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The Noncommutative Ward Metric*

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Abstract. We analyze the moduli-space metric in the static non-Abelian charge-two sector of the Moyal-deformed $\mathbb{C}P^1$ sigma model in $1+2$ dimensions. After carefully reviewing the commutative results of Ward and Ruback, the noncommutative Kähler potential is expanded in powers of dimensionless moduli. In two special cases we sum the perturbative series to analytic expressions. For any nonzero value of the noncommutativity parameter, the logarithmic singularity of the commutative metric is expelled from the origin of the moduli space and possibly altogether.

Key words: noncommutative geometry; $\mathbb{C}P^1$ sigma model

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1 Introduction and summary

The $\mathbb{C}P^1$ sigma model in $1+2$ dimensions is a paradigm for soliton studies [1, 2]. In particular, it provides the simplest example for a nontrivial dynamics of slowly-moving lumps, following the adiabatic approximation scheme of Manton [3]. In a slice of the charge-two sector, the moduli-space metric was worked out and the geodesic motion was analyzed by Ward [4]. The corresponding Kähler potential was then given by Ruback [5] (see also [6]).

In the case just mentioned, the (restricted) moduli space of static charge-two solutions is complex two-dimensional and contains ring-like as well as two-lump configurations. On the complex line where the lump size shrinks to zero, the metric develops a logarithmic singularity. Such divergencies can often be regulated by subjecting the system to a noncommutative deformation, which introduces a dimensionful deformation parameter θ. To explore this possibility, we analyze the Moyal-deformed $\mathbb{C}P^1$ model [7] in this paper.

In fact, the (restricted) moduli-space metric for the charge-two sector of this noncommutative model was already investigated in [8]. There, the authors show that the metric in question is flat for $\theta \to \infty$ (corresponding to vanishing values of the dimensionless moduli) and possesses a smooth $\theta \to 0$ limit (which is attained for infinite values of the dimensionless moduli). However, these findings do not establish the removal of the logarithmic singularity for finite values of θ or amount to an explicit computation of the Kähler potential.

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In this paper, we review the commutative results and present a power-series expansion of the deformed Kähler potential in the 'ring' regime of the moduli space. For the first time, this is achieved for arbitrary values of θ. We verify the commutative limit and sum up the perturbation series on the would-be singular line in the 'two-lump' domain via the Gel'fand–Yaglom method. There is a curious connection with the eigenvalues of the spheroidal wave equation. Around the origin of the moduli space, the Kähler potential is shown to be analytic, which substantiates the claim of [8]. Perturbative expressions for the moduli-space metric follow via differentiation, and the two-lump scattering behavior may be quantified.

2 The $\mathbb{C}P^1$ model and its solitons

The $\mathbb{C}P^1$ or, equivalently, the O(3) sigma model describes the dynamics of maps from $\mathbb{R}^{1,2}$ with a metric $(\eta_{\mu\nu}) = \mathrm{diag}(-1,+1,+1)$ into $\mathbb{C}P^1 \simeq \frac{\mathrm{SU}(2)}{\mathrm{U}(1)} \simeq S^2$. There are various ways to parametrize the target space, for instance by hermitian rank-one projectors P in $\mathbb{C}^2$,

$$P = P^\dagger = P^2 = T(T^\dagger T)^{-1}T^\dagger,$$

or else by vectors $T \in \mathbb{C}^2$ modulo complex scale,

$$T = \begin{pmatrix} p \\ q \end{pmatrix} \sim \begin{pmatrix} u \\ 1 \end{pmatrix} \qquad \text{with} \quad u = \frac{p}{q},$$

so that the field degree of freedom is a single function u taking values in the extended complex plane $\dot{\mathbb{C}} \simeq \mathbb{C}P^1$.

Introducing coordinates on $R^{1,2}$,

$$(x^\mu) = (t, x, y) \qquad \text{with} \quad \mu = 0,1,2 \qquad \text{and} \qquad z = x + \mathrm{i}y,$$

we can formulate the action as

$$\begin{aligned} S &= -4\int \mathrm{d}^3x \,\mathrm{tr}\,\eta^{\mu\nu}\partial_\mu P\,\partial_\nu P = -4\int \mathrm{d}^3x\, (T^\dagger T)^{-1}\eta^{\mu\nu}\partial_\mu T^\dagger(\mathbb{1}-P)\,\partial_\nu T \\ &= -4\int \mathrm{d}^3x\, (1+\bar{u}u)^{-2}\eta^{\mu\nu}\partial_\mu\bar{u}\,\partial_\nu u, \end{aligned}$$

where $\bar{u}$ is the complex conjugate of u, and only the last equality uses the commutativity of the functions. For later convenience, we also define the kinetic and potential energy density,

$$\begin{aligned} \mathcal{T} &= 4(T^\dagger T)^{-1}\dot{T}^\dagger(1-P)\dot{T} = \frac{4\dot{\bar{u}}\dot{u}}{(1+\bar{u}u)^2} \qquad \text{and} \\ \mathcal{V} &= 8(T^\dagger T)^{-1}\partial_{\bar{z}}T^\dagger(1-P)\partial_z T + (\partial_z \leftrightarrow \partial_{\bar{z}}) = \frac{8\partial_{\bar{z}}\bar{u}\partial_z u}{(1+\bar{u}u)^2} + (\partial_z \leftrightarrow \partial_{\bar{z}}), \end{aligned}$$

respectively, so that

$$S = \int \mathrm{d}^3x\,(\mathcal{T}-\mathcal{V}), \qquad \text{and} \qquad E = \int \mathrm{d}^2z\,(\mathcal{T}+\mathcal{V})$$

yields the total energy of the configuration, which is conserved in time. Clearly, action and energy are form-invariant under translations and rotations of the domain $\mathbb{R}^2$ (at fixed t),

$$z \mapsto z + \lambda \qquad \text{and} \qquad z \mapsto \mathrm{e}^{\mathrm{i}\mu}z,$$

as well as under global SO(3) rotations of the target,

$$u \mapsto \frac{au+b}{-\bar{b}u+\bar{a}} \qquad \text{with} \quad \bar{a}a + \bar{b}b = 1.$$

Classically, one is interested in the extrema of S whose energy is finite. Among the static configurations, $\dot{u}=0$ (hence $\mathcal{T}=0$), those are all well known:

$$\delta S=0 \quad\Rightarrow\quad \delta E=0 \quad\Rightarrow\quad \delta\int \mathrm{d}^2z\,\mathcal{V}=0 \quad\Rightarrow\quad u=u(z)\ \text{ or }\ u=u(\bar{z}),$$

with u being a rational function (of z or $\bar{z}$) to ensure finite energy. Each rational analytic (or anti-analytic) function $u=\frac{p}{q}$ is a soliton (or anti-soliton) with a topological charge given by its degree n (or $-n$) and with energy $E=8\pi|n|$. Hence, the soliton moduli space $\mathcal{M}_n$ for charge n has complex dimension $2n+1$. Some of these moduli, however, correspond to isometries of the domain or the target.

In this paper, we shall investigate only charge-one and charge-two solitons. Let us characterize their static moduli spaces. By employing the SO(3) target rotations, in the numerator p we remove the highest monomial and restrict the coefficient of the second-highest one to be real and non-negative. This is also true for the third-highest one by means of a domain rotation. Furthermore, by a common rescaling of p and q we set the coefficient of the highest monomial in the denominator q to unity. Finally, the domain translation isometry allows us to remove the second-highest monomial of q, which corresponds to picking a center-of-mass frame for our configuration. These choices fix all isometries except possibly for special values of the remaining moduli. Of course, the full moduli space is recovered by acting with all isometries. In the charge-one case, we thus get

$$T(z)=\begin{pmatrix}\beta\\ z\end{pmatrix} \quad\Rightarrow\quad \mathcal{V}=\frac{8|\beta|^2}{(|\beta|^2+|z|^2)^2}=\frac{8}{|\beta|^2}\frac{1}{(1+|z'|^2)^2}$$

with $\beta\in\mathbb{R}_{\geq 0}$ and $z=\beta z'$. This is a single lump of height $|\beta|^{-2}$ and width of order $|\beta|$.

For charge two, one finds

$$T(z)=\begin{pmatrix}\beta z+\gamma\\ z^2+\epsilon\end{pmatrix} \quad\Rightarrow\quad \mathcal{V}=\frac{8|\beta z^2+2\gamma z-\beta\epsilon|^2}{(|\beta z+\gamma|^2+|z^2+\epsilon|^2)^2}=\frac{8}{|\beta|^2}\frac{|z'^2+2\gamma' z'-\epsilon'|^2}{(|z'+\gamma'|^2+|z'^2+\epsilon'|^2)^2}$$

with $\beta,\gamma\in\mathbb{R}_{\geq 0}$ and $\epsilon\in\mathbb{C}$. In the last expression, we have introduced dimensionless quantities by the rescaling

$$z=\beta z', \qquad \gamma=\beta^2\gamma' \qquad\text{and}\qquad \epsilon=\beta^2\epsilon',$$

effectively putting $\beta=1$. A different situation arises for the special value $\beta=0$. Here, one can also rotate away the phase of ϵ and should rather use $z=\sqrt{\gamma}z'$ to arrive at

$$\mathcal{V}=\frac{32}{|\gamma|}\frac{|z'|^2}{\left(1+\left|z'^2+\frac{\epsilon}{\gamma}\right|^2\right)^2}.$$

One may check that $\mathcal{V}$ integrates to 16π in both cases. This energy density can take a variety of shapes, depending on the values of the moduli. Two well-separated lumps appear for $|\epsilon|>|\beta|^2$ and $|\epsilon|>|\gamma|$, while ring-like structures emerge in the regime $|\gamma|>|\beta|^2$ and $|\gamma|>|\epsilon|$.

3 Moduli space metric

So far, we have only considered static solutions to the sigma model. For dynamical issues, we must bring back the time dependence. Rather than attempting to solve the full equations of motion $\delta S=0$ for $u(t,z,\bar{z})$, we resort to the adiabatic approximation valid for slow motion [2],

$$u(t,z,\bar{z})\approx u(z\,|\,\alpha(t)),$$

where $u(z|\alpha)$ denotes a static soliton depending holomorphically on moduli parameters α. For simplicity we suppress here the moduli labels but let α represent the holomorphic set $\{\alpha\}$.[1] By allowing these moduli to vary with time, we approximate the true time-dependent solution by a sequence of snapshots of static solutions. In this way, the dynamics in the configuration space of maps, $u : \mathbb{R} \to \mathrm{maps}(\mathbb{C}, \mathbb{C}P^1)$ via $t \mapsto u(t, \cdot)$, gets projected to the 'mechanics' of a particle moving in the finite-dimensional moduli space for a fixed topological charge, $\alpha : \mathbb{R} \to \mathcal{M}_n$.

Since the potential energy of the soliton configurations is independent of α, the kinetic energy provides an action principle for $\alpha(t)$: the extrema of

$$\begin{aligned}\int \mathrm{d}^3x\, \mathcal{T}\big[u(\cdot\,|\,\alpha(t))\big] &= 4\int \mathrm{d}t \left[\int \mathrm{d}^2z\; (T^\dagger T)^{-1}\partial_{\bar\alpha}T^\dagger(1-P)\,\partial_\alpha T\right]\dot{\bar\alpha}\,\dot\alpha \\ &= 4\int \mathrm{d}t \left[\int \mathrm{d}^2z\; \frac{\partial_{\bar\alpha}\bar u\,\partial_\alpha u}{(1+\bar u u)^2}\right]\dot{\bar\alpha}\,\dot\alpha =: \frac{1}{2}\int \mathrm{d}t\; g_{\bar\alpha\alpha}(\alpha)\,\dot{\bar\alpha}\,\dot\alpha\end{aligned}$$

are just geodesics in $\mathcal{M}_n$ endowed with the induced Kähler metric

$$g_{\bar\alpha\alpha} = \partial_{\bar\alpha}\partial_\alpha\mathcal{K},$$

where

$$\mathcal{K} = 8\int \mathrm{d}^2z\; \ln T^\dagger T = 8\int \mathrm{d}^2z\; \ln(1+\bar u u)$$

computes the Kähler potential from the static soliton configurations $u = u(z|\alpha)$. We remark that the freedom of rescaling T reappears in the ambiguity of $\mathcal{K}$ due to Kähler transformations, $\mathcal{K} \sim \mathcal{K} + f(\alpha) + g(\bar\alpha)$, and so we may also use the more divergent formal expression

$$\mathcal{K} = 8\int \mathrm{d}^2z\; \ln(\bar p p + \bar q q).$$

It turns out that $g_{\bar\alpha\alpha}$ diverges for the modulus β (and also for the removed z^n coefficient in p). Hence, these particular moduli carry infinite inertia and do not participate in the dynamics, because changing their values requires an infinite amount of energy. Consequently, they get degraded to external parameters which are to be dialled by hand. In the charge-one case, no dynamics remains, which is consistent with the picture of a single lump sitting in its rest frame. Nevertheless, it is instructive to reinstate the translation moduli and verify the flat moduli space. With $T = \left(\begin{smallmatrix}\beta\\ z+\delta\end{smallmatrix}\right)$ we get

$$\mathcal{K} = 8\int \mathrm{d}^2z\; \ln\left(1 + \frac{|\beta|^2}{|z+\delta|^2}\right).$$

This is formally independent of δ (by shifting $z \mapsto z-\delta$) but it is logarithmically divergent, so we better compute its second derivatives

$$\partial_{\bar\beta}\partial_\beta\mathcal{K} = 8\int \mathrm{d}^2z\; \frac{|z|^2}{(|\beta|^2+|z|^2)^2} = \infty \qquad \text{and} \qquad \partial_{\bar\delta}\partial_\delta\mathcal{K} = 8\int \mathrm{d}^2z\; \frac{|\beta|^2}{(|\beta|^2+|z|^2)^2} = 8\pi,$$

as well as $\partial_{\bar\delta}\partial_\beta\mathcal{K} = -8\pi\delta/\beta$. Hence, we indeed get $\mathcal{K} = 8\pi\bar\delta\delta$. Since the center-of-mass motion decouples from the remaining dynamics, we shall suppress it from now on.

For charge two, the Kähler potential $\mathcal{K}$ reads

$$8|\beta|^2\int \mathrm{d}^2z'\; \ln\left(1 + \frac{|z'+\gamma'|^2}{|z'^2+\epsilon'|^2}\right) \qquad \text{or} \qquad 8|\gamma|\int \mathrm{d}^2z'\; \ln\left(1 + \frac{1}{|z'^2+\frac{\epsilon}{\gamma}|^2}\right)$$

[1] We apply isometries to undo possible phase restrictions on β or γ.

depending on whether β is chosen nonzero or not. In the first case, $\mathcal{K}$ is again divergent, and its derivatives are not elementary integrable. For the sake of simplicity, we therefore restrict ourselves to the second (special) case and put $\beta = 0$ from now on. The form of the relevant integral reveals that $\mathcal{K}$ is a function only of $|\gamma|$ and $|\epsilon|$ which, up to an overall dimensional factor, depends merely on their ratio. The last integral can in fact be executed to yield[2]

$$\mathcal{K} = 16\pi|\gamma| \int_0^{\pi/2} \mathrm{d}\theta \sqrt{1 + \left|\frac{\epsilon}{\gamma}\right|^2 \sin^2\theta} = 16\pi|\gamma| E\left(-\left|\frac{\epsilon}{\gamma}\right|^2\right) = 16\pi r|\cos\varphi| E\big(-\tan^2\varphi\big),$$

where $E(m = k^2)$ denotes the complete elliptic integral of the second kind as a function of its parameter m (k is called the elliptic modulus), and we have parametrized[3]

$$\epsilon = r\mathrm{e}^{\mathrm{i}\omega}\sin\varphi \qquad \text{and} \qquad \gamma = r\mathrm{e}^{\mathrm{i}\chi}\cos\varphi.$$

In the $(|\epsilon|, |\gamma|)$ plane, the Kähler potential grows linearly with the distance from the origin, with a slope varying between $8\pi^2$ (for $\epsilon = 0$) and 16π (for $\gamma = 0$). It is continuous but not smooth on the complex line $\gamma = 0$ ($\varphi = \frac{\pi}{2}$), which is the localization locus in the two-lump region because the lump width is of order $\frac{|\gamma|}{\sqrt{|\epsilon|}}$ at a lump separation of order $2\sqrt{|\epsilon|}$.

To investigate the two extreme situations, $|\epsilon| \ll |\gamma|$ and $|\gamma| \ll |\epsilon|$, we expand the Kähler potential in $|\frac{\epsilon}{\gamma}|$ and $|\frac{\gamma}{\epsilon}|$, respectively. For the 'ring' regime, $|\epsilon| \ll |\gamma|$, we have

$$\begin{aligned}
\mathcal{K} &= 8\pi^2|\gamma| \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{1-2\ell} \left[\frac{(2\ell-1)!!}{(2\ell)!!}\right]^2 \left|\frac{\epsilon}{\gamma}\right|^{2\ell} \\
&= 8\pi^2|\gamma| \left\{1 + \frac{1}{4}\left|\frac{\epsilon}{\gamma}\right|^2 - \frac{3}{64}\left|\frac{\epsilon}{\gamma}\right|^4 + \frac{5}{256}\left|\frac{\epsilon}{\gamma}\right|^6 + \cdots\right\},
\end{aligned} \tag{1}$$

while in the 'two-lump' domain, $|\gamma| \ll |\epsilon|$, we encounter logarithms,

$$\begin{aligned}
\mathcal{K} &= 16\pi|\epsilon| \left\{1 - \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{4(\ell+1)} \left[\frac{(2\ell-1)!!}{(2\ell)!!}\right]^2 c_\ell \left|\frac{\gamma}{\epsilon}\right|^{2(\ell+1)}\right\} \\
&\quad - 4\pi\left|\frac{\gamma^2}{\epsilon}\right| \ln\left|\frac{\gamma}{4\epsilon}\right|^2 {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 2; -\left|\frac{\gamma}{\epsilon}\right|^2\right) \\
&= 16\pi|\epsilon| \left\{1 - \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{4(\ell+1)} \left[\frac{(2\ell-1)!!}{(2\ell)!!}\right]^2 \left(c_\ell + \ln\left|\frac{\gamma}{4\epsilon}\right|^2\right) \left|\frac{\gamma}{\epsilon}\right|^{2(\ell+1)}\right\} \\
&= 16\pi|\epsilon| \left\{1 - \frac{1}{4}\left(-1 + \ln\left|\frac{\gamma}{4\epsilon}\right|^2\right)\left|\frac{\gamma}{\epsilon}\right|^2 + \frac{1}{32}\left(\frac{3}{2} + \ln\left|\frac{\gamma}{4\epsilon}\right|^2\right)\left|\frac{\gamma}{\epsilon}\right|^4 \right. \\
&\quad \left. - \frac{3}{256}\left(2 + \ln\left|\frac{\gamma}{4\epsilon}\right|^2\right)\left|\frac{\gamma}{\epsilon}\right|^6 + \cdots\right\}
\end{aligned} \tag{2}$$

with

$$\begin{aligned}
&c_0 = -1, \qquad c_1 = \frac{3}{2}, \qquad c_2 = 2, \\
&c_\ell = \frac{2}{\ell} + \frac{3}{\ell+1} + \frac{4}{\ell+2} + \frac{4}{\ell+3} + \cdots + \frac{4}{2\ell-1} \quad \text{for} \quad \ell \geq 3.
\end{aligned}$$

[2] Apart from the normalization, we differ from [5] by the absence of a term linear in $|\epsilon|$.

[3] There is some ambiguity in the range of the angles. We take $\varphi \in [0, 2\pi)$.

The metric coefficients

$$g_{\bar\gamma\gamma} = \int \frac{8|z^2+\epsilon|^2 \mathrm{d}^2 z}{(|\gamma|^2+|z^2+\epsilon|^2)^2}, \qquad g_{\bar\epsilon\epsilon} = \int \frac{8|\gamma|^2 \mathrm{d}^2 z}{(|\gamma|^2+|z^2+\epsilon|^2)^2}, \qquad g_{\bar\gamma\epsilon} = \int \frac{-8\gamma\bar\epsilon \mathrm{d}^2 z}{(|\gamma|^2+|z^2+\epsilon|^2)^2}$$

may of course be expressed in terms of complete elliptic integrals [4]. The geodesic motion in this metric cannot be found in closed form, except for special motions $\alpha(t)$,

$$\dot\omega = \dot\chi = \dot\varphi = 0 \quad \Rightarrow \quad (\mathrm{d}s)^2 = \frac{4\pi}{r}|\cos\varphi| E\big(-\tan^2\varphi\big)(\mathrm{d}r)^2,$$

which yields $r(t) = r_0 + h(\varphi)t^2$ with a specific function $h(\varphi)$.

4 Moyal deformation

The task of this paper is the Moyal deformation of the Ward metric and the Kähler potential presented in the previous section. One way to describe such a noncommutative deformation of the $z\bar z$ plane is by giving the following 'quantization rule':

$$\text{coordinates} \quad (z,\bar z) \quad \longmapsto \quad \text{operators} \quad (Z,\bar Z) \quad \text{with} \quad [Z,\bar Z] = 2\theta = \text{const},$$

where these operators may be realized as infinite matrices

$$Z = \sqrt{2\theta}\,a = \sqrt{2\theta}\begin{pmatrix} 0 & 0 & & & \\ \sqrt{1} & 0 & 0 & & \\ & \sqrt{2} & 0 & 0 & \\ & & \sqrt{3} & 0 & \ddots \\ & & & \ddots & \ddots \end{pmatrix}, \qquad \bar Z = \sqrt{2\theta}\,a^\dagger = \sqrt{2\theta}\begin{pmatrix} 0 & \sqrt{1} & & & \\ 0 & 0 & \sqrt{2} & & \\ & 0 & 0 & \sqrt{3} & \\ & & 0 & 0 & \ddots \\ & & & \ddots & \ddots \end{pmatrix}.$$

More generally,

$$\text{functions} \quad f(z,\bar z) \quad \longmapsto \quad \text{operators} \quad F = f(Z,\bar Z)\big|_{\text{sym}},$$

where 'sym' indicates a symmetric ordering of all monomials in $(Z,\bar Z)$. Naturally, derivatives turn into inner derivations,

$$\partial_z \quad \mapsto \quad \frac{1}{2\theta}[\,\cdot\,,\bar Z] = \frac{1}{\sqrt{2\theta}}[\,\cdot\,,a^\dagger] \qquad \text{and} \qquad \partial_{\bar z} \quad \mapsto \quad \frac{1}{2\theta}[Z,\,\cdot\,] = \frac{1}{\sqrt{2\theta}}[a,\,\cdot\,],$$

and the integral over the complex plane becomes a trace over the operator algebra,

$$\int \mathrm{d}^2 z\; f(z,\bar z) \quad \longmapsto \quad 2\pi\theta\,\mathrm{tr}\,F.$$

A highest-weight representation space $\mathcal{F}$ for the Heisenberg algebra, $[a,a^\dagger]=1$, is easily constructed from a vacuum $|0\rangle$,

$$a\,|0\rangle = 0 \quad \Rightarrow \quad \mathcal{F} = \text{span}\Big\{ |n\rangle = \frac{1}{\sqrt{n!}}(a^\dagger)^n|0\rangle \;\big|\; n=0,1,2,\dots \Big\},$$

where the basis states are the normalized eigenstates of the 'number operator' $N = a^\dagger a$,

$$N\,|n\rangle = n\,|n\rangle \qquad \text{and} \qquad \langle n|n\rangle = 1 \qquad \text{for} \quad n=0,1,2,\dots. \tag{3}$$

The Moyal-deformed $\mathbb{C}P^1$ model is defined by copying most definitions of the previous section, but taking the entries of P and T to be operator-valued. Since, in this context, q may not have an inverse, we avoid using u as a variable and work with p and q instead. Because the deformation has traded functions on the xy plane with operators on $\mathcal{F}$, densities such as $\mathcal{T}$ or $\mathcal{V}$ are less intuitive objects, but may still be visualized via the Moyal–Weyl map. The noncommutative solitons are found by taking T to be polynomial in a, i.e. both p and q are polynomial of degree n, and their moduli are identical to the commutative ones[4]. It is important to note that the deformation has introduced a new dimensionful parameter, θ. Therefore, we may relate all dimensional quantities to θ and pass to dimensionless parameters,

$$Z = \sqrt{2\theta}a, \qquad \beta = \sqrt{2\theta}b, \qquad \delta = \sqrt{2\theta}d, \qquad \gamma = 2\theta g, \qquad \epsilon = 2\theta e.$$

As a consequence, $\mathcal{K}$ and $g_{\bar{\alpha}\alpha}$ will depend on all moduli individually and not only on their ratios. Of course, in the commutative limit $\theta \to 0$, the ratios will again dominate.

As a warm-up, let us reconsider the charge-one soliton (with b frozen but including the translational moduli d), now given by

$$T = \begin{pmatrix} b \\ a+d \end{pmatrix} \quad \Rightarrow \quad \mathcal{K} = 16\pi\theta \operatorname{tr} \ln T^\dagger T = 16\pi\theta \operatorname{tr} \ln\bigl(\bar{b}b + \bigl(a^\dagger + \bar{d}\bigr)(a+d)\bigr).$$

Since by a unitary basis change in $\mathcal{F}$ we can shift $a \mapsto a - d$, this expression is again formally independent of d, but it is divergent:

$$\frac{\mathcal{K}}{16\pi\theta} = \sum_{n=0}^{\infty} \langle n| \ln(\bar{b}b + N)|n\rangle = \sum_{n=0}^{\infty} \ln(\bar{b}b + n) = -\ln\Gamma(\bar{b}b) + \lambda\bar{b}b + \mu, \tag{4}$$

where the divergence is hidden in the ambiguous coefficients λ and μ, which may depend on b and $\bar{b}$. To fix this ambiguity, we first take derivatives and then shift away the d dependence:

$$\begin{aligned}
\frac{g_{\bar{d}d}}{16\pi\theta} = \frac{\partial_{\bar{d}}\partial_d\mathcal{K}}{16\pi\theta} &= \operatorname{tr}\Bigl[(\bar{b}b+N)^{-1}\bigl(1 - a(\bar{b}b+N)^{-1}a^\dagger\bigr)\Bigr] \\
&= \operatorname{tr}\Bigl[(\bar{b}b+N)^{-1}\bigl(1 - (N+1)(\bar{b}b+N+1)^{-1}\bigr)\Bigr] \\
&= \bar{b}b \operatorname{tr}\Bigl[(\bar{b}b+N)^{-1}(\bar{b}b+N+1)^{-1}\Bigr] = \sum_{n\geq 0} \frac{\bar{b}b}{(n+\bar{b}b)(n+1+\bar{b}b)} = 1,
\end{aligned}$$

while $g_{\bar{b}b}$ is still infinite. Hence, λ remains arbitrary but $\mu = \bar{d}d$ up to irrelevant terms. With b fixed, we therefore get $\mathcal{K} = 16\pi\theta\bar{d}d = 8\pi\bar{\delta}\delta$, the same flat metric as in the commutative case. Note that even though the modulus b has infinite inertia, it is needed to regulate the Kähler potential (4), which blows up at $b = 0$.[5]

5 Deformed rings

We now turn to the nontrivial charge-two case with the choice of $\beta = 0$, defined by

$$\begin{aligned}
&T = \begin{pmatrix} g \\ a^2 + e \end{pmatrix} \quad \Rightarrow \quad \mathcal{K} = 16\pi\theta \operatorname{tr} \ln T^\dagger T \qquad \text{with} \\
&T^\dagger T = \bar{g}g + \bigl(a^{\dagger 2} + \bar{e}\bigr)\bigl(a^2 + e\bigr) = \bar{g}g + N(N-1) + ea^{\dagger 2} + \bar{e}a^2 + \bar{e}e.
\end{aligned}$$

Of course, $\mathcal{K}$ is divergent, but the singularity is removable, and $\partial_{\bar{g}g}\mathcal{K}$ and $\partial_{\bar{e}}\partial_e\mathcal{K}$ already converge. Like the modulus b in the previous section, here g plays the role of a regulator, but this time

[4]In addition to these 'non-Abelian' solitons, which smoothly deform the standard commutative solitons, there exist a plethora of 'Abelian' solitons, which are singular in the commutative limit [9, 10, 11].

[5]In the operator formalism, the inversion of operators is sometimes complicated due to zero modes, but may still be accomplished by means of partial isometries or Murray–von Neumann transformations (see, e.g. [12]).

its inertia is finite. This expression is not amenable to exact analytic computation, but we can attempt to establish power series expansions in $\bar{e}e$ or in $\bar{g}g$. In this section, we investigate the 'ring' regime $|e| \ll |g|$.

It is easy to set up an expansion around $e = 0$, since $T^\dagger T|_{e=0}$ is already diagonal in our basis (3). Note that no zero-mode issue arises since $\bar{g}g > 0$. Writing

$$T^\dagger T = G + E \quad \text{with} \quad G = \bar{g}g + N(N-1) \qquad \text{and} \qquad E = e a^{\dagger 2} + \bar{e} a^2 + \bar{e}e =: \nwarrow + \swarrow + \leftarrow,$$

the Taylor series of $\ln(1+x)$ unfolds to

$$\begin{aligned}
\frac{\mathcal{K}}{16\pi\theta} &= \mathrm{tr}\,\ln(G+E) = \mathrm{tr}\,\ln G - \sum_{k=1}^{\infty} \frac{(-1)^k}{k}\,\mathrm{tr}\left(G^{-1}E\right)^k \\
&= \mathrm{tr}\ln G + \bar{e}e\,\mathrm{tr}\frac{1}{G} - \frac{2}{2}\bar{e}e\,\mathrm{tr}\frac{1}{G}a^{\dagger 2}\frac{1}{G}a^2 - \frac{1}{2}(\bar{e}e)^2\mathrm{tr}\frac{1}{G}\frac{1}{G} + \frac{3}{3}(\bar{e}e)^2\mathrm{tr}\frac{1}{G}\frac{1}{G}a^{\dagger 2}\frac{1}{G}a^2 \\
&\quad + \frac{3}{3}(\bar{e}e)^2\mathrm{tr}\frac{1}{G}a^{\dagger 2}\frac{1}{G}\frac{1}{G}a^2 + \frac{1}{3}(\bar{e}e)^3\mathrm{tr}\frac{1}{G}\frac{1}{G}\frac{1}{G} - \frac{2}{4}(\bar{e}e)^2\mathrm{tr}\frac{1}{G}a^{\dagger 2}\frac{1}{G}a^2\frac{1}{G}a^{\dagger 2}\frac{1}{G}a^2 \\
&\quad - \frac{4}{4}(\bar{e}e)^2\mathrm{tr}\frac{1}{G}a^{\dagger 2}\frac{1}{G}a^{\dagger 2}\frac{1}{G}a^2\frac{1}{G}a^2 - \frac{4}{4}(\bar{e}e)^3\mathrm{tr}\frac{1}{G}\frac{1}{G}a^{\dagger 2}\frac{1}{G}\frac{1}{G}a^2 \\
&\quad - \frac{4}{4}(\bar{e}e)^3\mathrm{tr}\frac{1}{G}\frac{1}{G}\frac{1}{G}a^{\dagger 2}\frac{1}{G}a^2 - \frac{4}{4}(\bar{e}e)^3\mathrm{tr}\frac{1}{G}a^{\dagger 2}\frac{1}{G}\frac{1}{G}\frac{1}{G}a^2 - \frac{1}{4}(\bar{e}e)^4\mathrm{tr}\frac{1}{G}\frac{1}{G}\frac{1}{G}\frac{1}{G} \\
&\quad + \frac{5}{5}(\bar{e}e)^3\mathrm{tr}\frac{1}{G}\frac{1}{G}a^{\dagger 2}\frac{1}{G}a^2\frac{1}{G}a^{\dagger 2}\frac{1}{G}a^2 + \frac{5}{5}(\bar{e}e)^3\mathrm{tr}\frac{1}{G}a^{\dagger 2}\frac{1}{G}\frac{1}{G}a^2\frac{1}{G}a^{\dagger 2}\frac{1}{G}a^2 \\
&\quad + \frac{5}{5}(\bar{e}e)^3\mathrm{tr}\frac{1}{G}\frac{1}{G}a^{\dagger 2}\frac{1}{G}a^{\dagger 2}\frac{1}{G}a^2\frac{1}{G}a^2 + \frac{5}{5}(\bar{e}e)^3\mathrm{tr}\frac{1}{G}a^{\dagger 2}\frac{1}{G}\frac{1}{G}a^{\dagger 2}\frac{1}{G}a^2\frac{1}{G}a^2 \\
&\quad + \frac{5}{5}(\bar{e}e)^3\mathrm{tr}\frac{1}{G}a^{\dagger 2}\frac{1}{G}a^{\dagger 2}\frac{1}{G}\frac{1}{G}a^2\frac{1}{G}a^2 + \frac{5}{5}(\bar{e}e)^3\mathrm{tr}\frac{1}{G}a^{\dagger 2}\frac{1}{G}a^{\dagger 2}\frac{1}{G}a^2\frac{1}{G}\frac{1}{G}a^2 \\
&\quad + \frac{5}{5}(\bar{e}e)^4\mathrm{tr}\frac{1}{G}\frac{1}{G}\frac{1}{G}\frac{1}{G}a^{\dagger 2}\frac{1}{G}a^2 + \frac{5}{5}(\bar{e}e)^4\mathrm{tr}\frac{1}{G}\frac{1}{G}\frac{1}{G}a^{\dagger 2}\frac{1}{G}\frac{1}{G}a^2 \\
&\quad + \frac{5}{5}(\bar{e}e)^4\mathrm{tr}\frac{1}{G}\frac{1}{G}a^{\dagger 2}\frac{1}{G}\frac{1}{G}\frac{1}{G}a^2 + \frac{5}{5}(\bar{e}e)^4\mathrm{tr}\frac{1}{G}a^{\dagger 2}\frac{1}{G}\frac{1}{G}\frac{1}{G}\frac{1}{G}a^2 + \frac{1}{5}(\bar{e}e)^5\mathrm{tr}\frac{1}{G}\frac{1}{G}\frac{1}{G}\frac{1}{G}\frac{1}{G} \\
&\quad - \frac{2}{6}(\bar{e}e)^3\mathrm{tr}\frac{1}{G}a^{\dagger 2}\frac{1}{G}a^2\frac{1}{G}a^{\dagger 2}\frac{1}{G}a^2\frac{1}{G}a^{\dagger 2}\frac{1}{G}a^2 - \frac{6}{6}(\bar{e}e)^3\mathrm{tr}\frac{1}{G}a^{\dagger 2}\frac{1}{G}a^{\dagger 2}\frac{1}{G}a^{\dagger 2}\frac{1}{G}a^2\frac{1}{G}a^2\frac{1}{G}a^2 \\
&\quad - \frac{6}{6}(\bar{e}e)^3\mathrm{tr}\frac{1}{G}a^{\dagger 2}\frac{1}{G}a^{\dagger 2}\frac{1}{G}a^2\frac{1}{G}a^{\dagger 2}\frac{1}{G}a^2\frac{1}{G}a^2 - \frac{6}{6}(\bar{e}e)^3\mathrm{tr}\frac{1}{G}a^{\dagger 2}\frac{1}{G}a^2\frac{1}{G}a^{\dagger 2}\frac{1}{G}a^{\dagger 2}\frac{1}{G}a^2\frac{1}{G}a^2 \\
&\quad + \ O\big((\bar{e}e)^4\big),
\end{aligned}$$

displaying all terms to order $(G^{-1}E)^6$ and $(\bar{e}e)^3$. One sees that for a given power k of $G^{-1}E$, there is a sum over all cyclic paths of length k, where each step is either $\nwarrow$ or $\swarrow$ or $\leftarrow$, separated by a factor of $\frac{1}{G}$. All terms containing $\leftarrow$ can be resummed into the shift operator $\exp(\bar{e}e\,\partial_{\bar{g}g})$, which shortens the above to

$$\begin{aligned}
\frac{\mathcal{K}}{16\pi\theta} &= \exp\big(\bar{e}e\partial_{\bar{g}g}\big)\Big\{\mathrm{tr}\ln G - \bar{e}e\,\mathrm{tr}\frac{1}{G}a^{\dagger 2}\frac{1}{G}a^2 - \frac{1}{2}(\bar{e}e)^2\mathrm{tr}\frac{1}{G}a^{\dagger 2}\frac{1}{G}a^2\frac{1}{G}a^{\dagger 2}\frac{1}{G}a^2 \\
&\quad - (\bar{e}e)^2\mathrm{tr}\frac{1}{G}a^{\dagger 2}\frac{1}{G}a^{\dagger 2}\frac{1}{G}a^2\frac{1}{G}a^2 - \frac{1}{3}(\bar{e}e)^3\mathrm{tr}\frac{1}{G}a^{\dagger 2}\frac{1}{G}a^2\frac{1}{G}a^{\dagger 2}\frac{1}{G}a^2\frac{1}{G}a^{\dagger 2}\frac{1}{G}a^2 \\
&\quad - (\bar{e}e)^3\mathrm{tr}\frac{1}{G}a^{\dagger 2}\frac{1}{G}a^{\dagger 2}\frac{1}{G}a^{\dagger 2}\frac{1}{G}a^2\frac{1}{G}a^2\frac{1}{G}a^2 - (\bar{e}e)^3\mathrm{tr}\frac{1}{G}a^{\dagger 2}\frac{1}{G}a^{\dagger 2}\frac{1}{G}a^2\frac{1}{G}a^{\dagger 2}\frac{1}{G}a^2\frac{1}{G}a^2 \\
&\quad - (\bar{e}e)^3\mathrm{tr}\frac{1}{G}a^{\dagger 2}\frac{1}{G}a^2\frac{1}{G}a^{\dagger 2}\frac{1}{G}a^{\dagger 2}\frac{1}{G}a^2\frac{1}{G}a^2 + \cdots\Big\} \\
&= \exp\big(\bar{e}e\,\partial_{\bar{g}g}\big)\,\mathrm{tr}\Big\{\ln G - \bar{e}e\ \nwarrow\swarrow\ -(\bar{e}e)^2\Big(\frac{1}{2}\nwarrow\swarrow\nwarrow\swarrow + \nwarrow\nwarrow\swarrow\swarrow\Big) - (\bar{e}e)^3 \\
&\quad \times\Big(\frac{1}{3}\nwarrow\swarrow\nwarrow\swarrow\nwarrow\swarrow + \nwarrow\nwarrow\nwarrow\swarrow\swarrow\swarrow + \nwarrow\nwarrow\swarrow\nwarrow\swarrow\swarrow + \nwarrow\swarrow\nwarrow\nwarrow\swarrow\swarrow\Big) + \cdots\Big\}.
\end{aligned}$$

Using

$$a^{\dagger 2}|n\rangle = \sqrt{(n+1)(n+2)}\,|n+2\rangle, \qquad a^2|n\rangle = \sqrt{n(n-1)}\,|n-2\rangle \qquad \text{and}$$
$$\frac{1}{G}|n\rangle = \frac{1}{\bar{g}g + n(n-1)}|n\rangle,$$

the above traces convert into infinite sums of rational functions of n. After repeated partial fraction decomposition these sums can be evaluated to

$$\begin{aligned}
\frac{\mathcal{K}}{16\pi\theta} &= \ln\bar{g}g + \ln\cos W + \bar{e}e\pi^2\frac{\bar{g}g}{4\bar{g}g+3}\frac{\tan W}{W} \\
&\quad + (\bar{e}e)^2\pi^4\left\{\frac{48(\bar{g}g)^4+200(\bar{g}g)^3-33(\bar{g}g)^2+27\bar{g}g}{4(4\bar{g}g+3)^3(4\bar{g}g+15)}\frac{\tan W}{W^3} - \frac{(\bar{g}g)^2}{2\,(4\bar{g}g+3)^2}\frac{\sec^2 W}{W^2}\right\} \\
&\quad + (\bar{e}e)^3\pi^6\left\{\frac{1}{8(4\bar{g}g+3)^5(4\bar{g}g+15)^2(4\bar{g}g+35)}\big(10240(\bar{g}g)^8+171520(\bar{g}g)^7\right. \\
&\quad + 878336(\bar{g}g)^6+1161920(\bar{g}g)^5-354936(\bar{g}g)^4+549414(\bar{g}g)^3-13770(\bar{g}g)^2 \\
&\quad + 6075\bar{g}g\big)\frac{\tan W}{W^5} - \frac{48(\bar{g}g)^5+200(\bar{g}g)^4-33(\bar{g}g)^3+27(\bar{g}g)^2}{4(4\bar{g}g+3)^4(4\bar{g}g+15)}\frac{\sec^2 W}{W^4} \\
&\quad \left. + \frac{(\bar{g}g)^3}{3(4\bar{g}g+3)^3}\frac{\tan W\sec^2 W)}{W^3}\right\} + \ O\big((\bar{e}e)^4\big)
\end{aligned} \tag{5}$$

with the definition

$$W = \frac{\pi}{2}\sqrt{1-4\bar{g}g}.$$

The leading term was determined via

$$\partial_{\bar{g}g}\mathrm{tr}\ln G = \mathrm{tr}\,G^{-1} = \sum_{n\geq 0}\frac{1}{\bar{g}g+n(n-1)} = \frac{1}{\bar{g}g} + \frac{\pi^2}{2}\frac{\tan W}{W},$$

and a constant as well as the $\ln\bar{g}g$ term in $\mathcal{K}$ may be omitted.

To each order in $\bar{e}e$, the expression (5) is exact in $\bar{g}g$ and, hence, valid for arbitrary values of θ. For strong noncommutativity, when $g \to 0$ but $|\frac{e}{g}| \ll 1$ fixed, potential poles due to $\tan W \sim \sec W \sim (\pi\bar{g}g)^{-1}$ are always compensated by suitable powers of $\bar{g}g$. To check our computation, let us take the opposite, commutative limit,

$$\theta \;\to\; 0 \quad \text{with } \gamma,\ \epsilon \text{ fixed} \quad \Rightarrow \quad g,\ e \;\to\; \infty \quad \text{with } \frac{e}{g} = \frac{\epsilon}{\gamma} \quad \text{fixed.}$$

For $g \to \infty$, the expansion (5) takes the form

$$\begin{aligned}
\frac{\mathcal{K}}{16\pi\theta} &= \ln\bar{g}g - \ln 2 + \pi\sqrt{\bar{g}g}\left\{\left(1-\frac{1}{8\bar{g}g}+\cdots\right) + \left|\frac{e}{g}\right|^2\left(\frac{1}{4}-\frac{5}{32\bar{g}g}+\cdots\right)\right. \\
&\quad \left. + \left|\frac{e}{g}\right|^4\left(-\frac{3}{64}+\frac{35}{512\bar{g}g}+\cdots\right) + \left|\frac{e}{g}\right|^6\left(\frac{5}{256}-\frac{105}{2048\bar{g}g}+\cdots\right)+\cdots\right\} \\
&\simeq \frac{\pi|\gamma|}{2\theta}\left\{1+\frac{1}{4}\left|\frac{\epsilon}{\gamma}\right|^2 - \frac{3}{64}\left|\frac{\epsilon}{\gamma}\right|^4 + \frac{5}{256}\left|\frac{\epsilon}{\gamma}\right|^6 - \frac{175}{16384}\left|\frac{\epsilon}{\gamma}\right|^8 + \cdots\right\} + O(\theta),
\end{aligned}$$

after dropping the irrelevant logarithmic and constant terms through '$\simeq$'. Indeed, the leading contributions reproduce the commutative Kähler potential (1) in the $|\epsilon| \ll |\gamma|$ 'ring' regime.

6 Deformed lumps

More interesting however is the $|\gamma| \ll |\epsilon|$ 'two-lump' domain, which at $\theta = 0$ featured a weak logarithmic singularity for $\gamma \to 0$, where the two lumps are localized infinitely sharply. To analyze this situation, we need to expand $\mathcal{K}$ around $g = 0$, in powers and perhaps also logarithms of $\bar{g}g$, generalizing (2) to finite values of θ. To this end, we are interested in the eigenvalues of

$$T^\dagger T\big|_{g=0} = (a^{\dagger 2} + \bar{e})(a^2 + e) = N(N-1) + ea^{\dagger 2} + \bar{e}a^2 + \bar{e}e =: F.$$

Representing the noncommutative coordinates on $L_2(\mathbb{R}) \ni f : \mathbb{R} \to \mathbb{R}$,

$$a = \frac{1}{\sqrt{2}}(x + \partial_x) = \frac{1}{\sqrt{2}}\mathrm{e}^{-x^2/2}\partial_x \mathrm{e}^{x^2/2} \qquad \text{and}$$
$$a^\dagger = \frac{1}{\sqrt{2}}(x - \partial_x) = \frac{1}{\sqrt{2}}\mathrm{e}^{x^2/2}\partial_x \mathrm{e}^{-x^2/2},$$

one gets

$$\begin{aligned}\big[Ff\big](x) &\equiv \left[\frac{1}{4}\partial_x^4 - x\partial_x^3 + \left(x^2 - \frac{1}{2} + \frac{e+\bar{e}}{2}\right)\partial_x^2 - 2ex\partial_x + (2ex^2 - e + \bar{e}e)\right] f(x) \\ &= \lambda(e) f(x),\end{aligned}$$

which via Fourier transformation and change of variables is equivalent to

$$\left[-\partial_z(1-z^2)\partial_z + \frac{1}{1-z^2} + \bar{e}e\big(1-z^2\big)\right] f(z) = \lambda(e)\, f(z) \qquad \text{with} \quad z \in [-1,1]. \tag{6}$$

This equation matches with the one defining spheroidal (scalar) wave functions [13, 14],

$$\left[-\partial_z\big(1-z^2\big)\partial_z + \frac{m^2}{1-z^2} - \gamma^2\big(1-z^2\big)\right] f_{mn}(z) = \lambda_{mn}(\gamma) f_{mn}(z), \qquad \text{with} \quad m \in \mathbb{Z}$$

and $n = 0, 1, 2, \ldots$ (in our convention) counting the discrete spheroidal eigenvalues $\lambda_{mn}(\gamma)$. Clearly, we have $m = 1$ and $\gamma^2 = -\bar{e}e$ (the oblate case), hence $\lambda_n(e) = \lambda_{1n}(\mathrm{i}|e|)$. For small values of $\bar{e}e$ one finds the expansion [13]

$$\begin{aligned}\lambda_n(e) = n(n-1)\bigg\{1 &+ \frac{2}{(2n-3)(2n+1)}\bar{e}e \\ &+ \frac{2(4n^4 - 8n^3 - 35n^2 + 39n + 63)}{(2n-5)(2n-3)^3(2n+1)^3(2n+3)}(\bar{e}e)^2 + \cdots\bigg\} =: n(n-1)\tilde{\lambda}_n(e),\end{aligned}$$

where the two zero modes of F, namely $\lambda_0 = \lambda_1 = 0$, are explicit. Therefore, we may write

$$\begin{aligned}\frac{\mathcal{K}}{16\pi\theta} &= \sum_{n=0}^{\infty} \ln\big[\lambda_n(e) + \bar{g}g\big] = 2\ln\bar{g}g + \sum_{n=2}^{\infty}\ln\big[\lambda_n(e) + \bar{g}g\big] \\ &= 2\ln\bar{g}g + \sum_{n=2}^{\infty}\ln\big[n(n-1) + \bar{g}g\big] + \sum_{n=2}^{\infty}\ln\left[1 + \frac{n(n-1)}{n(n-1)+\bar{g}g}(\tilde{\lambda}_n(e) - 1)\right].\end{aligned} \tag{7}$$

The role of g as a regulator is obvious; the first term carries the F zero modes. After expanding the logarithm under the last sum, one can perform the sums and nicely reproduces all terms in (5).

We have not found an asymptotic expansion of the spheroidal eigenvalues around $|e| = \infty$, and so it is difficult to analyze the 'two-lump' domain in general. For a first impression, let us expand (5) in powers of $\bar{g}g$ and collect the e dependence of each term:

$$\begin{aligned}\frac{\mathcal{K}}{16\pi\theta} = 2\ln\bar{g}g &+ \left\{\ln\pi + \frac{2}{3}\bar{e}e - \frac{4}{45}(\bar{e}e)^2 + \frac{64}{2835}(\bar{e}e)^3 - \frac{32}{4725}(\bar{e}e)^4 + \cdots\right\} \\ &+ \bar{g}g\left\{1 - \frac{2}{9}\bar{e}e + \frac{56}{675}(\bar{e}e)^2 - \frac{656}{19845}(\bar{e}e)^3 + \frac{1216}{91125}(\bar{e}e)^4 + \cdots\right\} \\ &+ (\bar{g}g)^2\left\{\left(\frac{3}{2} - \frac{\pi^2}{6}\right) + \left(\frac{62}{27} - \frac{2\pi^2}{9}\right)\bar{e}e + \left(\frac{3742}{3375} - \frac{16\pi^2}{135}\right)(\bar{e}e)^2\right. \\ &+ \left.\left(\frac{3822944}{10418625} - \frac{32\pi^2}{945}\right)(\bar{e}e)^3 + \cdots\right\} + (\bar{g}g)^3\left\{\left(\frac{10}{3} - \frac{\pi^2}{3}\right) + \left(\frac{292}{81} - \frac{10\pi^2}{27}\right)\bar{e}e\right. \\ &+ \left.\left(\frac{254846}{151875} - \frac{112\pi^2}{675}\right)(\bar{e}e)^2 + \left(\frac{1235859892}{3281866875} - \frac{12176\pi^2}{297675}\right)(\bar{e}e)^3 + \cdots\right\} + O\big((\bar{g}g)^4\big).\end{aligned} \tag{8}$$

The $g \to 0$ singularity due to the two zero modes is visible in the first term, but it is inconsequential in a Kähler potential. Besides this, the expression is devoid of the commutative logarithmic small-g singularity[6]! From the pattern in (5) it is clear that this feature persists to all orders in the expansion. Apparently, the Moyal deformation has smoothed out the Kähler potential near $\gamma = \epsilon = 0$, where the two lumps collide.

To attain the analog of (5) for the 'two-lump' domain, one would have to sum the series in each pair of curly brackets in (8). This can actually be achieved for the first of these series (the g-independent contribution), as we shall demonstrate shortly. So let us concentrate on the dangerous $g = 0$ line from now on. Because $\tilde{\lambda}_n(0) = 1$ and

$$\frac{1}{16\pi\theta}\mathcal{K}\big|_{e=0} = 2\ln\bar{g}g + \ln\pi + O(\bar{g}g),$$

at $g = 0$ we may subtract this from the Kähler potential, and (7) simplifies to

$$\ln\frac{\det(a^{\dagger 2} + \bar{e})(a^2 + e)}{\det a^{\dagger 2}a^2} = \lim_{g\to 0}\frac{\mathcal{K} - \mathcal{K}|_{e=0}}{16\pi\theta} = \sum_{n=2}^{\infty}\ln\tilde{\lambda}_n(e) = -\ln\tilde{\lambda}_1(e).$$

The last equality is an observation we have checked to $O((\bar{e}e)^8)$ but do not know its origin[7]. It allows us to easily push the $\bar{e}e$ expansion to higher orders,

$$\begin{aligned}\lim_{g\to 0}\frac{\mathcal{K} - \mathcal{K}|_{e=0}}{16\pi\theta} &= \frac{2}{3}\bar{e}e - \frac{4}{45}(\bar{e}e)^2 + \frac{64}{2835}(\bar{e}e)^3 - \frac{32}{4725}(\bar{e}e)^4 + \frac{1024}{467775}(\bar{e}e)^5 \\ &- \frac{1415168}{1915538625}(\bar{e}e)^6 + \frac{32768}{127702575}(\bar{e}e)^7 - \frac{14815232}{162820783125}(\bar{e}e)^8 + \cdots .\end{aligned} \tag{9}$$

To identify the function behind this power series, we exploit the Gel'fand–Yaglom theorem [15, 16, 17]. Let us go back to the eigenvalue problem (6) and stretch the interval $[-1, 1]$ to $\mathbb{R}$ by the change of variables $z = \tanh y$, so that it becomes

$$\big[F(e)f\big](y) \equiv \left[-\cosh^2 y\,\partial_y^2 + \cosh^2 y + \frac{\bar{e}e}{\cosh^2 y}\right] f(y) = \lambda(e)f(y).$$

[6] The linear piece, $\mathcal{K}_{\text{lin}} = 16\pi\theta\big(\bar{g}g + \frac{2}{3}\bar{e}e\big)$, has already been found in [8]. Furthermore, $\sum_{n=2}^{\infty}\ln n(n-1) \overset{\text{reg}}{=} \ln\pi$.

[7] It amounts to $\prod_{n=1}^{\infty}\tilde{\lambda}_n(e) = \prod_{n=1}^{\infty}\frac{\lambda_n(e)}{\lambda_n(0)} = 1$, i.e. the formal product of all spheroidal eigenvalues is e independent.

The Gel'fand–Yaglom theorem states that

$$\frac{\det(a^{\dagger 2}+\bar{e})(a^2+e)}{\det a^{\dagger 2}a^2} \equiv \frac{\det F(e)}{\det F(0)} = \lim_{L\to\infty}\frac{\Phi(L)}{\Psi(L)}, \tag{10}$$

where the functions $\Phi(y)$ and $\Psi(y)$ satisfy the following set of equations and boundary conditions,

$$\begin{aligned} \big[F(e)\Phi\big](y) &= 0, \qquad \Phi(-L)=0, \qquad \Phi'(-L)=1, \\ \big[F(0)\Psi\big](y) &= 0, \qquad \Psi(-L)=0, \qquad \Psi'(-L)=1. \end{aligned} \tag{11}$$

respectively. The solution can be given analytically:

$$\begin{aligned} \Phi(y) &= \frac{1}{\sqrt{\bar{e}e}}\sinh\big[\sqrt{\bar{e}e}(\tanh y+\tanh L)\big]\cosh y\cosh L \\ &= \frac{1}{\sqrt{\bar{e}e}}\sinh\left[\sqrt{\bar{e}e}\frac{\sinh(y+L)}{\cosh y\cosh L}\right]\cosh y\cosh L, \\ \Psi(y) &= \sinh(y+L), \end{aligned}$$

which leads to

$$\frac{\Phi(L)}{\Psi(L)} = \frac{\sinh(2\sqrt{\bar{e}e}\tanh L)}{2\sqrt{\bar{e}e}\tanh L} \quad \stackrel{L\to\infty}{\longrightarrow} \quad \frac{\sinh(2\sqrt{\bar{e}e})}{2\sqrt{\bar{e}e}} = \frac{\det F(e)}{\det F(0)}.$$

Hence, we finally arrive at

$$\lim_{g\to 0}\frac{\mathcal{K}-\mathcal{K}|_{e=0}}{16\pi\theta} = \ln\frac{\sinh(2\sqrt{\bar{e}e})}{2\sqrt{\bar{e}e}} = \ln\left\{1+\sum_{\ell=1}^{\infty}\frac{(4\,\bar{e}e)^{\ell}}{(2\ell+1)!}\right\},$$

whose expansion indeed reproduces all terms in (9). This simple expression represents the full noncommutative Kähler potential at $g=0$ and provides an analytic formula for $\tilde{\lambda}_1(e)$. Moreover, it has the correct commutative limit $e\to\infty$,

$$\lim_{g\to 0}\frac{\mathcal{K}-\mathcal{K}|_{e=0}}{16\pi\theta} = 2|e|-\ln|e|-2\ln 2+O(\mathrm{e}^{-4|e|}) \quad \stackrel{2e=\epsilon/\theta}{\Longrightarrow} \quad \lim_{\theta\to 0}\mathcal{K}(\gamma=0) = 16\pi|\epsilon|,$$

again up to irrelevant constant and $\ln\bar{\epsilon}\epsilon$ terms. It is tempting to add the constant shift by $\bar{g}g$ in (10) and apply the Gel'fand–Yaglom technique for all values of g. However, we have not been able to solve (11) with this shift.

For completeness, we display the metric coefficients,

$$\begin{aligned} g_{\bar{\gamma}\gamma} &= \frac{4\pi}{\theta}\partial_{\bar{g}g}\left(\bar{g}g\partial_{\bar{g}g}\frac{\mathcal{K}}{16\pi\theta}\right) = \frac{4\pi}{\theta}\operatorname{tr}\big\{(T^{\dagger}T)^{-1}-\bar{g}g(T^{\dagger}T)^{-2}\big\}, \\ g_{\bar{\epsilon}\epsilon} &= \frac{4\pi}{\theta}\partial_{\bar{e}e}\left(\bar{e}e\partial_{\bar{e}e}\frac{\mathcal{K}}{16\pi\theta}\right) = \frac{4\pi}{\theta}\operatorname{tr}\big\{(T^{\dagger}T)^{-1}-(T^{\dagger}T)^{-1}(a^{\dagger 2}+\bar{e})(T^{\dagger}T)^{-1}(a^2+e)\big\}, \\ g_{\bar{\gamma}\epsilon} &= \frac{4\pi}{\theta}g\bar{e}\partial_{\bar{g}g}\partial_{\bar{e}e}\frac{\mathcal{K}}{16\pi\theta} = -\frac{4\pi}{\theta}g\operatorname{tr}\big\{(T^{\dagger}T)^{-2}(a^{\dagger 2}+\bar{e})\big\}. \end{aligned}$$

All these traces converge and should be finite in the entire $\gamma\epsilon$ plane. The coefficient $g_{\bar{\epsilon}\epsilon}$ may be read off (5) by replacing $(\bar{e}e)^k$ with $k^2(\bar{e}e)^{k-1}$ in the series. For the other two, one has to work out the derivatives.

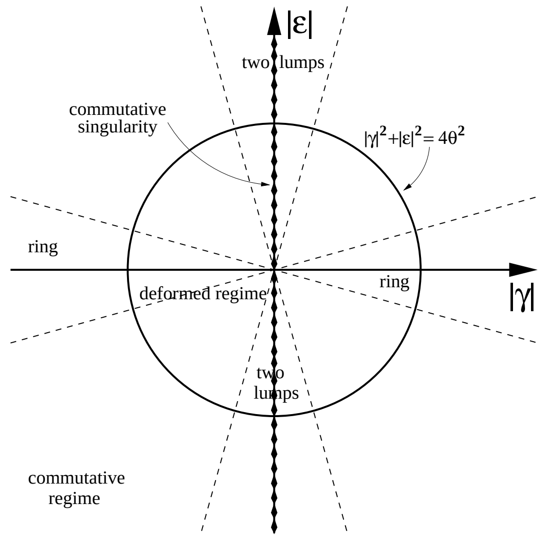

Figure 1. Modulus-of-moduli space.

7 Conclusions

We have investigated the charge-two moduli-space metric in the noncommutative $\mathbb{C}P^1$ sigma model in 2+1 dimensions. After decoupling the center of mass and a convenient dialling of frozen moduli, we find that the Kähler potential depends only on the combinations $\bar{g}g$ and $\bar{e}e$ of the dynamical complex-valued dimensionless moduli g and e. The noncommutativity strength $\sqrt{\theta}$ sets the single scale of the system. In the limit $|e| \ll |g| \to \infty$, where the solitonic energy density has a ring-like profile, our power series in $\bar{e}e$ matches with the known commutative Kähler potential, which depends only on the ratio $\frac{|e|}{|g|}$. In the complementary regime $|g| \ll |e|$, where the configuration splits into two lumps, we observe that the logarithmic singularity of the commutative Kähler potential is smoothed out by the deformation, which pushes it to the $\theta = 0$ boundary of the moduli space. The $(|\gamma|, |\epsilon|) = 2\theta(|g|, |e|)$ plane is depicted in Fig. 1.

We have expanded the Kähler potential to order $(\bar{e}e)^4$ and to any order in $\bar{g}g$, but an analytic expression remains a challenge, which amounts to computing the spectrum of the spheroidal wave equation for $m = 1$ but any e. However, at $g = 0$ we only needed the lowest (regularized) eigenvalue, and the $\bar{e}e$ series could be summed to an analytic function via the Gel'fand–Yaglom trick.

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