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On the regularity of solutions of quasilinear Poisson equations

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We study the Dirichlet problem for quasilinear partial differential equations of the form $\Delta u(z) = h(z) f(u(z))$ in the unit disk $\mathbb{D} \subset \mathbb{C}$ with continuous boundary data. Here, the function $h : \mathbb{D} \to \mathbb{R}$ belongs to the class $L^p(\mathbb{D})$, $p > 1$, and the continuous function $f : \mathbb{R} \to \mathbb{R}$ is assumed to have the nondecreasing $|f|$ of $|t|$ and such that $f(t)/t \to 0$ as $t \to \infty$. We prove the existence of a continuous solution u of the problem in the Sobolev class $W^{2,p}_{\mathrm{loc}}(\mathbb{D})$. Moreover, we show that if $p > 2$, then $u \in C^{1,\alpha}_{\mathrm{loc}}(\mathbb{D})$ with $\alpha = (p-2)/p$.

Keywords: *quasilinear Poisson equation, potential theory, logarithmic and Newtonian potentials, Dirichlet problem, Sobolev classes, quasiconformal mappings.*

1. Introduction. We study the existence of regular solutions to the Dirichlet problem for the quasilinear Poisson equation

$$\Delta u(z) = h(z) f(u(z)) \tag{1}$$

in the unit disk $\mathbb{D} = \{z : |z| < 1\}$ of the complex plane $\mathbb{C}$ with continuous boundary values. In general, we assume that the function $h : \mathbb{D} \to \mathbb{R}$ is in the class $L^p(\mathbb{D})$, $p > 1$, and the continuous function $f : \mathbb{R} \to \mathbb{R}$ is either bounded or has the non-decreasing $|f|$ of $|t|$ and such that

$$\lim_{t \to \infty} \frac{f(t)}{t} = 0, \tag{2}$$

without any assumptions on the sign and zeros of the right-hand side in (1). We analyze how the degree of regularity of solutions depends on the degree of integrability of the multiplier *h*.

On the one hand, the interest in this subject is well known both from a purely theoretical point of view, due to its deep relations to linear and nonlinear harmonic analysis, and because of numerous applications of equations of this type in various areas of physics, differential geometry, logistic problems, etc. In particular, in the excellent book by M. Marcus and L. Veron [1], the reader can find a comprehensive analysis of the Dirichlet problem for the semilinear equation

$$\Delta u(z) = f(z, u(z)) \tag{3}$$

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in smooth (C^2) domains D in $\mathbb{R}^n$, $n \geqslant 3$, with boundary data in L^1. Here, $t \to f(\cdot, t)$ is a continuous mapping from $\mathbb{R}$ to a weighted Lebesgue space $L^1(D, \rho)$, and $z \to f(z, \cdot)$ is a non-decreasing function for every $z \in D$, $f(z, 0) \equiv 0$, such that

$$\lim_{t\to\infty} \frac{f(z,t)}{t} = \infty, \tag{4}$$

uniformly with respect to the parameter z in compact subsets of D.

On the other hand, Eqs. (1) naturally arise under the study of some semilinear equations in the divergent form. Indeed, we have established [2] that, in arbitrary simply connected domains $D \subset \mathbb{C}$, solutions of the semilinear equations

$$\mathrm{div}[A(z)\nabla U(z)] = f(U(z)) \tag{5}$$

with suitable matrix functions $A(z)$ can be represented as the composition $U = u \circ \omega$, where ω is a quasiconformal mapping of D onto $\mathbb{D}$ associated with A, and u is a solution of Eq. (1) with $h = J$. Here, J stands for the Jacobian of the mapping ω^{-1}. Hence, the results on regular solutions for Eqs. (1) presented in this paper and the comprehensively developed theory of quasiconformal mappings in the plane, see, e.g., [3-5], are good basic tools for the further study of Eqs. (5). The latter opens up a new approach to the study of a number of semi-linear equations of mathematical physics in anisotropic and inhomogeneous media.

In Section 2, we give a necessary background for the Poisson equation $\Delta u(z) = g(z)$ due to the theory of the Newtonian potential and to the theory of singular integrals in $\mathbb{C}$. First, we recall that, correspondingly to the key fact of the potential theory, see Proposition 1, the Newtonian potential

$$N_g(z) := \frac{1}{2\pi} \int_{\mathbb{C}} (\ln|z-w|)\, g(w)\, dm(w) \tag{6}$$

of arbitrary integrable densities g of charge with compact support satisfies the Poisson equation in a distributional sense, see Corollary 1. Moreover, N_g is continuous for $g \in L^p(\mathbb{C})$, and, furthermore, the Newtonian operator $N : L^p(\mathbb{C}) \to C(\mathbb{C})$ is completely continuous for $p > 1$, see Theorem 1. The example in Remark 2 shows that N_g for $g \in L^1(\mathbb{C})$ can be not continuous and even not in $L^\infty_{\mathrm{loc}}(\mathbb{C})$. Theorem 2 describes additional regularity properties of N_g depending on a degree of integrability of g. Finally, resulting Corollary 2 states the existence, representation, and regularity of solutions to the Dirichlet problem for the Poisson equation with arbitrary continuous boundary data.

Section 3 contains the main result of the paper. It is well known that solutions to the quasilinear Poisson equation (1) in a unit disk $\mathbb{D}$ for arbitrary continuous boundary data belong to the Sobolev space $W^{1,q}_{\mathrm{loc}}(\mathbb{D})$ for some $q > 2$, and U is locally Hölder continuous in $\mathbb{D}$, see, e.g., [6]. If, in addition, φ is Hölder continuous, then U is Hölder continuous in $\overline{\mathbb{D}}$. In Theorem 3, we prove the existence of solutions in the Sobolev class $W^{2,p}_{\mathrm{loc}}(\mathbb{D})$, if the multiplier $h : \mathbb{D} \to \mathbb{R}$ is in the class $L^p(\mathbb{D})$, $p > 1$. Moreover, we show that if $p > 2$, then $U \in C^{1,\alpha}_{\mathrm{loc}}(\mathbb{D})$ with $\alpha = (p-2)/p$. The proof of Theorem 3 is realized through reducing the problem to the case of the linear Poisson equation by the Leray—Schauder approach.

2. Potentials and the Poisson equation. Let $\mathbb{D}$ be the unit disk in the complex plane $\mathbb{C}$. For z and $w \in \mathbb{D}$ with $z \neq w$, let

$$G(z,w) := \log\left|\frac{1-z\overline{w}}{z-w}\right| \quad \text{and} \quad P(z,e^{it}) := \frac{1-|z|^2}{|1-ze^{-it}|^2} \tag{7}$$

be the *Green function* and *Poisson kernel* in $\mathbb{D}$. If $\varphi \in C(\partial\mathbb{D})$ and $g \in C(\overline{\mathbb{D}})$, then a solution to the *Poisson equation*

$$\Delta f(z) = g(z) \tag{8}$$

satisfying the boundary condition $f|_{\partial\mathbb{D}} = \varphi$ is given by the formula

$$f(z) = \mathcal{P}_{\varphi}(z) - \mathcal{G}_g(z), \tag{9}$$

where

$$\mathcal{P}_{\varphi}(z) = \frac{1}{2\pi}\int_0^{2\pi} P(z,e^{it})\varphi(e^{-it})\,dt, \quad \mathcal{G}_g(z) = \int_{\mathbb{D}} G(z,w)g(w)\,dm(w), \tag{10}$$

see, e.g., [7], p. 118-120. Here, $m(w)$ denotes the Lebesgue measure in $\mathbb{C}$.

In this section, we give the representation of solutions of the Poisson equation in the form of the Newtonian (normalized antilogarithmic) potential that is more convenient for our research. On this basis, we prove the existence and representation theorem for solutions of the Dirichlet problem to the Poisson equation under the corresponding conditions of integrability of sources g.

Correspondingly to 3.1.1 in [8], given a finite Borel measure ν on $\mathbb{C}$ with compact support, its **potential** is the function $p_{\nu} : \mathbb{C} \to [-\infty,\infty)$ defined by

$$p_{\nu}(z) = \int_{\mathbb{C}} \ln|z-w|\,d\nu(w). \tag{11}$$

Remark 1. Note that the function p_{ν} is subharmonic by Theorem 3.1.2 in [8] and, consequently, it is locally integrable on $\mathbb{C}$ by Theorem 2.5.1 in [8]. Moreover, p_{ν} is harmonic outside the support of ν.

This definition can be extended to finite *charges* ν with compact support (named also *signed measures*), i. e., to real-valued sigma-additive functions on Borel sets in $\mathbb{C}$, because $\nu = \nu^+ - \nu^-$, where ν^+ and ν^- are Borel measures by the well-known Jordan decomposition.

The key fact is the following statement, see, e.g., Theorem 3.7.4 in [8].

Proposition 1. *Let ν be a finite charge with compact support in $\mathbb{C}$. Then*

$$\Delta p_{\nu} = 2\pi\nu \tag{12}$$

in the distributional sense, i. e.,

$$\int_{\mathbb{C}} p_{\nu}(z)\Delta\psi(z)\,dm(z) = 2\pi\int_{\mathbb{C}} \psi(z)\,d\nu(z) \quad \forall \psi \in C_0^{\infty}(\mathbb{C}). \tag{13}$$

As usual, $C_0^\infty(\mathbb{C})$ denotes the class of all infinitely differentiable functions $\psi:\mathbb{C}\to\mathbb{R}$ with compact support in $\mathbb{C}$, $\Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$ is the Laplace operator, and $dm(z)$ corresponds to the Lebesgue measure in $\mathbb{C}$.

Corollary 1. *In particular, if, for every Borel set B in $\mathbb{C}$,*

$$\nu(B):=\int_B g(z)\,dm(z), \tag{14}$$

where $g:\mathbb{C}\to\mathbb{R}$ is an integrable function with compact support, then

$$\Delta N_g = g, \tag{15}$$

where

$$N_g(z):=\frac{1}{2\pi}\int_{\mathbb{C}}(\ln|z-w|)\,g(w)\,dm(w) \tag{16}$$

in the distributional sense, i. e.,

$$\int_{\mathbb{C}} N_g(z)\Delta\psi(z)\,dm(z)=\int_{\mathbb{C}}\psi(z)g(z)\,dm(z)\quad \forall\psi\in C_0^\infty(\mathbb{C}). \tag{17}$$

Here, the function g is called a *density of charge* ν and the function N_g is said to be the *Newtonian potential* of g.

The next statement on continuity in the mean of functions $\psi:\mathbb{C}\to\mathbb{R}$ in $L^q(\mathbb{C})$, $q\in[1,\infty)$, with respect to shifts is useful for the study of the Newtonian potential, see, e.g., Theorem 1.4.3 in [9].

Lemma 1. *Let $\psi\in L^q(\mathbb{C})$, $q\in[1,\infty)$, have a compact support. Then*

$$\lim_{\Delta z\to 0}\int_{\mathbb{C}}|\psi(z+\Delta z)-\psi(z)|^q\,dm(z)=0. \tag{18}$$

Theorem 1. *Let $g:\mathbb{C}\to\mathbb{R}$ be in $L^p(\mathbb{C})$, $p>1$, with compact support. Then N_g is continuous. A collection $\{N_g\}$ is equicontinuous on compacta, if the collection $\{g\}$ is bounded by the norm in $L^p(\mathbb{C})$ with supports in a fixed disk K. Moreover, under these conditions, on each compact set in $\mathbb{C}$,*

$$\|N_g\|_C\leqslant M\cdot\|g\|_p\,. \tag{19}$$

Proof. By the Hölder inequality with $\frac{1}{q}+\frac{1}{p}=1$, we have

$$|N_g(z)-N_g(\zeta)|\leqslant\frac{\|g\|_p}{2\pi}\cdot\left[\int_K\big|\ln|z-w|-\ln|\zeta-w|\big|^q\,dm(w)\right]^{\frac{1}{q}}=$$

$$=\frac{\|g\|_p}{2\pi}\cdot\left[\int_{\mathbb{C}}|\psi_\zeta(\xi+\Delta z)-\psi_\zeta(\xi)|^q\,dm(\xi)\right]^{\frac{1}{q}},$$

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where $\xi = \zeta - w$, $\Delta z = z - \zeta$, $\psi_\zeta(\xi) := \chi_{K+\zeta}(\xi)\ln|\xi|$. Thus, the first conclusion follows from Lemma 1, because $\ln|\xi| \in L^q_{\rm loc}(\mathbb{C})$ for all $q \in [1, \infty)$.

The second conclusion follows by continuity of the integral on the right-hand side in the above estimate with respect to the parameter $\zeta \in \mathbb{C}$. Indeed,

$$\| \psi_\zeta - \psi_{\zeta_*} \|_q = \left\{ \int_\Delta |\ln|\xi||^q \, dm(\xi) \right\}^{\frac{1}{q}},$$

where Δ denotes the symmetric difference of the disks $K + \zeta$ and $K + \zeta_*$. Thus, the statement follows from the absolute continuity of the indefinite integral.

The third conclusion similarly follows through the direct estimate

$$|N_g(\zeta)| \leqslant \frac{\|g\|_p}{2\pi} \left[\int_K |\ln|\zeta - w||^q \, dm(w) \right]^{\frac{1}{q}} = \frac{\|g\|_p}{2\pi} \left[\int_{\mathbb{C}} |\psi_\zeta(\xi)|^q \, dm(\xi) \right]^{\frac{1}{q}}.$$

Remark 2. It is easy to verify that the function

$$g(z) := \omega(|z|), \quad z \in \overline{\mathbb{D}}, \quad g(z) \equiv 0, \quad z \in \mathbb{C} \setminus \overline{\mathbb{D}},$$

where

$$\omega(t) = 1/t^2(1 - \ln t)^\alpha, \quad t \in (0, 1], \quad \alpha \in (1, 2), \quad \omega(0) = \infty,$$

is in $L^1(\mathbb{C})$, and its potential N_g is not continuous. Furthermore, $N_g \notin L^\infty_{\rm loc}$.

The following theorem on the Newtonian potentials is important to obtain solutions of the Dirichlet problem to the Poisson equation of a higher regularity.

Theorem 2. *Let* $g : \mathbb{C} \to \mathbb{R}$ *have compact support. If* $g \in L^1(\mathbb{C})$, *then* $N_g \in L^r_{\rm loc}$ *for all* $r \in [1, \infty)$, $N_g \in W^{1,q}_{\rm loc}$ *for all* $q \in [1, 2)$, *moreover*, $N_g \in W^{2,1}_{\rm loc}$,

$$4\frac{\partial^2 N_g}{\partial z \partial \overline{z}} = \Delta N_g = 4\frac{\partial^2 N_g}{\partial \overline{z} \partial z} = g \quad a.e. \tag{20}$$

If $g \in L^p(\mathbb{C})$, $p > 1$, *then* $N_g \in W^{2,p}_{\rm loc}$, $\Delta N_g = g$ *a. e. and, moreover*, $N_g \in W^{1,q}_{\rm loc}$ *for* $q > 2$, *consequently*, N_g *is locally Hölder continuous. If* $g \in L^p(\mathbb{C})$, $p > 2$, *then* $N_g \in C^{1,\alpha}_{\rm loc}$, *where* $\alpha = (p-2)/p$.

In this connection, recall the definition of formal complex derivatives:

$$\frac{\partial}{\partial z} := \frac{1}{2}\left\{ \frac{\partial}{\partial x} - i\frac{\partial}{\partial y} \right\}, \quad \frac{\partial}{\partial \overline{z}} := \frac{1}{2}\left\{ \frac{\partial}{\partial x} + i\frac{\partial}{\partial y} \right\}, \quad z = x + iy.$$

The elementary algebraic calculations show that the Laplacian

$$\Delta := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z \partial \overline{z}} = 4\frac{\partial^2}{\partial \overline{z} \partial z}.$$

Proof. Note that N_g is the convolution $\psi * g$, where $\psi(\zeta) = \ln|\zeta|$, and, hence, $N_g \in L^r_{\mathrm{loc}}$ for all $r \in [1, \infty)$, see, e. g., Corollary 4.5.2 in [10]. Moreover, as is well known, $\frac{\partial \psi * g}{\partial z} = \frac{\partial \psi}{\partial z} * g$ and $\frac{\partial \psi * g}{\partial \overline{z}} = \frac{\partial \psi}{\partial \overline{z}} * g$, see, e. g., (4.2.5) in [10]. In addition by elementary calculations, we get

$$\frac{\partial}{\partial z} \ln|z - w| = \frac{1}{2}\frac{1}{z - w}, \quad \frac{\partial}{\partial \overline{z}} \ln|z - w| = \frac{1}{2}\frac{1}{\overline{z} - \overline{w}}.$$

Consequently,

$$\frac{\partial N_g(z)}{\partial z} = \frac{1}{4} Tg(z), \quad \frac{\partial N_g(z)}{\partial \overline{z}} = \frac{1}{4} \overline{T} g(z),$$

where Tg and $\overline{T}g$ are the well-known integral operators

$$Tg(z) := \frac{1}{\pi}\int_{\mathbb{C}} g(w)\frac{dm(w)}{z - w}, \quad \overline{T}g(z) := \frac{1}{\pi}\int_{\mathbb{C}} g(w)\frac{dm(w)}{\overline{z} - \overline{w}}.$$

Thus, all the rest conclusions for $g \in L^1(\mathbb{C})$ follow from Theorems 1.13-1.14 in [11]. If $g \in L^p(\mathbb{C})$, $p > 1$, then $N_g \in W^{1,q}_{\mathrm{loc}}$, $q > 2$, by Theorem 1.27, (6.27) in [11]. Consequently, N_g is locally Hölder continuous, see, e. g., Theorem 8.22 in [12], and $N_g \in W^{2,p}_{\mathrm{loc}}$ by Theorems 1.36-1.37 in [11]. If $g \in L^p(\mathbb{C})$, $p > 2$, then $N_g \in C^{1,\alpha}_{\mathrm{loc}}$ with $\alpha = \frac{p-2}{p}$ by Theorem 1.19 in [11].

By Theorem 2 and the known Poisson formula, see, e. g., I.D.2 in [13], we come to the following consequence on the existence, regularity, and representation of solutions for the Dirichlet problem to the Poisson equation in the unit disk $\mathbb{D}$, where we assume the charge density g to be extended by zero outside $\mathbb{D}$.

Corollary 2. *Let* $\varphi : \partial\mathbb{D} \to \mathbb{R}$ *be a continuous function and let* $g : \mathbb{D} \to \mathbb{R}$ *belong to the class* $L^p(\mathbb{D})$, $p > 1$. *Then the function* $U := N_g - \mathcal{P}_{N_g^*} + \mathcal{P}_\varphi$, $N_g^* := N_g|_{\partial\mathbb{D}}$, *is continuous in* $\overline{\mathbb{D}}$ *with* $U|_{\partial\mathbb{D}} = \varphi$, *belongs to the class* $W^{2,p}_{\mathrm{loc}}(\mathbb{D})$, *and* $\Delta U = g$ *a.e. in* $\mathbb{D}$. *Moreover,* $U \in W^{1,q}_{\mathrm{loc}}(\mathbb{D})$ *for some* $q > 2$ *and* U *is locally Hölder continuous. If, in addition,* φ *is Hölder continuous, then* U *is Hölder continuous in* $\overline{\mathbb{D}}$. *If* $g \in L^p(\mathbb{D})$, $p > 2$, *then* $U \in C^{1,\alpha}_{\mathrm{loc}}(\mathbb{D})$, *where* $\alpha = (p-2)/p$.

Here, the Hölder continuity of U for Hölder continuous φ follows from the corresponding result for integrals of the Cauchy type over the unit circle, see, e. g., Theorem 1.10 in [11], because the Poisson kernel has the representation $P(z, e^{it}) = \mathrm{Re}\frac{e^{it} + z}{e^{it} - z}$.

3. The case of quasilinear Poisson equations. The case is reduced to the Poisson equation by the Leray—Schauder approach.

Theorem 3. *Let* $\varphi : \partial\mathbb{D} \to \mathbb{R}$ *be a continuous function, let* $h : \mathbb{D} \to \mathbb{R}$ *be a function in the class* $L^p(\mathbb{D})$, $p > 1$, *and let* $f : \mathbb{R} \to \mathbb{R}$ *be a continuous function with the nondecreasing function* $|f|$ *of* $|t|$ *such that*

$$\lim_{t \to +\infty} \frac{f(t)}{t} = 0. \tag{21}$$

Then there is a continuous function $U:\overline{\mathbb{D}}\to\mathbb{R}$ *with* $U|_{\partial\mathbb{D}}=\varphi$ *, such that* $U\in W^{2,p}_{\rm loc}(\mathbb{D})$ *and*

$$\Delta U(z)=h(z)f(U(z)) \text{ for a. e. } z\in\mathbb{D}. \tag{22}$$

Moreover, $U\in W^{1,q}_{\rm loc}(\mathbb{D})$ *for some* $q>2$ *, and* U *is locally Hölder continuous in* $\mathbb{D}$ *. If, in addition,* φ *is Hölder continuous, then* U *is Hölder continuous in* $\overline{\mathbb{D}}$ *. Furthermore, if* $p>2$ *, then* $U\in C^{1,\alpha}_{\rm loc}(\mathbb{D})$ *where* $\alpha=(p-2)/p$ *. In particular,* $U\in C^{1,\alpha}_{\rm loc}(\mathbb{D})$ *for all* $\alpha\in(0,1)$ *if* $h\in L^{\infty}(\mathbb{D})$ *.*

Proof. If $\|h\|_p=0$ or $\|f\|_c=0$, then the Poisson integral $\mathcal{P}_\varphi$ gives the desired solution of the Dirichlet problem for Eq. (22), see, e. g., I.D.2 in [13]. Hence ,we may assume further that $\|h\|_p\neq 0$ and $\|f\|_c\neq 0$.

By Theorem 1 and the maximum principle for harmonic functions, we obtain the family of operators $F(g;\tau):L^p(\mathbb{D})\to L^p(\mathbb{D})$, $\tau\in[0,1]$:

$$F(g;\tau):=\tau h\cdot f(N_g-\mathcal{P}_{N_g^*}+\mathcal{P}_\varphi), \quad N_g^*:=N_g|_{\partial\mathbb{D}}, \quad \forall\,\tau\in[0,1], \tag{23}$$

which satisfies all hypotheses H1-H3 of Theorem 1 in [14]:

H1. First of all, $F(g;\tau)\in L^p(\mathbb{D})$ for all $\tau\in[0,1]$ and $g\in L^p(\mathbb{D})$, because by Theorem 1, $f(N_g-\mathcal{P}_{N_g^*}+\mathcal{P}_\varphi)$ is a continuous function and, moreover, by (19),

$$\|F(g;\tau)\|_p\leqslant\|h\|_p|f(2M\|g\|_p+\|\varphi\|_C)|<\infty \quad \forall\tau\in[0,1].$$

Thus, by Theorem 1 in combination with the Arzela—Ascoli theorem, see, e. g., Theorem IV.6.7 in [15], the operators $F(g;\tau)$ are completely continuous for each $\tau\in[0,1]$ and even uniformly continuous with respect to the parameter $\tau\in[0,1]$.

H2. The index of the operator $F(g;0)$ is obviously equal to 1 .

H3. By inequality (19) and the maximum principle for harmonic functions, we have the estimate for solutions $g\in L^p$ of the equations $g=F(g;\tau)$:

$$\|g\|_p\leqslant\|h\|_p|f(2M\|g\|_p+\|\varphi\|_C)|\leqslant\|h\|_p|f(3M\|g\|_p)|$$

whenever $\|g\|_p\geq\|\varphi\|_C/M$, i.e. then it should be

$$\frac{|f(3M\|g\|_p)|}{3M\|g\|_p}\geqslant\frac{1}{3M\|h\|_p}, \tag{24}$$

and, hence, $\|g\|_p$ should be bounded in view of condition (21).

Thus, by Theorem 1 in [14], there is a function $g\in L^p(\mathbb{D})$ such that $g=F(g;1)$. Consequently, by Corollaries 2, the function $U:=N_g-\mathcal{P}_{N_g^*}+\mathcal{P}_\varphi$ gives the desired solution of the Dirichlet problem to the quasilinear Poisson equation (22).

Remark 3. As is clear from the proof, Theorem 3 remains valid, if f is an arbitrary continuous bounded function. These results can be extended to arbitrary smooth domains and applied to the study of some physical problems.

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ПРО РЕГУЛЯРНІСТЬ РОЗВ'ЯЗКІВ
КВАЗІЛІНІЙНИХ РІВНЯНЬ ПУАССОНА

Вивчається задача Діріхле для квазілінійних диференціальних рівнянь у частинних похідних виду $\Delta u(z) = h(z) f(u(z))$ в одиничному колі $\mathbb{D} \subset \mathbb{C}$ з неперервними граничними умовами. Тут функція $h: \mathbb{D} \to \mathbb{R}$ належить класу $L^p(\mathbb{D})$, $p > 1$, і неперервна функція $f: \mathbb{R} \to \mathbb{R}$ припускається такою, що її $|f|$ як функція від $|t|$ є неспадною і такою, що $f(t)/t \to 0$ при $t \to \infty$. Доводиться існування неперервного розв'язку u даної проблеми в класі Соболєва $W^{2,p}_{\text{loc}}(\mathbb{D})$. Більш того, показано, що якщо $p > 2$, то $u \in C^{1,\alpha}_{\text{loc}}(\mathbb{D})$ з $\alpha = (p-2)/p$.

Ключові слова: *квазілінійне рівняння Пуассона, теорія потенціалу, логарифмічний та ньютонів потенціали, задачі Діріхле, класи Соболєва, квазіконформні відображення.*

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О РЕГУЛЯРНОСТИ РЕШЕНИЙ
КВАЗИЛИНЕЙНЫХ УРАВНЕНИЙ ПУАССОНА

Изучается задача Дирихле для квазилинейных дифференциальных уравнений в частных производных вида $\Delta u(z) = h(z) f(u(z))$ в единичном круге $\mathbb{D} \subset \mathbb{C}$ с непрерывными граничными условиями. Здесь функция $h : \mathbb{D} \to \mathbb{R}$ принадлежит классу $L^p(\mathbb{D})$, $p > 1$, и непрерывная функция $f : \mathbb{R} \to \mathbb{R}$ предполагается такой, что ее $|f|$ как функция от $|t|$ является неубывающей и такой, что $f(t)/t \to 0$ при $t \to \infty$. Доказывается существование непрерывного решения u рассматриваемой проблемы в классе Соболева $W^{2,p}_{\mathrm{loc}}(\mathbb{D})$. Более того, показано, что если $p > 2$, то $u \in C^{1,\alpha}_{\mathrm{loc}}(\mathbb{D})$ с $\alpha = (p-2)/p$.

Ключевые слова: *квазилинейное уравнение Пуассона, теория потенциала, логарифмический и ньютонов потенциалы, задача Дирихле, классы Соболева, квазиконформные отображения.*