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THE EFFECT OF HYDROSTATIC INITIAL STRESS ON THE PLANE WAVES IN A FIBER-REINFORCED MAGNETO-THERMOELASTIC MEDIUM WITH FRACTIONAL DERIVATIVE HEAT TRANSFER

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Abstract. An effect of three factors - the order of fractional derivative, the hydrostatic initial stress, and parameter of magnetic field – on the plane waves in the half-space made of fiber-reinforced material, that is described by the theory of generalized magneto-thermoelasticity, is studied. The problem is solved numerically using the normal mode analysis. The results correspond to the Lord-Shulman model and the model, that uses the fractional derivatives and are presented in the form of graphs. The findings show pronounced effect of mentioned three factors. The results are compared with the case, when the initial stress and magnetic field are absent.

Key words: generalized magneto-thermoelasticity, fractional derivative, fiber reinforced material, hydrostatic initial stress, normal mode analysis.

1. Introduction.

During recent years, by applying the fractional calculus several interesting models have been established successfully to study the physical processes particularly in the area of mechanics of solids, control theory, electricity, heat conduction, diffusion problems, viscoelasticity etc. It has been verified/examined that the use of fractional order derivatives/integrals lead to the formulation of certain physical problem which is more economical and useful than the classical approach. There are some materials (e.g. porous materials, biological materials/polymers and colloids, glassy etc.) and physical situations (like low-temperature, amorphous media and transient loading etc.) where the conventional coupled dynamical theory (CD) [1] based on the classical Fourier's law is unsuitable (see [2] for details). In such cases, one needs to use a generalized thermoelastic (and more generally thermo-viscoelastic) model based on an anomalous heat conduction theory involving fractional time-derivatives; see Ignaczak & Ostoja-Starzewski [3]. Recently, fractional calculus has also been employed in the area of thermoelasticity. Povstenko [4] has constructed a quasi-static uncoupled thermoelasticity model based on the heat conduction equation with fractional order time derivatives. He has used the Caputo fractional derivative (see [5] for details) and obtained the stress components corresponding to the fundamental solution of a Cauchy problem for the fractional order heat conduction equation in both the one-dimensional and two-dimensional cases. In 2010, a new theory of generalized thermoelasticity in the context of a new consideration of the heat conduction equation with fractional order time derivatives has been proposed by Youssef [6].

The uniqueness of the solution has also been proved in the same work. Youssef & Al-Lehaibi [7] have studied a problem on an elastic half space using this theory. Sherief *et al.* [8] and Ezzat & Fayik [9] have also constructed some model in generalized thermoelasticity by using fractional time-derivatives.

Fiber-reinforced composites are widely used in engineering structures, due to their superiority over the structural materials in applications requiring high strength and stiffness in lightweight components. A continuum model is used to explain the mechanical properties of such materials. A reinforced concrete member should be designed for all conditions of stresses that may occur and in accordance with the principles of mechanics. The characteristic property of a reinforced concrete member is that its components, namely concrete and steel, act together as a single unit as long as they remain in the elastic condition, i.e., the two components are bound together so that there can be no relative displacement between them.

In the linear case, the associated constitutive relations, relating infinitesimal stress and strain components, have five material constants. In the last three decades, the analysis of stress and deformation of fiber-reinforced composite materials has been an important research area of solid mechanics. Belfield et al. [10] has introduced the idea of continuous self-reinforcement at every point of an elastic solid. Spencer [11], Pipkin [12] and Rogers [13, 14] have done pioneering works on this subject. Fibers are assumed as an inherent material property, rather than some form of inclusion in such models, see [11] for details. One can find some work on transversely isotropic elasticity in the literatures [15-19].

The study of the magneto-thermoelastic interactions which deals with the interactions among the strain, temperature and the electromagnetic field in an elastic solid is of great practical importance due to its extensive uses in diverse field, such as geophysics (for understanding the effect of the Earth's magnetic field on seismic waves), damping of acoustic waves in a magnetic field, designing machine elements like heat exchangers, boiler tubes where the temperature induced elastic deformation occurs, biomedical engineering (problems involving thermal stress), emissions of the electromagnetic radiations from nuclear devices, development of a highly sensitive super conducting magnetometer, electrical power engineering, plasma physics etc. [20, 21]. Many works in generalized magneto-thermoelasticity can be found in the literatures Sarkar & Lahiri [22], Abbas et al. [23], Ezzat & Youssef [24, 25], Youssef [26], Ezzat & Abd Elall [27] and Xion & Tian [28].

The aim of the present paper is to investigate the influences of fractional order, hydrostatic initial stress and the magnetic field on the plane waves in a fiber-reinforced generalized thermoelastic solid half-space. The problem has been solved numerically using the normal mode analysis [23, 29 – 32]. Numerical results for the temperature, displacement components and the stresses are represented graphically for the Lord-Shulman (LS) and farctional order (FO) model of generalized thermoelasticity and analyze the results. The graphical results indicate that the effect of fractional order, hydrostatic initial stress and magnetic field on plane waves are very pronounced. Comparisons are made with the results in the absence of the hydrostatic initial stress and the magnetic field.

2. Formulation of the Problem and Basic Equations.

We consider the problem of a fiber-reinforced generalized thermoelastic half-space ($x \geq 0$). A magnetic field with a constant intensity $\vec{H} = (0, 0, H_0)$ acts parallel to the boundary plane (taken as the direction of the *z*-axis). We begin our consideration with linearized equations of electro-dynamics of a slowly moving medium [23]:

$$\vec{J} = \vec{\nabla} \times \vec{h} - \varepsilon_0 \dot{\vec{E}}, \tag{1}$$

$$\vec{\nabla} \times \vec{E} = -\mu_0 \dot{\vec{h}}, \tag{2}$$

$$\vec{E} = -\mu_0 \left(\dot{\vec{u}} \times \vec{H} \right), \tag{3}$$

$$\vec{\nabla}\cdot\vec{h}=0. \tag{4}$$

The above equations are supplemented by the displacement equations of the theory of elasticity, taking into consideration the Lorentz force to give

$$\sigma_{ij,j}+F_i=\rho\ddot{u}_i, \tag{5}$$

$$F_i=\mu_0\left(\vec{J}\times\vec{H}\right)_i. \tag{6}$$

The constitutive relations for a fiber-reinforced linearly thermoelastic isotropic medium with respect to the reinforcement direction a with an initial hydrostatic stress and without body forces and heat sources are given by Lord & Shulman [33], Montanaro [34] and Singh [16] as

$$\begin{aligned}\sigma_{ij}&=-P\left(\delta_{ij}+\omega_{ij}\right)+\lambda e_{kk}\delta_{ij}+2\mu_T e_{ij}+\alpha\left(a_k a_m e_{km}\delta_{ij}+a_i a_j e_{kk}\right)+\\&+2\left(\mu_L-\mu_T\right)\left(a_i a_k e_{ik}+a_j a_k e_{kj}\right)+\beta a_k a_m e_{km} a_i a_j-\gamma\left(T-T_0\right)\delta_{ij},\end{aligned} \tag{7}$$

$$e_{ij}=\frac{1}{2}\left(u_{i,j}+u_{j,i}\right), \tag{8}$$

$$\omega_{ij}=\frac{1}{2}\left(u_{i,j}-u_{j,i}\right). \tag{9}$$

The heat conduction equation with fractional derivative heat transfer heat transfer proposed by Ezzat & Fayik [9] is

$$k\nabla^2 T=\frac{\partial}{\partial t}\left(1+\frac{\tau_0^{\nu}}{\nu!}\frac{\partial^{\nu}}{\partial t^{\nu}}\right)\left(\rho C_E T+\gamma T_0 e\right),\ 0<\nu\le 1, \tag{10}$$

where $\vec{a}=(a_1,a_2,a_3)$, $a_1^2+a_2^2+a_3^2=1$ and

$$\frac{\partial^{\nu}}{\partial t^{\nu}}f(x,t):=\begin{cases} f(x,t)-f(x,0) & \text{when } \nu\to 0,\\ I^{1-\nu}\dfrac{\partial f(x,t)}{\partial t} & \text{when } 0<\nu<1,\\ \dfrac{\partial f(x,t)}{\partial t} & \text{when } \nu=1.\end{cases}$$

In the above definition, the *Riemann–Liouville* fractional integral operator I^{ν} is defined as

$$I^{\nu}[f(x,t)]:=\frac{1}{\Gamma(\nu)}\int_0^t (t-s)^{\nu-1}f(x,s)ds,$$

where $\Gamma(...)$ is the well-known Gamma function. The *comma* notation is used for spatial derivatives and *superimposed dot* represents time differentiation.

For plane strain deformation in the *xy*-plane, all the considered functions will be depend on the time t and the coordinates x and y and the displacement vector $\vec{u}$ will have the components

$$u=u_x=u(x,y,t),\ v=u_y=v(x,y,t),\ w=u_z=0. \tag{11}$$

We choose the fiber-direction as $\vec{a}=(1,0,0)$ so that the preferred direction is the x-axis, and Eq. (5)-(7) simplify, as given below,

$$\sigma_{xx} = -P + (\lambda + 2\alpha + 4\mu_L - 2\mu_T + \beta)\frac{\partial u}{\partial x} + (\lambda + \alpha)\frac{\partial v}{\partial y} - \gamma(T - T_0), \tag{12}$$

$$\sigma_{yy} = -P + (\lambda + \alpha)\frac{\partial u}{\partial x} + (\lambda + 2\mu_T)\frac{\partial v}{\partial y} - \gamma(T - T_0), \tag{13}$$

$$\sigma_{xy} = \left(\mu_L - \frac{P}{2}\right)\frac{\partial v}{\partial x} + \left(\mu_L + \frac{P}{2}\right)\frac{\partial u}{\partial y}, \tag{14}$$

$$\sigma_{yx} = \left(\mu_L + \frac{P}{2}\right)\frac{\partial v}{\partial x} + \left(\mu_L - \frac{P}{2}\right)\frac{\partial u}{\partial y}, \tag{15}$$

$$F_x = \mu_0 H_0^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x \partial y} - \varepsilon_0\mu_0\frac{\partial^2 u}{\partial t^2}\right), \tag{16}$$

$$F_y = \mu_0 H_0^2\left(\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 u}{\partial x \partial y} - \varepsilon_0\mu_0\frac{\partial^2 v}{\partial t^2}\right), \tag{17}$$

$$\left(A_{11} + \rho R_H^2\right)\frac{\partial^2 u}{\partial x^2} + \left(A_{12} + \mu_L - \frac{P}{2} + \rho R_H^2\right)\frac{\partial^2 v}{\partial x \partial y} + \left(\mu_L + \frac{P}{2}\right)\frac{\partial^2 u}{\partial y^2} - \gamma\frac{\partial T}{\partial x} = \rho\left(1 + \frac{R_H^2}{c^2}\right)\frac{\partial^2 u}{\partial t^2}, \tag{18}$$

$$\left(A_{22} + \rho R_H^2\right)\frac{\partial^2 v}{\partial y^2} + \left(A_{12} + \mu_L - \frac{P}{2} + \rho R_H^2\right)\frac{\partial^2 u}{\partial x \partial y} + \left(\mu_L + \frac{P}{2}\right)\frac{\partial^2 v}{\partial x^2} - \gamma\frac{\partial T}{\partial y} = \rho\left(1 + \frac{R_H^2}{c^2}\right)\frac{\partial^2 v}{\partial t^2}. \tag{19}$$

To transform the above equations in non-dimensional forms, we will use the following non-dimensional variables

$$(x', y', u', v') = c_1\eta(x, y, u, v), (t', \tau_0{}') = c_1^2\eta(t, \tau_0),$$

$$\sigma'_{ij} = \frac{\sigma_{ij}}{\rho c_1^2}, \theta = \frac{\gamma(T - T)_0}{\rho c_{1\,0}^2}, h' = \frac{h}{H_0}, \eta = \frac{\rho c_E}{k}.$$

Using the above non-dimensional variables, Eqs. (12) – (15), (18), (19) and (10) take the following forms (omitting the primes for convenience)

$$\sigma_{xx} = -2R_P + \frac{\partial u}{\partial x} + B_1\frac{\partial v}{\partial y} - \theta, \tag{20}$$

$$\sigma_{yy} = -2R_P + B_1\frac{\partial u}{\partial x} + B_2\frac{\partial v}{\partial y} - \theta, \tag{21}$$

$$\sigma_{xy} = (B_3 - R_P)\frac{\partial v}{\partial x} + (B_3 + R_P)\frac{\partial u}{\partial y}, \tag{22}$$

$$\sigma_{yx} = (B_3 + R_P)\frac{\partial v}{\partial x} + (B_3 - R_P)\frac{\partial u}{\partial y}, \tag{23}$$

$$(1 + M_1)\frac{\partial^2 u}{\partial x^2} + (B_1 + B_3 - R_P + M_1)\frac{\partial^2 v}{\partial x \partial y} + (B_3 + R_P)\frac{\partial^2 u}{\partial y^2} - \frac{\partial \theta}{\partial x} = M_2\frac{\partial^2 u}{\partial t^2}, \tag{24}$$

$$(B_2 + M_1)\frac{\partial^2 v}{\partial y^2} + (B_1 + B_3 - R_P + M_1)\frac{\partial^2 u}{\partial x \partial y} + (B_3 + R_P)\frac{\partial^2 v}{\partial x^2} - \frac{\partial \theta}{\partial y} = M_2\frac{\partial^2 v}{\partial t^2}, \tag{25}$$

$$\nabla^2\theta = \left(1 + \frac{\tau_0^\nu}{\nu!}\frac{\partial^\nu}{\partial t^\nu}\right)\left(\frac{\partial \theta}{\partial t} + \varepsilon_0\frac{\partial e}{\partial t}\right), \tag{26}$$

where $(B_1, B_2, B_3) = \frac{1}{A_1}(A_{11}, A_{22}, \mu_L), M_1 = \frac{R_H^2}{c_1^2}, M_2 = \left(1 + \frac{R_H^2}{c^2}\right), R_P = \frac{P}{2A_{11}}.$

3. Normal mode analysis.

The solution of the physical quantities can be decomposed in terms of *normal modes* in the following form:

$$\left[u,v,e,\theta,\sigma_{ij}\right](x.y.t)=\left[u^*,v^*,e^*,\theta^*,\sigma_{ij}^*\right](x)\exp\left(\omega t+imy\right), \tag{27}$$

where $u^*(x)$ etc. are the amplitude of the function $u(x, y, t)$ etc., i is the imaginary unit, ω (complex) is the time constant and m is the wave number in the y-direction.

By using (27), we can obtain the following equations from (24)-(26) respectively:

$$\begin{aligned}&\left(D^2-C_{41}\right)u^*(x)-C_{45}Dv^*(x)-C_{46}D\theta^*(x)=0,\\&C_{54}Du^*(x)-\left(D^2-C_{52}\right)v^*(x)+C_{53}\theta^*(x)=0,\\&C_{64}Du^*(x)+C_{62}v^*(x)-\left(D^2-C_{63}\right)\theta^*(x)=0,\end{aligned} \tag{28}$$

where

$$C_{41}=\frac{\left[m^2\left(B_3+R_P\right)+M_2\omega^2\right]}{1+M_1},\ C_{45}=\frac{im\left[R_P-B_1-B_3-M_1\right]}{1+M_1},\ C_{46}=\frac{1}{1+M_1},$$

$$C_{52}=\frac{\left[m^2\left(B_2+M_1\right)+M_2\omega^2\right]}{B_3+R_P},\ C_{53}=\frac{im}{B_3+R_P},\ C_{54}=\frac{im\left[R_P-B_1-B_3-M_1\right]}{B_3+R_P},$$

$$C_{62}=im\varepsilon\omega_1,\ C_{63}=m^2+\omega_1,\ C_{64}=\varepsilon\omega_1,\omega_1=\omega\left(1+\frac{\tau_0^{\nu}}{\nu!}\omega^*\right),\ \omega^*=e^{-\omega t}t^{-\nu}\sum_{n=1}^{\infty}\frac{\left(\omega t\right)^n}{\Gamma(n+1-\nu)}.$$

Eliminating $v^*(x)$ and $\theta^*(x)$ from Eqs. (28), we get after some simple computations the following sixth-order ordinary differential equation satisfied by $u^*(x)$

$$\left[D^6-g_1D^4+g_2D^2-g_3\right]u^*(x)=0 \tag{29}$$

where

$$g_1=\left[C_{41}+C_{52}+C_{63}+C_{45}C_{54}+C_{46}C_{64}\right],$$

$$g_2=\left[C_{41}C_{52}-C_{53}C_{62}-C_{46}C_{54}C_{62}+C_{41}C_{63}+C_{52}C_{63}+C_{45}C_{54}C_{63}+C_{46}C_{52}C_{64}-C_{45}C_{53}C_{64}\right],$$

$$g_3=\left[C_{41}C_{53}C_{62}-C_{41}C_{52}C_{63}\right].$$

In a similar manner, we can show that $v^*(x)$ and $\theta^*(x)$ satisfy the following equations

$$\left[D^6-g_1D^4+g_2D^2-g_3\right]\left\{v^*(x),\theta^*(x)\right\}=0. \tag{30}$$

The general solution of Eq. (29) which is regular at $x\to+\infty$ can be written as

$$u^*(x)=\sum_{j=1}^{3}R_j(m,\omega)\exp\left(-k_jx\right), \tag{31}$$

where $k_j\ (j=1,2,3)$, the roots (with positive real part) of the following characteristics equation

$$k^6 - g_1 k^4 + g_2 k^2 - g_3 = 0, \tag{32}$$

are given by

$$k_1^2 = \frac{1}{3}\left(2p\sin q + g_1\right),\ k_2^2 = \frac{-1}{3}\left(p[\sqrt{3}\cos q + \sin q] - g_1\right),\ k_3^2 = \frac{1}{3}\left(p[\sqrt{3}\cos q - \sin q] + g_1\right),$$

and

$$p = \sqrt{g_1^2 - 3g_2},\ q = \frac{\sin^{-1} r}{3},\ r = \frac{9g_1 g_2 - 2g_1^3 - 27g_3}{2p^3}.$$

Similarly, the solutions for $v^*(x)$ and $\theta^*(x)$ can be written as

$$v^*(x) = \sum_{j=1}^{3} R_j^{'}(m,\omega)\exp\left(-k_j x\right), \tag{33}$$

$$\theta^*(x) = \sum_{j=1}^{3} R_j^{''}(m,\omega)\exp\left(-k_j x\right). \tag{34}$$

Substituting from Eqs. (31), (33) and (34) into the Eqs. (28), we obtain the following relations

$$R_j^{'} = G_{1j} R_j,\ R_j^{''} = G_{2j} R_j, \tag{35}$$

$$G_{1j} = \frac{C_{53}\left(C_{41} - k_j^2\right) - k_j^2 C_{46} C_{54}}{k_j C_{45} C_{53} + k_j C_{46}\left(k_j^2 - C_{52}\right)}, G_{2j} = \frac{\left(k_j^2 - C_{52}\right)\left(C_{41} - k_j^2\right) + k_j^2 C_{45} C_{54}}{k_j C_{45} C_{53} + k_j C_{46}\left(k_j^2 - C_{52}\right)}, j = 1,2,3. \tag{36}$$

By using the relation (27), the solution for the Eqs. (24) – (26) can be written as

$$u(x,y,t) = \exp\left(\omega t + imy\right)\sum_{j=1}^{3} R_j(m,\omega)\exp\left(-k_j x\right), \tag{37}$$

$$v(x,y,t) = \exp\left(\omega t + imy\right)\sum_{j=1}^{3} G_{1j} R_j(m,\omega)\exp\left(-k_j x\right), \tag{38}$$

$$\theta(x,y,t) = \exp\left(\omega t + imy\right)\sum_{j=1}^{3} G_{2j} R_j(m,\omega)\exp\left(-k_j x\right), \tag{39}$$

Substituting from Eqs. (37) – (39) into Eqs. (20)-(23), we get the following expressions for the stress components

$$\sigma_{xx}(x,y,t) = -2R_P + \exp\left(\omega t + imy\right)\sum_{j=1}^{3} M_{1j} R_j \exp\left(-k_j x\right), \tag{40}$$

$$\sigma_{yy}(x,y,t) = -2R_P + \exp\left(\omega t + imy\right)\sum_{j=1}^{3} M_{2j} R_j \exp\left(-k_j x\right), \tag{41}$$

$$\sigma_{xy}(x,y,t) = \exp\left(\omega t + imy\right)\sum_{j=1}^{3} M_{3j} R_j \exp\left(-k_j x\right), \tag{42}$$

$$\sigma_{yx}(x,y,t) = \exp\left(\omega t + imy\right)\sum_{j=1}^{3} M_{4j} R_j \exp\left(-k_j x\right), \tag{43}$$

where

$$M_{1j} = \left[-k_j + imG_{ij}B_1 - G_{2j} \right],\ M_{2j} = \left[-k_jB_1 + imG_{ij}B_2 - G_{2j} \right],$$

$$M_{3j} = \left[-G_{1j}k_j(B_3 - R_P) + im(B_3 + R_P) \right],\ M_{4j} = \left[-G_{1j}k_j(B_3 + R_P) + im(B_3 - R_P) \right].$$

4. Application.

We consider the problem of a fiber-reinforced elastic half-space under hydrostatic initial stress which fills the region Ω defined as follows:

$$\Omega = \left\{ (x, y, z) : 0 \le x < \infty, -\infty < y < \infty, -\infty < z < \infty \right\}.$$

We apply the following boundary conditions for the present problem. The boundary conditions at the plane surface x=0 subjected to an arbitrary normal force P_1 are

$$\sigma_{xx}(0, y, t) = -P_1 \exp(\omega t + imy),\ \sigma_{xy}(0, y, t) = 0,\ \theta(0, y, t) = 0. \quad (44)$$

Substituting the expressions of the variables considered into the above boundary conditions, we obtain the following equations satisfied by the parameters R_j $(j = 1,2,3)$

$$\sum_{j=1}^{3} M_{1j}R_j = R_P^*, \quad (45)$$

$$\sum_{j=1}^{3} M_{3j}R_j = 0, \quad (46)$$

$$\sum_{j=1}^{3} G_{2j}R_j = 0\,, \quad (47)$$

where

$$R_P^* = 2R_P \exp(\omega t + imy) - P_1.$$

Solving Eqs. (45) – (47), we get the parameters R_j $(j = 1,2,3)$ with the following forms respectively

$$R_1 = \frac{R_P^*\left[G_{23}M_{32} - G_{22}M_{33}\right]}{\Delta}, R_2 = \frac{R_P^*\left[G_{21}M_{33} - G_{23}M_{31}\right]}{\Delta}, R_3 = \frac{R_P^*\left[G_{22}M_{31} - G_{21}M_{32}\right]}{\Delta}, \quad (48)$$

where

$$\Delta = M_{11}\left(G_{23}M_{32} - G_{22}M_{33}\right) + M_{12}\left(G_{21}M_{33} - G_{23}M_{31}\right) + M_{13}\left(G_{22}M_{31} - G_{21}M_{32}\right).$$

5. Particular cases.

(i) **Isotropic generalized magneto-thermoelastic medium with hydrostatic initial stress;**

Substituting $\mu_L = \mu_T = \mu$ and $\alpha = \beta = 0$ in Eqs. (37) – (43), we obtain the corresponding expressions of the temperature, the displacements and the stress distribution in isotropic generalized thermoelastic medium with hydrostatic initial stress and magnetic field.

(ii) **Isotropic generalized thermoelastic medium with hydrostatic initial stress;**

Substituting $\mu_L = \mu_T = \mu$, $\alpha = \beta = 0$ and $R_H = 0$ in Eqs. (37) – (43), we obtain the corresponding expressions of all the physical quantities in isotropic generalized thermoelastic medium with hydrostatic initial stress and without magnetic field.

(iii) **Isotropic generalized magneto-thermoelastic medium without hydrostatic initial stress;**

Substituting $\mu_L = \mu_T = \mu$, $\alpha = \beta = 0$ and $R_P \to 0$ in Eqs. (37) – (43), we obtain the corresponding expressions of all the physical quantities in an isotropic generalized magneto-thermoelastic medium without hydrostatic initial stress and with magnetic field.

(iv) **Isotropic generalized thermoelastic medium without hydrostatic initial stress and magnetic field;**

Substituting $\mu_L = \mu_T = \mu$, $\alpha = \beta = 0$, $R_P \to 0$ and $R_H = 0$ in Eqs. (37) – (43), for the above physical quantities in an isotropic generalized thermoelastic medium without hydrostatic initial stress and with magnetic field.

(v) **Fiber-reinforced generalized magneto-thermoelastic medium;**

Setting $R_P \to 0$, the expressions in Eqs. (37) – (43) reduce to the case of a fiber-reinforced generalized thermoelastic medium without hydrostatic initial stress and with magnetic field.

(vi) **Fiber-reinforced generalized thermoelastic medium;**

Setting $R_P \to 0$ and $R_H = 0$, the expressions in Eqs. (37) – (43) reduce to the case of a fiber-reinforced generalized thermoelastic medium without hydrostatic initial stress and magnetic field.

6. Special cases of thermoelasticity theory

(i) **Classical dynamical theory of thermoelasticity (CD-theory);**

Setting $\nu = 1$ and $\tau_0 = 0$, the equations of the CD-theory can be obtained.

(ii) **Lord-Shulman theory of thermoelasticity (LS-theory);**

Setting $\nu = 1$ where $\tau_0 > 0$, the equations of the LS-theory can be obtained.

(ii) **Fractional order theory of thermoelasticity (FO-theory);**

In this case, we take $\nu = 0.5$ where $\tau_0 > 0$.

7. Numerical results.

With the analytical procedure presented earlier, we consider a numerical example for which computational results are given. We use the following physical constants of a fiber-reinforced thermoelastic solid to study the effect of the reinforcement, fractional parameter and the magnetic field on the wave propagation:

$$\lambda = 5.65\times10^4\,\text{N.m}^{-2}, \mu_T = 2.46\times10^2\,\text{N.m}^{-2}, \mu_L = 5.66\times10^3\,\text{N.m}^{-2}, \alpha = -1.28\times10^2\,\text{N.m}^{-2},$$

$$\beta = 220.9\times10^3\,\text{N.m}^{-2}, \rho = 2660\text{kg.m}^{-3}, k = 0.3\,\text{J.m}^{-1}.s^{-1}.\text{K}^{-1}, c_E = 787\text{J.kg}^{-1}.\text{K}^{-1},$$

$$\alpha_t = 1.7810^{-5}\,\text{N.m}^{-2}, T_0 = 298\text{K}, \tau_0 = 0.05\text{s}, \text{H}_0 = 10^4\,\text{A.m}^{-1}, \varepsilon_0 = 0.03\,\text{F.m}^{-1}, \mu_0 = 0.04\,\text{H.m}^{-1}.$$

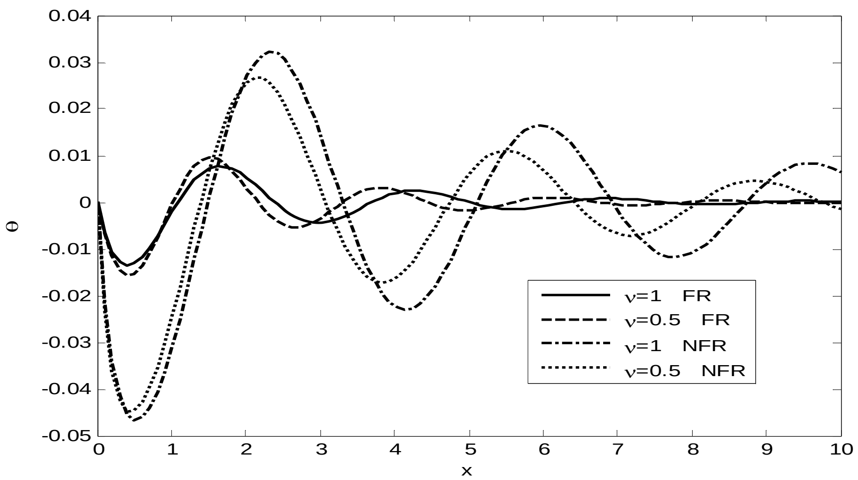

Fig. 1

Variation of the temperature distribution with x at $H_0 = 10^4, R_p = 10^3$.

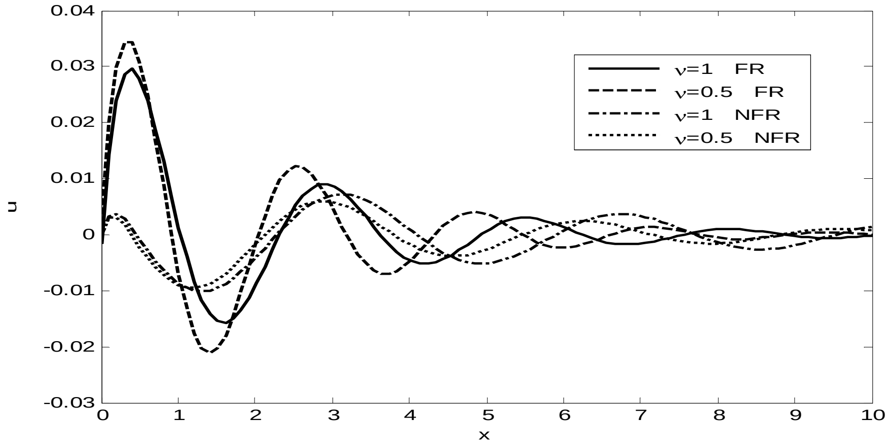

Fig. 2

Variation of the displacement distribution u with x at $H_0 = 10^4, R_p = 10^3$.

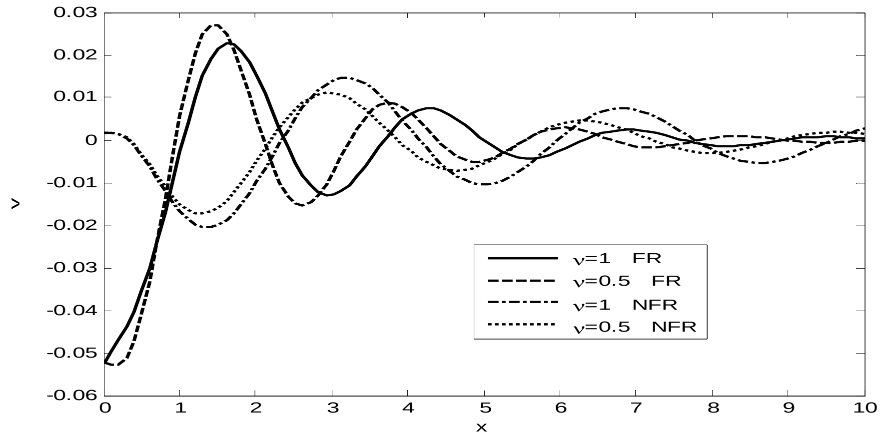

Fig. 3

Variation of the displacement distribution v with x at $H_0 = 10^4, R_p = 10^3$.

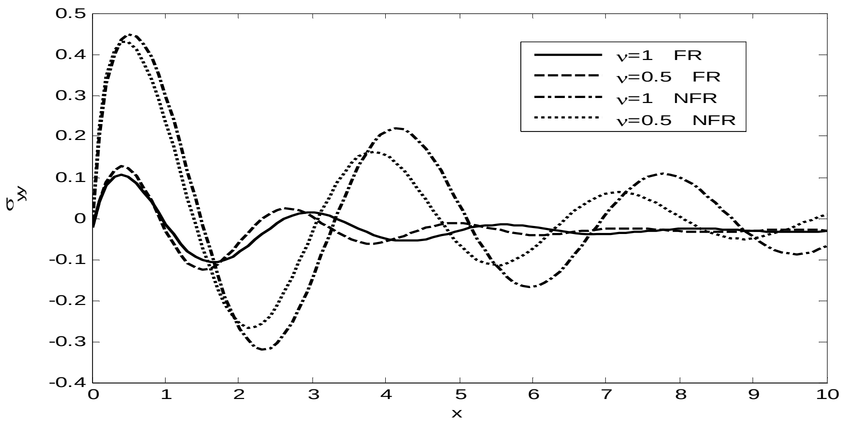

Fig. 4

Variation of the stress distribution σ_{yy} *with x at* $H_0 = 10^4, R_p = 10^3$.

The other constants of the problem may be taken as $\nu = 0.5, m = 3.6$, $\omega = \omega_0 + i\xi, \omega_0 = -2.5, \xi = 10, P = 0.1, P_1 = 100$. The computations are carried out on the surface y=1.5 at time t=0.3. The distribution of the real part of the non-dimensional temperature (θ), the displacement components (u, v) and the stress component (σ_{yy}) for the problem considered are shown in figs. 1 – 16 for three different cases. In the first case, we are investigating how the non-dimensional temperature (θ), the displacement components (u, v) and the stress component (σ_{yy}) vary with different values of the fractional parameter $\nu = 1.0$ and $\nu = 0.5$ against x for the fiber-reinforced (FR) and non-fiber-reinforced (NFR) elastic half-space when the initial hydrostatic stress and the magnetic field remain constant. In the second case, we will show how the non-dimensional temperature (θ), the displacements (u, v) and the stress (σ_{yy}) vary with different values of the fractional parameter $\nu = 1.0$ and $\nu = 0.5$ in the presence $(H_0 = 10^4)$ and absence $(H_0 = 0)$ of the magnetic field against x for the fiber-reinforced elastic half-space when the initial hydrostatic stress remains constant. The third case is investigating how the non-dimensional temperature θ, the displacements u and v and the stress (σ_{yy}) vary with different values of the fractional parameter $\nu = 1.0$ and $\nu = 0.5$ in the presence $(R_p = 10^3)$ and absence $(R_p = 0)$ of the initial hydrostatic stress against x for the fiber-reinforced elastic half-space when the magnetic field remains constant.

Figs. 1 – 4 depict the variety of the real part of the non-dimensional temperature (θ), the displacement components (u, v) and the stress component (σ_{yy}) for two different values of the fractional parameter (ν) for the fiber-reinforced (FR) and non-fiber-reinforced (NFR) elastic half-space. Fig. 1 and 4 show that the range of magnitude of the temperature (θ) and the stress (σ_{yy}) are greater in the NFR thermoelastic medium for $\nu = 1.0$. Fig. 2 exhibits that the normal displacement u starts with a zero value and shows the oscillatory nature and converges to the zero value rapidly with the increase of the distance x. Fig. 3 shows that the horizontal displacement v starts with a positive initial value for the FR case but with a negative value for the NFR case and vanishes identically with the increase of the distance x. It is also clearly depicted from figs. 2, 3 that the values of u and v are maximum in the FR elastic half-space for $\nu = 0.5$. Figs. 5 – 8 exhibit that as the value of x increases, the values of the non-dimensional temperature (θ), the displacements (u, v) and the stress component (σ_{yy}) approach rapidly to the zero value in the fiber-reinforced elastic half-space without the effect of the magnetic field. It is also clearly depicted that the values of all the physical quantities are maximum in the fiber-reinforced thermoelastic medium for $\nu = 0.5$ when the effect of the magnetic field is present. Figs. 9 – 12 display the distribution of the real part of the non-dimensional temperature (θ), the displacement components (u, v) and the stress component (σ_{yy}) for two different values of the fractional parameter ν in the presence and absence of the hydrostatic initial stress for the FR elastic medium. The values of all the non-dimensional physical quantities approach rapidly to the zero value in the fiber-reinforced elastic half-space when the effect of the hydrostatic initial stress is absent. It can also be noted that the values of all the physical quantities are maximum in the fiber-reinforced thermoelastic medium for $\nu = 0.5$ in the presence of the hydrostatic initial stress. Figs. 1, 5, 9 depict that the temperature (θ) is zero on the boundary surface x=0.0 for all values of y and t. The isothermal boundary condition (44) on the surface x=0 of the half-space $x \geq 0$ is thus found to be satisfied numerically. This is consistent with our theoretical result. Figs. 13 – 16 depict the three-dimensional distribution of the real part of the non-dimensional temperature (θ), the displacement components (u, v) and the stress component (σ_{yy}) for two different values of the fractional parameter (ν) for the fiber-reinforced elastic half-space in the presence of the hydrostatic initial stress and with the magnetic field effect.

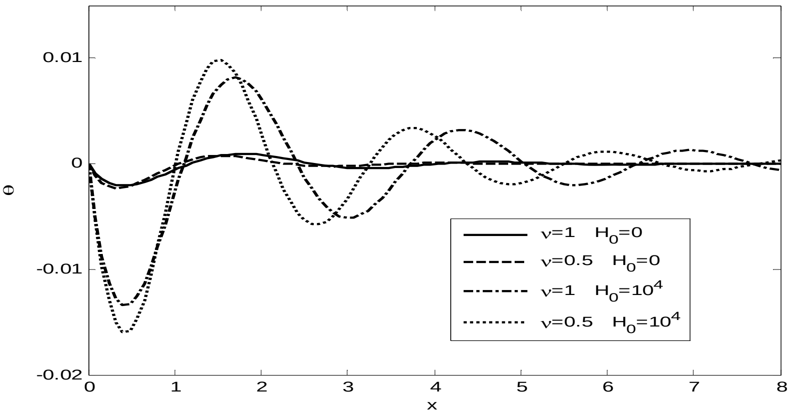

Fig. 5

Variation of the temperature distribution with x at $R_p = 10^3$.

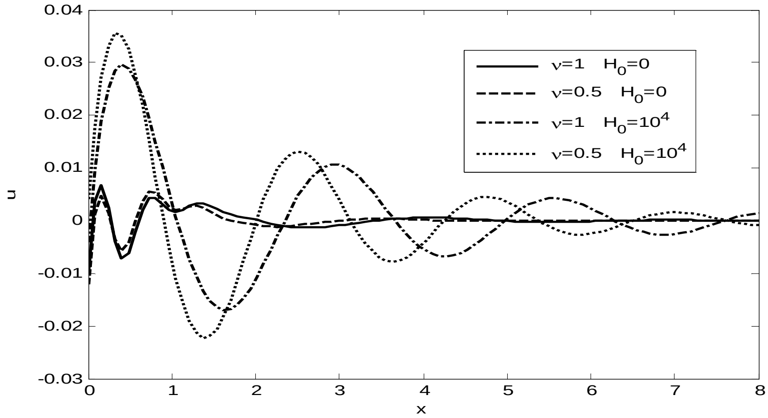

Fig. 6

Variation of the displacement distribution u with x at $R_p = 10^3$.

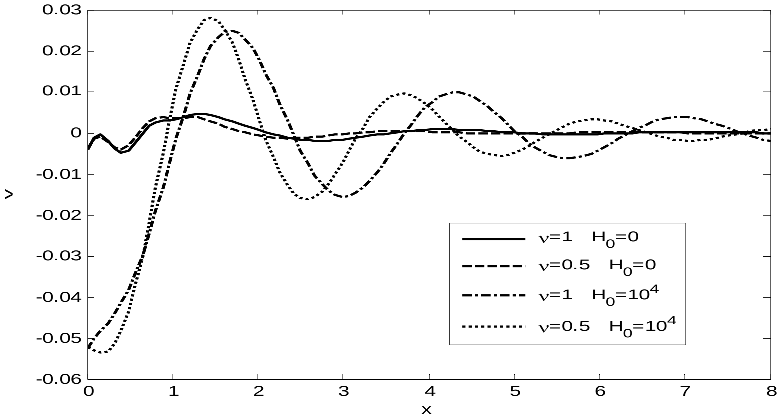

Fig. 7

Variation of the displacement distribution v with x at $R_p = 10^3$.

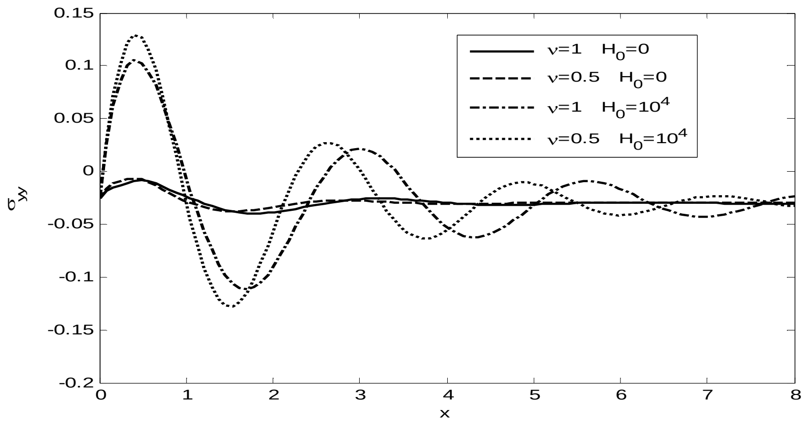

Fig. 8

Variation of the stress distribution σ_{yy} *with x at* $R_p = 10^3$.

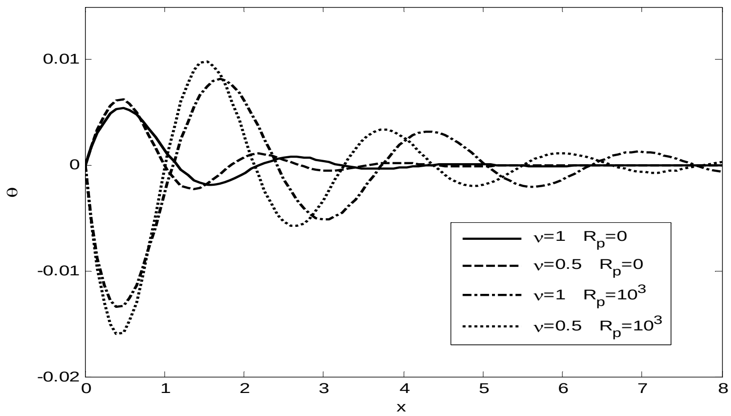

Fig. 9

Variation of the temperature distribution with x at $H_0 = 10^4$.

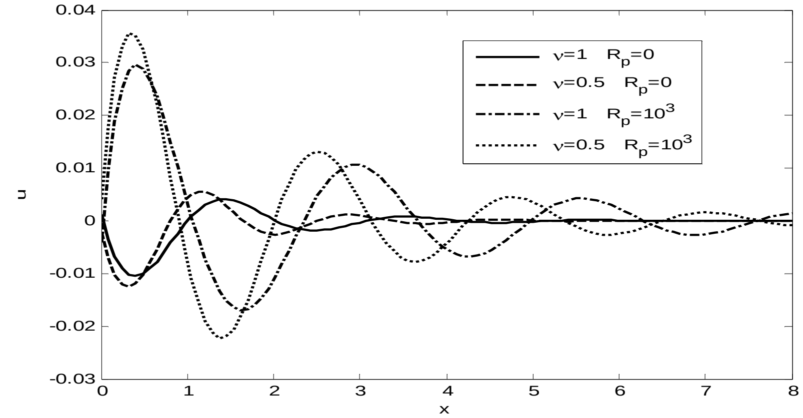

Fig. 10

Variation of the displacement distribution u with x at $H_0 = 10^4$.

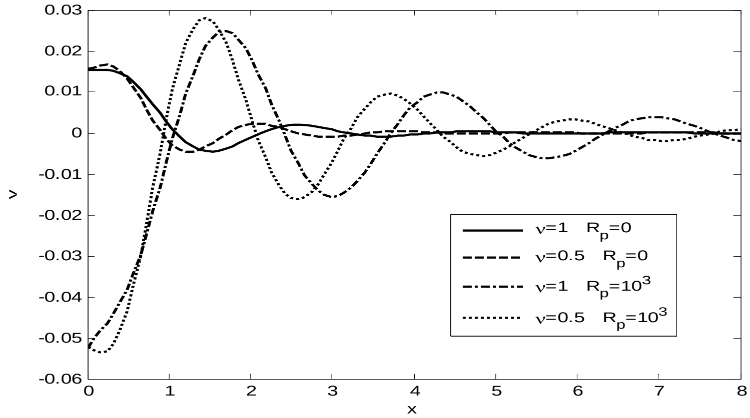

Fig. 11

Variation of the displacement distribution v with x at $H_0 = 10^4$.

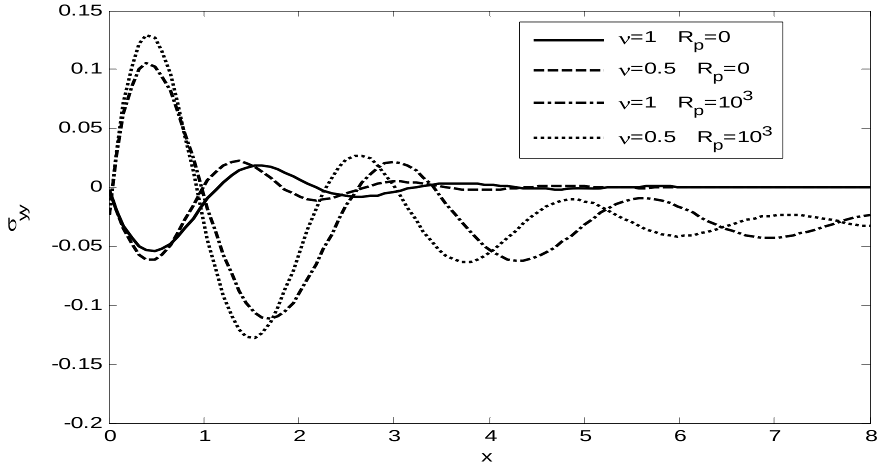

Fig. 12

Variation of the stress distribution σ_{yy} *with x at* $H_0 = 10^4$.

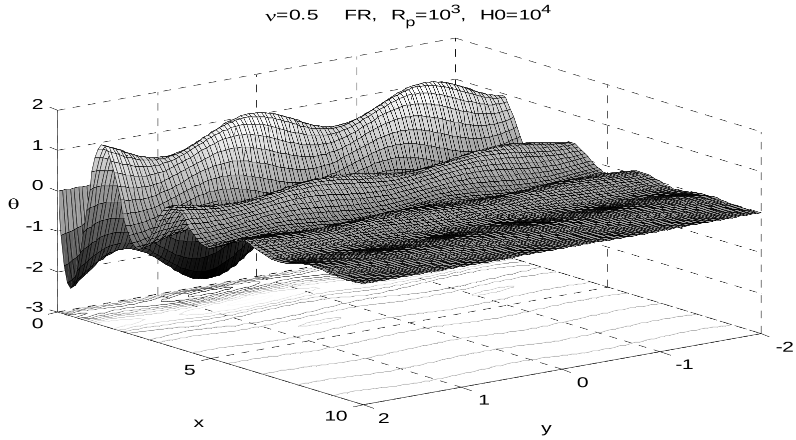

Fig. 13.

The three-dimensional temperature distribution with distance x and y.

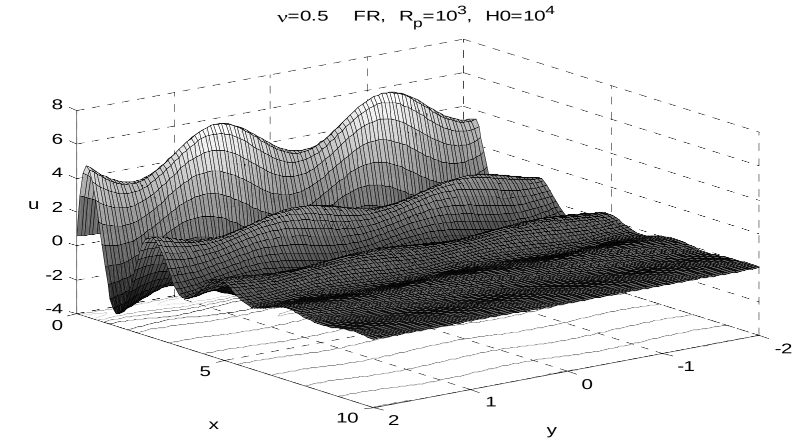

Fig. 14
The three-dimensional displacement distribution u with distance x and y.

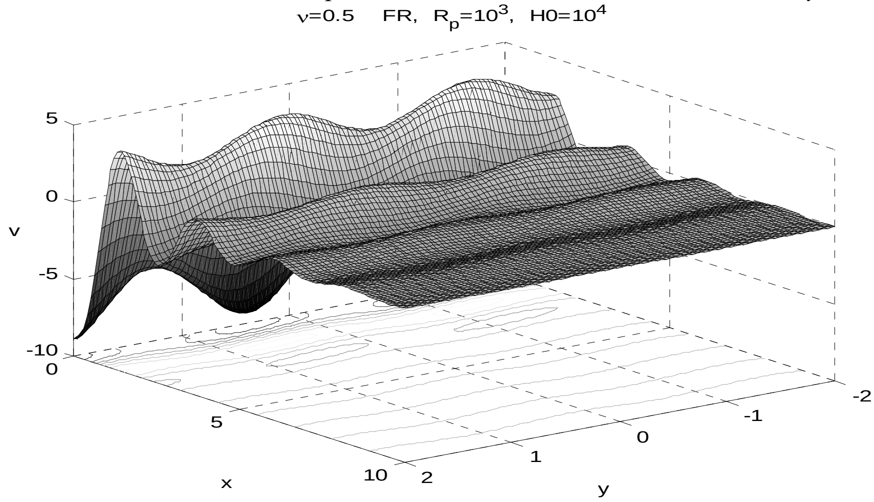

Fig. 15
The three-dimensional displacement distribution v with distance x and y.

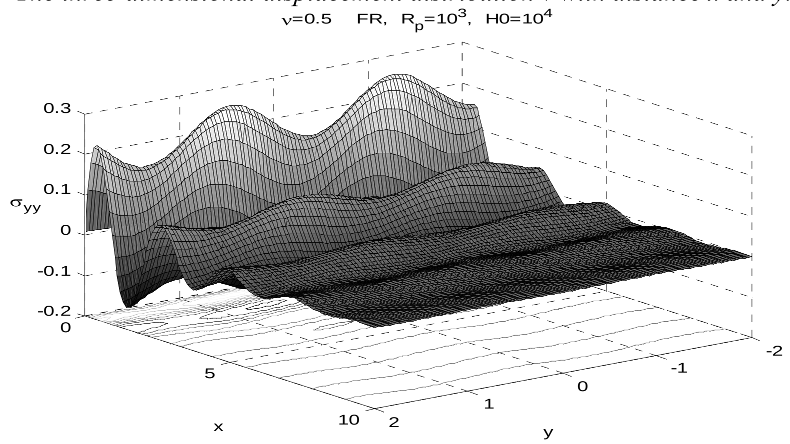

Fig. 16
The three-dimensional stress distribution σ_{yy} with distance x and y.

Fig. 13 also clearly shows that the temperature θ starts with a zero value which satisfies the boundary condition (44).

8. Concluding remarks.

According to the analysis above, we can conclude the following points:

1. The hydrostatic initial stress and the fractional parameter have a great effect on the distribution of the field quantities. The presence of the magnetic field plays a significant role in the field quantities.

2. It is clear from all the figures that all the distributions considered have a non-zero value only in a bounded region of the fiber-reinforced elastic half-space. Outside of this region, the values vanish identically and this means that the region has not felt a thermal disturbance yet.

3. The values of all the physical quantities converge to zero with increasing distance x.

4. All the physical quantities satisfy the boundary conditions.

5. Deformation of a body depends on the nature of the applied force as well as the type of boundary conditions.

6. The method that was used in the present article is applicable to a wide range of problems in thermodynamics and thermoelasticity.

7. Analytical solutions based upon normal mode analysis of the thermoelastic problem in solids have been developed and utilized.

8. From the temperature distributions, we have found wave type heat propagation with finite speeds in the medium.

9. The results presented in this paper should prove useful for researchers in material science, designers of new materials, low temperature physicists, as well as for those working on the development of a theory of hyperbolic thermoelasticity with a fractional derivative heat transfer. The introduction of the magnetic field and the fractional derivative heat transfer to the generalized thermoelastic medium provides a more realistic model for these studies.

Р Е З Ю М Е . Вивчено вплив трьох факторів – порядку дробових похідних, початкового гідростатичного напруження, параметру магнітного поля – на плоскі хвилі в півпросторі з армованого волокнами матеріалу, який описується теорією узагальненої магнітотермопружності. Задача розв’язана чисельно за допомогою аналізу нормальних мод. Результати аналізу відповідають моделі Лорда – Шульмана і моделі, що описується за допомогою дробових похідних, і представлені у вигляді графіків. Отримані результати показують добре виражений вплив вказаних трьох факторів. Також ці результати порівнюються з результатами, що отримані для випадку відсутності початкового гідростатичного напруження і магнітного поля.

Nomenclature

$\vec{H}$	applied magnetic field vector
$\vec{J}$	current density vector
$\vec{E}$	induced electric field vector
$\vec{h}$	induced magnetic field vector
ε_0	electric permeability
μ_0	magnetic permeability
λ, μ_L, μ_T	elastic parameters
$\alpha, \beta, (\mu_L - \mu_T), \gamma$	reinforcement parameters
ρ	acceleration due to gravity
c_E	specific heat of the solid at constant strain
τ_0	the thermal relaxation time parameter
ν	fractional parameter
σ_{ij}	components of the stress tensor
e_{ij}	components of the strain tensor
u_i	components of the displacement vector $\vec{u}$
e_{kk}	$=e$, cubical dilatation
t	time variable
x, y	space variables
T	absolute temperature
T_0	the temperature of the medium in it's natural state, assumed to be such that $\left\|\frac{\theta}{T_0}\right\| << 1$
γ	$= (3\lambda + 2\mu_T)\alpha_T$
α_T	coefficient of linear thermal expansion
k	thermal conductivity
P	the initial hydrostatic pressure
δ_{ij}	Kronecker delta
A_{11}	$= \lambda + 2\alpha + \beta + 4\mu_L - 2\mu_T$
A_{12}	$= \alpha + \lambda$
A_{22}	$= \lambda + 2\mu_T$
$R_{2,H}$	$= \frac{\mu_0 H_0^2}{\rho}$
c_1	$= \sqrt{\frac{A_{11}}{\rho}}$
c^2	$= \frac{1}{\varepsilon_0 \mu_0}$, speed of light
ε	$= \frac{\gamma^2 T_0}{A_{11}\rho c_E}$
D	$\equiv \frac{d}{dx}$

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