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A Formula for the Logarithm of the KZ Associator*

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Abstract. We prove that the logarithm of a group-like element in a free algebra coincides with its image by a certain linear map. We use this result and the formula of Le and Murakami for the Knizhnik–Zamolodchikov (KZ) associator Φ to derive a formula for $\log(\Phi)$ in terms of MZV's (multiple zeta values).

Key words: free Lie algebras; Campbell–Baker–Hausdorff series, Knizhnik–Zamolodchikov associator

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To the memory of Vadim Kuznetsov.

1 Logarithms of group-like elements

Let F_n be the free associative algebra generated by free variables $x_1, \dots, x_n$, let $\mathfrak{f}_n \subset F_n$ be the free Lie algebra with the same generators, and let $\widehat{\mathfrak{f}}_n$, $\widehat{F}_n$ be their degree completions (where $x_1, \dots, x_n$ have degree 1). A group-like element of $\widehat{F}_n$ is an element of the form $X = 1+$ (terms of degree > 0), such that $\Delta(X) = X \otimes X$, where Δ is the completion of the coproduct $F_n \to F_n^{\otimes 2}$, for which $x_1, \dots, x_n$ are primitive. It is well-known that the exponential defines a bijection exp: $\widehat{\mathfrak{f}}_n \to \{$group-like elements of $\widehat{F}_n\}$ (also denoted $x \mapsto e^x$). We denote by log the inverse bijection.

We denote by $\mathrm{CBH}_n(x_1, \dots, x_n)$ the multilinear part (in $x_1, \dots, x_n$) of $\log(e^{x_1} \cdots e^{x_n})$. Define a linear map

$$\mathrm{cbh}_n : \ F_n \to \mathfrak{f}_n$$

by $\mathrm{cbh}_n(1) = 0$ and $\mathrm{cbh}_n(x_{i_1} \cdots x_{i_k}) := \mathrm{CBH}_k(x_{i_1}, \dots, x_{i_k})$. This map extends to a linear map $\widehat{\mathrm{cbh}}_n$: $\widehat{F}_n \to \widehat{\mathfrak{f}}_n$.

Proposition 1. *If* $X \in \widehat{F}_n$ *is group-like, then* $\log(X) = \widehat{\mathrm{cbh}}_n(X)$.

Proof. It is known that $F_n = U(\mathfrak{f}_n)$, so that the symmetrization is an isomorphism sym: $S(\mathfrak{f}_n) \to F_n$. Denote by p_n: $F_n \to \mathfrak{f}_n$ the composition of sym^{-1} with the projection $S(\mathfrak{f}_n) = \oplus_{k \geq 0} S^k(\mathfrak{f}_n)$ onto $S^1(\mathfrak{f}_n) = \mathfrak{f}_n$. We first prove:

Lemma 1. $p_n = \mathrm{cbh}_n$.

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Proof. If $\mathfrak{g}$ is a Lie algebra, let $p_{\mathfrak{g}}\colon U(\mathfrak{g}) \to \mathfrak{g}$ be the composition of the inverse of the symmetrization $S(\mathfrak{g}) \to U(\mathfrak{g})$ with the projection onto the first component of $S(\mathfrak{g})$. If $\phi\colon \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra morphism, then we have a commutative diagram

$$\begin{array}{ccc} U(\mathfrak{g}) & \stackrel{p_{\mathfrak{g}}}{\to} & \mathfrak{g} \\ {\scriptstyle U(\phi)}\downarrow & & \downarrow\phi \\ U(\mathfrak{h}) & \stackrel{p_{\mathfrak{h}}}{\to} & \mathfrak{h} \end{array}$$

It follows from Lemma 3.10 of [4] that if $k \geq 0$ and F_k is the free algebra with generators $y_1, \ldots, y_k$, then $p_k(y_1, \ldots, y_k) = \mathrm{CBH}_k(y_1, \ldots, y_k)$. If now $\boldsymbol{i} = (i_1, \ldots, i_k)$ is a sequence of elements of $\{1, \ldots, n\}$, we have a unique morphism $\phi_{\boldsymbol{i}}\colon \mathfrak{f}_k \to \mathfrak{f}_n$, such that $y_1 \mapsto x_{i_1}, \ldots, y_k \mapsto x_{i_k}$.

Then

$$\begin{aligned} p_n(x_{i_1} \cdots x_{i_k}) &= p_{\mathfrak{f}_n} \circ U(\phi_{\boldsymbol{i}})(y_1 \cdots y_k) = \phi_{\boldsymbol{i}} \circ p_{\mathfrak{f}_k}(y_1 \cdots y_k) \\ &= \phi_{\boldsymbol{i}}(\mathrm{CBH}_k(y_1, \ldots, y_k)) = \mathrm{CBH}_k(x_{i_1}, \ldots, x_{i_k}) = \mathrm{cbh}_n(x_{i_1} \cdots x_{i_k}), \end{aligned}$$

which proves the lemma. ■

End of proof of Proposition 1. We denote by $\widehat{p}_n : \widehat{F}_n \to \widehat{\mathfrak{f}}_n$ the map similarly derived from the isomorphism $\widehat{F}_n \simeq \widehat{\oplus}_{k\geq 0} S^k(\widehat{\mathfrak{f}}_n)$ (where $\widehat{\oplus}$ is the direct product). Then $p_n = \mathrm{cbh}_n$ implies $\widehat{p}_n = \widehat{\mathrm{cbh}}_n$.

If now $X \in \widehat{F}_n$ is group-like, let $\ell := \log(X)$. We have $X = 1+\ell+\ell^2/2!+\cdots$; here $\ell^k \in S^k(\widehat{\mathfrak{f}}_n)$, so $\widehat{p}_n(X) = \ell$. Hence $\widehat{\mathrm{cbh}}_n(X) = \ell = \log(X)$. ■

2 Corollaries

The KZ associator is defined as follows. Let A_0, A_1 be noncommutative variables. Let F_2 be the free associative algebra generated by A_0 and A_1, let $\mathfrak{f}_2 \subset F_2$ be its (free) Lie subalgebra generated by A_0 and A_1. Let $\widehat{F}_2$ and $\widehat{\mathfrak{f}}_2$ be the degree completions of F_2 and $\mathfrak{f}_2$ (A_0 and A_1 have degree 1).

The KZ associator Φ is defined [1] as the renormalized holonomy from 0 to 1 of the differential equation

$$G'(z) = \left(\frac{A_0}{z} + \frac{A_1}{z-1}\right) G(z), \tag{1}$$

i.e., $\Phi = G_1 G_0^{-1}$, where $G_0, G_1 \in \widehat{F}_2 \widehat{\otimes} \mathcal{O}_{]0,1[}$ are the solutions of (1) with $G_0(z) \sim z^{A_0}$ as $z \to 0^+$ and $G_1(z) \sim (1-z)^{A_1}$ as $z \to 1^-$; here $\mathcal{O}_{]0,1[}$ is the ring of analytic functions on $]0,1[$, and $\widehat{F}_2 \widehat{\otimes} V$ is the completion of $F_2 \otimes V$ w.r.t. the topology defined by the $F_2^{\geq n} \otimes V$ (here $F_2^{\geq n}$ is the part of F_2 of degree $\geq n$).

We recall Le and Murakami's formula for Φ [3]. We say that a sequence $(a_1, \ldots, a_n) \in \{0,1\}^n$ is admissible if $a_1 = 1$ and $a_n = 0$. If $(a_1, \ldots, a_n)$ is admissible, we set

$$\omega_{a_1,\ldots,a_n} = \int_0^1 \omega_{a_1} \circ \cdots \circ \omega_{a_n},$$

where $\omega_0(t) = dt/t$, $\omega_1(t) = dt/(t-1)$ and $\int_a^b \alpha_1 \circ \cdots \circ \alpha_n = \int_{a\leq t_1 \leq \cdots \leq t_n \leq b} \alpha_1(t_1) \wedge \cdots \wedge \alpha_n(t_n)$. Up to sign, the $\omega_{a_1,\ldots,a_n}$ are MZV's (multiple zeta values).

If $(i_1, \ldots, i_n)$ is an arbitrary sequence in $\{0,1\}^n$, and $(a_1, \ldots, a_n)$ is an admissible sequence, define integers $C^{a_1,\ldots,a_n}_{i_1,\ldots,i_n}$ by the relation

$$\sum_{(i_1,\ldots,i_n)\in\{0,1\}^n} C^{a_1,\ldots,a_n}_{i_1,\ldots,i_n} A_{i_n}\cdots A_{i_1} = \sum_{\substack{S\subset\{\alpha|a_\alpha=0\},\\ T\subset\{\beta|a_\beta=1\}}} (-1)^{\mathrm{card}(S)+\mathrm{card}(T)} A_1^{\mathrm{card}(T)} A(a_1,\ldots,a_n)^{S,T} A_0^{\mathrm{card}(S)},$$

where for any $S \subset \{\alpha|a_\alpha = 0\}$, $T \subset \{\beta|a_\beta = 1\}$, $A(a_1, \ldots, a_n)^{S,T} := \prod_{\alpha\in[1,n]\setminus(S\cup T)} A_{a_\alpha}$ (the product is taken in decreasing order of the α's).

Theorem 1 (**[3]**).

$$\Phi = 1 + \sum_{n\geq 1} \sum_{\substack{(a_1,\ldots,a_n)\text{ admissible}\\ (i_1,\ldots,i_n)\in\{0,1\}^n}} \omega_{a_1,\ldots,a_n} C^{a_1,\ldots,a_n}_{i_1,\ldots,i_n} A_{i_n}\cdots A_{i_1}.$$

Since $\Phi \in \widehat{F}_2$ is a group-like element, Proposition 1 implies that $\log(\Phi) = \widehat{\mathrm{cbh}}_2(\Phi)$, therefore:

Corollary 1.

$$\log(\Phi) = \sum_{n\geq 1} \sum_{\substack{(a_1,\ldots,a_n)\text{ admissible}\\ (i_1,\ldots,i_n)\in\{0,1\}^n}} \omega_{a_1,\ldots,a_n} C^{a_1,\ldots,a_n}_{i_1,\ldots,i_n} \mathrm{CBH}_n(A_{i_n},\ldots,A_{i_1}).$$

Using the explicit formula of [2], one computes similarly the logarithm of the analogue Ψ of the KZ associator of the equation $G'(z) = \big(A/z + \sum_{\zeta|\zeta^n=1} b_\zeta/(z-\zeta)\big)G(z)$.

Proposition 1 also implies:

Lemma 2. *Let $\mathfrak{g}$ be a nilpotent Lie algebra, G be the associated Lie group, let $a < b \in \mathbb{R}$. Fix $h(z) \in C^0([a,b],\mathfrak{g})$ and let H be the holonomy from a to b of the differential equation $H'(z) = h(z)H(z)$, where $H(z) \in C^1([a,b],G)$. Then*

$$\log(H) = \sum_{n\geq 1} \int_{a\leq z_1\leq\cdots\leq z_n\leq b} \mathrm{CBH}_n(h(z_n),\ldots,h(z_1))dz_1\cdots dz_n.$$

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We first established the formula for $\log(\Phi)$ in Corollary 1 by analytic computations (using a direct proof of Lemma 2). It was the referee who remarked its formal similarity with the formula of Le and Murakami (Theorem 1); this remark can be expressed as the equality $\log(\Phi) = \widehat{\mathrm{cbh}}_2(\Phi)$. This led us to try and understand whether this formula followed from the group-likeness of Φ, which is indeed the case (Proposition 1). C. Reutenauer then pointed out that a part of our argument is a result in his book.

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