

On the Abstract Inverse Scattering Problem for Trace Class Perturbations

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The scattering problem for a pair of selfadjoint operators $\{L_0, L\}$, where $L - L_0$ is of trace-class, is studied. The explicit form of the scattering matrix and its properties are defined. The equation for the inverse problem is obtained.

Key words: scattering matrix, Friedrichs–Faddeev model, inverse scattering problem.

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Introduction

The scattering theory is not only the source of new non-traditional mathematical methods in the realm of perturbations of linear operators but it is also an important instrument for solving some problems of mathematical physics and performing calculus [1, 2]. Along with the problems of scattering theory, such as the existence of wave operators and their completeness, the properties of scattering matrix etc. (the Kato–Birman theory), the inverse problem, contained in the reconstruction of the initial operator system by the physically measured scattering data, plays an important, if not the main, role [1–6].

In this paper, the scattering problem for the pair of selfadjoint operators $\{L_0, L\}$ is solved under the assumption that $L - L_0$ is a trace-class operator. Similar problems were also studied in [8–10]. For simplicity sake (although this does not complicate significantly the considerations mentioned below), the operator

L_0 is assumed to be bounded. The main goal of the paper is to construct the scattering operator S for the given pair $\{L_0, L\}$ and solve the inverse problem. The paper consists of 5 sections, in the first of which the wave operators W_{\pm} and the scattering operator S are calculated using the methods of [2]. In Section 2, for the operator system $\{L_0, L\}$ under study, the Friedrichs–Faddeev model is constructed and the scattering matrix $S(x)$ is calculated. The main properties of $S(x)$ are proven and the link between $S(x)$ and the function of spectral shift $\xi(x)$ is determined (the Birman–Krein Theorem). Section 3 is dedicated to the study of the Friedrichs–Faddeev model $\{Q, \mathcal{L}\}$. The eigenfunctions of the operator \mathcal{L} are described and the type of spectral projectors of absolutely continuous part of the spectrum of the operator \mathcal{L} is defined. In Section 4, the main integral equation of the inverse problem (an analogue of the Marchenko equation) is obtained. The last section is dedicated to the solution of the inverse Riemann scattering problem for the case where $\dim E = 1$. Similar problems for one-dimensional perturbations are studied in [19–21]. Notice that the statement of the problem is not traditional: the Friedrichs–Faddeev model is unambiguously restored by the scattering data. The solution of the inverse problem for the case $\dim E > 1$ will be the subject of further publications.

1. Scattering Operator

I. In a separable Hilbert space H we consider a pair of bounded linear selfadjoint operators $\{L_0, L\}$ such that

$$L \stackrel{\text{def}}{=} L_0 - \varphi^* R \varphi, \tag{1}$$

where $\varphi : H \rightarrow E$ (E is a separable Hilbert space); $R = R^*$ is an operator in E . It follows from (1) that

$$R_0(\lambda) = R(\lambda) - R_0(\lambda) \varphi^* R \varphi R_0(\lambda) \quad (\lambda \notin \sigma(L_0) \cup \sigma(L)),$$

where $R_0(\lambda) = (L_0 - \lambda I)^{-1}$ and $R(\lambda) = (L - \lambda I)^{-1}$, and thus

$$\varphi R(\lambda) = (I - V(\lambda)R)^{-1} \varphi R_0(\lambda),$$

where $V(\lambda)$ is a Nevanlinna [9] operator-function in E ,

$$V(\lambda) \stackrel{\text{def}}{=} \varphi R_0(\lambda) \varphi^*. \tag{2}$$

Thus

$$R(\lambda) = R_0(\lambda) + R_0(\lambda) \varphi^* N(\lambda) \varphi R_0(\lambda), \tag{3}$$

where

$$N(\lambda) \stackrel{\text{def}}{=} R(I - V(\lambda)R)^{-1}. \tag{4}$$

Theorem 1. *The function $N(\lambda)$ (4) is Nevanlinna [12], $(N(\lambda) - R)$ is holomorphic at the infinitely distant point and the following statements are true:*

- 1) $N(\lambda) = (I - RV(\lambda))^{-1}R$;
- 2) $N^*(\lambda) = N(\bar{\lambda})$;
- 3) $N(\lambda) - N(w) = N(\lambda)[V(\lambda) - V(w)]N(w)$
 $= (\lambda - w)N(\lambda)\varphi R_0(\lambda)R_0(w)\varphi^*N(w)$; (5)
- 4) *the representation*

$$N(\lambda) = R + \int_a^b \frac{d\sigma(t)}{t - \lambda}$$

is true, where λ, w belong to the analyticity domain of $N(\lambda)$; $V(\lambda)$ is given by (2); $R_0(\lambda) = (L_0 - \lambda I)^{-1}$; $\sigma(t)$ is a nondecreasing operator-function of bounded variation in E ; $-\infty < a < b < \infty$.

P r o o f. Relation 1) follows from the identity $(I - RV(\lambda))R = R(I - V(\lambda)R)$. Equality 2) follows from 1) and from the fact that $R = R^*$, $V^*(\lambda) = V(\bar{\lambda})$. Since

$$\begin{aligned} N(\lambda) - N(w) &= (I - RV(\lambda))^{-1}R[V(\lambda) - V(w)]R(I - V(w)R)^{-1} \\ &= N(\lambda)[V(\lambda) - V(w)]N(w), \end{aligned}$$

3) is a corollary of $V(\lambda) - V(w) = (\lambda - w)\varphi R_0(\lambda)R_0(w)\varphi^*$. Taking into account that

$$N(\lambda) = R + RV(\lambda)R(I - V(\lambda)R)^{-1}, \quad \sup_{y>1} \|yV(iy)\| < \infty,$$

we obtain 4) in view of the well-known [12] representation for the Nevanlinna functions. ■

For a selfadjoint operator L in H and for all $\varepsilon > 0$,

$$\int_{\mathbb{R}_+} e^{i\lambda t} e^{-itL} e^{-\varepsilon t} dt = -iR(\lambda + i\varepsilon), \quad \int_{\mathbb{R}_+} e^{-i\lambda t} e^{itL} e^{-\varepsilon t} dt = iR(\lambda - i\varepsilon) \quad (6)$$

take place, where $\lambda \in \mathbb{R}$ and $R(\lambda) = (L - \lambda I)^{-1}$. Consider an operator-function

$$\delta(\lambda, \varepsilon) \stackrel{\text{def}}{=} \frac{1}{2\pi i} \{R(\lambda + i\varepsilon) - R(\lambda - i\varepsilon)\} = \frac{\varepsilon}{\pi} R(\lambda + i\varepsilon)R(\lambda - i\varepsilon) \quad (7)$$

($\lambda \in \mathbb{R}$, $\varepsilon > 0$), then [2]

$$\int_{\mathbb{R}} \delta(\lambda, \varepsilon) d\lambda = I. \quad (8)$$

II. We define the wave operators [1 – 3],

$$W_{\pm} \stackrel{\text{def}}{=} s - \lim_{t \rightarrow \mp\infty} \exp\{itL\} \cdot \exp\{-itL_0\} P_0^a, \quad (9)$$

where P_0^a is an orthoprojector on an absolutely continuous subspace of the operator L_0 . The Kato–Rosenblum Theorem [1, 2] yields that if R is of **trace-class**, then the limits (9) exist and W_{\pm} are complete, i. e., $W_{\pm}H = P^aH$, where P^a is an orthoprojector on an absolutely continuous subspace of the operator L . The operators W_{\pm} (9) are partial isometries from P_0^aH on P^aH [1, 2] and they intertwine the absolutely continuous parts of the operators L_0 and L ,

$$L^a W_{\pm} = W_{\pm} L_0^a, \quad (10)$$

where $L_0^a = L_0 P_0^a$, $L^a = L P^a$. The existence of W_{\pm} (9) yields that the limits W_{\pm} exist in the weak convergence sense and these limits coincide. Use the weak convergence by Abel [2].

Theorem 2. *If the limits W_{\pm} (9) exist, then*

$$\langle W_{\pm} f_0, f \rangle = \lim_{\varepsilon \rightarrow +0} \int_{\mathbb{R}_+} 2\varepsilon e^{-2\varepsilon t} \langle \exp\{\mp itL\} \exp\{\pm itL_0\} f_0, f \rangle dt, \quad (11)$$

$\forall f_0 \in P_0^a H, \forall f \in P^a H$.

P r o o f. Let us show that $A \rightarrow 0$ as $\varepsilon \rightarrow +0$, where

$$\begin{aligned} A &\stackrel{\text{def}}{=} \langle W_- f_0, f \rangle - 2\varepsilon \int_{\mathbb{R}_+} e^{-2\varepsilon t} \langle e^{itL} e^{-itL_0} f_0, f \rangle dt = \\ &= 2\varepsilon \int_{\mathbb{R}_+} e^{-2\varepsilon t} \langle (W_- - e^{itL} e^{-itL_0}) f_0, f \rangle dt. \end{aligned}$$

For all $\delta > 0$ there exists $N = N(\delta, f_0, f)$ such that $|\langle (W_- e^{itL} e^{-itL_0}) f_0, f \rangle| < \delta$ takes place for $t > N$, therefore

$$\left| 2\varepsilon \int_N^{\infty} e^{-2\varepsilon t} \langle (W_- - e^{itL} e^{-itL_0}) f_0, f \rangle dt \right| < \delta e^{-2\varepsilon N}.$$

Taking into account the unitarity of e^{itL} , e^{itL_0} and the isometricity of W_- , we obtain

$$\left| 2\varepsilon \int_0^N e^{-2\varepsilon t} \langle (W_- - e^{itL} e^{-itL_0}) f_0, f \rangle dt \right| \leq 2 \|f_0\| \cdot \|f\| (1 - e^{-2\varepsilon N}).$$

Using the obvious inequality $1 - e^{-\delta} \leq \delta$, which is true for all $\delta \in \mathbb{R}_+$, we have $|A| < \delta e^{-2\varepsilon N} + 4\varepsilon N \|f_0\| \cdot \|f\|$. Let ε be such that $\varepsilon N < \delta$. Then $|A| < C\delta$ ($C = \max\{1, 4\|f_0\| \cdot \|f\|\}$). ■

Theorem 3. *Let R in (1) be of trace-class, then for the wave operators W_{\pm} (9), the following representations are true:*

$$\langle W_{\pm} f_0, f \rangle = \langle f_0, f \rangle + \lim_{\varepsilon \rightarrow +0} \int_{\mathbb{R}} \langle R_0(\lambda \pm i\varepsilon) \varphi^* N(\lambda \pm i\varepsilon) \varphi \delta_0(\lambda, \varepsilon) f_0, f \rangle dt, \quad (12)$$

where $f_0 \in P_0^a H$, $f \in P^a H$; the functions $N(\lambda)$ and $\delta(\lambda, \varepsilon)$ are given by (4) and (7), respectively; $R_0(\lambda) = (L_0 - \lambda I)^{-1}$.

P r o o f. Since

$$\langle W_- f_0, f \rangle = \lim_{\varepsilon \rightarrow +0} \int_{\mathbb{R}_+} e^{-2\varepsilon t} \langle e^{-itL_0} f_0, e^{-itL} f \rangle dt$$

in view of (11), then using the Parseval inequality for the Fourier transform and (6), we obtain

$$\langle W_- f_0, f \rangle = \lim_{\varepsilon \rightarrow +0} \frac{\varepsilon}{\pi} \int_{\mathbb{R}} \langle R_0(\lambda + i\varepsilon) f_0, R(\lambda + i\varepsilon) f \rangle d\lambda.$$

Taking into account (3) and (7), we have

$$\begin{aligned} & \langle W_- f_0, f \rangle = \\ & = \lim_{\varepsilon \rightarrow +0} \left\{ \int_{\mathbb{R}} \langle \delta_0(\lambda, \varepsilon) f_0, f \rangle d\lambda + \int_{\mathbb{R}} \langle R_0(\lambda - i\varepsilon) \varphi^* N(\lambda - i\varepsilon) \varphi \delta_0(\lambda, \varepsilon) f_0, f \rangle d\lambda \right\}, \end{aligned}$$

which, in view of (8), gives us (12). For W_+ , formula (12) is proved in a similar way. ■

Formula (12) implies

$$\langle W_{\mp}^* f, f_0 \rangle = \langle f, f_0 \rangle + \lim_{\varepsilon \rightarrow +0} \int_{\mathbb{R}} \langle \delta_0(\mu, \varepsilon) \varphi^* N(\mu \pm i\varepsilon) \varphi R_0(\mu \pm i\varepsilon) f, f_0 \rangle d\mu. \quad (13)$$

Define the scattering operator [1–3],

$$S \stackrel{\text{def}}{=} W_+^* W_-. \quad (14)$$

The operator S maps $P_0^a H$ onto itself isometrically, and

$$SL_0^a = L_0^a S \tag{15}$$

in view of (10), where L_0^a is an absolutely continuous part of L_0 .

Theorem 4. *The scattering operator S (14) of the pair $\{L_0, L\}$ (L is given by (1), R is of trace-class) is given by the formula*

$$\langle (S - I)f_0, g_0 \rangle = -2\pi i \lim_{\varepsilon \rightarrow +0} \int_{\mathbb{R}} \langle \delta_0(\lambda, \varepsilon) \varphi^* N(\lambda - i\varepsilon) \varphi \delta_0(\lambda, \varepsilon) f_0, g_0 \rangle d\lambda, \tag{16}$$

where $f_0, g_0 \in P_0^a H$; $N(\lambda)$ and $\delta_0(\lambda, \varepsilon)$ are given by (4) and (7), respectively.

P r o o f. Formulas (12), (13) imply

$$\begin{aligned} \langle S f_0, g_0 \rangle &= \langle W_- f_0, W_+ g_0 \rangle = \langle f_0, W_+ g_0 \rangle + \lim_{\varepsilon \rightarrow +0} \int_{\mathbb{R}} \langle R_0(\lambda - i\varepsilon) \varphi^* N(\lambda - i\varepsilon) \varphi \\ &\quad \times \delta_0(\lambda, \varepsilon) f_0, W_+ g_0 \rangle d\lambda = \langle f_0, g_0 \rangle + \lim_{\varepsilon_1 \rightarrow +0} \int_{\mathbb{R}} \langle \delta_0(\mu, \varepsilon_1) \varphi^* N(\mu - i\varepsilon_1) \varphi \\ &\quad \times R_0(\mu - i\varepsilon_1) f_0, g_0 \rangle d\mu + \lim_{\varepsilon \rightarrow +0} \int_{\mathbb{R}} \langle R_0(\lambda - i\varepsilon) \varphi^* N(\lambda - i\varepsilon) \varphi \delta_0(\lambda, \varepsilon) f_0, g_0 \rangle d\lambda \\ &+ \lim_{\varepsilon_1 \rightarrow +0} \lim_{\varepsilon \rightarrow +0} \int_{\mathbb{R}} d\lambda \int_{\mathbb{R}} d\mu \langle \delta_0(\mu, \varepsilon_1) \varphi^* N(\mu - i\varepsilon_1) \varphi R_0(\mu - i\varepsilon_1) R_0(\lambda - i\varepsilon) \varphi^* \\ &\quad \times N(\lambda - i\varepsilon) \varphi \delta_0(\lambda, \varepsilon) f_0, g_0 \rangle. \end{aligned}$$

The double integral, in view of 3) (5), is equal to

$$B \stackrel{\text{def}}{=} \int_{\mathbb{R}} d\lambda \int_{\mathbb{R}} d\mu \left\langle \delta_0(\mu, \varepsilon_1) \varphi^* \frac{N(\mu - i\varepsilon_1) - N(\lambda - i\varepsilon)}{\mu - \lambda + i(\varepsilon - \varepsilon_1)} \varphi \delta_0(\lambda, \varepsilon) f_0, g_0 \right\rangle.$$

Calculate the expression

$$\begin{aligned} \int_{\mathbb{R}} \frac{d\lambda}{\mu - \lambda + i(\varepsilon - \varepsilon_1)} \delta_0(\lambda, \varepsilon) &= \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{d\lambda}{\mu - \lambda + i(\varepsilon_1 - \varepsilon)} \left\{ \int_{\mathbb{R}} \frac{dE_x}{x - \lambda - i\varepsilon} \right. \\ &\quad \left. - \int_{\mathbb{R}} \frac{dE_x}{x - \lambda + i\varepsilon} \right\} = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{dE_x}{x - \mu + i(\varepsilon_1 - 2\varepsilon)} \left\{ \int_{\mathbb{R}} \left[\frac{1}{\lambda - x + i\varepsilon} \right. \right. \end{aligned}$$

$$-\frac{1}{\lambda - \mu + i(\varepsilon_1 - \varepsilon)} d\lambda \Big\} - \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{dE_x}{x - \mu + i\varepsilon_1} \left\{ \int_{\mathbb{R}} \left[\frac{1}{\lambda - x - i\varepsilon} - \frac{1}{\mu - \lambda + i(\varepsilon_1 - \varepsilon)} \right] d\lambda \right\}.$$

If $\varepsilon > \varepsilon_1$, then

$$\int_{\mathbb{R}} \left[\frac{1}{\lambda - x + i\varepsilon} - \frac{1}{\lambda - \mu + i(\varepsilon_1 - \varepsilon)} \right] d\lambda = -2\pi i,$$

$$\int_{\mathbb{R}} \left[\frac{1}{\lambda - x - i\varepsilon} - \frac{1}{\mu - \lambda + i(\varepsilon_1 - \varepsilon)} \right] d\lambda = 0,$$

and

$$\int_{\mathbb{R}} \frac{d\lambda}{\mu - \lambda + i(\varepsilon - \varepsilon_1)} \delta_0(\mu, \varepsilon) = -R_0(\mu - i(2\varepsilon - \varepsilon_1)) \quad (\varepsilon > \varepsilon_1).$$

Analogously,

$$\int_{\mathbb{R}} \frac{d\mu}{\mu - \lambda + i(\varepsilon - \varepsilon_1)} \delta_0(\mu, \varepsilon_1) = R_0(\lambda_1 - \varepsilon) \quad (\varepsilon > \varepsilon_1).$$

Calculate the double integral

$$B = - \int_{\mathbb{R}} d\mu \langle \delta_0(\mu, \varepsilon_1) \varphi^* R_0(\mu - i(2\varepsilon - \varepsilon_1)) f_0, g_0 \rangle - \int_{\mathbb{R}} d\lambda \langle R_0(\lambda - i\varepsilon) \varphi^* N(\lambda - i\varepsilon) \varphi \delta_0(\lambda, \varepsilon) f_0, g_0 \rangle.$$

Substituting this expression into the sum for $\langle S f_0, g_0 \rangle$, we obtain

$$\begin{aligned} \langle (S - I) f_0, g_0 \rangle &= \lim_{\varepsilon, \varepsilon_1 \rightarrow +0} \int_{\mathbb{R}} d\mu \langle \delta(\mu, \varepsilon_1) \varphi^* N(\mu - i\varepsilon_1) \varphi \times \\ &\quad \times \{ R_0(\mu - i\varepsilon_1) - R_0(\mu + i(2\varepsilon - \varepsilon_1)) \} f_0, g_0 \rangle. \end{aligned}$$

Proceeding to the limit as $\varepsilon \rightarrow \varepsilon_1 + 0$ ($\varepsilon > \varepsilon_1$) and taking into account (7), we obtain (16). For $\varepsilon_1 > \varepsilon$, the considerations are similar and we again obtain formula (16). ■

2. The Friedrichs–Faddeev Model

III. Consider the subspace $H_1 (\subseteq H)$,

$$H_1 \stackrel{\text{def}}{=} \text{span} \{R_0(\lambda)\varphi^*E : \lambda \in \mathbb{C} \setminus \sigma(L_0)\}, \tag{17}$$

where $R_0(\lambda) = (L_0 - \lambda I)^{-1}$. The subspace H_1 reduces the operators L_0 and L_1 (1). Since $H_1^\perp \subset \text{Ker } \varphi$, then $L|_{H_1^\perp} = L_0|_{H_0^\perp}$ and the wave operators W_\pm (9) corresponding to these restrictions coincide with the projector P_1^a (P_1^a is a projector on the absolutely continuous subspace of $L_0|_{H_1^\perp}$). The scattering operator S (14) on H_1^\perp is a trivial one; $S|_{H_1^\perp} = P_1^a$. Therefore the supposition that $H = H_1$ (17) is reasonable, and thus $\overline{\varphi^*E}$ is **the generating subspace** of the operator L_0 [13].

In E , specify a non-descending operator-function, $F(x) = \varphi E_x \varphi^*$, where E_x is the resolution of identity of the operator L_0 . The operator L_0 being bounded, the support of the measure $dF(x)$ is finite. Define the Hilbert space

$$L_{\mathbb{R}}^2(E, dF(x)) \stackrel{\text{def}}{=} \left\{ f(x) \in E : \int_{\mathbb{R}} \langle dF(x)f(x), f(x) \rangle < \infty \right\} \tag{18}$$

obtained as a result of the closure of the class of continuous vector-functions $f(x)$ from E . If $H_1 = H$, then the operator $U : L_{\mathbb{R}}^2(E, dF(x)) \rightarrow H$, given by the formula

$$f = Uf(x), \quad f \stackrel{\text{def}}{=} \int_{\mathbb{R}} dE_x \varphi^* f(x) \tag{19}$$

($f \in H, f(x) \in L_{\mathbb{R}}^2(E, dF(x))$), specifies the unitary isomorphism between H ($= H_1$) and $L_{\mathbb{R}}^2(E, dF(x))$ (18).

Theorem 5. Let $H = H_1$ (17) and $U : L_{\mathbb{R}}^2(E, dF(x)) \rightarrow H$ be given by (19), then the operators $\mathcal{Q} = U^*L_0U$ and $\mathcal{L} = U^*LU$ are

$$(\mathcal{Q}f)(x) \stackrel{\text{def}}{=} xf(x), \quad (\mathcal{L}f) \stackrel{\text{def}}{=} xf(x) - R \int_{\mathbb{R}} dF(t)f(t), \tag{20}$$

where $f(x) \in L_{\mathbb{R}}^2(E, dF(x))$.

P r o o f. The representation of \mathcal{Q} (20) follows from the equality

$$L_0Uf(x) = L_0 \int_{\mathbb{R}} dE_x \varphi^* f(x) = \int_{\mathbb{R}} dE_x \varphi^* (xf(x)) = U\mathcal{Q}f(x).$$

Since $\tilde{\varphi} = \varphi U$ is given by

$$\tilde{\varphi}f(x) = \int_{\mathbb{R}} dF(x)f(x),$$

then, taking into account that $\tilde{\varphi}^*$ is the operator of enclosure of E into $L^2_{\mathbb{R}}(E, dF(x))$, we obtain the representation of \mathcal{L} in view of (1). ■

Formulas (20), well-known in the scattering theory [2, 4], are called **the Friedrichs–Faddeev model**.

The scattering operator S (14) corresponds to the operator of multiplication by the operator-function $S(x) = U^*SU$ in E (U is given by (19)), which is commonly [1, 2] named a **scattering matrix**. The above follows from the commutativity of U^*SU and \mathcal{Q} (20) in view of (15).

Further, we will suppose that the operator φ from (1) belongs to the Hilbert–Schmidt class. Denote the function $F(x) = \varphi E_x \varphi^*$ which also belongs to this class for all $x \in \mathbb{R}$. Moreover, we suppose that the function $F(x)$ is absolutely continuous in the sense that there exists a nonnegative integrable (by Bochner) operator-function $a(\cdot) : \mathbb{R} \rightarrow B(E)$, ($B(E)$ is the space of bounded operators in E), such that

$$F(\Delta) = \int_{\Delta} a(x) dx \quad (a(x) \geq 0)$$

for any Borel set Δ from \mathbb{R} . Formally, we write this condition of absolute continuity in the form

$$dF(x) = a(x) dx. \tag{21}$$

Theorem 6. *Let $H = H_1$ (17), and the function $F(x) = \varphi E_x \varphi^*$ be absolutely continuous, moreover, we suppose that R in (1) is of trace-class and φ belongs to the Hilbert–Schmidt class. Then the scattering matrix $S(x) = U^*SU$ corresponding to S (14) is given by*

$$S(x) = I - 2\pi i N_-(x) a(x) \quad (x \in \mathbb{R}), \tag{22}$$

where $N_-(x) = N(x - i0)$ are the boundary values on \mathbb{R} from \mathbb{C}_- of the function $N(\lambda)$ (4).

P r o o f. Let

$$f_0 = \int_{\mathbb{R}} dE_x \varphi^* f_0(x) \quad (f_0 \in L^2_{\mathbb{R}}(E, dF(x))). \tag{23}$$

Then (16) implies

$$\begin{aligned} & \langle (S - I)f_0, g_0 \rangle \\ &= -2\pi i \lim_{\varepsilon \rightarrow +0} \int_{\mathbb{R}} \left\langle \delta_0(\lambda, \varepsilon) \varphi^* N(\lambda - i\varepsilon) \varphi \delta_0(\lambda, \varepsilon) \int_{\mathbb{R}} dE_x \varphi^* f_0(x), g_0 \right\rangle d\lambda. \end{aligned}$$

Since

$$\begin{aligned} \varphi\delta_0(\lambda, \varepsilon) \int_{\mathbb{R}} dE_x \varphi^* f_0(x) &= \frac{\varepsilon}{\pi} \varphi \int_{\mathbb{R}} dE_x \varphi^* f_0(x) \\ &= \frac{\varepsilon}{\pi} \int_{\mathbb{R}} \frac{dF(x) f_0(x)}{(x - \lambda)^2 + \varepsilon^2} = \frac{1}{\pi} \int_{\mathbb{R}} \frac{\varepsilon}{(x - \lambda)^2 + \varepsilon^2} a(x) f_0(x) dx, \end{aligned}$$

then, using the properties of the Poisson kernel [2, 14, 15], we obtain

$$\varphi\delta_0(\lambda, x) \int_{\mathbb{R}} dE_x \varphi^* f_0(x) \rightarrow a(\lambda) f_0(\lambda) \quad (\varepsilon \rightarrow +0).$$

Taking into account [13, 14],

$$\langle \delta_0(\lambda, \varepsilon) f, g \rangle \rightarrow \frac{d}{d\lambda} \langle E_\lambda f, g \rangle \quad (\varepsilon \rightarrow +0),$$

and the existence of the boundary values $N(\lambda - i\varepsilon) \rightarrow N_-(\lambda)$ ($= N(\lambda - i0)$) as $\varepsilon \rightarrow +0$ [11, 12], we obtain

$$\langle (S - I) f_0, g_0 \rangle = -2\pi i \int_{\mathbb{R}} \langle dE_\lambda \varphi^* N_-(\lambda) a(\lambda) f_0(\lambda), g_0 \rangle.$$

Hence,

$$S f_0 = \int_{\mathbb{R}} dE_\lambda \varphi^* \{I - 2\pi i N_-(\lambda) a(\lambda)\} f_0(\lambda) = U \{I - 2\pi i N_-(\lambda) a(\lambda)\} f_0(\lambda),$$

which gives us (22). ■

Let

$$S^+ = S^* \stackrel{\text{def}}{=} W_-^* W_+. \tag{24}$$

Then, similarly to (16),

$$\langle (S^+ - I) f_0, g_0 \rangle = 2\pi i \lim_{\varepsilon \rightarrow +0} \int_{\mathbb{R}} \langle \delta_0(\lambda, \varepsilon) \varphi^* N(\lambda + i\varepsilon) \varphi \delta_0(\lambda, \varepsilon) f_0, g_0 \rangle d\lambda,$$

and thus the scattering matrix $S^+(x) = U^* S^+ U$ can be written in the form

$$S^+(x) = I + 2\pi i N_+(x) a(x) \quad (x \in \mathbb{R}), \tag{25}$$

where $N_+(x) = N(x + i0)$ are the boundary values on \mathbb{R} from \mathbb{C}_+ of the function $N(\lambda)$ (4).

It is obvious that as $x \notin \overline{\text{supp } a(x)}$ the matrices $S(x)$ (22) and $S^+(x)$ (25) are equal to the identity operator, moreover,

$$a(x)S^+(x) = S^*(x)a(x) \quad (x \in \mathbb{R}). \tag{26}$$

Hence, $S^+(x)$ is adjoint to $S(x)$ in the metric of $L^2_{\mathbb{R}}(E, dF(x))$.

Let us find the functional realizations of the wave operators W_{\pm} (12). Substitute f_0 (23) into representation (12) for W_- , then

$$\begin{aligned} & \langle (W_- - I) f_0, f \rangle \\ &= \lim_{\varepsilon \rightarrow +0} \int_{\mathbb{R}} \left\langle R_0(\lambda - i\varepsilon) \varphi^* N(\lambda - i\varepsilon) \varphi \delta_0(\lambda, \varepsilon) \int_{\mathbb{R}} dE_x \varphi^* f_0(x), f \right\rangle d\lambda \\ &= \lim_{\varepsilon \rightarrow +0} \int_{\mathbb{R}} \left\langle dE_y \varphi^* \int_{\mathbb{R}} \frac{N(\lambda - i\varepsilon)}{y - \lambda + i\varepsilon} \varphi \delta_0(\lambda, \varepsilon) \int_{\mathbb{R}} dE_x \varphi^* f_0(x), f \right\rangle d\lambda, \end{aligned}$$

which gives

$$\langle (W_- - I) f_0, f \rangle = - \int_{\mathbb{R}} \left\langle dE_x \varphi^* \int_{\mathbb{R}} \frac{N_-(\lambda)}{\lambda - x - i0} a(\lambda) f(\lambda) d\lambda, f \right\rangle.$$

Theorem 7. *If $H = H_1$ (17), the function $F(x) = \varphi E_x \varphi^*$ is absolutely continuous (21), the operator R from (1) is of trace-class, and φ is a Hilbert-Schmidt operator, then the wave operators $\widetilde{W}_{\pm} = U^* W_{\pm} U$ are*

$$\left(\widetilde{W}_{\mp} f \right) (x) = f(x) - \int_{\mathbb{R}} \frac{N_{\mp}(\lambda)}{\lambda - x \mp i0} a(\lambda) f(\lambda) d\lambda, \tag{27}$$

where $f(x) \in L^2_{\mathbb{R}}(E, dF(x))$.

Using (22), (24), we obtain

$$\begin{aligned} \left(\widetilde{W}_- f \right) (x) &= f(x) - \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{I - S(\lambda)}{\lambda - x - i0} f(\lambda) d\lambda, \\ \left(\widetilde{W}_+ f \right) (x) &= f(x) - \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{S^+(\lambda) - I}{\lambda - x + i0} f(\lambda) d\lambda, \end{aligned} \tag{28}$$

which, by the Sokhotsky formulas [16], gives us the representations

$$\begin{aligned} \left(\widetilde{W}_- f \right) (x) &= \frac{1}{2} (I + S(\lambda)) f(\lambda) - \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{I - S(\lambda)}{\lambda - x} f(\lambda) d\lambda, \\ \left(\widetilde{W}_+ f \right) (x) &= \frac{1}{2} (S^+(\lambda) + I) f(\lambda) - \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{S^+(\lambda) - I}{\lambda - x} d\lambda. \end{aligned} \tag{29}$$

The intertwining conditions follow directly from (10),

$$\mathcal{L}^a \widetilde{W}_\pm = \widetilde{W}_\pm \mathcal{Q}^a. \tag{30}$$

Construct a direct proof. Using (27), we get

$$\begin{aligned} (\widetilde{W}_- \mathcal{Q}^a f)(x) &= x f(x) - \pi i N_-(x) a(x) x f(x) - \int_{\mathbb{R}} \frac{N_-(\lambda)}{\lambda - x} a(\lambda) \lambda f(\lambda) \\ &= x (\widetilde{W}_- f)(x) - \int_{\mathbb{R}} N_-(\lambda) a(\lambda) f(\lambda) d\lambda. \end{aligned}$$

Taking into account the identity $N(\lambda) = R + RV(\lambda)N(\lambda)$, we obtain

$$\begin{aligned} (\widetilde{W}_- \mathcal{Q}^a f)(x) &= x (\widetilde{W}_- f)(x) - R \int_{\mathbb{R}} a(\lambda) f(\lambda) - R \int_{\mathbb{R}} \left\{ -\pi i a(\lambda) + \int_{\mathbb{R}} \frac{a(t) dt}{t - \lambda} \right\} \\ &\times N_-(\lambda) a(\lambda) f(\lambda) d\lambda = x (\widetilde{W}_- f)(x) - R \int_{\mathbb{R}} a(\lambda) \{ f(\lambda) - \pi i N_-(\lambda) a(\lambda) f(\lambda) \\ &- \int_{\mathbb{R}} \frac{N_-(x)}{x - \lambda} a(x) f(x) dx \} d\lambda = \mathcal{L}^a (\widetilde{W}_- f)(x). \end{aligned}$$

Equalities (30) show that \widetilde{W}_\pm (27) are the **transformation operators** [4, 5].

IV. The absolute continuity of $F(x)$ (21) and (7) imply

$$a(x) = \lim_{\varepsilon \rightarrow +0} \varphi \delta_0(\lambda, \varepsilon) \varphi = \frac{1}{2\pi i} \lim_{\varepsilon \rightarrow +0} [V(x + i\varepsilon) - V(x - i\varepsilon)],$$

thus

$$V_+(x) - V_-(x) = 2\pi i a(x), \tag{31}$$

where $V_\pm(x)$ are the boundary values on \mathbb{R} from \mathbb{C}_\pm of the function $V(\lambda)$ (2). Equality (31) coincides with the Stieltjes–Perron inversion formula [2, 12] and may also follow from the Sokhotsky formula [16] for $V_\pm(x)$.

Using 3) (5) and (31), we obtain

$$\begin{aligned} N_+(x) - N_-(x) &= 2\pi i N_+(x) a(x) N_-(x), \\ N_+(x) - N_-(x) &= 2\pi i N_-(x) a(x) N_+(x), \end{aligned} \tag{32}$$

where $N_\pm(x)$ are the boundary values on \mathbb{R} from \mathbb{C}_\pm of the function $N(\lambda)$ (4).

Theorem 8. *If the representing measure $F(x)$ of the function $V(\lambda)$ (2) is absolutely continuous (21) and the suppositions of Theorem 6 take place, then*

$$\begin{aligned} 1) \quad & S^*(x)a(x)S(x) = a(x), \quad (S^+(x))^* a(x)S^+(x) = a(x), \\ 2) \quad & S^+(x)S(x) = S(x)S^+(x) = I, \end{aligned} \tag{33}$$

where $S(x)$ and $S^+(x)$ are given by (22) and (25).

P r o o f. The definition of $S(x)$ from (22) implies

$$\begin{aligned} S^*(x)a(x)S(x) &= a(x) - 2\pi ia(x)N_-(x)a(x) + 2\pi ia(x)N_+(x)a(x) - \\ &\quad - (2\pi i)^2 a(x)N_+(x)a(x)N_-(x)a(x) = a(x) \end{aligned}$$

in view of (32). We prove the second relation from 1) in a similar way. Taking into account (22), (25), and (32), we obtain

$$S^+(x)S(x) = I - 2\pi i N_-(x)a(x) + 2\pi i N_+(x)a(x) - (2\pi i)^2 N_+(x)a(x)N_-(x)a(x) = I$$

in view of (32). ■

Corollary 1. *The equalities 1) (33) prove that the reciprocal matrix-functions $S(x)$ and $S^+(x)$ are unitary in the metric of the space $L^2_{\mathbb{R}}(E, dF(x))$ (18) when $dF(x) = a(x)dx$.*

Using (22) and (31), we obtain

$$S(x) = I - (I - RV_-(x))^{-1} R(V_+(x) - V_-(x)) = (I - RV_-(x))^{-1} (I - RV_+(x)),$$

therefore

$$S(x) = (I - RV_-(x))^{-1} (I - RV_+(x)).$$

Thus,

$$S(x) (I - RV_+(x))^{-1} = (I - RV_-(x))^{-1}, \tag{34}$$

which, in view of 1) (5), gives us

$$S(x)N_+(x) = N_-(x). \tag{35}$$

Theorem 9. *Let the suppositions of Theorem 6 be true. Then the Nevanlinna function $N(\lambda)$ (4) is the solution of the Riemann boundary value problem [16], (35), the coefficient of which is the scattering matrix $S(x)$ (22).*

The equalities 2) (33) are dual to (34),

$$S^+(x)N_-(x) = N_+(x), \tag{36}$$

where $S^+(x)$ is given by (25).

V. From (3) and the fact that $N(\lambda)$ (4) is of trace-class we have

$$\operatorname{tr} \{R(\lambda) - R_0(\lambda)\} = \operatorname{tr} \{\varphi R_0^2(\lambda) \varphi^* N(\lambda)\} = \operatorname{tr} \{V'(\lambda) N(\lambda)\}.$$

Using (4) and the formula from [17],

$$\frac{d}{d\lambda} \ln \det(I - A(\lambda)) = -\operatorname{tr} \{A'(\lambda)(I - A(\lambda))^{-1}\},$$

we obtain

$$\operatorname{tr} \{V'(\lambda) N(\lambda)\} = -\frac{d}{d\lambda} D(\lambda),$$

where

$$D(\lambda) \stackrel{\text{def}}{=} \det(I - RV(\lambda)) \tag{37}$$

is the **perturbation operator** [17] of the pair of operators \mathcal{Q} , \mathcal{L} (20). The existence of (37) is determined by the fact that R is of trace-class.

The following statement belongs to M. G. Krein [2, 18].

Theorem 10. *Let R be of trace-class. Then for $D(\lambda)$ (37),*

$$\ln D(\lambda) = -\int_{\mathbb{R}} \frac{\xi(x)}{x - \lambda} dx \quad (\operatorname{Im} \lambda \neq 0) \tag{38}$$

takes place, where $\xi(x)$ is the **spectral shift function** [2, 18],

$$\xi(x) \stackrel{\text{def}}{=} -\frac{1}{\pi} \lim_{\varepsilon \rightarrow +0} \arg D(x + i\varepsilon). \tag{39}$$

The limit (38) exists almost everywhere for $x \in \mathbb{R}$, and

$$\int_{\mathbb{R}} |\xi(x)| dx \leq \|R\|_1; \quad \int_{\mathbb{R}} \xi(x) dx = \operatorname{tr} R$$

($\|R\|_1$ is the trace-class norm of R). Moreover, $\xi(x) \leq k_+$ ($\xi(x) \geq k_-$) almost everywhere if R has exactly k_+ positive (k_- negative) eigenvalues.

Using this theorem, we obtain

$$\operatorname{tr} \{R(\lambda) - R_0(\lambda)\} = \int_{\mathbb{R}} \frac{\xi(x)}{(x - \lambda)^2} dx \quad (\operatorname{Im} \lambda \neq 0),$$

which implies the well-known **Krein trace formula** [2, 18].

Theorem 11 (M. G. Krein). *Let R be of trace-class and $f(\lambda)$ be a holomorphic function as $\lambda \in \sigma(L_0) \cup \sigma(L_1)$, then*

$$\operatorname{tr} \{f(L) - f(L_0)\} = \int_{\mathbb{R}} \xi(x) f'(x) dx.$$

The link between a scattering matrix and a function of spectral shift was established by M. Sh. Birman and M. G. Krein [2].

Theorem 12 (M. Sh. Birman, M. G. Krein). *If R is of trace-class, then for almost all $x \in \mathbb{R}$ the formula*

$$\det S(x) = \exp\{-2\pi i\xi(x)\} \tag{40}$$

is true, where $S(x)$ is the scattering matrix (22), and $\xi(x)$ is the function of spectral shift (39).

P r o o f. Equality (34) yields

$$\det S(x) = D(x + i0)D^{-1}(x - i0),$$

where $D(\lambda)$ is given by (37). To conclude the proof it remains to use definition (39) of the function $\xi(x)$. ■

3. Scattering Matrix

VI. We proceed to the analysis of the Friedrichs–Faddeev model (20) assuming that $F(x)$ is absolutely continuous (21). To find the resolvent $R_{\mathcal{L}}(\lambda) = (\mathcal{L} - \lambda I)^{-1}$, we denote $(\mathcal{L} - \lambda I)^{-1}f(x) = g(x)$. Then

$$(x - \lambda)g(x) - R \int_{\mathbb{R}} a(t)g(t)dt = f(x).$$

In terms of $G(x) = (x - \lambda)g(x)$ this equation takes the form

$$G(x) - R \int_{\mathbb{R}} a(t) \frac{G(t)}{t - \lambda} dt = f(x).$$

Multiplying this equality by $a(x)(x - \lambda)^{-1}$ and integrating it, we obtain

$$\int_{\mathbb{R}} a(x) \frac{G(x)}{x - \lambda} dx - V(\lambda)R \int_{\mathbb{R}} a(t) \frac{G(t)}{t - \lambda} dt = \int_{\mathbb{R}} a(x) \frac{f(x)}{x - \lambda} dx.$$

Hence,

$$\int_{\mathbb{R}} a(x) \frac{G(x)}{x - \lambda} dx = (I - V(\lambda)R)^{-1} \int_{\mathbb{R}} a(x) \frac{f(x)}{x - \lambda} dx.$$

Theorem 13. *The resolvent $R_{\mathcal{L}}(\lambda) = (\mathcal{L} - \lambda I)^{-1}$ of the operator \mathcal{L} (20) is given by*

$$(R_{\mathcal{L}}(\lambda)f)(x) = \frac{f(x)}{x - \lambda} + \frac{N(\lambda)}{x - \lambda} \int_{\mathbb{R}} a(t) \frac{f(t)}{t - \lambda} dt, \quad (41)$$

where $f(x) \in L^2_{\mathbb{R}}(E, dF(x))$, the function $F(x)$ is absolutely continuous (21), and $N(\lambda)$ is given by formula (4).

The description of the discrete spectrum $\sigma_p(\mathcal{L})$ of the operator \mathcal{L} (20) lies in the following.

Theorem 14. *Suppose that the function $F(x) = \varphi E_x \varphi^*$ is absolutely continuous and R is of trace-class. Then the eigenfunctions $\mathcal{L}f_{\lambda}(x) = \lambda f_{\lambda}(x)$ of the operator \mathcal{L} (20) are given by*

$$f_{\lambda}(x) = \frac{h_{\lambda}}{x - \lambda}, \quad (42)$$

where $h_{\lambda} \in \text{Ker}(I - RV(\lambda))$ and it does not depend on x , besides, $\dim \text{Ker}(I - RV(\lambda)) < \infty$; the eigenvalues λ lie outside the $\text{supp } a(x)$ and are zeroes of the function $D(\lambda)$ (36), $\overline{D(\lambda)} = D(\bar{\lambda})$; the order of the values λ is $\dim \text{Ker}(I - RV(\lambda))$; the eigenvalues $\lambda \in \sigma_p(\mathcal{L})$ have the limiting points of the “absolutely continuous closure” [8] of the set $\sigma_0 := \{x \in \mathbb{R} : x \in \overline{\text{supp } a(x)}\}$ as their accumulation points. The operator R on the subspace $\text{Ker}(I - RV(\lambda))$ is invertible.

P r o o f. Equality $\mathcal{L}f_{\lambda}(x) = \lambda f_{\lambda}(x)$ implies

$$(x - \lambda)f_{\lambda}(x) - R \int_{\mathbb{R}} a(t)f_{\lambda}(t)dt = 0,$$

and for $h_{\lambda}(x) = (x - \lambda)f_{\lambda}(x)$ we obtain

$$h_{\lambda}(x) - R \int_{\mathbb{R}} a(t) \frac{h_{\lambda}(t)}{t - \lambda} dt = 0.$$

It is seen from the relation that $h_{\lambda}(x) = h_{\lambda}$ does not depend on x and $[I - RV(\lambda)]h_{\lambda} = 0$. The finite dimensionality of $\text{Ker}(I - RV(\lambda))$ provides the total continuity of the operator R . Taking into account that $(I - RV(\lambda))^* = I - V(\bar{\lambda})R$ (in view of $R = R^*$, $V^*(\lambda) = V(\bar{\lambda})$) and using $\det(I - AB) = \det(I - BA)$ [17], we can prove that $\overline{D(\lambda)} = D(\bar{\lambda})$. There are no complex roots of $D(\lambda)$ (37) as \mathcal{L} is selfadjoint.

It remains to prove that R is not degenerate on $\text{Ker}(I - RV(\lambda))$ in view of $\dim \text{Ker}(I - RV(\lambda)) < \infty$. If $h_{\lambda} \in \text{Ker}(I - RV(\lambda))$, then $h_{\lambda} = RV(\lambda)h_{\lambda}$. Assuming the contrary, $Rh_{\lambda} = 0$, we obtain

$$\|h_\lambda\|^2 = \langle h_\lambda, h_\lambda \rangle = \langle h_\lambda, RV(\lambda)h_\lambda \rangle = \langle Rh_\lambda, V(\lambda)h_\lambda \rangle = 0,$$

and hence $h_\lambda = 0$. ■

The singularities of $N(\lambda)$ (4) consist of an absolutely continuous part coinciding with $\sigma_0 := \{x \in \mathbb{R} : x \in \overline{\text{supp } a(x)}\}$ (32), ($\sigma_0 \subseteq [a, b]$, $-\infty < a < b < \infty$, in view of the boundedness of L_0), and a discrete part which is the eigenvalues of \mathcal{L} that are zeroes of $D(\lambda)$ (37) and lie outside $\{x \in \mathbb{R} : \overline{\text{supp } a(x)}\}$. Thus,

$$\sigma(\mathcal{L}) = \sigma_0 \cup \sigma_p, \tag{43}$$

where

$$\sigma_p \stackrel{\text{def}}{=} \sigma_p(\mathcal{L}) = \{\varkappa_n \in \mathbb{R} : D(\varkappa_n) = 0\}. \tag{44}$$

The sets σ_0 and σ_p (44) do not intersect with each other except the limit points. Notice that if σ_0 consists of the finite (denumerable) number of not-intersecting segments from $[a, b]$, then the point spectrum (44) of the semiaxes $(-\infty, a)$ and (b, ∞) is in the lacunas between these segments and probably has some limit points of these intervals as its accumulation points.

If $\varkappa_n \in \sigma_p$ (44), then the subspace

$$F_n \stackrel{\text{def}}{=} \text{Ker}(I - RV(\varkappa_n)) \tag{45}$$

is nontrivial and finite-dimensional, $m_n \stackrel{\text{def}}{=} \dim F_n < \infty$.

Lemma 1. *Suppose that*

$$1) \overline{\text{supp } a(x)} = [a, b]; \quad 2) \bigcap_{x \in [a, b]} \text{Ker } a(x) = \{0\}. \tag{46}$$

Then the subspaces F_n (45) do not intersect each other, $F_n \cap F_k = \{0\}$ ($k \neq n$).

P r o o f. Let there be a nonzero vector $h \in F_k \cap F_n$ ($k \neq n$). Then $R^{-1}h = V(\varkappa_n)h = V(\varkappa_k)h$, in view of the invertibility of R on $\mathfrak{B} F_n$ (Theorem 14). Therefore, $\langle \{V(\varkappa_n) - V(\varkappa_k)\}h, h \rangle = 0$, i. e.,

$$(\varkappa_n - \varkappa_k) \int_{\mathbb{R}} \frac{\langle a(x)h, h \rangle}{(x - \varkappa_n)(x - \varkappa_k)} dx = 0.$$

Since $\varkappa_n \neq \varkappa_k$ ($n \neq k$) and the integrand has a constant sign ($\varkappa_n, \varkappa_k \notin [a, b]$), then $\langle a(x)h, h \rangle = 0$ for all $x \in [a, b]$. Taking into account that $a(x) \geq 0$, we obtain $a(x)h = 0$ for all $x \in [a, b]$, which contradicts condition 2) in (46). ■

Corollary 2. *If condition 1) (46) does not hold, then the statement of the lemma is not true. Let for $\dim E = 1$ and $\text{supp } a(x) = [a, c] \cup [d, b]$ ($a < c < d < b$), the equation $1 = RV(\lambda)$ always have two roots: $\varkappa_1 \in (-\infty, a)$ and $\varkappa_2 \in (c, d)$ as $R > 0$, or $\varkappa_1 \in (c, d)$ and $\varkappa_2 \in (b, \infty)$ as $R < 0$. Then the equality*

$$\int_a^c \frac{a(x)dx}{(x - \varkappa_1)(x - \varkappa_2)} + \int_d^b \frac{a(x)dx}{(x - \varkappa_1)(x - \varkappa_2)} = 0$$

does not imply $a(x) = 0$.

Let $\varkappa_n \in \sigma_p$ (44). Then the eigenfunctions of the operator \mathcal{L} (Theorem 14) have the form

$$f_{\varkappa_n}^s(x) = \frac{h_n^s}{x - \varkappa_n} \quad (1 \leq s \leq m_n, n \in \mathbb{N}),$$

where $\{h_n^s\}_1^{m_n}$ is the orthonormal basis in F_n (45), and $D(\varkappa_n) = 0$ ($D(\lambda)$ is given by (36)). It is obvious that

$$\langle f_{\varkappa_n}^s(x), f_{\varkappa_n}^p(x) \rangle_{L^2} = \langle V'(\varkappa_n) h_n^s, h_n^p \rangle_E \quad (1 \leq s, p \leq m_n),$$

where $V'(\varkappa_n)$ is the value of the derivative of the function $V(\lambda)$ as $\lambda = \varkappa_n$. Show that $V'(\varkappa_n)$ is non-degenerate on F_n (45). Assuming the contrary, we suppose that there is a vector $h \in F_n$ such that $V'(\varkappa_n)h = 0$. Since $R^{-1}h = V(\varkappa_n)h$, then for all $f \in E$ we have

$$\begin{aligned} |\langle R^{-1}h, f \rangle|^2 &= |\langle V(\varkappa_n)h, f \rangle|^2 = \left| \int_{\mathbb{R}} \frac{\langle a(x)h, f \rangle}{x - \varkappa_n} dx \right|^2 \\ &\leq \int_{\mathbb{R}} \frac{\langle a(x)h, h \rangle}{(x - \varkappa_n)^2} dx \cdot \int_{\mathbb{R}} \langle a(x)f, f \rangle dx = \langle V'(\varkappa_n)h, h \rangle \int_{\mathbb{R}} \langle a(x)f, f \rangle dx = 0. \end{aligned}$$

The arbitrariness of $f \in E$ implies $R^{-1}h = 0$. Consequently, $h = 0$ (R is invertible on F_n by Theorem 14).

Hence, the operator $P_n V'(\varkappa_n) P_n$ is positive and invertible (P_n is an orthoprojector on F_n (45)). Therefore there exist

$$M_n \stackrel{\text{def}}{=} \{P_n V'(\varkappa_n) P_n\}^{-\frac{1}{2}} \quad (n \in \mathbb{N}), \tag{47}$$

where $M_n > 0$ and $\text{rank } M_n = m_n$. Then the vector-functions

$$f^s(\varkappa_n, x) = \frac{M_n h_n^s}{x - \varkappa_n} \quad (1 \leq s \leq m_n, n \in \mathbb{N}) \tag{48}$$

form the orthonormal basis in $L^2_{\mathbb{R}}(E, a(x)dx)$ corresponding to the eigenvalue \varkappa_n (for all $n \in \mathbb{N}$).

The functions $f^s(\varkappa_n, x)$ (48) are orthogonal for different \varkappa_n as the eigenfunctions of the operator \mathcal{L} . Therefore,

$$\int_{\mathbb{R}} \frac{1}{(x - \varkappa_n)(x - \varkappa_l)} \left\langle a(x)M_n h_n^s, M_l h_l^k \right\rangle dx = 0,$$

where $\{h_n^s\}_1^{m_n}$ and $\{h_l^k\}_1^{m_l}$ are orthonormal bases in F_n and F_l (44), respectively. Hence,

$$M_l \{V(\varkappa_n) - V(\varkappa_l)\} M_l = 0 \quad (\forall \varkappa_n \neq \varkappa_l).$$

Show that this equality holds automatically. Formula (45) implies $P_n = RV(\varkappa_n)P_n$ or $R^{-1}M_n = V(\varkappa_n)M_n$ in view of the invertibility of R on F_n . Taking into account $M_l R^{-1} = M_l V(\varkappa_l)$, we have

$$M_l \{V(\varkappa_n) - V(\varkappa_l)\} M_n = M_l R^{-1}M_n - M_l R^{-1}M_n = 0.$$

Theorem 15. *Suppose that the operator R is of trace-class; the representing measure $dF(x)$ of the function $V(\lambda)$ (2) is absolutely continuous (21); $\{\varkappa_n\}_1^\infty$ are the set of zeroes of the function $D(\lambda)$ (37), $D(\varkappa_n) = 0$, and $\varkappa_n \notin \sigma_0$.*

Then the function $N(\lambda)$ (4) is given by

$$N(\lambda) = R + \sum_{n=1}^{\infty} \frac{M_n^2}{\varkappa_n - \lambda} + \int_{\sigma_0} \frac{b(x)}{x - \lambda} dx, \quad (49)$$

where $b(x) \geq 0$, $\overline{\text{supp } b(x)} = \sigma_0$, and M_n are given by (47). Moreover,

$$\int_{\sigma_0} b(x) dx < \infty; \quad \sum_{n=1}^{\infty} M_n^2 < \infty. \quad (50)$$

P r o o f. Let us turn back to representation 4) (5) of the function $N(\lambda)$ (4). Then the absolute continuity of $d\sigma(t)$ on σ_0 follows from (32) and the Kato–Rosenblum theorem [2]. It is necessary to prove that discontinuities of the measure $d\sigma(t)$ at the points \varkappa_n coincide with M_n^2 .

Write the Laurent expansion of $N(\lambda)$ (4) at the point $\varkappa_n \in \sigma_p$ (44),

$$N(\lambda) = \frac{\sigma_n}{\varkappa_n - \lambda} + R + \dots,$$

where $\sigma_n \geq 0$, and substitute the obtained equality into the identity $N(\lambda) = R + RV(\lambda)N(\lambda)$. Then

$$\frac{\sigma_n}{\varkappa_n - \lambda} + R + \dots = R + RV(\lambda) \left(\frac{\sigma_n}{\varkappa_n - \lambda} + R + \dots \right).$$

Multiplying the last equality by $(\varkappa_n - \lambda)$ and passing to limit as $\lambda \rightarrow \varkappa_n$, we obtain that $\sigma_n = RV(\varkappa_n)\sigma_n$ ($V(\varkappa_n)$ exist since $\varkappa_n \notin \sigma_0$). Show that $\sigma_n : F_n \rightarrow F_n$ and $\sigma_n F_n^\perp = 0$. The definition of F_n (45) implies $P_n = RV(\varkappa_n)P_n$ (P_n is an orthoprojector on F_n). Then

$$\sigma_n = RV(\varkappa_n)\sigma_n = RV(\varkappa_n) [P_n + P_n^\perp] \sigma_n = P_n \sigma_n + RV(\varkappa_n) P_n^\perp \sigma_n,$$

where $P_n^\perp = I - P_n$. Therefore, $P_n^\perp \sigma_n = RV(\varkappa_n) P_n^\perp \sigma_n$, or $(I - RV(\varkappa_n)) P_n^\perp \sigma_n = 0$. Thus, $P_n^\perp \sigma_n f \in F_n$ for all $f \in F_n$ and $P_n^\perp \sigma_n P_n = 0$, which proves that $\sigma_n : F_n \rightarrow F_n$ and $\sigma_n|_{F_n^\perp} = 0$, in view of the self-adjointness of σ_n .

Now we have to prove that $\sigma_n = M_n^2$ (47). First, we multiply the equality $R = (I - RV(\lambda))N(\lambda)$ from the right by P_n . Then, taking into account the invertibility of R on F_n , we obtain

$$\frac{R^{-1} - V(\lambda)}{\varkappa_n - \lambda} (\varkappa_n - \lambda) N(\lambda) P_n = P_n.$$

Using the Laurent expansion of $N(\lambda)$ at the point $\lambda = \varkappa_n$ and passing to limit as $\lambda \rightarrow \varkappa_n$, we have $V'(\varkappa_n) \sigma_n P_n = P_n$, which gives us $M_n^{-2} \sigma_n = P_n$ ($\sigma_n P_n = P_n \sigma_n$). The convergence of the integral and the series in (50) follows from the finiteness of variation of the measure $d\sigma(t)$ at representation 4) (5) of the function $N(\lambda)$. ■

VII. Find the resolution of the identity E_λ^a corresponding to the absolutely continuous part of \mathcal{L}^a of the operator \mathcal{L} . Then

$$\frac{d}{d\lambda} \langle E_\lambda^a f, f \rangle = \lim_{\varepsilon \rightarrow +0} \langle \delta(\lambda, \varepsilon) f, f \rangle,$$

where $\delta(\lambda, \varepsilon)$ (7) corresponds to the operator \mathcal{L} . Using (41), we obtain

$$\begin{aligned} \langle \delta(\lambda, \varepsilon) f, f \rangle &= \frac{1}{2\pi i} \{ \langle R_{\mathcal{L}}(\lambda + i\varepsilon) f, f \rangle - \langle R_{\mathcal{L}}(\lambda - i\varepsilon) f, f \rangle \} = \frac{1}{2\pi i} \left\langle \frac{f(x)}{x - \lambda - i\varepsilon} \right. \\ &+ \frac{N(\lambda + i\varepsilon)}{x - \lambda - i\varepsilon} \int_{\mathbb{R}} \frac{a(t)}{t - \lambda - i\varepsilon} f(t) dt, f(x) \left. \right\rangle - \frac{1}{2\pi i} \left\langle \frac{f(x)}{x - \lambda + i\varepsilon} - \frac{N(\lambda - i\varepsilon)}{x - \lambda + i\varepsilon} \right. \\ &\left. \times \int_{\mathbb{R}} \frac{a(t)}{t - \lambda + i\varepsilon} f(t) dt, f(x) \right\rangle = A_\varepsilon + B_\varepsilon + \overline{B_\varepsilon}, \end{aligned}$$

where,

$$A_\varepsilon \stackrel{\text{def}}{=} \frac{1}{\pi} \int_{\mathbb{R}} \frac{\varepsilon}{(x-\lambda)^2 + \varepsilon^2} \langle a(x)f(x), f(x) \rangle dx,$$

$$B_\varepsilon \stackrel{\text{def}}{=} \frac{1}{\pi} \int_{\mathbb{R}} \left\langle N(\lambda + i\varepsilon) \int_{\mathbb{R}} \frac{a(t)}{t - \lambda - i\varepsilon} f(t) dt, \int_{\mathbb{R}} \frac{a(x)}{x - \lambda + i\varepsilon} f(x) dx \right\rangle.$$

Lemma 2. For all $\lambda \in \mathbb{R}$, the formula

$$A_0 + B_0 + \overline{B_0} = \langle a(\lambda)f(\lambda), f(\lambda) \rangle - \pi^2 \langle a(\lambda)N_-(\lambda)a(\lambda)f(\lambda), N_-(\lambda)a(\lambda)f(\lambda) \rangle$$

$$+ \operatorname{Re} \left\langle [N_+(\lambda) + N_-(\lambda)] a(\lambda)f(\lambda), \int_{\mathbb{R}} \frac{a(x)}{t - \lambda} f(t) dt \right\rangle \quad (51)$$

$$+ \left\langle a(\lambda)N_-(\lambda) \int_{\mathbb{R}} \frac{a(t)}{t - \lambda} f(t) dt, N_-(\lambda) \int_{\mathbb{R}} \frac{a(t)}{t - \lambda} f(t) dt \right\rangle$$

is true, where $A_0 (B_0) = A_\varepsilon (B_\varepsilon)|_{\varepsilon=+0}$, and $N_\pm(\lambda)$ are the boundary values on \mathbb{R} from \mathbb{C}_\pm of the function $N(\lambda)$ (4).

P r o o f. The property of the Poisson kernel [14, 15] yields

$$A_\varepsilon \rightarrow A_0 = \langle a(\lambda)f(\lambda), f(\lambda) \rangle \quad (\varepsilon \rightarrow +0).$$

Using the Sokhotsky formulas [16], we have

$$B_0 = -2\pi i \left\langle N_+(\lambda) \left\{ \frac{1}{2} a(\lambda)f(\lambda) + \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{a(t)}{t - \lambda} f(t) dt \right\}, \right.$$

$$\left. \left\{ -\frac{1}{2} a(\lambda)f(\lambda) + \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{a(t)}{t - \lambda} f(t) dt \right\} \right\rangle.$$

Finally, after elementary transformations, taking into account (32), we obtain (51). ■

Theorem 16. Suppose that the suppositions of Theorem 6 take place. Then for the identity resolution E_λ^a of the absolutely continuous part \mathcal{L}^a of the operator \mathcal{L} (20), the representation

$$\frac{d}{d\lambda} \langle E_\lambda^a f, f \rangle = \left\langle a(\lambda) \left\{ \frac{1}{2} (I + S^+(\lambda)) f(\lambda) + N_+(\lambda) \int_{\mathbb{R}} \frac{a(t)}{t - \lambda} f(t) dt \right\}, \right.$$

$$\left. \left\{ \frac{1}{2} (I + S^+(\lambda)) f(\lambda) + N_+(\lambda) \int_{\mathbb{R}} \frac{a(t)}{t - \lambda} f(t) dt \right\} \right\rangle \quad (52)$$

is true, where $S^+(\lambda)$ is the scattering matrix (25), and $N_+(\lambda)$ is the boundary value on \mathbb{R} from \mathbb{C}_+ of the function $N(\lambda)$ (4).

P r o o f. Since $N_-(\lambda) = N_+(\lambda) - 2\pi i N_+(\lambda) a(\lambda) N_-(\lambda)$ (32), then the third summand from (51) is given by

$$\frac{1}{2} \left\langle [N_+(\lambda) + N_-(\lambda)] \int_{\mathbb{R}} \frac{a(t)}{t-\lambda} f(t) dt, a(\lambda) f(\lambda) \right\rangle = \left\langle N_+(\lambda) \int_{\mathbb{R}} \frac{a(t)}{t-\lambda} f(t) dt, a(\lambda) f(\lambda) \right\rangle + \left\langle a(\lambda) N_-(\lambda) \int_{\mathbb{R}} \frac{a(t)}{t-\lambda} f(t) dt, \pi i N_-(\lambda) a(\lambda) f(\lambda) \right\rangle.$$

Substituting this expression in (51) and taking into account that $S(\lambda) N_+(\lambda) = N_-(\lambda)$ (35) and $S^*(\lambda) a(\lambda) S(\lambda) = a(\lambda) 1$ (33), we obtain

$$\begin{aligned} A_0 + B_0 + \overline{B_0} &= \langle a(\lambda) f(\lambda), f(\lambda) \rangle - \langle a(\lambda) \pi i N_+(\lambda) a(\lambda) f(\lambda), \pi i N_+(\lambda) a(\lambda) f(\lambda) \rangle \\ &+ \left\langle N_+(\lambda) \int_{\mathbb{R}} \frac{a(t)}{t-\lambda} f(t) dt, a(\lambda) f(\lambda) \right\rangle + \left\langle a(\lambda) N_+(\lambda) \int_{\mathbb{R}} \frac{a(t)}{t-\lambda} f(t) dt, \pi i N_+(\lambda) a(\lambda) f(\lambda) \right\rangle \\ &+ \left\langle a(\lambda) f(\lambda), N_+(\lambda) \int_{\mathbb{R}} \frac{a(t)}{t-\lambda} f(t) dt \right\rangle \\ &+ \left\langle a(\lambda) \pi i N_+(\lambda) a(\lambda) f(\lambda), N_+(\lambda) \int_{\mathbb{R}} \frac{a(t)}{t-\lambda} f(t) dt \right\rangle \\ &+ \left\langle a(\lambda) N_+(\lambda) \int_{\mathbb{R}} \frac{a(t)}{t-\lambda} f(t) dt, N_+(\lambda) \int_{\mathbb{R}} \frac{a(t)}{t-\lambda} f(t) dt \right\rangle. \end{aligned}$$

Hence, after elementary transformations we obtain

$$\begin{aligned} A_0 + B_0 + \overline{B_0} &= \left\langle a(\lambda) \left\{ f(\lambda) + \pi i N_+(\lambda) a(\lambda) f(\lambda) + N_+(\lambda) \int_{\mathbb{R}} \frac{a(t)}{t-\lambda} f(t) dt \right\}, \right. \\ &\left. \left\{ f(\lambda) + \pi i N_+(\lambda) a(\lambda) f(\lambda) + N_+(\lambda) \int_{\mathbb{R}} \frac{a(t)}{t-\lambda} f(t) dt \right\} \right\rangle, \end{aligned}$$

which gives us (52) in view of (25). ■

4. Equation of the Inverse Scattering Problem

VIII. Associate every function $f(x) \in L^2_{\mathbb{R}}(E, dF(x))$ (18) with the Fourier coefficients,

$$f^s(\varkappa_n) \stackrel{\text{def}}{=} \langle f(x), f^s(\varkappa_n, x) \rangle \quad (1 \leq s \leq m_n, n \in \mathbb{N}), \quad (53)$$

corresponding to the discrete spectrum $\sigma_p(\mathcal{L})$ (44) of the operator \mathcal{L} , where $f^s(\varkappa_n, x)$ are given by (48). Then

$$f^s(\varkappa_n) = \left\langle \int_{\mathbb{R}} \frac{a(x)}{x - \varkappa_n} f(x) dx, M_n h_n^s \right\rangle, \quad (54)$$

where $\{h_n^s\}_1^{m_n}$ are the orthonormal basis in F_n (45), and M_n are given by (47).

In the case of absolutely continuous spectrum of the operator \mathcal{L} ,

$$\tilde{f}(\lambda) \stackrel{\text{def}}{=} \frac{1}{2} (I + S^+(\lambda)) f(\lambda) + N_+(\lambda) \int_{\mathbb{R}} \frac{a(t)}{t - \lambda} f(t) dt \quad (55)$$

are the ‘‘Fourier coefficients’’ of the function $f(x) \in L^2_{\mathbb{R}}(E, dF(x))$ by Theorem 16. The space $L^2_{\mathbb{R}}(E, dF(x))$ is the orthogonal sum of two subspaces, the first of which is the eigenfunctions (associated states) of the operator \mathcal{L} , and the second subspace corresponds to the absolutely continuous spectrum of \mathcal{L} . Write the Parseval equality [4],

$$\langle f(x), g(x) \rangle_{L^2_{\mathbb{R}}(E, dF(x))} = \sum_{n=1}^{\infty} \sum_{s=1}^{m_n} f^s(\varkappa_n) \overline{g^s(\varkappa_n)} + \langle \tilde{f}(\lambda), \tilde{g}(\lambda) \rangle_{L^2_{\mathbb{R}}(E, dF(\lambda))}, \quad (56)$$

where $f, g \in L^2_{\mathbb{R}}(E, dF(x))$. (54) implies

$$\sum_{s=1}^{m_n} f^s(\varkappa_n) \overline{g^s(\varkappa_n)} = \int_{\mathbb{R}} dx \int_{\mathbb{R}} dy \left\langle M_n^2 \frac{a(x)}{x - \varkappa_n} f(x), \frac{a(y)}{y - \varkappa_n} g(y) \right\rangle$$

since $\{h_n^s\}_1^{m_n}$ is the orthonormal basis in F_n . Using

$$\frac{1}{(x - \varkappa_n)(y - \varkappa_n)} = \frac{1}{x - y} \left[\frac{1}{y - \varkappa_n} - \frac{1}{x - \varkappa_n} \right],$$

we obtain

$$\sum_{n=1}^{\infty} \sum_{s=1}^{m_n} f^s(\varkappa_n) \overline{g^s(\varkappa_n)} = - \int_{\mathbb{R}} dx \int_{\mathbb{R}} dy \left\langle \frac{P(x) - P(y)}{x - y} a(x) f(x), a(y) f(y) \right\rangle,$$

where

$$P(x) \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} \frac{M_n^2}{x - \varkappa_n}. \tag{57}$$

By the condition of (50), the series of (57) converges.

Equality (55) implies

$$\begin{aligned} B &\stackrel{\text{def}}{=} \left\langle \tilde{f}(\lambda), \tilde{g}(\lambda) \right\rangle_{L_{\mathbb{R}}^2(E, dF(\lambda))} = \int_{\mathbb{R}} \left\langle a(\lambda) \left\{ \frac{1}{2} (I + S^+(\lambda)) f(\lambda) \right. \right. \\ &+ N_+(\lambda) \int_{\mathbb{R}} \frac{a(t)}{t - \lambda} f(t) dt \left. \left. \right\}, \left\{ \frac{1}{2} (I + S^+(\lambda)) g(\lambda) + N_+(\lambda) \int_{\mathbb{R}} \frac{a(t)}{t - \lambda} g(t) dt \right\} \right\rangle \\ &= \frac{1}{4} \int_{\mathbb{R}} \left\langle [2a(\lambda) + a(\lambda)S^+(\lambda) + (S^+(\lambda))^* a(\lambda)] f(\lambda), g(\lambda) \right\rangle d\lambda \\ &\quad + \frac{1}{2} \int_{\mathbb{R}} \left\langle a(\lambda) (I + S^+(\lambda)) f(\lambda), N_+(\lambda) \int_{\mathbb{R}} \frac{a(t)}{t - \lambda} g(t) dt \right\rangle \\ &\quad + \frac{1}{2} \int_{\mathbb{R}} \left\langle a(\lambda) N_+(\lambda) \int_{\mathbb{R}} \frac{a(t)}{t - \lambda} f(t) dt, (I + S^+(\lambda)) g(\lambda) \right\rangle d\lambda \\ &\quad + \int_{\mathbb{R}} \left\langle N_-(\lambda) a(\lambda) N_+(\lambda) \int_{\mathbb{R}} \frac{a(t)}{t - \lambda} f(t) dt, \int_{\mathbb{R}} \frac{a(t)g(t)}{t - \lambda} dt \right\rangle d\lambda. \end{aligned}$$

Using (26), (32), (33), we obtain

$$\begin{aligned} B &= \langle f(x), g(x) \rangle_{L_{\mathbb{R}}^2(E, dF(x))} + \frac{1}{4} \int_{\mathbb{R}} \langle a(\lambda) [S^+(\lambda) + S(\lambda) - 2I] f(\lambda), g(\lambda) \rangle d\lambda \\ &\quad + \frac{1}{2} \int_{\mathbb{R}} \left\langle N_-(\lambda) [I + S^*(\lambda)] a(\lambda) f(\lambda), \int_{\mathbb{R}} \frac{a(t)}{t - \lambda} g(t) dt \right\rangle d\lambda \\ &\quad + \frac{1}{2} \int_{\mathbb{R}} \left\langle a(\lambda) [I + S(\lambda)] N_+(\lambda) \int_{\mathbb{R}} \frac{a(t)}{t - \lambda} f(t) dt, g(\lambda) \right\rangle d\lambda \\ &\quad + \frac{1}{2\pi i} \int_{\mathbb{R}} \left\langle [N_+(\lambda) - N_-(\lambda)] \int_{\mathbb{R}} \frac{a(t)}{t - \lambda} f(t) dt, \int_{\mathbb{R}} \frac{a(t)}{t - \lambda} g(t) dt \right\rangle d\lambda. \end{aligned}$$

Taking into account (22), (25), and (35), we have

$$\begin{aligned}
 B &= \langle f(x), g(x) \rangle_{L^2_{\mathbb{R}}(E, dF(x))} + \frac{\pi i}{2} \int_{\mathbb{R}} \langle a(\lambda)[I - S(\lambda)]N_+(\lambda)a(\lambda)f(\lambda), g(\lambda) \rangle d\lambda \\
 &\quad - \frac{1}{2} \int_{\mathbb{R}} \left\langle a(\lambda) \int_{\mathbb{R}} \frac{(I + S(t))N_+(t) - (I + S(\lambda))N(\lambda)}{t - \lambda} a(t)f(t)dt, g(\lambda) \right\rangle d\lambda \\
 &\quad + \frac{1}{2\pi i} \int_{\mathbb{R}} \left\langle [I - S(\lambda)]N_+(\lambda) \int_{\mathbb{R}} \frac{a(t)}{t - \lambda} f(t)dt, \int_{\mathbb{R}} \frac{a(t)}{t - \lambda} g(t)dt \right\rangle d\lambda.
 \end{aligned}$$

The last integral from this sum may be written in the form

$$C \stackrel{\text{def}}{=} \frac{1}{2\pi i} \int_{\mathbb{R}} dx \int_{\mathbb{R}} \frac{d\lambda}{x - \lambda} \left\langle (I - S(\lambda))N_+(\lambda) \int_{\mathbb{R}} \frac{a(t)}{t - \lambda} f(t)dt, a(x)g(x) \right\rangle.$$

Using the Poincaré–Bertrand formula [16] for rearranging the order of integration in improper integrals,

$$\frac{1}{\pi i} \int_{\mathbb{R}} \frac{d\tau}{\tau - t} \int_{\mathbb{R}} \frac{\varphi(\tau, \xi)}{\xi - \tau} d\xi = \pi i \varphi(t, t) + \int_{\mathbb{R}} d\xi \int_{\mathbb{R}} \frac{\varphi(\tau, \xi)}{(\tau - t)(\xi - \tau)} d\tau,$$

we obtain

$$\begin{aligned}
 \frac{1}{\pi i} \int_{\mathbb{R}} \frac{d\lambda}{\lambda - x} \int_{\mathbb{R}} \frac{(I - S(\lambda))N_+(\lambda)}{t - \lambda} a(t)f(t)dt &= \pi i [I - S(x)]N_+(x)a(x)f(x) \\
 &\quad + \frac{1}{\pi i} \int_{\mathbb{R}} dt \int_{\mathbb{R}} \frac{(I - S(\lambda))N_+(\lambda)}{(t - \lambda)(\lambda - x)} a(t)f(t)d\lambda.
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 C &= -\frac{\pi i}{2} \int_{\mathbb{R}} \langle [I - S(x)]N_+(x)a(x)f(x), a(x)g(x) \rangle dx \\
 &\quad + \frac{1}{2\pi i} \int_{\mathbb{R}} dx \left\langle \int_{\mathbb{R}} \frac{M(x) - M(t)}{x - t} a(t)f(t)dt, a(x)g(x) \right\rangle,
 \end{aligned}$$

where

$$M(x) \stackrel{\text{def}}{=} \int_{\mathbb{R}} \frac{(I - S(\lambda))N_+(\lambda)}{\lambda - x} d\lambda. \tag{58}$$

Finally, we have

$$\begin{aligned} \left\langle \tilde{f}(\lambda), \tilde{g}(\lambda) \right\rangle_{L^2_{\mathbb{R}}(E, dF(\lambda))} &= \langle f(x), g(x) \rangle_{L^2_{\mathbb{R}}(E, dF(x))} \\ &- \frac{1}{2} \int_{\mathbb{R}} \left\langle a(\lambda) \int_{\mathbb{R}} \frac{[I + S(t)]N_+(t) - [I + S(\lambda)]N_+(\lambda)}{t - \lambda} a(t)f(t)dt, g(\lambda) \right\rangle d\lambda \\ &+ \frac{1}{2\pi i} \int_{\mathbb{R}} \left\langle a(\lambda) \int_{\mathbb{R}} \frac{M(t) - M(\lambda)}{t - \lambda} a(t)f(t)dt, g(\lambda) \right\rangle d\lambda. \end{aligned}$$

Substituting the obtained expressions in the Parseval equality (56), we have

$$\begin{aligned} &- \int_{\mathbb{R}} d\lambda \left\langle a(\lambda) \int_{\mathbb{R}} \frac{P(x) - P(\lambda)}{x - \lambda} a(x)f(x)dx, g(\lambda) \right\rangle \\ &- \frac{1}{2} \int_{\mathbb{R}} d\lambda \left\langle a(\lambda) \int_{\mathbb{R}} \frac{[I + S(x)]N_+(x) - [I + S(\lambda)]N_+(\lambda)}{x - \lambda} a(x)f(x)dx, g(\lambda) \right\rangle \\ &+ \frac{1}{2\pi i} \int_{\mathbb{R}} d\lambda \left\langle a(\lambda) \int_{\mathbb{R}} \frac{M(x) - M(\lambda)}{x - \lambda} a(x)f(x)dx, g(\lambda) \right\rangle = 0. \end{aligned}$$

Let

$$G(\lambda) \stackrel{\text{def}}{=} -P(\lambda) - \frac{1}{2}[I + S(\lambda)]N_+(\lambda) + \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{[I - S(x)]N_+(x)}{x - \lambda} dx. \quad (59)$$

Then the last equality takes the form

$$\int_{\mathbb{R}} \left\langle a(\lambda) \int_{\mathbb{R}} \frac{G(x) - G(\lambda)}{x - \lambda} a(x)f(x)dx, g(\lambda) \right\rangle d\lambda = 0,$$

that, in view of the arbitrariness of $g(x) \in L^2_{\mathbb{R}}(E, a(x)dx)$, gives

$$\int_{\mathbb{R}} \frac{G(x) - G(\lambda)}{x - \lambda} a(x)f(x)dx = 0$$

for all $f(x) \in L^2_{\mathbb{R}}(E, a(x)dx)$. This implies

$$\int_{\mathbb{R}} \left\langle a(\lambda)f(\lambda), \frac{G(x) - G(\lambda)}{x - \lambda} h \right\rangle = 0$$

for an arbitrary $h \in E$, in view of the self-adjointness of $G(\lambda)$ (59), since $S(\lambda)N_+(\lambda) = N_-(\lambda)$ (35). Thus,

$$\frac{G(x) - G(\lambda)}{x - \lambda} h = 0 \quad (\forall x, \lambda),$$

and $G(\lambda)$ is a constant operator

$$G(\lambda) = B (= B^*).$$

Since

$$G(\lambda) = -P(\lambda) - N_+(\lambda) + \frac{1}{2} [I - S(\lambda)] N_+(\lambda) + \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{[I - S(x)] N_+(x)}{x - \lambda} dx,$$

then

$$G(\lambda) = -P(\lambda) - N_+(\lambda) + \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{[I - S(x)] N_+(x)}{x - \lambda - i0} dx \quad (\lambda \in \mathbb{R})$$

in view of the Sokhotsky formula [16], therefore

$$G(\lambda) = -P(\lambda) - N(\lambda) + \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{[I - S(x)] N_+(x)}{x - \lambda} dx \quad (\lambda \in \mathbb{C}_+). \quad (60)$$

Let $\lambda = iy$ ($y > 0$), then the first and the third members of this sum converge to zero as $y \rightarrow +\infty$ and $N(iy) \rightarrow R$ ($y \rightarrow \infty$) in view of 4) (5), and we get $B = -R$.

Theorem 17. Let $f, g \in L^2_{\mathbb{R}}(E, dF(x))$, where $F(x)$ is absolutely continuous (21), R is of trace-class, φ is a Hilbert-Schmidt operator, and $f^s(\varkappa_n)$, $\tilde{f}(\lambda)$, and $g^s(\varkappa_n)$, $\tilde{g}(\lambda)$, are the corresponding Fourier coefficients (54), (55) of these functions.

For the Parseval equality (56) to be fulfilled, it is necessary and sufficient that the relation

$$R - P(\lambda) - \frac{1}{2} [I + S(\lambda)] N_+(\lambda) + \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{[I - S(x)] N_+(x)}{x - \lambda} dx = 0 \quad (\lambda \in \mathbb{R}) \quad (61)$$

hold, where $P(\lambda)$ is given by (57); \varkappa_n are zeroes of $D(\lambda)$ (37); M_n are the finite-dimensional positive invertible operators (47) such that series (50) converges; $N_+(\lambda)$ are the boundary values on \mathbb{R} from \mathbb{C}_+ of the function $N(\lambda)$ (4); $S(\lambda)$ is the scattering matrix (22).

Corollary 3. Taking into account $S(\lambda)N_+(\lambda) = N_-(\lambda)$ (35), we write (61) in the form

$$R - F(\lambda) - \frac{1}{2} [N_+(\lambda) + N_-(\lambda)] + \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{N_+(x) - N_-(x)}{x - \lambda} dx = 0. \quad (62)$$

In view of the Sokhotsky formula [16], (49) implies

$$N_{\pm}(\lambda) = R + \sum_{n=1}^{\infty} \frac{M_n^2}{\varkappa_n - \lambda} \pm \pi i b(\lambda) + \int_{\mathbb{R}} \frac{b(x)}{x - \lambda} dx \quad (\lambda \in \mathbb{R}). \quad (63)$$

Substituting $N_{\pm}(\lambda)$ (63) into (62), we obtain an identity.

Relation (61) should be considered as an equation with respect to $N_+(\lambda)$ which is an analogue of the Marchenko equation for the inverse scattering problem [4–7]. Besides, we ought to search solution (61) in the class of Nevanlinna functions 4) (5).

Let $N_+(\lambda)$ be the solution of equation (61), and $N(\lambda)$ be its analytic continuation into \mathbb{C}_+ in view of (60). Continue $N(\lambda)$ into \mathbb{C}_- in a natural way, then, using the Sokhotsky formula and (61), we obtain

$$N_{\pm}(\lambda) = R - F(\lambda) \pm \frac{1}{2}(I - S(\lambda))N_+(\lambda) + \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{[I - S(x)]N_+(x)}{x - \lambda} dx. \quad (64)$$

This implies the statement.

Theorem 18. *Suppose that the suppositions of Theorem 6 are true, $N_+(\lambda)$ is the solution of equation (61), $N(\lambda)$ is its holomorphic extension into \mathbb{C}_+ , and $N_-(\lambda)$ is a boundary value on \mathbb{R} from \mathbb{C}_- after the natural extension of $N(\lambda)$ into \mathbb{C}_- . Then the Riemann boundary value problem $S(\lambda)N_+(\lambda) = N_-(\lambda)$ (38) takes place. If*

$$\frac{1}{2i}(I - S(\lambda))N_+(\lambda) \geq 0 \quad (\lambda \in \mathbb{R}), \quad (65)$$

then $N_-(\lambda) = N_+^*(\lambda)$.

P r o o f. The Riemann problem follows from (64) after the subtraction of $N_-(\lambda) - N_+(\lambda)$. Condition (65) is natural in view of (32). The fact that (65) implies $N_-(\lambda) = N_+^*(\lambda)$ is obvious in view of (64). ■

5. Inverse Problem

IX. Let $\dim E = 1$ and σ_0 be the segment $\sigma_0 = [a, b]$ ($-\infty < a < b < \infty$). The discrete spectrum σ_p (44) is formed by the zeroes of the equation $1 = RV(\lambda)$. Taking into account monotonicity of $V(\lambda)$ as $\lambda \in \mathbb{R} \setminus [a, b]$, we see that the root $1 = RV(\varkappa)$ is unique since $\varkappa \in (-\infty, a)$ for $R > 0$, and $\varkappa \in (b, \infty)$ for $R < 0$. Thus, $\sigma_p = \{\varkappa \notin [a, b]\}$.

Corollary 4. *The set σ_p can be empty depending on the number R and on the behavior of the representing measure $dF(x) = a(x)dx$ of the function $V(\lambda)$ (2) at*

the end points of the segment $[a, b]$. If, for example, $a(x) = \alpha(x - a)$ ($\alpha > 0$), then

$$V(\lambda) = \alpha(b - a) + \alpha(\lambda - a) \int_a^b \frac{dx}{x - \lambda}.$$

The equation $1 = RV(\lambda)$ has no roots when $R > \{\alpha(b - a)\}^{-1}$ as $V(a - 0) = \alpha(b - a)$.

In this case, the Riemann boundary value problem (34) can be written as

$$S(\lambda) \frac{\lambda + i}{\lambda - i} \frac{\lambda - \varkappa}{\lambda + i} (1 - RV_+(\lambda))^{-1} = \frac{\lambda - \varkappa}{\lambda - i} (1 - RV_-(\lambda))^{-1}. \quad (66)$$

Each function $\frac{\lambda - \varkappa}{\lambda \pm i} (1 - RV_{\pm}(\lambda))^{-1}$ is holomorphically extendable in \mathbb{C}_{\pm} and has no zeroes and singularities in $\overline{\mathbb{C}_{\pm}}$, besides, it tends to 1 at the infinity. Using the standard method [16] of the solution of the boundary problem (66) and the Sokhotsky formulas [16], we obtain

$$\frac{\lambda - \varkappa}{\lambda + i} (1 - RV(\lambda)) = \exp \left\{ -\frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\ln \left[S(x) \frac{x + i}{x - i} \right]}{x - \lambda} dx \right\} \quad (\lambda \in \mathbb{C}_+). \quad (67)$$

Let

$$\psi(x) = \frac{1}{\pi} \operatorname{arctg} \frac{1}{x} \quad (x \in \mathbb{R}, -\frac{1}{2} \leq \psi(x) \leq \frac{1}{2}). \quad (68)$$

Then, taking into account (39), we have

$$S(x) \frac{x + i}{x - i} = \exp\{-2\pi i(\xi(x) - \psi(x))\},$$

where $\xi(x)$ is the function of spectral shift (39). Equality (67) can be written in the form

$$(1 - RV(\lambda))^{-1} = \frac{\lambda + i}{\lambda - \varkappa} \exp \left\{ \int_{\mathbb{R}} \frac{\xi(x) - \psi(x)}{x - \lambda} dx \right\} \quad (\lambda \in \mathbb{C}_+). \quad (69)$$

The function $(1 - RV(\lambda))^{-1}$ maps \mathbb{C}_+ into \mathbb{C}_+ as $R > 0$, and $-\mathbb{C}_-$ into \mathbb{C}_- as $R < 0$. For definiteness sake, take $R > 0$. Then the argument on the right-hand side of equality (69) belongs to $[0, \pi]$ as $\lambda \in \mathbb{C}_+$. This argument is

$$\arg \frac{\lambda + i}{\lambda - \varkappa} + \int_{\mathbb{R}} \frac{y}{(t - x)^2 + y^2} (\xi(t) - \psi(t)) dt, \quad (70)$$

where $\lambda = x + iy \in \mathbb{C}_+$. Each of the summands in this sum is a harmonic function in \mathbb{C}_+ , in view of the maximum principle, and its maximum and minimum values are achieved on the boundary $\lambda = x + i0 \in \mathbb{R}$. Taking into account the properties of the Poisson kernel [12], we obtain that the boundary value on \mathbb{R} from \mathbb{C}_+ of sum (70) is $\pi\psi(x) + \pi(\xi(x) - \psi(x)) = \pi\xi(x)$. And thus $0 \leq \xi(x) \leq 1$ for all $x \in \mathbb{R}$ when $R > 0$.

Theorem 19. *Let $\dim E = 1$, σ_0 and σ_p (44) be given by $\sigma_0 = [a, b]$ ($-\infty < a < b < \infty$) and $\sigma_p = \{\varkappa \in \mathbb{R} \setminus [a, b]\}$, and the scattering matrix $S(x) = \exp\{-2\pi\xi(x)\}$, where $\xi(x)$ is the function of spectral shift (39).*

Then $V(\lambda)$ (2) is given by

$$V(\lambda) = R^{-1} \left\{ 1 - \frac{\lambda - \varkappa}{\lambda + i} \exp \left\{ \int_{\mathbb{R}} \frac{\psi(x) - \xi(x)}{x - \lambda} dx \right\} \right\} \quad (\lambda \in \mathbb{C}_+), \quad (71)$$

where

$$0 \leq \xi(x) \leq 1 \quad (R > 0) \quad (-1 \leq \xi(x) \leq 0, R < 0), \quad (72)$$

the function $\varphi(x)$ is given by (68), and $R^{-1} = V(\varkappa)$.

It should be noticed that in (66) we could choose ia ($a > 0$) instead of i . Either for ia or i the final result is the same: formula (71) does not depend on a .

By the scattering data, the scattering matrix $S(x) = \exp\{-2\pi i\xi(x)\}$, and the number $\varkappa \in \sigma_p$, using (71), we construct the Nevanlinna function $V(\lambda)$ where the constant $R = V(r)$. Next, knowing $V(\lambda)$, we find its representing measure $dF(x) = a(x)dx$, using which we define the space $L^2_{\mathbb{R}}(E, dF(x))$ (18) and the pair of operators $\{\mathcal{Q}, \mathcal{L}\}$ (20). Thus, the Friedrichs–Faddeev model is unambiguously defined by the scattering data $\{S(x), \varkappa\}$ (20).

X. Let $\dim E = 1$ and σ_0 be given by

$$\sigma_0 = [a, c_1] \cup [c_2, c_3] \cup \dots \cup [c_m, b] \quad (73)$$

($-\infty < a < c_1 < c_2 < \dots < c_m < b < \infty$, $m \in \mathbb{N}$). Then the discrete spectrum σ_p (44) may consist of m points $\{\varkappa_k\}_1^m$ that are zeroes of the equation $1 = RV(\lambda)$ and are in the lacunas of the set σ_0 (73). Besides, there are two possibilities: either $\varkappa_1 \in (-\infty, a)$, $\varkappa_2 \in (c_1, c_2), \dots, \varkappa_m \in (c_{m-1}, c_m)$ as $R > 0$; or $\varkappa_1 \in (c_1, c_2)$, $\varkappa_2 \in (c_3, c_4), \dots, \varkappa_m \in (b, \infty)$, when $R < 0$. The roots $\{\varkappa_k\}_1^m$ in the lacunas of the set σ_0 (73) depend on the behavior of the representing measure $dF(x) = a(x)dx$ of the function $V(\lambda)$ (2) at the extremities of zones, i. e., at the points $a < c_1 < \dots < c_m < b$, as well as on the value of number R .

Similarly to (66), in this case, we write the Riemann boundary value problem (34) as

$$S(\lambda) \left(\frac{\lambda + i}{\lambda - i} \right)^m \prod_{k=1}^m \frac{\lambda - \varkappa_k}{\lambda + i} (1 - RV_+(\lambda))^{-1} = \prod_{k=1}^m \frac{\lambda - \lambda_k}{\lambda - i} (1 - RV_-(\lambda))^{-1}. \quad (74)$$

Using again the considerations of the previous section, we obtain the unique solution for this boundary problem,

$$(1 - RV(\lambda))^{-1} = \prod_{k=1}^m \frac{\lambda + i}{\lambda - \varkappa_k} \exp \left\{ \int_{\mathbb{R}} \frac{\xi(x) - m\varphi(x)}{x - \lambda} dx \right\} \quad (\lambda \in \mathbb{C}_+), \quad (75)$$

where $\xi(x)$ is the function of spectral shift (39), and $\psi(x)$ is given by (68). In this case, Theorem 19 can be formulated as follows.

Theorem 20. *Let $\dim E = 1$ and σ_0 be given by (73), and the set σ_p (44) be $\sigma_p = \{\varkappa_k \in \mathbb{R} \setminus \sigma_0, 1 \leq k \leq m\}$. Then the function $V(\lambda)$ (2) is defined unambiguously by the collection $\{S(x, \{\varkappa_k\}_1^m)\}$,*

$$V(\lambda) = R^{-1} \left\{ 1 - \prod_{k=1}^m \frac{\lambda - \varkappa_k}{\lambda + i} \exp \left\{ \int_{\mathbb{R}} \frac{m\psi(x) - \xi(x)}{x - \lambda} dx \right\} \right\} \quad (\lambda \in \mathbb{C}_+), \quad (76)$$

$\xi(x)$ is the function of spectral shift (39) ($S(x) = \exp\{-2\pi i\xi(x)\}$) and

$$0 \leq \xi(x) \leq 1 \quad (R > 0); \quad (-1 \leq \xi(x) \leq 0, R < 0), \quad (77)$$

where $\varphi(x)$ is given by (68), and $R^{-1} = V(\varkappa_n)$ ($1 \leq k \leq m$).

We can conclude that the Friedrichs–Faddeev model $\{\mathcal{Q}, \mathcal{L}\}$ (20) is defined unambiguously by the scattering matrix $S(x)$ and the collection of numbers $\{\varkappa_k\}_1^m$.

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