}^\nu$:

$$\nabla_\mu T^{\mu\nu}(y) = \int d\tau\, (\dot{x}^\mu \nabla^y_\mu)\delta^4(y - x(\tau)) m \dot{x}^\nu(\tau) = \int d\tau\, m\frac{\nabla^2 x^\nu}{d\tau^2}(\tau)\delta^4(y - x(\tau))$$

$$= \int d\tau q^a(\tau) F^{a\nu}{}_\mu(x(\tau)) \dot{x}^\mu(\tau) \delta^4(y - x(\tau)) = F^{a\nu}{}_\mu(y) j^{a\mu}(y).$$

Similarly, substitution of the charge transport equation $\frac{Dq^a}{d\tau} = 0$ into the gauge covariant divergence of the current density gives

$$D_\mu j^{a\mu}(y) = \int d\tau (\dot{x}^\mu D^y_\mu) \delta^4(y - x(\tau)) q^a(\tau) = \int d\tau \delta^4(y - x(\tau)) \frac{Dq^a(\tau)}{d\tau} = 0.$$

Therefore the continuum version of equations (A.2) reads

$$\nabla_\nu T^{\nu\mu} = F^{a\mu}{}_\nu j^{a\nu}, \qquad \text{where} \qquad D_\mu j^{a\mu} = 0. \tag{A.4}$$

These relations may be called *continuum Lorentz–Yang–Mills equations*.

General derivation of the continuum equations. Actually equations (A.4) do not only hold for point particles but in a rather general context as will be shown in the sequel. To this end let us consider an arbitrary action functional

$$S = S[\phi; g_{\mu\nu}, A^a_\mu],$$

where $(g_{\mu\nu})$ and (A^a_μ) denote a fixed 4-geometry and Yang–Mills potential respectively, whereas ϕ denotes arbitrary dynamical matter fields. Taking the action S to be gauge invariant entails the vanishing of its gauge variation:

$$0 = \delta_\lambda S = \int \left(\frac{\delta S}{\delta \phi} \delta_\lambda \phi + \frac{\delta S}{\delta A^a_\mu} \delta_\lambda A^a_\mu \right).$$

Together with the matter field equations of motion $\delta S / \delta \phi = 0$ and the gauge variation of the Yang–Mills connection, $\delta_\lambda A^a_\mu = D_\mu \lambda^a$, this implies

$$D_\mu j^\mu_a = 0, \qquad \text{where} \qquad j^\mu_a(x) \equiv \frac{\delta S}{\delta A^a_\mu(x)}, \tag{A.5}$$

i.e. the second of equations (A.4).

The fact that S is geometrically well defined is reflected by its invariance under general coordinate transformations (diffeomorphisms). The latter are generated by a generic vector field $\xi \equiv \xi^\mu \partial_\mu$. Thus, we have

$$0 = \delta_\xi S = \int \left(\frac{\delta S}{\delta \phi} \delta_\xi \phi + \frac{\delta S}{\delta g_{\mu\nu}} \delta_\xi g_{\mu\nu} + \frac{\delta S}{\delta A^a_\mu} \delta_\xi A^a_\mu \right), \tag{A.6}$$

where the matter field equations again imply the vanishing of the first term. The metric tensor field and the Yang–Mills connection 1-form $A \equiv A_\mu dx^\mu \equiv A^a_\mu T^a dx^\mu$ transform [40] with the Lie derivative with respect to the vector field ξ:

$$\delta_\xi g_{\mu\nu} = (L_\xi g)_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu,$$

$$\delta_\xi A_\mu = (L_\xi A)_\mu \equiv ((i_\xi d + d i_\xi)A)_\mu = (i_\xi(dA - i\frac{g}{2}[A, A]) - ig[A, i_\xi A] + d i_\xi A)_\mu$$

$$= \xi^\nu F_{\nu\mu} + D_\mu(\xi^\nu A_\nu). \tag{A.7}$$

Here, i_ξ denotes the inner product of differential forms with the vector field ξ and $F \equiv dA - i\frac{g}{2}[A, A] \equiv \frac{1}{2}F_{\mu\nu}dx^\mu \wedge dx^\nu$ the Yang–Mills curvature 2-form. Substitution of the variations (A.7) into (A.6) and use of relation (A.5) now yields

$$\nabla_\mu T^{\mu\nu} = F^{a\nu}{}_\mu j^\mu_a, \qquad \text{where} \qquad T^{\mu\nu}(x) \equiv 2\frac{\delta S}{\delta g_{\mu\nu}(x)},$$

i.e. the first of equations (A.4), thereby completing the proof of our claim.

Acknowledgements

References

[1] Acatrinei C., Comments on noncommutative particle dynamics, hep-th/0106141.

[2] Acatrinei C., Lagrangian versus quantization, *J. Phys. A: Math. Gen.* **37** (2004), 1225–1230, hep-th/0212321.

[3] Acatrinei C.S., Surprises in noncommutative dynamics, *Ann. U. Craiova Phys.* **21** (2011), 107–116.

[4] Adorno T.C., Gitman D.M., Shabad A.E., Vassilevich D.V., Classical noncommutative electrodynamics with external source, *Phys. Rev. D* **84** (2011), 065003, 13 pages, arXiv:1106.0639.

[5] Bahns D., Doplicher S., Fredenhagen K., Piacitelli G., On the unitarity problem in space/time noncommutative theories, *Phys. Lett. B* **533** (2002), 178–181, hep-th/0201222.

[6] Balachandran A.P., Borchardt S., Stern A., Lagrangian and Hamiltonian descriptions of Yang–Mills particles, *Phys. Rev. D* **17** (1978), 3247–3256.

[7] Balachandran A.P., Marmo G., Skagerstam B.S., Stern A., Gauge symmetries and fibre bundles. Applications to particle dynamics, *Lecture Notes in Physics*, Vol. 188, Springer-Verlag, Berlin, 1983.

[8] Blaschke D.N., Kronberger E., Sedmik R.I.P., Wohlgenannt M., Gauge theories on deformed spaces, *SIGMA* **6** (2010), 062, 70 pages, arXiv:1004.2127.

[9] Brown L.S., Weisberger W.I., Vacuum polarization in uniform nonabelian gauge fields, *Nuclear Phys. B* **157** (1979), 285–326, Errata, *Nuclear Phys. B*, **172** (1980), 544.

[10] Chamseddine A.H., Connes A., Marcolli M., Gravity and the standard model with neutrino mixing, *Adv. Theor. Math. Phys.* **11** (2007), 991–1089, hep-th/0610241.

[11] Delduc F., Duret Q., Gieres F., Lefrançois M., Magnetic fields in noncommutative quantum mechanics, *J. Phys. Conf. Ser.* **103** (2008), 012020, 26 pages, arXiv:0710.2239.

[12] Deriglazov A.A., Poincare covariant mechanics on noncommutative space, *J. High Energy Phys.* **2003** (2003), no. 3, 021, 9 pages, hep-th/0211105.

[13] Dixon W.G., Dynamics of extended bodies in general relativity. I. Momentum and angular momentum, *Proc. Roy. Soc. London Ser. A* **314** (1970), 499–527.

[14] Dubois-Violette M., Kerner R., Madore J., Noncommutative differential geometry and new models of gauge theory, *J. Math. Phys.* **31** (1990), 323–330.

[15] Duval C., On the prequantum description of spinning particles in an external gauge field, in Differential Geometrical Methods in Mathematical Physics (Proc. Conf., Aix-en-Provence/Salamanca, 1979), *Lecture Notes in Math.*, Vol. 836, Springer, Berlin, 1980, 49–66.

[16] Duval C., Horváthy P., Particles with internal structure: the geometry of classical motions and conservation laws, *Ann. Physics* **142** (1982), 10–33.

[17] Duval C., Horváthy P.A., The 'Peierls substitution' and the exotic Galilei group, *Phys. Lett. B* **479** (2000), 284–290, hep-th/0002233.

[18] Duval C., Horváthy P.A., Exotic Galilean symmetry in the non-commutative plane and the Hall effect, *J. Phys. A: Math. Gen.* **34** (2001), 10097–10107, hep-th/0106089.

[19] Estienne B., Haaker S.M., Schoutens K., Particles in non-Abelian gauge potentials: Landau problem and insertion of non-Abelian flux, *New J. Phys.* **13** (2011), 045012, 24 pages, arXiv:1102.0176.

[20] Fatollahi A.H., Mohammadzadeh H., On the classical dynamics of charges in non-commutative QED, *Eur. Phys. J. C* **36** (2004), 113–116, hep-th/0404209.

[21] Gracia-Bondía J.M., Várilly J.C., Algebras of distributions suitable for phase-space quantum mechanics. I, *J. Math. Phys.* **29** (1988), 869–879.

[22] Horváthy P.A., The non-commutative Landau problem, *Ann. Physics* **299** (2002), 128–140, hep-th/0201007.

[23] Horváthy P.A., Exotic Galilean symmetry and non-commutative mechanics in mathematical and in condensed matter physics, Talk given at International Conference on Noncommutative Geometry and Quantum Physics (January 4–10, 2006, Kolkata, India), hep-th/0602133.

[24] Horváthy P.A., Martina L., Stichel P.C., Exotic Galilean symmetry and non-commutative mechanics, *SIGMA* **6** (2010), 060, 26 pages, arXiv:1002.4772.

[25] Kerner R., Generalization of the Kaluza–Klein theory for an arbitrary non-abelian gauge group, *Ann. Inst. H. Poincaré Sect. A (N.S.)* **9** (1968), 143–152.

[26] Kosyakov B.P., Exact solutions in the Yang–Mills–Wong theory, *Phys. Rev. D* **57** (1998), 5032–5048, hep-th/9902039.

[27] Kosyakov B.P., Introduction to the classical theory of particles and fields, Springer-Verlag, Berlin, 2007.

[28] Linden N., Macfarlane A.J., van Holten J.W., Particle motion in a Yang–Mills field: Wong's equations and spin-$\frac{1}{2}$ analogues, *Czechoslovak J. Phys.* **46** (1996), 209–215, hep-th/9512071.

[29] Marsden J.E., Ratiu T.S., Introduction to mechanics and symmetry. A basic exposition of classical mechanical systems, *Texts in Applied Mathematics*, Vol. 17, 2nd ed., Springer-Verlag, New York, 1999.

[30] Mathisson M., Neue Mechanik materieller Systeme, *Acta Phys. Polon.* **6** (1937), 163–200.

[31] Mezincescu L., Star operation in quantum mechanics, hep-th/0007046.

[32] Nair V.P., Polychronakos A.P., Quantum mechanics on the noncommutative plane and sphere, *Phys. Lett. B* **505** (2001), 267–274, hep-th/0011172.

[33] Papapetrou A., Spinning test-particles in general relativity. I, *Proc. Roy. Soc. London. Ser. A.* **209** (1951), 248–258.

[34] Ramond P., Field theory: a modern primer, *Frontiers in Physics*, Vol. 51, Benjamin/Cummings Publishing Co., Inc., Reading, Mass., 1981.

[35] Rivasseau V., Non-commutative renormalization, in Quantum Spaces, *Prog. Math. Phys.*, Vol. 53, Editors B. Duplantier, V. Rivasseau, Birkhäuser, Basel, 2007, 19–107, arXiv:0705.0705.

[36] Sikivie P., Weiss N., Classical Yang–Mills theory in the presence of external sources, *Phys. Rev. D* **18** (1978), 3809–3821.

[37] Souriau J.M., Structure of dynamical systems. A symplectic view of physics, *Progress in Mathematics*, Vol. 149, Birkhäuser Boston, Inc., Boston, MA, 1997.

[38] Stern A., Noncommutative point sources, *Phys. Rev. Lett.* **100** (2008), 061601, 4 pages, arXiv:0709.3831.

[39] Szabo R.J., Quantum field theory on noncommutative spaces, *Phys. Rep.* **378** (2003), 207–299, hep-th/0109162.

[40] Thirring W., Classical mathematical physics. Dynamical systems and field theories, 3rd ed., Springer-Verlag, New York, 1997.

[41] Vanhecke F.J., Sigaud C., da Silva A.R., Noncommutative configuration space. Classical and quantum mechanical aspects, *Braz. J. Phys.* **36** (2006), 194–207, math-ph/0502003.

[42] Várilly J.C., Gracia-Bondía J.M., Algebras of distributions suitable for phase-space quantum mechanics. II. Topologies on the Moyal algebra, *J. Math. Phys.* **29** (1988), 880–887.

[43] Vignes-Tourneret F., Renormalisation des theories de champs non commutatives, Ph.D. Thesis, Université Paris 11, 2006, math-ph/0612014.

[44] Wipf A.W., Nonrelativistic Yang–Mills particles in a spherically symmetric monopole field, *J. Phys. A: Math. Gen.* **18** (1985), 2379–2384.

[45] Wong S.K., Field and particle equations for the classical Yang–Mills field and particles with isotopic spin, *Nuovo Cimento A* **65** (1970), 689–694.