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The Lax Integrable Differential-Difference Dynamical Systems on Extended Phase Spaces*

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Abstract. The Hamiltonian representation for the hierarchy of Lax-type flows on a dual space to the Lie algebra of shift operators coupled with suitable eigenfunctions and adjoint eigenfunctions evolutions of associated spectral problems is found by means of a specially constructed Bäcklund transformation. The Hamiltonian description for the corresponding set of squared eigenfunction symmetry hierarchies is represented. The relation of these hierarchies with Lax integrable $(2+1)$-dimensional differential-difference systems and their triple Lax-type linearizations is analysed. The existence problem of a Hamiltonian representation for the coupled Lax-type hierarchy on a dual space to the central extension of the shift operator Lie algebra is solved also.

Key words: Lax integrable differential-difference systems; Bäcklund transformation; squared eigenfunction symmetries

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1 Introduction

The first papers on the Lie-algebraic interpretation of Lax integrable differential-difference systems were published by Adler [2], Kostant [21] and Symes [42]. They considered non-periodic lattices of a Toda type [10, 33, 37, 41] related to coadjoint orbits of solvable matrix Lie algebras.

The $\mathcal{R}$-matrix approach [5, 14, 29, 35, 37, 40] being useful for the Lie-algebraic description of Lax integrable nonlinear dynamical systems on functional manifolds [1, 23] turned out to be suitable for the Lie-algebraic description of Lax integrable $(1+1)$-dimensional lattice and nonlocal differential-difference systems by means of the Lie algebra of shift operators [4, 11, 22, 27, 31].

The Lax integrable $(2+1)$-dimensional differential-difference systems were obtained via the Sato procedure [39] in [32, 43, 44] whereas in papers [6, 7, 8, 17, 20, 34] such differential-difference systems were considered as Hamiltonian flows on the dual spaces to the central extensions by the Maurer–Cartan 2-cocycle of shift operator Lie algebras.

Taking into account that every flow from the Lax-type hierarchy on the dual space to the shift operator Lie algebra or its central extension can be written as a compatibility condition of the spectral relationship for the corresponding operator and the suitable eigenfunction evolution an important problem of finding the Hamiltonian representation for the hierarchy of Lax-type flows coupled with the evolutions of eigenfunctions and appropriate adjoint eigenfunctions naturally arises. In the case when the spectral relationship admits a finite set of eigenvalues it was partly solved in the papers [16, 18, 19, 30, 36] for the Lie algebra of integral-differential operators [30, 36]

*This paper is a contribution to the Proceedings of the Eighth International Conference "Symmetry in Nonlinear Mathematical Physics" (June 21–27, 2009, Kyiv, Ukraine). The full collection is available at http://www.emis.de/journals/SIGMA/symmetry2009.html and its supergeneralizations [16, 19] as well as for the corresponding central extension [18] by means of the variational property of Casimir functionals under some Bäcklund transformation.

Section 2 deals with a general Lie-algebraic scheme for constructing the hierarchy of Lax-type flows as Hamiltonian ones on a dual space to the Lie algebra of shift operators [4, 8].

In Section 3 the Hamiltonian structure for the related coupled Lax-type hierarchy is obtained by means of the Bäcklund transformation technique developed in [16, 18, 19, 36].

In Section 4 the corresponding hierarchies of squared eigenfunction symmetries [3, 12, 13, 28, 44, 45] for the coupled Lax-type flows are established to be Hamiltonian also. It is proved that the additional hierarchy of Hamiltonian flows is generated by the Poisson structure being obtained from the tensor product of the $\mathcal{R}$-deformed canonical Lie–Poisson bracket with the standard Poisson bracket on the related eigenfunctions and adjoint eigenfunctions superspace [16, 18, 19, 36] and the corresponding natural powers of a suitable eigenvalue are their Hamiltonians. These hierarchies are applied to constructing Lax integrable $(2+1)$-dimensional differential-difference systems and their triple Lax-type linearizations.

In Section 5 the results obtained in Section 3 are generalized for the centrally extended Lie algebra of shift operators [6, 8].

2 The Lie-algebraic structure of Lax integrable $(1+1)$-dimensional differential-difference systems

Let us consider the Lie algebra $\mathcal{G}$ of linear operators [4, 11, 22, 27, 31]

$$A := \mathcal{E}^m + \sum_{j<m,\ j\in\mathbb{Z}} a_j(n)\mathcal{E}^j, \qquad m\in\mathbb{N}, \tag{1}$$

where coefficients a_j belong to the Schwarz space $\mathcal{S}(\mathbb{Z};\mathbb{C})$ of quickly decreasing sequences, $j\in\mathbb{Z}$, which are generated by the shift operator $\mathcal{E}$, satisfying the following rule

$$\mathcal{E}^j a = (\mathcal{E}^j a)\mathcal{E}^j,$$

with the standard commutator

$$[A,B] = AB - BA, \qquad A,B\in\mathcal{G}.$$

On the Lie algebra $\mathcal{G}$ there exists the ad-invariant nondegenerate symmetric bilinear form:

$$(A,B) := \operatorname{Tr}(AB), \qquad A,B\in\mathcal{G}, \tag{2}$$

where

$$\operatorname{Tr} A := \sum_{n\in\mathbb{Z}} a_0(n)$$

for any operator $A\in\mathcal{G}$ in the form (1).

With taking into account (2) the dual space to the Lie algebra $\mathcal{G}$ can be identified with the Lie algebra, that is, $\mathcal{G}^*\simeq\mathcal{G}$. The linear subspaces $\mathcal{G}_+\subset\mathcal{G}$ and $\mathcal{G}_-\subset\mathcal{G}$ such as

$$\mathcal{G}_+ := \left\{ a = \mathcal{E}^m + \sum_{0\le j<m,\ j\in\mathbb{Z}} a_j\mathcal{E}^j : \quad a_j\in\mathcal{S}(\mathbb{Z};\mathbb{C}),\ m\in\mathbb{N} \right\},$$

$$\mathcal{G}_- := \left\{ b = \sum_{\ell\in\mathbb{N}}^{\infty} b_j\mathcal{E}^{-\ell} : \quad b_j\in\mathcal{S}(\mathbb{Z};\mathbb{C}) \right\}, \tag{3}$$

are Lie subalgebras in $\mathcal{G}$ and

$$\mathcal{G} = \mathcal{G}_+ \oplus \mathcal{G}_-.$$

The following spaces can be identified as

$$\mathcal{G}_+^* \simeq \mathcal{G}_- \circ \mathcal{E}, \qquad \mathcal{G}_-^* \simeq \mathcal{G}_+ \circ \mathcal{E}.$$

Owing to the splitting $\mathcal{G}$ into the direct sum of its Lie subalgebras (3), one can construct a Lie–Poisson structure on $\mathcal{G}^*$ by use of the special linear endomorphism $\mathcal{R}$ of $\mathcal{G}$:

$$\mathcal{R} = (P_+ - P_-)/2, \qquad P_\pm \mathcal{G} = \mathcal{G}_\pm, \qquad P_\pm \mathcal{G}_\mp = 0.$$

The Lie–Poisson bracket on $\mathcal{G}^*$ is given as

$$\{\gamma, \mu\}_{\mathcal{R}}(l) = (l, [\nabla\gamma(l), \nabla\mu(l)]_{\mathcal{R}}), \tag{4}$$

where $\gamma, \mu \in \mathcal{D}(\mathcal{G}^*)$, $\mathcal{D}(\mathcal{G}^*)$ is a space of Frechet-smooth functionals on $\mathcal{G}^*$, $l \in \mathcal{G}^*$ and for all $A, B \in \mathcal{G}$ the $\mathcal{R}$-deformed commutator has the form [5, 14, 29, 31, 35, 37, 40]

$$[a, b]_{\mathcal{R}} = [\mathcal{R}a, b] + [a, \mathcal{R}b].$$

Based on the scalar product (2) the gradient $\nabla\gamma(l) \in \mathcal{G}$ of some functional $\gamma \in \mathcal{D}(\mathcal{G}^*)$ at the point $l \in \mathcal{G}^*$ is naturally defined as

$$\delta\gamma(l) = (\nabla\gamma(l), \delta l).$$

Let $I(\mathcal{G}^*)$ be a set of Casimir functionals on $\mathcal{G}^*$, being invariant with respect to Ad^*-action of the abstract Lie group G corresponding to the Lie algebra $\mathcal{G}$ by definition. Every Casimir functional $\gamma \in I(\mathcal{G}^*)$ obeys the following condition at the point $l \in \mathcal{G}^*$:

$$[l, \nabla\gamma(l)] = 0. \tag{5}$$

The relationship (5) is satisfied by the hierarchy of functionals $\gamma_n \in I(\mathcal{G}^*)$, $n \in \mathbb{N}$, taking the forms [4, 5]

$$\gamma_n(l) = \frac{1}{n+1}(l, l^n). \tag{6}$$

The Lie–Poisson bracket (4) generates the hierarchy of Hamiltonian dynamical systems on $\mathcal{G}^*$ with Casimir functionals $\gamma_n \in I(\mathcal{G}^*)$, $n \in \mathbb{N}$, as Hamiltonian functions:

$$dl/dt_n = [\mathcal{R}\nabla\gamma_n(l), l] = [(\nabla\gamma_n(l))_+, l]. \tag{7}$$

where the subscript "+" denotes a projection on the Lie subalgebra $\mathcal{G}_+$.

The latter equation is equivalent to the usual commutator Lax-type representation, which can be considered as a compatibility condition for the spectral problem

$$(lf) = \lambda f, \tag{8}$$

where $f \in W := L_2(\mathbb{Z}; \mathbb{C})$, $\lambda \in \mathbb{C}$ is a spectral parameter, and the following evolution equation:

$$df/dt_n = ((\nabla\gamma_n(l))_+ f). \tag{9}$$

The corresponding evolution for the adjoint eigenfunction $f^* \in W^* \simeq W$ takes the form

$$df^*/dt_n = -((\nabla\gamma_n(l))_+^* f^*). \tag{10}$$

Further one will assume that the spectral relationship (8) admits $N \in \mathbb{N}$ different eigenvalues $\lambda_i \in \mathbb{C}$, $i = \overline{1,N} := 1, \ldots, N$, and study algebraic properties of equation (7) combined with $N \in \mathbb{N}$ copies of (9):

$$df_i/dt_n = ((\nabla \gamma_n(l))_+ f_i), \tag{11}$$

for the corresponding eigenfunctions $f_i \in W$, $i = \overline{1,N}$, and the same number of copies of (10):

$$df_i^*/dt_n = -((\nabla \gamma_n(l))_+^* f_i^*), \tag{12}$$

for the suitable adjoint eigenfunctions $f_i^* \in W^*$, being considered as a coupled evolution system on the space $\mathcal{G}^* \oplus W^{2N}$.

3 The Poisson bracket on the extended phase space of a Lax integrable (1 + 1)-dimensional differential-difference system

To give the description below in a compact form one will use the following notation of the gradient vector:

$$\nabla \overline{\gamma}(\tilde{l}, \tilde{\mathrm{f}}, \tilde{\mathrm{f}}^*) := (\delta\overline{\gamma}/\delta\tilde{l},\ \delta\overline{\gamma}/\delta\tilde{\mathrm{f}},\ \delta\overline{\gamma}/\delta\tilde{\mathrm{f}}^*)^\top,$$

where $\tilde{\mathrm{f}} := (\tilde{f}_1, \ldots, \tilde{f}_N)^\top$, $\tilde{\mathrm{f}}^* := (\tilde{f}_1^*, \ldots, \tilde{f}_N^*)^\top$ and $\delta\overline{\gamma}/\delta\tilde{\mathrm{f}} := (\delta\overline{\gamma}/\delta\tilde{f}_1, \ldots, \delta\overline{\gamma}/\delta\tilde{f}_N)^\top$, $\delta\overline{\gamma}/\delta\tilde{\mathrm{f}}^* := (\delta\overline{\gamma}/\delta f_1^*, \ldots, \delta\overline{\gamma}/\delta\tilde{f}_N^*)^\top$, at the point $(\tilde{l}, \tilde{\mathrm{f}}, \mathrm{f}^*)^\top \in \mathcal{G}^* \oplus W^{2N}$ for any Frechet-smooth functional $\overline{\gamma} \in \mathcal{D}(\mathcal{G}^* \oplus W^{2N})$.

On the spaces $\mathcal{G}^*$ and W^{2N} there exist canonical Poisson structures in the forms

$$\delta\overline{\gamma}/\delta\tilde{l} \stackrel{\tilde{\theta}}{\mapsto} [\tilde{l}, (\delta\overline{\gamma}/\delta\tilde{l})_+] - [\tilde{l}, \delta\overline{\gamma}/\delta\tilde{l}]_{>0}, \tag{13}$$

where $\tilde{\theta} : \mathcal{T}^*(\mathcal{G}^*) \to \mathcal{T}(\mathcal{G}^*)$ is an implectic operator corresponding to (4) at the point $\tilde{l} \in \mathcal{G}^*$, $A_{>0} := A_+ - A_0$, $A_0 := a_0$, for an arbitrary operator $A \in \mathcal{G}$ in the form (1), and

$$(\delta\overline{\gamma}/\delta\tilde{\mathrm{f}},\ \delta\overline{\gamma}/\delta\tilde{\mathrm{f}}^*)^\top \stackrel{\tilde{J}}{\mapsto} (-\delta\overline{\gamma}/\delta\tilde{\mathrm{f}}^*,\ \delta\overline{\gamma}/\delta\tilde{\mathrm{f}})^\top, \tag{14}$$

where $\tilde{J} : \mathcal{T}^*(W^{2N}) \to \mathcal{T}(W^{2N})$ is an implectic operator corresponding to the symplectic form $\omega^{(2)} = \sum_{i=1}^N d\tilde{f}_i^* \wedge d\tilde{f}_i$ at the point $(\tilde{\mathrm{f}}, \tilde{\mathrm{f}}^*) \in W^{2N}$. It should be noted here that Poisson structure (13) generates equation (7) for any Casimir functional $\gamma \in I(\mathcal{G}^*)$.

Thus, one can obtain a Poisson structure on the extended phase space $\mathcal{G}^* \oplus W^{2N}$ as the tensor product $\tilde{\Theta} := \tilde{\theta} \otimes \tilde{J}$ of (13) and (14).

To find a Hamiltonian representation for the coupled dynamical systems (7), (11) and (12) one will make use of an approach, described in papers [16, 18, 19, 35, 36], and will consider the following Bäcklund transformation:

$$(\tilde{l}, \tilde{\mathrm{f}}, \tilde{\mathrm{f}}^*)^\top \stackrel{B}{\mapsto} (l(\tilde{l}, \tilde{\mathrm{f}}, \tilde{\mathrm{f}}^*),\ \mathrm{f} = \tilde{\mathrm{f}},\ \mathrm{f}^* = \tilde{\mathrm{f}}^*)^\top, \tag{15}$$

generating some Poisson structure $\Theta : \mathcal{T}^*(\mathcal{G}^* \oplus W^{2N}) \to \mathcal{T}(\mathcal{G}^* \oplus W^{2N})$ on $\mathcal{G}^* \oplus W^{2N}$. The main condition imposed on mapping (15) is the coincidence of the resulting dynamical system

$$(dl/dt_n,\ d\mathrm{f}/dt_n,\ d\mathrm{f}^*/dt_n)^\top := -\Theta\, \nabla \overline{\gamma}_n(l, \mathrm{f}, \mathrm{f}^*) \tag{16}$$

with equations (7), (11) and (12) in the case of $\overline{\gamma}_n \in I(\mathcal{G}^*)$, $n \in \mathbb{N}$, independent of variables $(\mathrm{f}, \mathrm{f}^*) \in W^{2N}$.

To satisfy that condition one will find a variation of a Casimir functional $\overline{\gamma}_n := \gamma_n|_{l=l(\tilde{l},\mathrm{f},\mathrm{f}^*)} \in \mathcal{D}(\mathcal{G}^* \oplus W^{2N})$, $n \in \mathbb{N}$, under $\delta\tilde{l} = 0$, taking into account evolutions (11), (12) and Bäcklund transformation definition (15). There follows

$$\begin{aligned}
\delta\overline{\gamma}_n(\tilde{l},\tilde{\mathrm{f}},\tilde{\mathrm{f}}^*)\big|_{\delta\tilde{l}=0} &= \sum_{i=1}^{N}\big(\langle\delta\overline{\gamma}_n/\delta\tilde{f}_i,\delta\tilde{f}_i\rangle + \langle\delta\overline{\gamma}_n/\delta\tilde{f}_i^*,\delta\tilde{f}_i^*\rangle\big) \\
&= \sum_{i=1}^{N}\big(\langle -d\tilde{f}_i^*/dt_n,\delta\tilde{f}_i\rangle + \langle d\tilde{f}_i/dt_n,\delta\tilde{f}_i^*\rangle\big)\big|_{\tilde{\mathrm{f}}=\mathrm{f},\,\tilde{\mathrm{f}}^*=\mathrm{f}^*} \\
&= \sum_{i=1}^{N}\big(\langle(\delta\gamma_n/\delta l)_+^* f_i^*,\delta f_i\rangle + \langle(\delta\gamma_n/\delta l)_+ f_i,\delta f_i^*\rangle\big) \\
&= \sum_{i=1}^{N}\big(\langle f_i^*,(\delta\gamma_n/\delta l)_+\delta f_i\rangle + \langle(\delta\gamma_n/\delta l)_+ f_i,\delta f_i^*\rangle\big) \\
&= \sum_{i=1}^{N}\big((\delta\gamma_n/\delta l,(\delta f_i)\mathcal{E}(\mathcal{E}-1)^{-1}f_i^*) + (\delta\gamma_n/\delta l, f_i\mathcal{E}(\mathcal{E}-1)^{-1}\delta f_i^*)\big) \\
&= \left(\delta\gamma_n/\delta l,\delta\sum_{i=1}^{N} f_i\mathcal{E}(\mathcal{E}-1)^{-1}f_i^*\right) = (\delta\gamma_n/\delta l,\delta l), \qquad (17)
\end{aligned}$$

where $\gamma_n \in I(\mathcal{G}^*)$, $n \in \mathbb{N}$ and the brackets $\langle\cdot,\cdot\rangle$ denote a scalar product on W.

As a result of expression (17) one obtains the relationship:

$$\delta l|_{\delta\tilde{l}=0} = \sum_{i=1}^{N}\delta\big(f_i\mathcal{E}(\mathcal{E}-1)^{-1}f_i^*\big). \qquad (18)$$

From (18) it follows directly that

$$l = \mathcal{K}(\tilde{l}) + \sum_{i=1}^{N} f_i\mathcal{E}(\mathcal{E}-1)^{-1}f_i^*,$$

where $\mathcal{K}$ is an arbitrary Frechet-smooth operator on $\mathcal{G}^*$. If $\mathcal{K}(\tilde{l}) = \tilde{l}$ for any $\tilde{l} \in \mathcal{G}^*$ then

$$l = \tilde{l} + \sum_{i=1}^{N} f_i\mathcal{E}(\mathcal{E}-1)^{-1}f_i^*. \qquad (19)$$

Thus, Bäcklund transformation (15) can be written as

$$(\tilde{l},\tilde{\mathrm{f}},\tilde{\mathrm{f}}^*)^\top \stackrel{B}{\mapsto} (l = \tilde{l} + \sum_{i=1}^{N} f_i\mathcal{E}(\mathcal{E}-1)^{-1}f_i^*,\mathrm{f},\mathrm{f}^*)^\top. \qquad (20)$$

The existence of Bäcklund transformation (20) enables the following theorem to be proved.

Theorem 1. *Under the Bäcklund transformation (20) the dynamical system (16) on $\mathcal{G}^* \oplus W^{2N}$ is equivalent to the system of evolution equations:*

$$\begin{aligned}
&d\tilde{l}/dt_n = [(\nabla\overline{\gamma}_n(\tilde{l}))_+,\tilde{l}] - [\nabla\overline{\gamma}_n(\tilde{l}),\tilde{l}]_{>0}, \\
&d\tilde{\mathrm{f}}/dt_n = \delta\overline{\gamma}_n/\delta\tilde{\mathrm{f}}^*, \qquad d\tilde{\mathrm{f}}^*/dt_n = -\delta\overline{\gamma}_n/\delta\tilde{\mathrm{f}},
\end{aligned}$$

where $\overline{\gamma}_n := \gamma_n|_{l=l(\tilde{l},\mathrm{f},\mathrm{f}^*)} \in \mathcal{D}(\mathcal{G}^* \oplus W^{2N})$ and $\gamma_n \in I(\mathcal{G}^*)$ is a Casimir functional at the point $l \in \mathcal{G}^*$ for every $n \in \mathbb{N}$.

The Frechet derivative $B' : \mathcal{T}(\mathcal{G}^* \oplus W^{2N}) \to \mathcal{T}(\mathcal{G}^* \oplus W^{2N})$ of the Bäcklund transformation (20) and the corresponding conjugate operator $B'^* : \mathcal{T}^*(\mathcal{G}^* \oplus W^{2N}) \to \mathcal{T}^*(\mathcal{G}^* \oplus W^{2N})$ take the following forms:

$$\begin{pmatrix} h \\ \alpha \\ \beta \end{pmatrix} \stackrel{B'}{\mapsto} \begin{pmatrix} h + \sum_{i=1}^{N} (\alpha_i \mathcal{E}(\mathcal{E}-1)^{-1} f_i^* + f_i \mathcal{E}(\mathcal{E}-1)^{-1} \beta_i) \\ \alpha \\ \beta \end{pmatrix},$$

$$\begin{pmatrix} r \\ \chi \\ \rho \end{pmatrix} \stackrel{B'^*}{\mapsto} \begin{pmatrix} r \\ \chi + (r_+^* \mathrm{f}^*) \\ \rho + (r_+ \mathrm{f}) \end{pmatrix},$$

where $(h, \alpha, \beta)^\top \in \mathcal{T}_{(l,\mathrm{f},\mathrm{f}^*)}(\mathcal{G}^* \oplus W^{2N})$ and $(r, \chi, \rho)^\top \in \mathcal{T}^*_{(l,\mathrm{f},\mathrm{f}^*)}(\mathcal{G}^* \oplus W^{2N})$ at the point $(l, \mathrm{f}, \mathrm{f}^*)^\top \in \mathcal{G}^* \oplus W^{2N}$, $\alpha = (\alpha_1, \ldots, \alpha_N)^\top$, $\beta = (\beta_1, \ldots, \beta_N)^\top$, $\chi = (\chi_1, \ldots, \chi_N)^\top$, $\rho = (\rho_1, \ldots, \rho_N)^\top$.

By means of calculations via the formula (see [15, 35]):

$$\Theta = B' \tilde{\Theta} B'^*, \tag{21}$$

one finds the Bäcklund transformed Poisson structure Θ on $\mathcal{G}^* \oplus W^{2N}$:

$$\nabla \overline{\gamma}(l, \mathrm{f}, \mathrm{f}^*) \stackrel{\Theta}{\mapsto} \begin{pmatrix} [l, (\delta\overline{\gamma}/\delta l)_+] - [l, \delta\overline{\gamma}/\delta l]_{>0} + \\ + \sum\limits_{i=1}^{N} (f_i \mathcal{E}(\mathcal{E}-1)^{-1} (\delta\overline{\gamma}/\delta f_i) - (\delta\overline{\gamma}/\delta f_i^*) \mathcal{E}(\mathcal{E}-1)^{-1} f_i^*) \\ -\delta\overline{\gamma}/\delta \mathrm{f}^* - ((\delta\overline{\gamma}/\delta l)_+ \mathrm{f}) \\ \delta\overline{\gamma}/\delta \mathrm{f} + ((\delta\overline{\gamma}/\delta l)_+^* \mathrm{f}^*) \end{pmatrix}, \tag{22}$$

where $\overline{\gamma} \in \mathcal{D}(\mathcal{G}^* \oplus W^{2N})$ is an arbitrary Frechet-smooth functional. Thereby, one can formulate the following theorem.

Theorem 2. *The hierarchy of dynamical systems* (7), (11) *and* (12) *is Hamiltonian with respect to the Poisson structure* Θ *in the form* (22) *and the Casimir functionals* $\gamma_n \in I(\mathcal{G}^*)$, $n \in \mathbb{N}$, *as Hamiltonian functions.*

Based on the expression (16) one can construct a new hierarchy of Hamiltonian evolution equations generated by involutive with respect to the Lie–Poisson bracket (4) Casimir invariants $\gamma_n \in I(\mathcal{G}^*)$, $n \in \mathbb{N}$, in the form (6) on the extended phase space $\mathcal{G}^* \oplus W^{2N}$. On the coadjoint orbits of the Lie algebra $\mathcal{G}$ they give rise to the Lax representations for some $(1+1)$-dimensional differential-difference systems [1, 4, 5, 14, 22, 25, 41].

4 The additional symmetry hierarchies associated with the Lax integrable $(1+1)$-dimensional differential-difference systems

The hierarchy of coupled evolution equations (7), (11) and (12) possesses another natural set of invariants including all higher powers of the eigenvalues λ_k, $k = \overline{1, N}$. They can be considered as Frechet-smooth functionals on the extended phase space $\mathcal{G}^* \oplus W^{2N}$, owing to the evident representation :

$$\lambda_k^s = \langle f_k^*, l^s f_k \rangle, \qquad s \in \mathbb{N}, \tag{23}$$

holding for every $k = \overline{1, N}$ under the normalizing constraint

$$\langle f_k^*, f_k \rangle = 1.$$

The Frechet-smooth functionals $\mu_i \in \mathcal{D}(\mathcal{G}^* \oplus W^{2N})$, $i=\overline{1,N}$:

$$\mu_i := \langle f_i^*, f_i \rangle,$$

are invariant with respect to the dynamical systems (7), (11) and (12).

In the case of Bäcklund transformation (19), where

$$l := l_{>0} + \sum_{i=1}^{N} f_i \mathcal{E}(\mathcal{E}-1)^{-1} f_i^*, \tag{24}$$

formula (23) gives rise to the following variation of the functionals $\lambda_k^s \in \mathcal{D}(\mathcal{G}^* \oplus W^{2N})$ for every $k=\overline{1,N}$ and $s \in \mathbb{N}$:

$$\begin{aligned}\delta\lambda_k^s &= \langle \delta f_k^*, l^s f_k \rangle + \langle f_k^*, \delta(l^s f_k) \rangle + \langle f_k^*, l^s(\delta f_k) \rangle \\ &= (\delta(l^s)_{>0}, M_k^s) + \sum_{i=1}^{N} \langle \delta f_i, (-M_k^s + \delta_k^i l^s)^* f_i^* \rangle + \sum_{i=1}^{N} \langle \delta f_i^*, (-M_k^s + \delta_k^i l^s) f_i \rangle,\end{aligned} \tag{25}$$

where δ_k^i is the Kronecker symbol and the operators M_k^s are determined as

$$M_k^s := \sum_{p=0}^{s-1} ((l^p f_k)(\mathcal{E}-1)^{-1}((l^*)^{s-1-p} f_k^*)).$$

It should be noted that the sth power of the operator (24) takes the form

$$l^s = (l^s)_{>0} + \sum_{i=1}^{N} \sum_{p=0}^{s-1} \big((l^p f_i)\mathcal{E}(\mathcal{E}-1)^{-1}\big((l^*)^{s-1-p} f_i^*\big)\big).$$

By means of the representation (25) one obtains the exact forms of gradients for the functionals $\lambda_k^s \in \mathcal{D}(\mathcal{G}^* \oplus W^{2N})$, $k=\overline{1,N}$:

$$\nabla\lambda_k^s(l_{>0}, \mathrm{f}, \mathrm{f}^*) = \big(M_k^s,\ (-M_k^s + \delta_k^i l^s)^* f_i^*, (-M_k^s + \delta_k^i l^s) f_i :\ i=\overline{1,N}\big)^\top. \tag{26}$$

The tensor product $\tilde{\Theta}$ of the Poisson structures (13) and (14) together with the relationships (26) generates a new hierarchy of coupled evolution equations on $\mathcal{G}^* \oplus W^{2N}$:

$$dl_{>0}/d\tau_{s,k} = -[M_k^s, l_{>0}]_{>0}, \tag{27}$$

$$df_i/d\tau_{s,k} = (-M_k^s + \delta_k^i l^s) f_i, \tag{28}$$

$$df_i^*/d\tau_{s,k} = (M_k^s - \delta_k^i l^s)^* f_i^*, \tag{29}$$

where $i=\overline{1,N}$ and $\tau_{s,k} \in \mathbb{R}$, $s \in \mathbb{N}$, $k=\overline{1,N}$, are evolution parameters. Owing to Bäcklund transformation (20), equation (27) can be rewritten as the following equivalent commutator relationship:

$$dl/d\tau_{s,k} = -[M_k^s, l] = -s\lambda_k^{s-1}[M_k^1, l] = s\lambda_k^{s-1} dl/d\tau_{1,k}. \tag{30}$$

Since the functionals $\mu_i \in \mathcal{D}(\mathcal{G}^* \oplus W^{2N})$, $i=\overline{1,N}$, are invariant with respect to the dynamical systems (30), (28) and (29) one can formulate the following theorem.

Theorem 3. *For every $k=\overline{1,N}$ and all $s \in \mathbb{N}$ the Hamiltonian representations of the dynamical systems (30), (28) and (29) on their invariant subspace $M_k \subset \mathcal{G}^* \oplus W^{2N}$:*

$$M_k := \{(l, f, f^*)^\top \in \mathcal{G}^* \oplus W^{2N} :\ \mu_k = 1\},$$

are given by the Poisson structure Θ in the form (22) and the functionals $\lambda_k^s \in \mathcal{D}(\mathcal{G}^* \oplus W^{2N})$ as Hamiltonian functions taken both to be reduced on M_k.

On the subspace of the operators $l \in \mathcal{G}^*$ in the forms (24) one has the following representation for the flows d/dt_n, $n \in \mathbb{N}$:

$$d/dt_n = n \sum_{k=1}^{N} \lambda_k^{n-1} d/d\tau_{1,k}. \tag{31}$$

Theorem 4. *The dynamical systems (30), (28) and (29) describe flows on $\mathcal{G}^* \oplus W^{2N}$ commuting both with each other and the hierarchy of Lax-type dynamical systems (7), (11) and (12).*

Proof. With taking into account the representation (31) it is sufficient to show that

$$[d/d\tau_{1,k}, d/d\tau_{1,q}] = 0,$$

where $k, q = \overline{1, N}$, $k \neq q$ and $n \in \mathbb{N}$. This equality follows from the identity

$$dM_k^1/d\tau_{1,q} - dM_q^1/d\tau_{1,k} = [M_k^1, M_q^1],$$

holding because of the relationship

$$\begin{aligned} M_k^1 M_q^1 &= (M_k^1 f_q)(\mathcal{E} - 1)^{-1} f_q^* + f_k(\mathcal{E} - 1)^{-1}((M_q^1)^* f_k^*) \\ &= -df_q/d\tau_{1,k}(\mathcal{E} - 1)^{-1} f_q^* + f_k(\mathcal{E} - 1)^{-1}(df_k^*/d\tau_{1,q}). \end{aligned}$$

■

Thus, for every $k = \overline{1, N}$ and all $s \in \mathbb{N}$ dynamical systems (30), (28) and (29) on $\mathcal{G}^* \oplus W^{2N}$ form a hierarchy of additional homogeneous, or so called squared eigenfunction, symmetries [3, 12, 13, 28, 44, 45] for Lax-type flows (7), (11) and (12).

Earlier the squared eigenfunction symmetry hierarchies associated with the Lie algebras of integral-differential and super-integral-differential operators in commutator forms as well as their relations to some (2+1)-, (2|1+1)- and (2|2+1)-dimensional Lax integrable nonlinear dynamical systems on functional manifolds and supermanifolds were investigated in [3, 12, 13, 16, 18, 19, 28]. In paper [44] the commutator-type squared eigenfunction symmetries were considered for the shift operator Lie algebra.

In the case when $N \geq 2$ one can obtain a new class of nontrivial independent flows $d/dT_{n,K} := d/dt_n + \sum_{k=1}^{K} d/d\tau_{n,k}$, $K = \overline{1, [N/2]}$, $n \in \mathbb{N}$, on $\mathcal{G}^* \oplus W^{2N}$ in the Lax-type forms by use of the considered above invariants of the shift operator Lie algebra $\mathcal{G}$. These flows are Hamiltonian on their invariant subspaces $\bigcap_{k=1}^{K} M_k \subset \mathcal{G}^* \oplus W^{2N}$ because the following relationship

$$\{\mu_i, \mu_q\}_{\tilde{J}} = 0,$$

where $\mu_i \in \mathcal{D}(\mathcal{G}^* \oplus W^{2N})$ and $\{\cdot,\cdot\}_{\tilde{J}}$ is a Poisson bracket on W^{2N}, related to the implectic operator (14), holds for all $i, q = \overline{1, N}$.

Acting on the eigenfunctions $(f_i, f_i^*) \in W^{2N}$, $i = \overline{1, N}$, the flows $d/dT_{n,K}$, $K = \overline{1, [N/2]}$, $n \in \mathbb{N}$, generate some Lax integrable $((1+K)+1)$-dimensional differential-difference dynamical systems. For example, in the case of the element

$$l := \mathcal{E} + f_1 \mathcal{E}(\mathcal{E} - 1)^{-1} f_1^* + f_2 \mathcal{E}(\mathcal{E} - 1)^{-1} f_2^* \in \mathcal{G}^*$$

with $(f_1, f_2, f_1^*, f_2^*) \in W^4$ the flows $d/d\tau := d/d\tau_{1,1}$ and $d/dT := d/dT_{2,1} := d/dt_2 + d/d\tau_{2,1}$ on $\mathcal{G}^* \oplus W^4$ give rise to such dynamical systems as

$$\begin{aligned} f_{1,\tau} &= (\mathcal{E} f_1) + f_1^2 f_1^* + f_1 f_2 f_2^* + \bar{u} f_2, \\ f_{1,\tau}^* &= -((\mathcal{E}^{-1} f_1^*) + f_1 (f_1^*)^2 + f_1^* f_2 f_2^* - (\mathcal{E} u) f_2^*), \\ f_{2,\tau} &= -u f_1, \qquad f_{2,\tau}^* = -(\mathcal{E}\bar{u}) f_1^*, \end{aligned} \tag{32}$$

and

$$
\begin{aligned}
&f_{1,T} = f_{1,\tau\tau} + (\mathcal{E}^2 f_1) + w_1(\mathcal{E} f_1) + w_0 f_1 + 2(f_1(\mathcal{E}^{-1} f_1^*) + u\bar{u}) f_1, \\
&f_{1,T}^* = -f_{1,\tau\tau}^* - (\mathcal{E}^{-2} f_1^*) - (\mathcal{E} w_1 f_1^*) - w_0 f_1^* - 2(f_1(\mathcal{E}^{-1} f_1^*) + u\bar{u}) f_1^*, \\
&f_{2,T} = (\mathcal{E}^2 f_2) + w_1(\mathcal{E} f_2) + w_0 f_2 - u f_{1,\tau} + u_\tau f_1, \\
&f_{2,T}^* = -(\mathcal{E}^{-2} f_2^*) - (\mathcal{E} w_1 f_2^*) - w_0 f_2^* + \bar{u} f_{1,\tau}^* - \bar{u}_\tau f_1^*, \\
&(\mathcal{E} - 1) u = f_1^* f_2, \qquad (\mathcal{E} - 1)\bar{u} = f_1 f_2^*,
\end{aligned}
\tag{33}
$$

where one puts $(\nabla\gamma_2(l))_+ := \mathcal{E}^2 + w_1\mathcal{E} + w_0$ and $w_0, w_1, u, \bar{u} \in \mathcal{S}(\mathbb{Z};\mathbb{C})$ are some functions, depending parametrically on variables $\tau, T \in \mathbb{R}$.

Together systems (32) and (33) represent some $(2+1)$-dimensional nonlinear dynamical system with an infinite sequence of conservation laws in the form (6). Its Lax-type linearization is given by spectral problem (7) and the following evolution equations:

$$
f_\tau = -M_1^1 f, \tag{34}
$$

$$
f_T = ((\nabla\gamma_2(l))_+ - M_1^2) f, \tag{35}
$$

for an arbitrary eigenfunction $f \in W$. From the compatibility condition of the relationships (34) and (35), being equivalent to the commutability of flows $d/d\tau$ and d/dT on the subspace of the operators $l \in \mathcal{G}^*$ in the forms (24), one has the equality

$$
d(\nabla\gamma_2(l))_+/d\tau_{1,k} = [(\nabla\gamma_2(l))_+, M_1^1]_+,
$$

which leads to the additional nonlinear constraints

$$
\begin{aligned}
&w_{0,\tau} = (\mathcal{E}^2 f_1) f_1^* - f_1(\mathcal{E}^{-2} f_1^*) + w_1(\mathcal{E} f_1) f_1^* - f_1(\mathcal{E}^{-1} w_1 f_1^*), \\
&w_{1,\tau} = (\mathcal{E}^2 f_1)(\mathcal{E} f_1^*) - f_1(\mathcal{E}^{-1} f_1^*),
\end{aligned}
$$

for the dynamical system (32), (33).

The results obtained in this section can be applied to constructing a wide class of integrable $(2+1)$-dimensional differential-difference systems with triple Lax-type linearizations.

5 The Hamiltonian structure of the Lax integrable (2 + 1)-dimensional differential-difference system on the extended phase space

One will assume that the Lie algebra $\mathcal{G}$ depends on the parameter $y \in \mathbb{S}^1 \simeq \mathbb{R}/2\pi\mathbb{Z}$, which generates the loop Lie algebra $\hat{\mathcal{G}} := C^\infty(\mathbb{S}^1;\mathcal{G})$ with the ad-invariant nondegenerate symmetric bilinear form:

$$
(A, B) := \int_0^{2\pi} dy \, \mathrm{Tr}\,(AB), \qquad A, B \in \hat{\mathcal{G}}.
$$

The Lie algebra $\hat{\mathcal{G}}$ can be extended via the central extension procedure [6, 7, 8, 14, 17, 34] to the Lie algebra $\hat{\mathcal{G}}_c := \hat{\mathcal{G}} \oplus \mathbb{C}$ with the commutator

$$
[(A,\alpha),(B,\beta)] = ([A,B], \omega_2(A,B)), \qquad A, B \in \hat{\mathcal{G}}, \qquad \alpha, \beta \in \mathbb{C},
$$

where $\omega_2(\cdot,\cdot)$ is a standard Maurer–Cartan 2-cocycle on $\hat{\mathcal{G}}$ such that

$$
\omega_2(A,B) := (A, \partial B/\partial y), \qquad A, B \in \hat{\mathcal{G}},
$$

and the scalar product takes the form

$$((A,\alpha),(B,\beta)) := (A,B) + \alpha\beta. \tag{36}$$

The $\mathcal{R}$-deformed commutator [6, 7, 8, 14, 17, 34] on $\hat{\mathcal{G}}_c$ such that

$$[(A,\alpha),(B,\beta)]_{\mathcal{R}} = ([A,B]_{\mathcal{R}}, \omega_{2,\mathcal{R}}(A,B)), \qquad A,B \in \hat{\mathcal{G}},$$

where

$$\omega_{2,\mathcal{R}}(A,B) = \omega_2(\mathcal{R}A,B) + \omega_2(A,\mathcal{R}B),$$

leads to the Lie–Poisson bracket

$$\{\gamma,\mu\}_{\mathcal{R}}(l) = (l, [\nabla\gamma(l), \nabla\mu(l)]_{\mathcal{R}}) + c\,\omega_{2,\mathcal{R}}(\nabla\gamma(l), \nabla\mu(l)), \tag{37}$$

where $\gamma,\mu \in \mathcal{D}(\hat{\mathcal{G}}_c^*)$ are some Frechet-smooth functionals, $l \in \hat{\mathcal{G}}^*$ and $c \in \mathbb{C}$, on the dual space $\hat{\mathcal{G}}_c^* \simeq \hat{\mathcal{G}}_c$ to $\hat{\mathcal{G}}_c$ with respect to the scalar product (36).

The corresponding Casimir functionals $\gamma_n \in I(\hat{\mathcal{G}}_c^*)$, $n \in \mathbb{N}$, obey the relationship:

$$[l - c\partial/\partial y, \nabla\gamma(l)] = 0,$$

which can be solved by use of the following expansions [34]

$$\nabla\gamma_n(l) := \sum_{n-j\in\mathbb{Z}_+} u_j \mathcal{E}^j. \tag{38}$$

The Lie–Poisson bracket (37) together with the Casimir functionals $\gamma_n \in I(\hat{\mathcal{G}}_c^*)$, $n \in \mathbb{N}$, generates the Hamiltonian flows

$$dl/dt_n := [\mathcal{R}\nabla\gamma_n(l),\ l - c\partial/\partial y] = [(\nabla\gamma_n(l))_+,\ l - c\partial/\partial y]. \tag{39}$$

Every Hamiltonian flow in (39) can be considered as a compatibility condition for the spectral problem

$$((l - c\partial/\partial y)f) = \lambda f, \tag{40}$$

where $f \in \bar{W} := L_2(\mathbb{Z}\times\mathbb{S}^1;\mathbb{C})$, $\lambda \in \mathbb{C}$ is a spectral parameter, and the following evolution equation:

$$df/dt_n = ((\nabla\gamma_n(l))_+ f). \tag{41}$$

The corresponding evolution for the adjoint eigenfunction $f^* \in \bar{W}^* \simeq \bar{W}$ takes the form:

$$df^*/dt_n = -((\nabla\gamma_n(l))_+^* f^*). \tag{42}$$

One will investigate the existence problem of a Hamiltonian representation for equation (39) coupled with $N \in \mathbb{N}$ copies of (41):

$$df_i/dt_n = ((\nabla\gamma_n(l))_+ f_i), \tag{43}$$

for the corresponding eigenfunctions $f_i \in \bar{W}$, $i = \overline{1,N}$, and the same number of copies of (42):

$$df_i^*/dt_n = -((\nabla\gamma_n(l))_+^* f_i^*), \tag{44}$$

for the suitable adjoint eigenfunctions $f_i^* \in \bar{W}^*$, in the case when the spectral relationship (40) admits $N \in \mathbb{N}$ different eigenvalues $\lambda_i \in \mathbb{C}$, $i = \overline{1, N}$.

On the spaces $\hat{\mathcal{G}}_c^*$ and $\bar{W}^{2N}$ there exist canonical Poisson structures in the corresponding forms

$$\delta\overline{\gamma}/\delta\tilde{l} \overset{\tilde{\theta}}{\mapsto} [\tilde{l} - c\partial/\partial y, (\delta\overline{\gamma}/\delta\tilde{l})_+] - [\tilde{l} - c\partial/\partial y, \delta\overline{\gamma}/\delta\tilde{l}]_{>0}, \tag{45}$$

where $\tilde{\theta} : \mathcal{T}^*(\hat{\mathcal{G}}_c^*) \to \mathcal{T}(\hat{\mathcal{G}}_c^*)$ is an implectic operator related to (37) at the point $\tilde{l} \in \hat{\mathcal{G}}^*$, and

$$(\delta\overline{\gamma}/\delta\tilde{\mathrm{f}},\, \delta\overline{\gamma}/\delta\tilde{\mathrm{f}}^*)^\top \overset{\tilde{J}}{\mapsto} (-\delta\overline{\gamma}/\delta\tilde{\mathrm{f}}^*,\, \delta\gamma/\delta\tilde{\mathrm{f}})^\top, \tag{46}$$

where $\tilde{J} : \mathcal{T}^*(\bar{W}^{2N}) \to \mathcal{T}(\bar{W}^{2N})$ at the point $(\tilde{\mathrm{f}}, \tilde{\mathrm{f}}^*) \in \bar{W}^{2N}$, for any Frechet-smooth functional $\overline{\gamma} \in \mathcal{D}(\hat{\mathcal{G}}_c^* \oplus \bar{W}^{2N})$.

The tensor product $\tilde{\Theta} := \tilde{\theta} \otimes \tilde{J}$ of (45) and (46) can be considered as a Poisson structure on the extended phase space $\hat{\mathcal{G}}_c^* \oplus \bar{W}^{2N}$.

Applying the procedure described in Section 3 to the coupled dynamical system (39), (43) and (44) one obtains a Bäcklund transformation on the extended phase space $\hat{\mathcal{G}}_c^* \oplus \bar{W}^{2N}$ in the form (20). Thus, the following theorem can be formulated.

Theorem 5. *Under the Bäcklund transformation (20) the dynamical system (16) on $\hat{\mathcal{G}}_c^* \oplus \bar{W}^{2N}$ is equivalent to the system of evolution equations:*

$$\begin{aligned}
&d\tilde{l}/dt_n = [(\nabla\overline{\gamma}_n(\tilde{l}))_+, \tilde{l}] - [\nabla\overline{\gamma}_n(\tilde{l}), \tilde{l}]_{>0} + c(\partial/\partial y(\nabla\overline{\gamma}_n(\tilde{l}))_0),\\
&d\tilde{\mathrm{f}}/dt_n = \delta\overline{\gamma}_n/\delta\tilde{\mathrm{f}}^*, \qquad d\tilde{\mathrm{f}}^*/dt_n = -\delta\overline{\gamma}_n/\delta\tilde{\mathrm{f}},
\end{aligned}$$

where $\overline{\gamma}_n := \gamma_n|_{l=l(\tilde{l},\mathrm{f},\mathrm{f}^*)} \in \mathcal{D}(\hat{\mathcal{G}}_c^* \oplus \bar{W}^{2N})$ and $\gamma_n \in I(\hat{\mathcal{G}}_c^*)$ is a Casimir functional at the point $l \in \mathcal{G}^*$ for every $n \in \mathbb{N}$.

By means of calculations via the formula (21) one can find the following form of the Bäcklund transformed Poisson structure Θ on $\hat{\mathcal{G}}_c^* \oplus \bar{W}^{2N}$

$$\nabla\overline{\gamma}(l, \mathrm{f}, \mathrm{f}^*) \overset{\Theta}{\mapsto} \begin{pmatrix} [l - c\partial/\partial y, (\delta\overline{\gamma}/\delta l)_+] - [l - c\partial/\partial y, \delta\overline{\gamma}/\delta l]_{>0} + \\ + \sum\limits_{i=1}^{N} (f_i\mathcal{E}(\mathcal{E}-1)^{-1}(\delta\overline{\gamma}/\delta f_i) - (\delta\overline{\gamma}/\delta f_i^*)\mathcal{E}(\mathcal{E}-1)^{-1} f_i^*) \\ -\delta\overline{\gamma}/\delta\mathrm{f}^* - ((\delta\overline{\gamma}/\delta l)_+\mathrm{f}) \\ \delta\overline{\gamma}/\delta\mathrm{f} + ((\delta\overline{\gamma}/\delta l)_+^*\mathrm{f}^*) \end{pmatrix}, \tag{47}$$

where $\overline{\gamma} \in \mathcal{D}(\hat{\mathcal{G}}_c^* \oplus \bar{W}^{2N})$ is an arbitrary Frechet-smooth functional. Thus, the following theorem holds.

Theorem 6. *The hierarchy of dynamical systems (39), (43) and (44) is Hamiltonian with respect to the Poisson structure Θ in the form (47) and the Casimir functionals $\gamma_n \in I(\hat{\mathcal{G}}_c^*)$, $n \in \mathbb{N}$, as Hamiltonian functions.*

On the coadjoint orbits of the Lie algebra $\hat{\mathcal{G}}_c$ the Hamiltonian evolution equations (16) generated by Casimir invariants $\gamma_n \in I(\hat{\mathcal{G}}_c^*)$, $n \in \mathbb{N}$, in the form (38), being involutive with respect to the Lie–Poisson bracket (37), on the extended phase space $\hat{\mathcal{G}}_c^* \oplus W^{2N}$ give rise to the Lax representations for some $(2+1)$-dimensional differential-difference systems [6, 17, 20, 34]. The Lie-algebraic structure of the corresponding Hamiltonian additional homogeneous symmetry hierarchies can be described by means of the approaches developed in Section 4 and in the paper [18].

6 Conclusion

In this paper the method of solving the existence problem of Hamiltonian representations for the coupled Lax-type hierarchies on extended phase spaces which was proposed in [16, 18, 19, 36] have been developed for Lax integrable $(1 + 1)$- and $(2 + 1)$-dimensional differential-difference systems of a lattice type associated with the Lie algebra of shift operators and its central extension by the Maurer–Cartan 2-cocycle correspondingly.

It is based on the invariance property of Casimir functionals under some specially constructed Bäcklund transformation on a dual space to the related operator Lie algebra whose a structure is strongly depending on a Lie algebra splitting into a direct sum of Lie subalgebras. Another possibility [4, 17, 31, 34] of choosing a such splitting can give rise to a different Bäcklund transformation.

For the coupled Lax-type hierarchy on the extended phase space of the shift operator Lie algebra the Hamiltonian representations for the additional homogeneous symmetry hierarchies have been obtained by using the Bäcklund transformation mentioned above. It has been shown that these hierarchies generate a new class of $(2 + 1)$-dimensional differential-difference systems of a lattice type which possess infinite sequences of conservation laws and triple Lax-type linearizations. The latter makes it possible to apply the reduction procedure upon invariant solution subspaces [5, 17, 24, 25] to this class of systems.

The approaches described in the paper can be used to solve analogous problems for nonlocal differential-difference systems [4, 9, 17, 34].

For the considered classes of Lax integrable lattice and nonlocal differential-difference systems it is still an open problem to develop the appropriate Darboux–Bäcklund transformation method [26, 38] which has been effective for solving Lax integrable nonlinear dynamical systems on functional manifolds.

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