Invariant Varieties of Periodic Points for the Discrete Euler Top*

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Abstract. The behaviour of periodic points of discrete Euler top is studied. We derive invariant varieties of periodic points explicitly. When the top is axially symmetric they are specified by some particular values of the angular velocity along the axis of symmetry, different for each period.

Key words: invariant varieties of periodic points; discrete Euler top; integrable map

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To the memory of Professor Vadim B. Kuznetsov

1 Introduction

The Kowalevski workshop on mathematical methods of regular dynamics was organized by Professor Vadim Kuznetsov in April 2000 at the University of Leeds [1]. In his introductory talk about the Kowalevski top, Professor Kuznetzov [2] had shown his strong interest on the subject and motivated the authors to work on classical tops.

In our recent paper [3] we have studied the behaviour of periodic points of a rational map and found that they form a variety for each period specified by invariants of the map if the map is integrable, while they form a set of isolated points dependent on the invariants otherwise. It is apparent that an application of our theorem to the problems of a classical top is quite interesting and will be fruitful. We investigate the discrete Euler top, in this article, to see how the invariant varieties of periodic points look like in this particular example. In conclusion we will show that there is no periodic points of period 2 and 4 if the top is not axially symmetric. In the case of period 3 we derive explicitly an algebraic variety of dimension two as an invariant variety of periodic points. When the top is axially symmetric, the angular velocity of a periodic map along the symmetry axis is quantized to some special values determined by the period and the shape of the top. The other components of the angular velocity are free, thus form an invariant variety of periodic points separately for each period.

To start with let us briefly review our theorem of [3]. We consider a rational map on $\hat{\mathbf{C}}^d$, where $\hat{\mathbf{C}} = \{\mathbf{C}, \infty\}$,

$$\mathbf{x} = (x_1, x_2, \ldots, x_d) \quad \rightarrow \quad \mathbf{X} = (X_1, X_2, \ldots, X_d) =: \mathbf{X}^{(1)}. \tag{1}$$

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We are interested in the behaviour of the sequence: $\mathbf{x} \to \mathbf{X}^{(1)} \to \mathbf{X}^{(2)} \to \cdots$. In particular we pay attention to the behaviour of periodic points of rational maps. If the map is nonintegrable we shall find a set of isolated points with fractal structure as a higher dimensional counterpart of the Julia set. Our question in [3] was what object appears when the map is integrable.

We assume that the map has p (≥ 0) invariants $H_1(\mathbf{x}), H_2(\mathbf{x}), \ldots H_p(\mathbf{x})$. If $h_1, h_2, \ldots, h_p$ are the values of the invariants given by the initial values of the map, the orbit of the map is constrained on the $d-p$ dimensional variety determined by the conditions $H_i(\mathbf{x}) = h_i$, $i = 1, 2, \ldots, p$, which we denote by $V(h)$. Now let us consider the periodicity conditions $\mathbf{X}^{(n)} = \mathbf{x}$ of period n. We can eliminate p variables out of $\mathbf{x}$ and only $d-p$ periodicity conditions remain. If they are independent, we obtain a set of isolated periodic points on $V(h)$ in general.

It may happen, however, that some of the $d-p$ conditions impose some relations on $h_1, h_2, \ldots,$ h_p instead of fixing all of the $d-p$ variables. If m is the number of the conditions which determine the values of the variables, $d-p-m$ variables are left free and the periodic points of period n form a subvariety of dimension $d-p-m$ on $V(h)$, instead of a set of isolated points. In this case we say that the periodicity conditions are 'correlated'. If l is the number of the periodicity conditions which relate the invariants, $p-l$ invariants remain undetermined. This means that the periodic points of period n form a $d-l-m$ dimensional subvariety in $\hat{\mathbf{C}}^d$. We have proven in [3] the following lemma:

Lemma 1 ([3]). *A set of correlated periodicity conditions satisfying* $\min\{p,\ d-p\} \geq l+m$ *and a set of uncorrelated periodicity conditions of a different period do not exist in one map simultaneously.*

When $m = 0$ the periodicity conditions determine none of the variables but impose l relations among the invariants. In this particular case all points of $V(h)$ are the points of period n, while the variety $V(h)$ itself is constrained by the relations among the invariants. We call the periodicity conditions are 'fully correlated' in this case. The periodic points form a subvariety of dimension $d-l$ in the space $\hat{\mathbf{C}}^d$, which we call 'an invariant variety of periodic points'. Every point of this variety can be an initial point of the n period map, whose orbit stays on it. Since the condition $\min\{p,\ d-p\} \geq l+m$ is automatically satisfied, our theorem follows to the Lemma 1 immediately:

Theorem 1 ([3]). *If there is an invariant variety of periodic points of some period, there is no set of isolated periodic points of other period in the map.*

The Theorem 1 doesn't tell us directly whether the map is integrable or nonintegrable. There is, however, some evidence to believe that the periodic points of a nonintegrable map, if they exist, form a fractal set of isolated points. Therefore it is reasonable to adopt the following proposition as our working hypothesis:

If a map is nonintegrable, there is a set of uncorrelated periodicity conditions of some period.

The Julia set, which is the source of chaotic orbits, is a subset of the closure of all isolated periodic points. We emphasize that our hypothesis does not require that all of the periodicity conditions of a nonintegrable map are uncorrelated but requires only one at least. On the other hand our theorem shows that the existence of an invariant variety of periodic points excludes a set of isolated periodic points in the map and vice versa. This means that if a set of periodic points of some period forms an invariant variety there is no Julia set, thus suggesting the following statement:

Conjecture ([3]). *If there is an invariant variety of periodic points of some period, the map is integrable.*

Note that this does not exclude possibilities that some integrable maps do not have an invariant variety of periodic points. For example an integrable map with no invariant does not have an invariant variety. On the other hand there are d dimensional maps which have $d-1$ invariants but reduce to the logistic map after elimination of $d-1$ variables by using the invariants. Therefore the situation is quite different from the continuous time Hamiltonian flow, whose integrability is guaranteed by the Liouville theorem if there are sufficient number of invariants.

In order to support our conjecture we have studied in [3] various maps, such as the QRT maps, the Lotka–Volterra maps and the Toda maps, and found that all periodic points form invariant varieties of periodic points if the map is integrable and periodic points exist, while no such property has been found otherwise. We also studied the q-Painlevé maps which are integrable but not volume preserving in general. We found invariant varieties only when the parameters are restricted so that the maps have sufficient number of invariants.

We introduce the discrete Euler top in Section 2. To find an invariant variety of periodic points for the Euler top the general scheme developed in [3] is applied in Sections 3 and 4. In the final section we study explicitly the nature of the invariant surfaces of an axially symmetric Euler top.

2 Discrete Euler top

When the time is continuous, the equation of motion for the Euler top is given by

$$I_1\frac{d\omega_1}{dt}=(I_2-I_3)\omega_2\omega_3,\qquad I_2\frac{d\omega_2}{dt}=(I_3-I_1)\omega_3\omega_1,\qquad I_3\frac{d\omega_3}{dt}=(I_1-I_2)\omega_1\omega_2,$$

where $(\omega_1,\omega_2,\omega_3)$ are the angular velocity in the body fixed frame and (I_1,I_2,I_3) are the corresponding moments of inertia. The system has two invariants, $\frac{1}{2}(I_1\omega_1^2+I_2\omega_2^2+I_3\omega_3^2)$ and $I_1^2\omega_1^2+I_2^2\omega_2^2+I_3^2\omega_3^2$, corresponding to the total kinetic energy and the square of angular momentum, hence is integrable.

A discretization of the Euler equation, which preserves integrability, was first obtained by Bobenko et al. in [4], and then discussed by other authors [5, 6, 7, 8]. We adopt here an explicit version of the discretization proposed by Hirota et al. [5, 6]. After the discretization we write the angular velocity as (x_1,x_2,x_3) instead of $(\omega_1,\omega_2,\omega_3)$ and consider the map $(x_1,x_2,x_3)\to(X_1,X_2,X_3)$ defined by

$$\begin{aligned}
I_1(X_1-x_1)&=\frac{\delta}{2}(I_2-I_3)(X_2x_3+x_2X_3),\\
I_2(X_2-x_2)&=\frac{\delta}{2}(I_3-I_1)(X_3x_1+x_3X_1),\\
I_3(X_3-x_3)&=\frac{\delta}{2}(I_1-I_2)(X_1x_2+x_1X_2).
\end{aligned}\tag{2}$$

The continuous limit corresponds to $\delta\to 0$.

Solving the equation (2) for (X_1,X_2,X_3) we find

$$\begin{aligned}
X_1&=\frac{x_1(1-\alpha_2\alpha_3x_1^2+\alpha_3\alpha_1x_2^2+\alpha_1\alpha_2x_3^2)+2\alpha_1x_2x_3}{1-2\alpha_1\alpha_2\alpha_3x_1x_2x_3-\alpha_2\alpha_3x_1^2-\alpha_3\alpha_1x_2^2-\alpha_1\alpha_2x_3^2},\\
X_2&=\frac{x_2(1+\alpha_2\alpha_3x_1^2-\alpha_3\alpha_1x_2^2+\alpha_1\alpha_2x_3^2)+2\alpha_2x_3x_1}{1-2\alpha_1\alpha_2\alpha_3x_1x_2x_3-\alpha_2\alpha_3x_1^2-\alpha_3\alpha_1x_2^2-\alpha_1\alpha_2x_3^2},\\
X_3&=\frac{x_3(1+\alpha_2\alpha_3x_1^2+\alpha_3\alpha_1x_2^2-\alpha_1\alpha_2x_3^2)+2\alpha_3x_1x_2}{1-2\alpha_1\alpha_2\alpha_3x_1x_2x_3-\alpha_2\alpha_3x_1^2-\alpha_3\alpha_1x_2^2-\alpha_1\alpha_2x_3^2},
\end{aligned}\tag{3}$$

where we used the notations

$$\alpha_1=\delta\frac{I_2-I_3}{2I_1},\qquad \alpha_2=\delta\frac{I_3-I_1}{2I_2},\qquad \alpha_3=\delta\frac{I_1-I_2}{2I_3}.$$

We notice that, when the top is axially symmetric, the map (3) is nothing but a two dimensional linear transformation. For example if we assume $I_2 = I_3$, the map becomes

$$\begin{aligned} X_1 &= x_1, \\ X_2 &= x_2 \cos\Omega + x_3 \sin\Omega, \\ X_3 &= x_3 \cos\Omega - x_2 \sin\Omega, \end{aligned} \tag{4}$$

where

$$\cos\Omega = \frac{4I_2^2 - (I_2 - I_1)^2 x_1^2}{4I_2^2 + (I_2 - I_1)^2 x_1^2}.$$

Therefore we discuss the axially symmetric top separately from the generic case in Section 5.

The invariants of the map (3) are

$$H_1 = \frac{I_1 x_1^2 + I_2 x_2^2 + I_3 x_3^2}{1 - \alpha_2\alpha_3 x_1^2}, \qquad H_2 = \frac{I_1^2 x_1^2 + I_2^2 x_2^2 + I_3^2 x_3^2}{1 - \alpha_2\alpha_3 x_1^2}, \tag{5}$$

as it will be checked by a direct substitution of (3). They coincide with the invariants of the continuous case in the limit $\delta \to 0$. We fix the value of δ at 1 hereafter, since it is irrelevant in the following discussions.

If we denote by x one of the three variables (x_1, x_2, x_3), the elimination of other two variables from the map (2) yields

$$S(X, x; \mathbf{q}) = 0 \tag{6}$$

with

$$S(X, x; \mathbf{q}) := aX^2x^2 + bXx(X + x) + c(X - x)^2 + dXx + e(X + x) + f. \tag{7}$$

The parameters $\mathbf{q} = (\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}, \mathbf{f})$ are given by

$$\begin{aligned}
\begin{pmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \\ e_1 \\ f_1 \end{pmatrix} &= \begin{pmatrix} -4\alpha_2\alpha_3(A_0 - A_2)(A_0 + A_3) \\ 0 \\ (A_0 - A_2 + A_3)^2 \\ 4\Big(A_1^2 - A_2(A_0 + A_3) + A_3(A_0 - A_2)\Big) \\ 0 \\ (4/\alpha_2\alpha_3)A_2A_3 \end{pmatrix}, \\
\begin{pmatrix} a_2 \\ b_2 \\ c_2 \\ d_2 \\ e_2 \\ f_2 \end{pmatrix} &= \begin{pmatrix} -4\alpha_3\alpha_1 A_0(A_0 - A_2) \\ 0 \\ (A_0 - A_2 - A_3)^2 \\ 4\Big(A_2^2 - A_0(A_2 + A_3) - A_3(A_0 - A_2)\Big) \\ 0 \\ (4/\alpha_3\alpha_1)A_3A_1 \end{pmatrix}, \\
\begin{pmatrix} a_3 \\ b_3 \\ c_3 \\ d_3 \\ e_3 \\ f_3 \end{pmatrix} &= \begin{pmatrix} -4\alpha_1\alpha_2 A_0(A_0 + A_3) \\ 0 \\ (A_0 + A_2 + A_3)^2 \\ 4\Big(A_3^2 + A_0(A_2 + A_3) + A_2(A_0 + A_3)\Big) \\ 0 \\ (4/\alpha_1\alpha_2)A_1A_2 \end{pmatrix}
\end{aligned} \tag{8}$$

corresponding, respectively, to $x = x_1, x_2, x_3$. Here we introduced the notations

$$A_0 = 4I_1I_2I_3, \qquad A_1 = (I_2 - I_3)(I_1H_1 - H_2), \qquad A_2 = (I_3 - I_1)(I_2H_1 - H_2),$$
$$A_3 = (I_1 - I_2)(I_3H_1 - H_2), \qquad \big(A_1 + A_2 + A_3 = 0\big).$$

One might wonder that the map $x \to X$ defined by (6) does not determine an image of the map uniquely. Since the function $S(X, x; \mathbf{q})$ of (7) is symmetric under the exchange of the variables x and X, we see that the two solutions of (6) correspond to the forward and the backward maps, which we denote $X^{(1)}$ and $X^{(-1)}$. If we apply the map to $X^{(1)}$ we should get x and $X^{(2)}$. In this way we shall obtain a chain of images of the map into two directions:

$$\cdots \longleftarrow X^{(-2)} \longleftarrow X^{(-1)} \longleftarrow x \longrightarrow X^{(1)} \longrightarrow X^{(2)} \longrightarrow \cdots.$$

The number of branches of the map does not increase, but remains two, at every step of the map. Therefore a map of the form (6) is well defined in general.

3 Iteration of the map

We studied in [3] the map defined by (6) and called it a 'biquadratic map'. An iteration of the map yields the biquadratic map again but with new parameters $\mathbf{q}^{(2)}$:

$$\begin{aligned}
a^{(2)} &:= (ae - cb)^2 - (ad - 2ac - b^2)(be - cd + 2c^2),\\
b^{(2)} &:= (ae - cb)(2af - be + cd - 4c^2) - (ad - 2ac - b^2)(bf - ce),\\
c^{(2)} &:= (af - c^2)^2 - (ae - bc)(bf - ce),\\
d^{(2)} &:= 4(af - c^2)^2 - 2(ae - bc)(bf - ce) - (be - cd + 2c^2)^2\\
&\qquad - (ad - 2ac - b^2)(df - 2cf - e^2),\\
e^{(2)} &:= (fb - ce)(2af - be + cd - 4c^2) - (fd - 2fc - e^2)(ea - cb),\\
f^{(2)} &:= (fb - ce)^2 - (fd - 2fc - e^2)(be - cd + 2c^2).
\end{aligned} \tag{9}$$

If we repeat the map further we obtain a series of biquadratic maps whose parameters can be determined iteratively from the previous ones as follows:

$$\begin{aligned}
a^{(n+1)} &= \frac{1}{a^{(n-1)}}\big((a_\wedge c)_n^2 - (a_\wedge b)_n(b_\wedge c)_n\big),\\
b^{(n+1)} &= \frac{1}{a^{(n-1)}}\Bigg(\frac{b^{(n-1)}}{a^{(n-1)}}\big((a_\wedge b)_n(b_\wedge c)_n - (a_\wedge c)_n^2\big) + (a_\wedge c)_n\big((a_\wedge e)_n + 2(b_\wedge c)_n\big)\\
&\qquad - \frac{1}{2}\big((a_\wedge b)_n(b_\wedge e)_n - (a_\wedge b)_n(c_\wedge d)_n + (a_\wedge d)_n(b_\wedge c)_n\big)\Bigg),\\
c^{(n+1)} &= \frac{1}{2c^{(n-1)}}\Big((ce^{(n)} - bf^{(n)})(ae^{(n)} - bc^{(n)}) + (cb^{(n)} - ea^{(n)})(fb^{(n)} - ec^{(n)})\\
&\qquad + (af^{(n)} - cc^{(n)})^2 + (fa^{(n)} - cc^{(n)})^2\Big),\\
d^{(n+1)} &= \frac{1}{d^{(n-1)}}\Big(-f^{(n-1)}a^{(n+1)} - a^{(n-1)}f^{(n+1)} - 4b^{(n-1)}e^{(n+1)} - 4e^{(n-1)}b^{(n+1)} + (a_\wedge f)_n^2\\
&\qquad + (c_\wedge d)_n^2 - (a_\wedge b)_n(e_\wedge f)_n - (b_\wedge c)_n(c_\wedge e)_n + (a_\wedge d)_n(d_\wedge f)_n + 2(b_\wedge e)_n(a_\wedge f)_n\\
&\qquad - \big(3(c_\wedge e)_n - (b_\wedge f)_n - (d_\wedge e)_n\big)\big(3(b_\wedge c)_n - (a_\wedge e)_n - (b_\wedge d)_n\big)\\
&\qquad + 2\big((a_\wedge d)_n - (a_\wedge c)_n\big)\big((c_\wedge f)_n - (d_\wedge f)_n\big) + 2\big((b_\wedge c)_n + (a_\wedge e)_n\big)\big((b_\wedge f)_n + (c_\wedge e)_n\big)\Big),
\end{aligned} \tag{10}$$

$$e^{(n+1)} = \frac{1}{f^{(n-1)}}\left(\frac{e^{(n-1)}}{f^{(n-1)}}\big((f_\wedge e)_n(e_\wedge c)_n - (f_\wedge c)_n^2\big) + (f_\wedge c)_n\big((f_\wedge b)_n + 2(e_\wedge c)_n\big)\right.$$
$$\left. - \frac{1}{2}\big((f_\wedge e)_n(e_\wedge b)_n - (f_\wedge e)_n(c_\wedge d)_n + (f_\wedge d)_n(e_\wedge c)_n\big)\right),$$
$$f^{(n+1)} = \frac{1}{f^{(n-1)}}\big((f_\wedge c)_n^2 - (f_\wedge e)_n(e_\wedge c)_n\big),$$

where we used the notation $(g_\wedge g')_n = g{g'}^{(n)} - g'g^{(n)}$.

Despite the complicated expression of the relation (10), we observe a special dependence on the nth parameters $\mathbf{q}^{(n)}$. Besides $c^{(n+1)}$, the dependence of the $(n+1)$th parameters on the nth ones is always in the form $(g_\wedge g')_n = g{g'}^{(n)} - g'g^{(n)}$. They all vanish simultaneously when the parameters $\mathbf{q}^{(n)}$ are 'fully correlated', that is, if there exists a function $\gamma^{(n+1)}(\mathbf{q})$ such that

$$\mathbf{q}^{(n)} = \epsilon\mathbf{q} + \gamma^{(n+1)}(\mathbf{q})\hat{\mathbf{q}}^{(n)}, \tag{11}$$

where ϵ is an arbitrary constant. In fact we obtain, after some manipulation,

$$(a_\wedge b)_2 = (af - eb - 3c^2 + cd)(2a^2e - abd + b^3),$$
$$(a_\wedge c)_2 = (af - eb - 3c^2 + cd)(a^2f + ac^2 - acd + b^2c),$$
$$(b_\wedge c)_2 = (af - eb - 3c^2 + cd)(2ace - abf - bc^2),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$(e_\wedge f)_2 = (af - eb - 3c^2 + cd)(edf - e^3 - 2bf^2),$$

from which we find $\gamma^{(3)}(\mathbf{q})$:

$$\gamma^{(3)}(\mathbf{q}) = af - be - 3c^2 + cd. \tag{12}$$

The formula (10) enables us to find a series of $\gamma^{(n)}(\mathbf{q})$ iteratively, as follows:

$$\gamma^{(4)}(\mathbf{q}) = 2acf - adf + b^2f + ae^2 - 2c^3 + c^2d - 2bce, \tag{13}$$

$$\begin{aligned}\gamma^{(5)}(\mathbf{q}) = {} & a^3f^3 + \Big(-cf^2d + 2cfe^2 + fde^2 - 3ebf^2 - e^4 - c^2f^2\Big)a^2 \\ & + \Big(-13c^4f + 18c^3fd + de^3b + 2cf^2b^2 + 7dc^2e^2 - ce^2d^2 - 2ce^3b \\ & + 2c^2feb - 7fd^2c^2 - 14c^3e^2 + cd^3f + fb^2e^2 + f^2db^2 - ebd^2f\Big)a \\ & - cd^2b^2f - b^3e^3 - 4c^3deb + cdb^2e^2 + 13ec^4b - f^2b^4 + 7fb^2c^2d \\ & + c^4d^2 - 5c^5d + 5c^6 - 2fb^3ec - e^2c^2b^2 + eb^3df - 14fb^2c^3,\end{aligned} \tag{14}$$

and so on.

When (11) holds, the equation $S(Q, x; \mathbf{q}_{n+1}) = 0$ can be written as

$$c^{(n+1)}(Q - x)^2 + (\gamma^{(n+1)}(\mathbf{q}))^2K_{n+1}(Q, x) = 0, \qquad n = 2, 3, 4, \ldots. \tag{15}$$

Here

$$K_{n+1}(Q, x) = \hat{a}^{(n+1)}Q^2x^2 + \hat{b}^{(n+1)}(Q + x)Qx + \hat{d}^{(n+1)}Qx + \hat{e}^{(n+1)}(Q + x) + \hat{f}^{(n+1)},$$

and $\hat{a}^{(n+1)}$, for instance, is obtained from $a^{(n+1)}$ simply replacing $(g_\wedge g')_n$ by $(\hat{g}_\wedge\hat{g}')_n$. If Q is a point of period $n+1$, the first term of (15) vanishes. Hence the periodicity condition requires for the second term to vanish. This is certainly satisfied for arbitrary x if $\gamma^{(n+1)}(\mathbf{q}) = 0$, namely when the periodicity conditions for the parameters $\mathbf{q}^{(n)}$ are fully correlated. The other possible solutions obtained by solving $K_{n+1}(x, x) = 0$ do not correspond to the points of period $n+1$, but represent the fixed points or the points of periods which divide $n+1$.

4 Invariant varieties of periodic points for the discrete Euler top

We are ready to study the periodicity conditions for the discrete Euler top. Throughout this section we will not consider axially symmetric cases, which we discuss in the next section. The direct calculation of the periodicity conditions $X_j^{(n)} = x_j$ is not easy to carry out by a small computer. Therefore we use the method we developed in the previous section. Before starting, however, let us first search the fixed points of the map. If we remember that the variables (x_1, x_2, x_3) are the discrete analog of the angular velocity $(\omega_1, \omega_2, \omega_3)$, a fixed point of the map corresponds to the motion of the top which does not change the angular velocity in all directions of the body fixed frame.

Needless to say the fixed points are nothing to do with the invariants of the map. To find them we go back to the map (2) and see immediately that they are

$$\text{fixed points}: \quad \{\mathbf{x} \mid x_2 = x_3 = 0 \cup x_3 = x_1 = 0 \cup x_1 = x_2 = 0\}.$$

This result shows that a fixed point is realized as a steady rotation along one of the three axes. The value of the angular velocity along the direction is arbitrary, while the angular velocities are zero along the other two directions.

The method we developed in the previous section enables us to get information of the periodicity conditions of period greater than three. The case of period 2 must be considered separately. From the general expression (9) the parameters in $S(X^{(2)}, x; \mathbf{q}_2)$ are not proportional to a common factor. But if we substitute (8) into (9), we find that they have the following form

$$\begin{aligned} &a^{(2)} = (\gamma^{(2)}(\mathbf{q}))^2 \hat{a}^{(2)}, \qquad b^{(2)} = 0, \qquad d^{(2)} = (\gamma^{(2)}(\mathbf{q}))^2 \hat{d}^{(2)}, \\ &e^{(2)} = 0, \qquad f^{(2)} = (\gamma^{(2)}(\mathbf{q}))^2 \hat{f}^{(2)} \end{aligned}$$

with

$$\gamma_1^{(2)}(\mathbf{q}) = A_0 - A_2 + A_3, \qquad \gamma_2^{(2)}(\mathbf{q}) = A_0 - A_2 - A_3, \qquad \gamma_3^{(2)}(\mathbf{q}) = A_0 + A_2 + A_3, \tag{16}$$

corresponding to $x = x_1,\ x_2,\ x_3$, respectively. For the higher periods we can apply the formulae (12) and (13) to obtain

$$\begin{aligned} \gamma_1^{(3)} &= \big(A_1^2 + 2A_0(A_2 - A_3) - 3A_0^2\big)\big((A_1 + A_0)^2 + 4A_0A_3\big), \\ \gamma_2^{(3)} &= \big(A_1^2 + 2A_0(A_2 - A_3) - 3A_0^2\big)\big((A_1 + A_0)^2 + 4A_3A_1\big), \\ \gamma_3^{(3)} &= \big(A_1^2 + 2A_0(A_2 - A_3) - 3A_0^2\big)\big((A_1 - A_0)^2 + 4A_1A_2\big), \end{aligned} \tag{17}$$

$$\begin{aligned} \gamma_1^{(4)} &= 2(A_1 - A_0)(A_0 - A_2 - A_3)\big((A_1 - A_0)^4 - 8A_2(A_0 + A_3)(A_1^2 + A_0^2)\big), \\ \gamma_2^{(4)} &= 2(A_0 - A_1)(A_0 - A_2 + A_3)\big((A_1 + A_0)^4 + 16A_0A_1A_3(A_0 - A_2)\big), \\ \gamma_3^{(4)} &= 2(A_0 + A_1)(A_0 - A_2 + A_3)\big((A_1 - A_0)^4 + 16A_0A_1A_2(A_0 + A_3)\big), \\ &\vdots \end{aligned} \tag{18}$$

From the expressions (16), (17), (18) it is clear that the periodicity conditions do not determine points but impose relations among the invariants of the map. This owes to the fact that the initial parameters $\mathbf{q}$ are dependent on the invariants of the map, as we see in (8). After iteration of the map n times the new parameters $\mathbf{q}^{(n)}$ are also dependent on the invariants. Therefore the periodicity condition $\gamma^{(n)}(\mathbf{q}) = 0$ imposes relations among the invariants.

The periodicity conditions of period n are satisfied when $\gamma_1^{(n)} = 0$, $\gamma_2^{(n)} = 0$, $\gamma_3^{(n)} = 0$ are satisfied simultaneously. By an inspection of (16), (17), (18) we notice that the conditions are satisfied by the single condition

$$A_1^2 + 2A_0(A_2 - A_3) - 3A_0^2 = 0 \tag{19}$$

in the case of $n = 3$. All other cases impose further relations among the invariants.

In order to find where the periodic points are in the (x_1, x_2, x_3) space, we simply substitute the formulae (5) into the conditions $\gamma^{(n)}(\mathbf{q}) = 0$. In the case of (19) we obtain

$$v^{(3)} = \left\{\mathbf{x} \,\middle|\, (1 + \xi_1 + \xi_2 + \xi_3)^2 - 4(1 + \xi_1\xi_2 + \xi_2\xi_3 + \xi_3\xi_1) = 0\right\}, \tag{20}$$

in terms of the new variables

$$\xi_1 = \frac{(I_3 - I_1)(I_1 - I_2)}{4I_2I_3}x_1^2, \qquad \xi_2 = \frac{(I_1 - I_2)(I_2 - I_3)}{4I_3I_1}x_2^2, \qquad \xi_3 = \frac{(I_2 - I_3)(I_3 - I_1)}{4I_1I_2}x_3^2.$$

The set of points satisfying (20) form a variety of periodic points of period 3 in the space of (x_1, x_2, x_3). Every point on this variety is a point of period 3. We called this type of variety 'an invariant variety of periodic points', because it is determined uniquely by the invariants of the map alone. The dimension of the variety is two, which is the number of the invariants. In the case of (20) the invariant variety is an algebraic variety of degree 4, symmetric in the three variables x_1, x_2, x_3.

Now let us pause a while. The discrete Euler top (2) has been known being satisfied by elliptic functions as special solutions. The map generates an elliptic curve. This curve, however, is not the invariant variety of periodic points in our consideration, since the map is not controlled, in general, by the periodicity conditions of some fixed period. In fact the invariant variety of (20) is not a curve but a surface. Once an initial point is chosen on the surface, the orbit stays on it before it returns to the initial point. The invariant variety (20) tells us where the map of period 3 should start. Every point on (20) is a candidate of the period 3 map. The elliptic curve is embedded in this invariant variety as a set of 3 points, if the initial point is on it. We can view this variety as a subspace of the set of all elliptic curves, which are restricted to 3 periodic motion. It is a highly nontrivial observation that the intersections form a surface characterized by certain specific relations among the invariants of the map alone. The existence of such an variety in any integrable map has not been known, to our knowledge, in the literature. The claim of our conjecture is that if there exists an invariant variety of some period, the map is guaranteed being integrable. Therefore the existence of the surface (20) of period 3 is sufficient to guarantee the integrability of the discrete Euler top. This is true irrespective whether some solutions are known or not known explicitly.

To see other conditions, let us present all expressions of (16), (17), (18) after the substitution of (5):

$$\gamma_1^{(2)} = (1 + \xi_1 - \xi_2 - \xi_3), \quad \gamma_2^{(2)} = (1 - \xi_1 + \xi_2 - \xi_3), \quad \gamma_3^{(2)} = (1 - \xi_1 - \xi_2 + \xi_3), \tag{21}$$

$$\begin{aligned}
\gamma_1^{(3)} &= \big((1 + \xi_1 + \xi_2 + \xi_3)^2 - 4(1 + \xi_1\xi_2 + \xi_2\xi_3 + \xi_3\xi_1)\big) \\
&\quad \times \big((1 + \xi_1 - \xi_2 - \xi_3)^2 - 4(\xi_1^2 - \xi_1\xi_2 + \xi_2\xi_3 - \xi_3\xi_1)\big), \\
\gamma_2^{(3)} &= \big((1 + \xi_1 + \xi_2 + \xi_3)^2 - 4(1 + \xi_1\xi_2 + \xi_2\xi_3 + \xi_3\xi_1)\big) \\
&\quad \times \big((1 - \xi_1 + \xi_2 - \xi_3)^2 - 4(\xi_2^2 - \xi_1\xi_2 - \xi_2\xi_3 + \xi_3\xi_1)\big), \\
\gamma_3^{(3)} &= \big((1 + \xi_1 + \xi_2 + \xi_3)^2 - 4(1 + \xi_1\xi_2 + \xi_2\xi_3 + \xi_3\xi_1)\big) \\
&\quad \times \big((1 - \xi_1 - \xi_2 + \xi_3)^2 - 4(\xi_3^2 + \xi_1\xi_2 - \xi_2\xi_3 - \xi_3\xi_1)\big),
\end{aligned} \tag{22}$$

$$\begin{aligned}\gamma_1^{(4)} &= \big((1-\xi_1)^2-(\xi_2-\xi_3)^2\big)\big((\xi_1-1)^2\big(2(1+\xi_1-\xi_2-\xi_3)^2-(\xi_1-1)^2\big)\\ &\quad+(\xi_2-\xi_3)^2\big((2+2\xi_1-\xi_2-\xi_3)^2+4\xi_3\xi_2-8\xi_1\big)\big),\\ \gamma_2^{(4)} &= \big((1-\xi_2)^2-(\xi_3-\xi_1)^2\big)\Big((\xi_2-1)^2\big(2(1-\xi_1+\xi_2-\xi_3)^2-(\xi_2-1)^2\big)\\ &\quad+(\xi_3-\xi_1)^2\big((2+2\xi_2-\xi_3-\xi_1)^2+4\xi_1\xi_3-8\xi_2\big)\big),\\ \gamma_3^{(4)} &= \big((1-\xi_3)^2-(\xi_1-\xi_2)^2\big)\big((\xi_3-1)^2\big(2(1-\xi_1-\xi_2+\xi_3)^2-(\xi_3-1)^2\big)\\ &\quad+(\xi_1-\xi_2)^2\big((2+2\xi_3-\xi_1-\xi_2)^2+4\xi_1\xi_2-8\xi_3\big)\big).\end{aligned} \tag{23}$$

The conditions (21) for the period 2 impose

$$\xi_1=\xi_2=\xi_3=1. \tag{24}$$

The second factors of $\gamma_1^{(3)}$, $\gamma_2^{(3)}$, $\gamma_3^{(3)}$ vanish simultaneously only when the point (x_1,x_2,x_3) is on the lines defined by

$$\{\mathbf{x}\,|\,\xi_1=1\ \cap\ \xi_2=\xi_3\}\ \cup\ \{\mathbf{x}\,|\,\xi_2=1\ \cap\ \xi_3=\xi_1\}\ \cup\ \{\mathbf{x}\,|\,\xi_3=1\ \cap\ \xi_1=\xi_2\}. \tag{25}$$

After some manipulation we find that $\gamma_1^{(4)}$, $\gamma_2^{(4)}$, $\gamma_3^{(4)}$ also vanish simultaneously iff the point is on these lines. Therefore every periodic point of period 2, 3 and 4 are on the lines of (25), if it is not on the invariant variety (20) of period 3. We now notice that the points on these lines (25) vanish the denominator of the map (3), or equivalently the function $(1-\xi_1-\xi_2-\xi_3)^2-4\xi_1\xi_2\xi_3$.

From this observation we are convinced that the Euler top has no periodic point of period 2 and 4 as long as the top is not axially symmetric, whereas the periodic points of period 3 form the invariant variety $v^{(3)}$ of (20).

5 Axially symmetric top

By studying the discrete Euler top we have found an invariant variety of periodic points in the case of period 3. If we adopt our conjecture in Section 1, this means that the system is integrable. We also found that there is no periodic point of period 2 and 4 if the top is not axially symmetric. Our method enables us to search the periodic points of larger period. Instead of carrying out further the cumbersome algebraic analysis, however, we conclude this paper by studying the cases of symmetric top.

If the top is totally symmetric, i.e., $I_1=I_2=I_3$, the equations (3) show that (x_1,x_2,x_3) remain constants. This is a top which never changes its angular velocity in all directions. When the top is axially symmetric, such as $I_2=I_3$, the motion is governed by (4). As we repeat the map n times we get

$$\begin{aligned}X_1^{(n)} &= x_1,\\ X_2^{(n)} &= x_2\cos(n\Omega)+x_3\sin(n\Omega),\\ X_3^{(n)} &= x_3\cos(n\Omega)-x_2\sin(n\Omega).\end{aligned} \tag{26}$$

The periodicity condition of period n in this map can be read off directly from (26) as $\cos(n\Omega)=1$, or $\Omega=\frac{2\pi}{n}$. This condition fixes the values of x_1 for each period, according to the rule

$$x_1=\pm\mu_n\frac{2I_2}{I_2-I_1},\qquad \mu_n=\sqrt{\frac{1-\cos(2\pi/n)}{1+\cos(2\pi/n)}},\qquad n=1,2,3,\ldots. \tag{27}$$

For small n's we have

$$\mu_1 = 0, \quad \mu_3 = \sqrt{3}, \quad \mu_4 = 1, \quad \mu_5 = \sqrt{5 - 2\sqrt{5}}, \quad \mu_6 = 2 - \sqrt{3}, \quad \ldots.$$

The conditions determine planes which are orthogonal to the axis of symmetry and intersect the axis at certain points defined by (27), different for each period. These planes

$$v^{(n)}_{\text{axial symm}} = \left\{ \mathbf{x} \,\middle|\, x_1^2 = \mu_n^2 \frac{4I_2^2}{(I_1 - I_2)^2} \right\}, \qquad n = 2, 3, 4, \ldots \tag{28}$$

are the invariant varieties of periodic points characterized by the relations among the invariants:

$$I_2 H_1 - H_2 = \frac{\mu_n^2}{1 + \mu_n^2} \frac{4 I_1 I_2^2}{I_2 - I_1}.$$

In terms of the variables (ξ_1, ξ_2, ξ_3), the conditions $I_2 = I_3$ and (27) are equivalent to $(\xi_1, \xi_2, \xi_3) = (-\mu_n^2, 0, 0)$. We notice that $v^{(3)}_{\text{axial symm}}$ is a special case of $v^{(3)}$ in (20). When $n = 4$ there is no invariant variety for generic values of (I_1, I_2, I_3), hence we are not able to derive $v^{(4)}_{\text{axial symm}}$ as a special case. We notice that the periodicity conditions $(\xi_1, \xi_2, \xi_3) = (-1, 0, 0)$ in the case of period 4 are compatible with the conditions $\gamma_2^{(4)} = \gamma_3^{(4)} = 0$ of (23).

The meaning of the planes presented in (28) is quite interesting. Because of the symmetry the angular velocity along the symmetry axis is constant as it is expected naturally. An interesting feature is that the value of this angular velocity x_1 is 'quantized' to some specific values determined by the shape of the top and different for each period, such that the 'angular velocity' Ω is quantized to $2\pi/n$ to generate the periodic maps. This is true irrespective to the values of other angular velocities x_2 and x_3. The generation of the invariant varieties $v^{(n)}_{\text{axial symm}}$ follows to this fact.

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