Journal of Mathematical Physics, Analysis, Geometry
2016, vol. 12, No. 4, pp. 315–337

On Submanifolds of Pseudo-Hyperbolic Space with 1-Type Pseudo-Hyperbolic Gauss Map

R. Yeğin

Istanbul Technical University, Faculty of Science and Letters
Department of Mathematics
34469 Maslak, Istanbul-Turkey

E-mail: ryegin@itu.edu.tr

U. Dursun

Işık University, Faculty of Arts and Sciences, Department of Mathematics
34980 Şile, Istanbul-Turkey

E-mail: ugur.dursun@isikun.edu.tr

Received October 16, 2015, revised May 4, 2016

In this paper, we examine pseudo-Riemannian submanifolds of a pseudo-hyperbolic space $\mathbb{H}_s^{m-1}(-1) \subset \mathbb{E}_{s+1}^m$ with finite type pseudo-hyperbolic Gauss map. We begin by providing a characterization of pseudo-Riemannian submanifolds in $\mathbb{H}_s^{m-1}(-1)$ with 1-type pseudo-hyperbolic Gauss map, and we obtain the classification of maximal surfaces in $\mathbb{H}_2^{m-1}(-1) \subset \mathbb{E}_3^m$ with 1-type pseudo-hyperbolic Gauss map. Then we investigate the submanifolds of $\mathbb{H}_s^{m-1}(-1)$ with 1-type pseudo-hyperbolic Gauss map containing nonzero constant component in its spectral decomposition.

Key words: finite type map, pseudo-hyperbolic Gauss map, pseudo-Riemannian submanifolds, Lorentzian hypersurfaces.

Mathematics Subject Classification 2010: 53B25 (primary), 53C50.

1. Introduction

The notion of a finite type submanifold of the Euclidean space was introduced by B.-Y. Chen in the late 1970s. Since then the finite type submanifolds of Euclidean spaces or pseudo-Euclidean spaces have been studied extensively, and many important results have been obtained ([3, 4, 6, 7], etc.).

In [8], Chen and Piccinni extended the notion of finite type to differentiable maps, in particular, to the Gauss map of submanifolds. A smooth map ϕ from

a compact Riemannian manifold M into a Euclidean space $\mathbb{E}^m$ is said to be of *finite type* if ϕ can be expressed as a finite sum of $\mathbb{E}^m$-valued eigenfunctions of the Laplacian Δ of M, that is,

$$\phi = \phi_0 + \phi_1 + \phi_2 + \cdots + \phi_k, \tag{1.1}$$

where ϕ_0 is a constant map, $\phi_1, \ldots, \phi_k$ are non-constant maps such that $\Delta\phi_i = \lambda_{p_i}\phi_i$, $\lambda_{p_i} \in \mathbb{R}$, $i = 1, \ldots, k$. If $\lambda_{p_1}, \ldots, \lambda_{p_k}$ are mutually distinct, then the map ϕ is said to be of k-type. If ϕ is an isometric immersion, then M is called a submanifold of finite type (or of k-type) if ϕ is. In the spectral decomposition of the immersion ϕ on a compact manifold, the constant vector ϕ_0 is the center of mass.

Chen and Piccinni characterized and classified compact hypersurfaces with 1-type Gauss map. They also provided the complete classification of minimal surfaces of $\mathbb{S}^{m-1}(1)$ with 2-type Gauss map.

Let $\mathbb{S}^{m-1}(1) \subset \mathbb{E}^m$ denote the unit hypersphere of $\mathbb{E}^m$ centered at the origin $\mathbb{E}^m$. A spherical finite type map $\phi : M^n \to \mathbb{S}^{m-1}(1) \subset \mathbb{E}^m$ of a compact Riemannian manifold M^n into $\mathbb{S}^{m-1}(1)$ is called *mass-symmetric* if the vector ϕ_0 in its spectral decomposition is the center of $\mathbb{S}^{m-1}(1)$ (which is the origin of $\mathbb{E}^m$). Otherwise, ϕ is called non-mass-symmetric.

If M^n is not compact, we cannot make the spectral decomposition of a map on M^n in general. However, it is possible to define the notion of a map of finite type on a non-compact manifold [6, page 124]. When M^n is non-compact, the vector ϕ_0 in the spectral decomposition in (1.1) is not necessarily a constant vector.

Let $\mathbf{x} : M_t^n \to \mathbb{E}_s^m$ be an oriented isometric immersion from a pseudo-Riemannian n-manifold M_t^n into a pseudo-Euclidean m-space $\mathbb{E}_s^m$. Let $G(n, m)$ denote the Grassmannian manifold consisting of all oriented n-planes of $\mathbb{E}_s^m$. The *classical Gauss map* $\nu : M_t^n \to G(n, m)$ associated with $\mathbf{x}$ is a map which carries each point $p \in M_t^n$ to the oriented n-plane of $\mathbb{E}_s^m$ obtained by parallel displacement of the tangent space $T_pM_t^n$ to the origin of $\mathbb{E}_s^m$. Since $G(n, m)$ can be canonically imbedded in the vector space $\bigwedge^n \mathbb{E}_s^m = \mathbb{E}_q^N$ for some integer q, the classical Gauss map ν gives rise to a well-defined map from M_t^n into the pseudo-Euclidean N-space $\mathbb{E}_q^N$, where $N = \binom{m}{n}$ and $\bigwedge^n \mathbb{E}_s^m$ is the vector space obtained by the exterior products of n vectors in $\mathbb{E}_s^m$ [17].

An isometric immersion from a Riemannian n-manifold M^n into an $(m-1)$-sphere $\mathbb{S}^{m-1}(1)$ can be viewed as one into a Euclidean m-space, and therefore the Gauss map associated with such an immersion can be determined in the ordinary sense. However, for the Gauss map to reflect the properties of the immersion into a sphere, instead of into the Euclidean space, Obata modified the definition of the Gauss map appropriately, [19].

Let $\mathbf{x} : M^n \to \widetilde{M}^m$ be an isometric immersion from a Riemannian n-manifold M^n into a simply-connected complete m-space $\widetilde{M}^m$ of constant curvature. The

generalized Gauss map in Obata's sense is a map which assigns to each $p \in M^n$ the totally geodesic n-space tangent to $\mathbf{x}(M^n)$ at $\mathbf{x}(p)$. In the case $\widetilde{M}^m = \mathbb{S}^m(1)$ (or resp. $\widetilde{M}^m = \mathbb{H}^m(-1)$), the generalized Gauss map is also called the *spherical Gauss map* (or resp. the *hyperbolic Gauss map*).

Later, in [15], Ishihara studied the Gauss map in a generalized sense of pseudo-Riemannian submanifolds of pseudo-Riemannian manifolds that also gives the Gauss map in Obata's sense.

Let $\widetilde{M}_s^{m-1}$ denote the pseudo-sphere $\mathbb{S}_s^{m-1}(1) \subset \mathbb{E}_s^m$ or the pseudo-hyperbolic space $\mathbb{H}_s^{m-1}(-1) \subset \mathbb{E}_{s+1}^m$. Let $\mathbf{x} : M_t^n \to \widetilde{M}_s^{m-1}$ be an oriented isometric immersion from a pseudo-Riemannian n-manifold M_t^n with index t into the complete pseudo-Riemannian $(m-1)$-space $\widetilde{M}_s^{m-1}$ of constant curvature. The generalized Gauss map in Obata's sense is a map associated to $\mathbf{x}$, which assigns to each $p \in M_t^n$ a totally geodesic n-subspace of $\widetilde{M}_s^{m-1}$ tangent to $\mathbf{x}(M_t^n)$ at $\mathbf{x}(p)$. Since the totally geodesic n-subspace of $\widetilde{M}_s^{m-1}$ tangent to $\mathbf{x}(M_t^n)$ at $\mathbf{x}(p)$ is the pseudo-sphere $\mathbb{S}_t^n(1)$ or the pseudo-hyperbolic space $\mathbb{H}_t^n(-1)$, it determines a unique oriented $(n+1)$-plane containing $\mathbb{S}_t^n(1)$ or $\mathbb{H}_t^n(-1)$. Thus, the generalized Gauss map in Obata's sense can be extended to a map $\hat{\nu}$ of M_t^n into the Grassmannian manifold $G(n+1, m)$ in the natural way, and the composition $\tilde{\nu}$ of $\hat{\nu}$ followed by the natural inclusion of $G(n+1, m)$ into a pseudo-Euclidean N-space $\mathbb{E}_q^N$, $N = \binom{m}{n+1}$, for some integer q is the pseudo-spherical Gauss map or the pseudo-hyperbolic Gauss map according to $\widetilde{M}_s^{m-1} = \mathbb{S}_s^{m-1}(1)$ or $\widetilde{M}_s^{m-1} = \mathbb{H}_s^{m-1}(-1)$, respectively.

In [9], Chen and Lue studied spherical submanifolds with finite type spherical Gauss map, and they obtained some characterization and classification results. In particular, they proved that Veronese surface and equilateral minimal torus are the only minimal surfaces in $\mathbb{S}^{m-1}(1)$ with 2-type spherical Gauss map. As it was explained in [9], the geometric behavior of the classical Gauss map differs from that of the spherical Gauss map. For example, the classical Gauss map of every compact Euclidean submanifold is mass-symmetric, but the spherical Gauss map of a spherical compact submanifold is not mass-symmetric in general.

In [14], the first author and Bektaş determined submanifolds of the unit sphere $\mathbb{S}^{m-1}(1)$ with non-mass-symmetric 1-type spherical Gauss map, and they also classified surfaces in $\mathbb{S}^3(1)$ with constant mean curvature and mass-symmetric 2-type spherical Gauss map.

There are many results obtained on the finite type submanifolds of hyperbolic spaces, pseudo-spheres and pseudo-hyperbolic spaces [4–6]. In [12], the first author studied hypersurfaces of hyperbolic space with 1-type Gauss map, and he provided the classification of hypersurfaces of a hyperbolic space canonically imbedded in Lorentz–Minkowski space $\mathbb{E}_1^m$ with at most two distinct principal curvatures and 1-type Gauss map.

Recently, in [13], we investigated submanifolds of hyperbolic spaces with finite type hyperbolic Gauss map. We characterized and classified submanifolds of the hyperbolic m-space $\mathbb{H}^m(-1)$ with 1-type hyperbolic Gauss map, and we obtained some results on hypersurfaces of $\mathbb{H}^m(-1)$ with 2-type hyperbolic Gauss map.

In this work, we study the pseudo-Riemannian submanifold of pseudo-hyperbolic space $\mathbb{H}_s^{m-1}(-1) \subset \mathbb{E}_{s+1}^m$ with finite type pseudo-hyperbolic Gauss map. We mainly obtain the following results:

1) An oriented pseudo-Riemannian submanifold M_t^n with index t of a pseudo-hyperbolic space $\mathbb{H}_s^{m-1}(-1) \subset \mathbb{E}_{s+1}^m$ has 1-type pseudo-hyperbolic Gauss map if and only if M_t^n has a zero mean curvature in $\mathbb{H}_s^{m-1}(-1)$, a constant scalar curvature and a flat normal bundle.

2) Let M be an oriented space-like surface in a pseudo-hyperbolic space $\mathbb{H}_s^{m-1}(-1) \subset \mathbb{E}_{s+1}^m$ for some values of s and m. Then, M has 1-type pseudo-hyperbolic Gauss map if and only if M is congruent to an open part of maximal surface $\mathbb{H}^1(-2) \times \mathbb{H}^1(-2)$ lying in $\mathbb{H}_1^3(-1) \subset \mathbb{H}_s^{m-1}(-1) \subset \mathbb{E}_{s+1}^m$ or the totally geodesic space $\mathbb{H}^2(-1)$ in $\mathbb{H}_s^{m-1}(-1) \subset \mathbb{E}_{s+1}^m$.

3) An oriented n-dimensional pseudo-Riemannian submanifold M_t^n with index t and non-zero mean curvature vector $\hat{H}$ of a pseudo-hyperbolic space $\mathbb{H}_s^{m-1}(-1) \subset \mathbb{E}_{s+1}^m$ has a 1-type pseudo-hyperbolic Gauss map with nonzero constant component in its spectral decomposition if and only if M_t^n is an open part of a non-flat, non-totally geodesic and totally umbilical pseudo-Riemannian hypersurface of a totally geodesic pseudo-hyperbolic space $\mathbb{H}_{s^*}^{n+1}(-1) \subset \mathbb{H}_s^{m-1}(-1) \subset \mathbb{E}_{s+1}^m$ for $s^* = t \leq s$ or $s^* = t+1 \leq s$, that is, M_t^n is an open part of $\mathbb{H}_t^n(-c) \subset \mathbb{H}_{t+1}^{n+1}(-1)$ of curvature $-c$ for $c > 1$ or $\mathbb{H}_t^n(-c) \subset \mathbb{H}_t^{n+1}(-1)$ of curvature $-c$ for $0 < c < 1$ or $\mathbb{S}_t^n(c) \subset \mathbb{H}_t^{n+1}(-1)$ of curvature $c > 0$.

2. Preliminaries

Let $\mathbb{E}_s^m$ denote the pseudo-Euclidean m-space with the canonical pseudo-Euclidean metric of index s given by

$$g_0 = \sum_{i=1}^{m-s} dx_i^2 - \sum_{j=m-s+1}^{m} dx_j^2, \tag{2.1}$$

where $(x_1, x_2, \ldots, x_m)$ is a rectangular coordinate system of $\mathbb{E}_s^m$. We put

$$\mathbb{S}_s^{m-1}(x_0, c) = \{x = (x_1, x_2, \ldots, x_m) \in \mathbb{E}_s^m | \, \langle x - x_0, x - x_0 \rangle = \frac{1}{c} > 0\},$$
$$\mathbb{H}_s^{m-1}(x_0, -c) = \{x = (x_1, x_2, \ldots, x_m) \in \mathbb{E}_{s+1}^m | \, \langle x - x_0, x - x_0 \rangle = -\frac{1}{c} < 0\},$$

where $\langle , \rangle$ is the indefinite inner product on $\mathbb{E}_s^m$ and c is a positive real number. Then $\mathbb{S}_s^{m-1}(x_0, c)$ and $\mathbb{H}_s^{m-1}(x_0, -c)$ are pseudo-Riemannian manifolds with index

s and of constant curvatures c and $-c$ called pseudo-sphere and pseudo-hyperbolic space, respectively. For $x_m > 0$ and $s = 0$, $\mathbb{H}^{m-1}(x_0, -c) = \mathbb{H}_0^{m-1}(x_0, -c)$ is called a hyperbolic space of curvature $-c$ centered at x_0. The manifolds $\mathbb{E}_s^m$, $\mathbb{S}_s^{m-1}(c)$ and $\mathbb{H}_s^{m-1}(-c)$ are known as indefinite space forms. In particular, $\mathbb{E}_1^m$, $\mathbb{S}_1^{m-1}(c)$ and $\mathbb{H}_1^{m-1}(-c)$ are called Minkowski space, de Sitter space and anti-de Sitter space in relativity, respectively. In order to simplify our notation, we will denote $\mathbb{S}_s^m(x_0, 1)$, $\mathbb{H}_s^m(x_0, -1)$ and $\mathbb{H}^m(x_0, -1)$ by $\mathbb{S}_s^m(1)$, $\mathbb{H}_s^m(-1)$ and $\mathbb{H}^m(-1)$, respectively, when x_0 is the origin.

Let M_t^n be an oriented n-dimensional pseudo-Riemannian submanifold with index t in an m-dimensional pseudo-Riemannian manifold $\widetilde{M}_s^m$ with index s. We choose a local orthonormal frame $e_1, \ldots, e_n,\ e_{n+1}, \ldots, e_m$ with signatures $\varepsilon_A = \langle e_A, e_A \rangle = \mp 1$, $A = 1, 2, \ldots, m$, on M_t^n such that the vectors $e_1, e_2, \ldots, e_n$ are tangent to M_t^n, and the vectors $e_{n+1}, \ldots, e_m$ are normal to M_t^n. We use the following convention on the ranges of indices:

$$1 \leq A, B, C, \ldots, \leq m; \quad 1 \leq i, j, k, \ldots, \leq n; \quad n+1 \leq r, s, t, \ldots, \leq m.$$

Let $\{\omega_A\}$ be the dual 1-forms of $\{e_A\}$ defined by $\omega_A(X) = \langle e_A, X \rangle$, and $\{\omega_{AB}\}$ the connection forms with $\omega_{AB} + \omega_{BA} = 0$ according to the chosen frame field $\{e_A\}$. Let ∇ and $\widetilde{\nabla}$ denote the Levi Civita connections on M_t^n and $\widetilde{M}_t^m$, respectively. Therefore, the Gauss and Weingarten formulas are given as

$$\widetilde{\nabla}_{e_k} e_i = \sum_{j=1}^{n} \varepsilon_j \omega_{ij}(e_k) e_j + \sum_{r=n+1}^{m} \varepsilon_r h_{ik}^r e_r \tag{2.2}$$

and

$$\widetilde{\nabla}_{e_k} e_s = -A_s(e_k) + \sum_{r=n+1}^{m} \varepsilon_r \omega_{sr}(e_k) e_r, \tag{2.3}$$

respectively, where h_{ik}^r's are the coefficients of the second fundamental form h, A_s is the Weingarten map in direction e_s, and ω_{rs} are the normal connection forms. Also, the normal connection is defined by $D_{e_i} e_r = \sum_{s=n+1}^{m} \varepsilon_s \omega_{rs}(e_i) e_s$.

The mean curvature vector H of M_t^n in $\widetilde{M}_s^m$ is defined by

$$H = \frac{1}{n} \sum_{r=n+1}^{m} \varepsilon_r \mathrm{tr} A_r e_r = \frac{1}{n} \sum_{i,r} \varepsilon_i \varepsilon_r h_{ii}^r e_r. \tag{2.4}$$

The squared norm $\|h\|^2$ of the second fundamental form h of M_t^n in $\widetilde{M}_s^m$ is defined by

$$\|h\|^2 = \sum_{i,j=1}^{n} \sum_{r=n+1}^{m} \varepsilon_i \varepsilon_j \varepsilon_r h_{ij}^r h_{ji}^r. \tag{2.5}$$

The Codazzi and Ricci equations of M_t^n are defined by

$$\begin{aligned} h^r_{ij;k} =& h^r_{ik;j} \\ h^r_{ij;k} =& e_k(h^r_{ij}) - \sum_{\ell=1}^{n} \varepsilon_\ell \Big(h^r_{i\ell}\omega_{j\ell}(e_k) + h^r_{j\ell}\omega_{i\ell}(e_k) \Big) + \sum_{s=n+1}^{m} \varepsilon_s h^s_{ij}\omega_{sr}(e_k) \end{aligned} \tag{2.6}$$

and

$$R^D(e_j, e_k; e_r, e_s) = \langle [A_{e_r}, A_{e_s}]e_j, e_k \rangle = \sum_{i=1}^{n} \varepsilon_i \left(h^r_{ki}h^s_{ij} - h^r_{ji}h^s_{ik} \right), \tag{2.7}$$

where R^D is the normal curvature tensor associated with the normal connection D. If the ambient space $\widetilde{M}_s^m$ is the pseudo-Euclidean space $\mathbb{E}_s^m$, then the scalar curvature S of M_t^n is given by

$$S = n^2 \langle H, H \rangle - \|h\|^2. \tag{2.8}$$

If M_t^n is immersed in the pseudo-hyperbolic space $\mathbb{H}_s^{m-1}(-1) \subset \mathbb{E}_{s+1}^m$, then (2.8) gives

$$S = -n(n-1) + n^2\langle \hat{H}, \hat{H} \rangle - \|\hat{h}\|^2, \tag{2.9}$$

where $\hat{H}$ and $\hat{h}$ are the mean curvature vector and the second fundamental form of M_t^n in $\mathbb{H}_s^{m-1}(-1)$. For M_t^n in $\mathbb{H}_s^{m-1}(-1) \subset \mathbb{E}_{s+1}^m$ we also have

$$H = \hat{H} + \mathbf{x}, \quad h(X,Y) = \hat{h}(X,Y) + \langle X, Y \rangle \mathbf{x}. \tag{2.10}$$

A point on a pseudo-Riemannian submanifold M_t^n of a pseudo-Riemannian manifold $\widetilde{M}_s^m$ is called *isotropic* if, at each point $p \in M_t^n$, $\langle h(X,X), h(X,X) \rangle$ is constant for any unit tangent vector X at p.

A pseudo-Riemannian hypersurface M_t^n of a pseudo-Riemannian manifold $\widetilde{M}_s^{n+1}$ is called *proper* if the shape operator A_ξ in a unit normal direction ξ can be expressed by a real diagonal matrix with respect to an orthonormal frame at each point of M_t^n.

A proper hypersurface M_t^n in $\mathbb{H}_s^{n+1}(-1)$ is said to be *isoparametric* if it has constant principal curvatures.

3. Pseudo-Hyperbolic Gauss Map

Let $\mathbf{x} : M_t^n \longrightarrow \mathbb{H}_s^{m-1}(-1) \subset \mathbb{E}_{s+1}^m$ be an oriented isometric immersion from a pseudo-Riemannian n-manifold M_t^n with index t into a pseudo-hyperbolic $(m-1)$-space $\mathbb{H}_s^{m-1}(-1) \subset \mathbb{E}_{s+1}^m$. The pseudo-hyperbolic Gauss map in Obata's sense $\hat{\nu} : M_t^n \longrightarrow G(n+1, m)$ of an immersion $\mathbf{x}$ into the Grassmannian manifold

$G(n+1,m)$ is a map which to each point p of M_t^n assigns the great pseudo-hyperbolic n-space $\mathbb{H}_t^n(-1)$ of $\mathbb{H}_s^{m-1}(-1)$ tangent to $\mathbf{x}(M_t^n)$ at $\mathbf{x}(p)$. The great pseudo-hyperbolic n-spaces $\mathbb{H}_t^n(-1)$ in $\mathbb{H}_s^{m-1}(-1)$ are naturally identified with the Grassmannian manifold of oriented $(n+1)$-planes through the center of $\mathbb{H}_s^{m-1}(-1)$ in $\mathbb{E}_{s+1}^m$ since such $(n+1)$-planes determine unique great pseudo-hyperbolic n-spaces and vice versa.

On the other hand, since the Grassmannian manifold $G(n+1,m)$ can be canonically imbedded in a pseudo-Euclidean space $\bigwedge^{n+1}\mathbb{E}_{s+1}^m \cong \mathbb{E}_q^N$ obtained by the exterior products of $n+1$ vectors in $\mathbb{E}_{s+1}^m$ for some positive integer q, the composite $\tilde{\nu}$ of $\hat{\nu}$ followed by the natural inclusion of $G(n+1,m)$ in $\mathbb{E}_q^N$ is also called *the pseudo-hyperbolic Gauss map* where $N=\binom{m}{n+1}$.

For each point $p \in M_t^n$, let $e_1,\ldots,e_n$ be an orthonormal basis of $T_pM_t^n$ with the signatures $\varepsilon_i = \langle e_i, e_i\rangle = \mp 1,\ i=1,\ldots,n$. Then the $n+1$ vectors $\mathbf{x}(p), e_1,\ldots,e_n$ determine a linear $(n+1)$-subspace in $\mathbb{E}_{s+1}^m$. The intersection of this linear subspace and $\mathbb{H}_s^{m-1}(-1)$ is a totally geodesic pseudo-hyperbolic n-space $\mathbb{H}_t^n(-1)$ determined by $T_pM_t^n$.

Let $\mathbb{E}_{s+1}^m$ be a pseudo-Euclidean space with index $s+1$. Let $f_{i_1}\wedge\cdots\wedge f_{i_{n+1}}$ and $g_{i_1}\wedge\cdots\wedge g_{i_{n+1}}$ be two vectors in $\bigwedge^{n+1}\mathbb{E}_{s+1}^m$, where $\{f_1,f_2,\ldots,f_m\}$ and $\{g_1,g_2,\ldots,g_m\}$ are two orthonormal bases of $\mathbb{E}_{s+1}^m$. Define an indefinite inner product $\langle\langle,\rangle\rangle$ on $\bigwedge^{n+1}\mathbb{E}_{s+1}^m$ by

$$\big\langle\big\langle f_{i_1}\wedge\cdots\wedge f_{i_{n+1}}, g_{j_1}\wedge\cdots\wedge g_{j_{n+1}}\big\rangle\big\rangle = \det(\langle f_{i_\ell}, g_{j_k}\rangle). \tag{3.1}$$

Therefore, we can identify $\bigwedge^{n+1}\mathbb{E}_{s+1}^m$ with some pseudo-Euclidean space $\mathbb{E}_q^N$ for some positive integer q where $N=\binom{m}{n+1}$ [17].

For an oriented immersion $\mathbf{x}: M_t^n \to \mathbb{H}_s^{m-1}(-1)\subset\mathbb{E}_{s+1}^m$, the map in Obata's sense can be considered as $\hat{\nu}: M_t^n \to G(n+1,m)$ which carries each $p\in M_t^n$ to $\hat{\nu}(p) = (\mathbf{x}\wedge e_1\wedge e_2\wedge\cdots\wedge e_n)(p)$. Since $\langle\langle\hat{\nu},\hat{\nu}\rangle\rangle = -\varepsilon_1\varepsilon_2\cdots\varepsilon_n = \mp 1$, the Grassmannian manifold $G(n+1,m)$ is a submanifold of $\mathbb{S}_q^{N-1}(1)\subset\mathbb{E}_q^N$ or $\mathbb{H}_{q-1}^{N-1}(-1)\subset\mathbb{E}_q^N$. Thus, considering the natural inclusion of $G(n+1,m)$ into $\mathbb{E}_q^N$, the pseudo-hyperbolic Gauss map $\tilde{\nu}$ associated with $\mathbf{x}$ is given by

$$\tilde{\nu} = \mathbf{x}\wedge e_1\wedge e_2\wedge\cdots\wedge e_n : M_t^n \to G(n+1,m)\subset\mathbb{E}_q^N. \tag{3.2}$$

Now, by differentiating $\tilde{\nu}$ from (3.2), we find

$$e_i(\tilde{\nu}) = \sum_{r=n+1}^{m-1}\sum_{j=1}^{n}\varepsilon_r h_{ij}^r \mathbf{x}\wedge e_1\wedge\cdots\wedge\underbrace{e_r}_{j-th}\wedge\cdots\wedge e_n. \tag{3.3}$$

Since $\nabla_{e_k}e_i = \sum_{j=1}^n \varepsilon_j\omega_{ij}(e_k)e_j$, we have

$$(\nabla_{e_i}e_i)\tilde{\nu} = \sum_{r=n+1}^{m-1}\sum_{j,k=1}^{n}\varepsilon_k\varepsilon_r\omega_{ik}(e_i)h_{kj}^r \mathbf{x}\wedge e_1\wedge\cdots\wedge\underbrace{e_r}_{j-th}\wedge\cdots\wedge e_n. \tag{3.4}$$

Considering that the Laplacian of $\tilde{\nu}$ is defined by

$$\Delta\tilde{\nu} = \sum_{i=1}^{n} \varepsilon_i (\nabla_{e_i} e_i - e_i e_i)\tilde{\nu}, \tag{3.5}$$

by a direct calculation, we obtain that

$$\begin{aligned}\Delta\tilde{\nu} &= \|\hat{h}\|^2\tilde{\nu} + n\hat{H} \wedge e_1 \wedge \cdots \wedge e_n \\ &\quad - n\sum_{k=1}^{n} \mathbf{x} \wedge e_1 \wedge \cdots \wedge \underbrace{D_{e_k}\hat{H}}_{k-th} \wedge \cdots \wedge e_n \\ &\quad + \sum_{\substack{j,k=1 \\ j\neq k}}^{n} \sum_{\substack{r,s=n+1 \\ s<r}}^{m-1} \varepsilon_r\varepsilon_s R^r_{sjk}\mathbf{x} \wedge e_1 \wedge \cdots \wedge \underbrace{e_s}_{k-th} \wedge \cdots \wedge \underbrace{e_r}_{j-th} \wedge \cdots \wedge e_n,\end{aligned} \tag{3.6}$$

where $R^r_{sjk} = R^D(e_j, e_k; e_r, e_s)$.

In [4], Chen studied non-compact finite type pseudo-Riemannian submanifold of a pseudo-Riemannian sphere $\mathbb{S}^{m-1}_s(1)$ or a pseudo-hyperbolic space $\mathbb{H}^{m-1}_s(-1)$, and the definition of spectral decomposition of an immersion was stated without a constant component.

A smooth map $\phi : M^n_t \to \mathbb{S}^{m-1}_s(1) \subset \mathbb{E}^m_s$ (resp., $\phi : M^n_t \to \mathbb{H}^{m-1}_s(-1) \subset \mathbb{E}^m_{s+1}$) from a pseudo-Riemannian manifold M^n_t into a pseudo-Riemannian sphere $\mathbb{S}^{m-1}_s(1)$ (resp., into a pseudo-hyperbolic space $\mathbb{H}^{m-1}_s(-1)$) is called of finite type in $\mathbb{S}^{m-1}_s(1)$ (resp., in $\mathbb{H}^{m-1}_s(-1)$) if the map ϕ has the spectral decomposition

$$\phi = \phi_1 + \cdots + \phi_k, \tag{3.7}$$

where ϕ_i's are non-constant $\mathbb{E}^m_s$-valued maps on M^n_t such that $\Delta\phi_i = \lambda_{p_i}\phi_i$ with $\lambda_{p_i} \in \mathbb{R}, i = 1, \ldots, k$. If the spectral decomposition (3.7) contains exactly k non-constant components, the map ϕ is called of k-type, [12].

For a finite type map, one of the components in its spectral decomposition may still be constant. A criteria for finite type maps was given in [13] as follows:

Theorem 3.1. *Let $\phi : M^n_t \longrightarrow \mathbb{E}^m_s$ be a smooth map from a pseudo-Riemannian manifold M^n_t with index t into a pseudo-Euclidean space $\mathbb{E}^m_s$, and let $\tau = \operatorname{div}(\mathrm{d}\phi)$ be the tension field of ϕ. Then,*

(i) If there is a non-trivial polynomial Q such that $Q(\Delta)\tau = 0$, then ϕ is either of infinite type or of finite type with type number $k \leqslant deg(Q) + 1$;

(ii) If there is a non-trivial polynomial P with simple roots such that $P(\Delta)\tau = 0$, then ϕ is of finite type with type number $k \leqslant deg(P)$.

A smooth map ϕ between two pseudo-Riemannian manifolds is said to be *harmonic* if its tension field $\tau = \operatorname{div}(d\phi)$ vanishes identically. For a harmonic pseudo-hyperbolic Gauss map we have the following.

Proposition 3.1. *Let* $\mathbf{x} : (M_t^n, g) \longrightarrow \mathbb{H}_s^{m-1}(-1) \subset \mathbb{E}_{s+1}^m$ *be an isometric immersion from a pseudo-Riemannian n-manifold* M_t^n *with metric g into a pseudo-hyperbolic space* $\mathbb{H}_s^{m-1}(-1) \subset \mathbb{E}_{s+1}^m$. *Then we have the following:*

(*i*) *the Obata map* $\hat{\nu} : (M_t^n, g) \longrightarrow G(n+1, m)$ *is a harmonic map if and only if the immersion* $\mathbf{x} : (M_t^n, g) \longrightarrow \mathbb{H}_s^{m-1}(-1)$ *has a zero mean curvature*;

(*ii*) *the pseudo-hyperbolic Gauss map* $\tilde{\nu} : (M_t^n, g) \longrightarrow \mathbb{E}_q^N$ *with* $N = \binom{m}{n+1}$ *and for some positive integer q is a harmonic map if and only if* M_t^n *has a zero mean curvature in* $\mathbb{H}_s^{m-1}(-1)$, *a flat normal bundle and the scalar curvature* $S = -n(n-1)$.

P r o o f. The proof of (i) is similar to that of Proposition 3.2 given in [9], and the proof of (ii) comes from (2.9) and (3.6). ■

4. Submanifolds with 1-Type Pseudo-Hyperbolic Gauss Map

In this section, we examine submanifolds of a pseudo-hyperbolic space $\mathbb{H}_s^{m-1}(-1) \subset \mathbb{E}_{s+1}^m$ with 1-type pseudo-hyperbolic Gauss map $\tilde{\nu}$.

If the pseudo-hyperbolic Gauss map $\tilde{\nu}$ is of 1-type, then we have $\Delta\tilde{\nu} = \lambda_p\tilde{\nu}$ from (3.7).

Theorem 4.1. *A pseudo-Riemannian oriented submanifold* M_t^n *with index t of a pseudo-hyperbolic space* $\mathbb{H}_s^{m-1}(-1) \subset \mathbb{E}_{s+1}^m$ *has a 1-type pseudo-hyperbolic Gauss map if and only if* M_t^n *has a zero mean curvature in* $\mathbb{H}_s^{m-1}(-1)$, *a constant scalar curvature and a flat normal bundle.*

P r o o f. Assume that a pseudo-Riemannian oriented submanifold M_t^n in $\mathbb{H}_s^{m-1}(-1)$ has a 1-type pseudo-hyperbolic Gauss map $\tilde{\nu}$ in $\mathbb{H}_s^{m-1}(-1)$, that is, $\Delta\tilde{\nu} = \lambda_p\tilde{\nu}$ for some nonzero constant $\lambda_p \in \mathbb{R}$. Therefore, from (3.6) we obtain that $\tilde{\nu}$ is of 1-type if and only if $\hat{H} = R^D = 0$, and $\|\hat{h}\|^2$ is a nonzero constant, i.e., M_t^n has a zero mean curvature in $\mathbb{H}_s^{m-1}(-1)$, the normal bundle of M_t^n is flat, and from (2.9) the scalar curvature is constant. ■

Corollary 4.2. *Totally geodesic pseudo-Riemannian oriented submanifolds of* $\mathbb{H}_s^{m-1}(-1) \subset \mathbb{E}_{s+1}^m$ *have a harmonic pseudo-hyperbolic Gauss map which is of 1-type.*

Corollary 4.3. *Let M_t^n be an n-dimensional pseudo-Riemannian oriented hypersurface with index t in a pseudo-hyperbolic space $\mathbb{H}_s^{n+1}(-1) \subset \mathbb{E}_{s+1}^{n+2}$. Then M_t^n has a 1-type pseudo-hyperbolic Gauss map if and only if M_t^n has a zero mean curvature in $\mathbb{H}_s^{n+1}(-1)$ and a constant scalar curvature.*

Corollary 4.4. *Isoparametric proper pseudo-Riemannian oriented hypersurfaces of $\mathbb{H}_s^{m-1}(-1) \subset \mathbb{E}_{s+1}^m$ with zero mean curvature in $\mathbb{H}_s^{m-1}(-1)$ have a 1-type pseudo-hyperbolic Gauss map.*

In [2], Zhen-qi and Xian-hua determined a space-like isoparametric hypersurface M in $\mathbb{H}_1^{n+1}(-1) \subset \mathbb{E}_2^{n+2}$. They showed that a space-like isoparametric hypersurface M in $\mathbb{H}_1^{n+1}(-1)$ can have at most two distinct principal curvatures. Moreover, they showed that M is congruent to an open subset of the umbilical hypersurface $\mathbb{H}^n(-c) \subset \mathbb{H}_1^{n+1}(-1)$ where $c > 0$ or the product of two hyperbolic spaces

$$\mathbb{H}^k(-c_1) \times \mathbb{H}^{n-k}(-c_2) = \{(x, y) \in \mathbb{E}_1^{k+1} \times \mathbb{E}_1^{n-k+1} : \langle x, x\rangle = -\frac{1}{c_1},\ \langle y, y\rangle = -\frac{1}{c_2}\},$$

where $c_1, c_2 > 0$.

In [11], Cheng gave the following corollary.

Corollary 4.5. *Let M be a complete isoparametric maximal space-like hypersurface in an anti-de Sitter space $\mathbb{H}_1^{n+1}(-c)$. Then $M = \mathbb{H}^n(-c)$ or $M = \mathbb{H}^{n_1}\left(\frac{-n}{n_1 c}\right) \times \mathbb{H}^{n-n_1}\left(\frac{-n}{(n-n_1)c}\right)$ for $(n > n_1 \geq 1)$.*

Therefore, we obtain the following corollary using Corollary 4.3 and Corollary 4.5.

Corollary 4.6. *A totally geodesic hyperbolic space $\mathbb{H}^n(-1)$ and the product hypersurface $M = \mathbb{H}^{n_1}\left(\frac{-n}{n_1 c}\right) \times \mathbb{H}^{n-n_1}\left(\frac{-n}{(n-n_1)c}\right)$ for $(n > n_1 \geq 1)$ in $\mathbb{H}_1^{n+1}(-1)$ are the only maximal isoparametric hypersurfaces with 1-type pseudo-hyperbolic Gauss map.*

We need the connection forms of the following surface to be used later:

E x a m p l e 4.7. (*Maximal space-like surface in $\mathbb{H}_1^3(-1)$*)

Let $\mathbf{x} : M = \mathbb{H}^1(-a^{-2}) \times \mathbb{H}^1(-b^{-2}) \longrightarrow \mathbb{H}_1^3(-1) \subset \mathbb{E}_2^4$ be an oriented isometric immersion from the space-like surface M into the anti-de Sitter space $\mathbb{H}_1^3(-1)$ defined by

$$\mathbf{x}(u, v) = (a \sinh u, b \sinh v, a \cosh u, b \cosh v)$$

with $a^2 + b^2 = 1$. If we put $e_1 = \frac{1}{a}\frac{\partial}{\partial u},\ e_2 = \frac{1}{b}\frac{\partial}{\partial v}$,

$$e_3 = (b \sinh u, -a \sinh v, b \cosh u, -a \cosh v),\ e_4 = \mathbf{x},$$

then $\{e_1, e_2, e_3, e_4\}$ form an orthonormal frame field on M in $\mathbb{E}_2^4$. A straightforward computation gives

$$\omega_{12} = \omega_{34} = 0,\ \omega_{13} = -\frac{b}{a}\omega_1,\ \omega_{23} = \frac{a}{b}\omega_2,\ \omega_{14} = -\omega_1,\ \omega_{24} = -\omega_2. \tag{4.1}$$

It follows from (4.1) that $\hat{H} = \frac{a^2-b^2}{2ab}e_3$, which implies that M is a maximal surface if and only if $a = b = \frac{1}{\sqrt{2}}$. Therefore, $\mathbb{H}^1(-2) \times \mathbb{H}^1(-2) \subset \mathbb{H}_1^3(-1) \subset \mathbb{E}_2^4$ is a maximal and flat surface, and hence $\mathbb{H}^1(-2) \times \mathbb{H}^1(-2)$ has a 1-type pseudo-hyperbolic Gauss map by Theorem 4.1.

Theorem 4.8. *Let M be a space-like oriented surface in a pseudo-hyperbolic space $\mathbb{H}_s^{m-1}(-1) \subset \mathbb{E}_{s+1}^m$ for some values of s and m. Then M has a 1-type pseudo-hyperbolic Gauss map if and only if M is congruent to an open part of maximal surface $\mathbb{H}^1(-2) \times \mathbb{H}^1(-2)$ lying in $\mathbb{H}_1^3(-1) \subset \mathbb{H}_s^{m-1}(-1) \subset \mathbb{E}_{s+1}^m$ or the totally geodesic space $\mathbb{H}^2(-1)$ in $\mathbb{H}_s^{m-1}(-1) \subset \mathbb{E}_{s+1}^m$.*

P r o o f. Let M be a space-like oriented surface in a pseudo-hyperbolic space $\mathbb{H}_s^{m-1}(-1) \subset \mathbb{E}_{s+1}^m$ with 1-type pseudo-hyperbolic Gauss map. Then, from Theorem 4.1, for $t = 0$, we obtain that M is a maximal surface in $\mathbb{H}_s^{m-1}(-1)$ with constant scalar curvature and flat normal bundle. Thus, (2.9) yields that $\|\hat{h}\|^2$ is constant.

Let $\mathbf{x}$ be the position vector of M in $\mathbb{E}_{s+1}^m$. Since M is maximal, we may choose an orthonormal tangent frame $\{e_1, e_2\}$ and an orthonormal normal frame $\{e_3, \ldots, e_{m-1}, e_m = \mathbf{x}\}$ of M such that

$$A_3 = \begin{pmatrix} h_{11}^3 & 0 \\ 0 & -h_{11}^3 \end{pmatrix},\ A_4 = \begin{pmatrix} 0 & h_{12}^4 \\ h_{12}^4 & 0 \end{pmatrix}, A_5 = \cdots = A_{m-1} = 0, A_m = -I,$$

where I is the 2×2 identity matrix. Hence we obtain that

$$\|\hat{h}\|^2 = 2\varepsilon_3(h_{11}^3)^2 + 2\varepsilon_4(h_{12}^4)^2. \tag{4.2}$$

On the other hand, as $K^D = -2h_{11}^3 h_{12}^4 = 0$, we have either $h_{11}^3 = 0$ or $h_{12}^4 = 0$.

Case (a): $h_{11}^3 = 0$. Then the first normal space of M is spanned by e_4, and hence M lies in a totally geodesic anti-de Sitter space $\mathbb{H}_1^3(-1) \subset \mathbb{H}_s^{m-1}(-1)$ or a totally geodesic hyperbolic 3-space $\mathbb{H}^3(-1) \subset \mathbb{H}_s^{m-1}(-1)$. From (4.2), we have $\|\hat{h}\|^2 = 2\varepsilon_4(h_{12}^4)^2$, which implies that h_{12}^4 is constant. It follows from the Codazzi equation that $\omega_{12}(e_j)h_{12}^4 = 0$ for $j = 1, 2$, which gives that either $\omega_{12}(e_j) = 0$ for $j = 1, 2$ or $h_{12}^4 = 0$.

Case (a.1): $\omega_{12}(e_j) = 0$ for $j = 1, 2$. Then the Gaussian curvature K is zero. On the other hand, from the Gauss equation we have

$$K = -1 + \varepsilon_3 \det A_3 + \varepsilon_4 \det A_4 = -1 - \varepsilon_4(h_{12}^4)^2 = 0. \tag{4.3}$$

Therefore, $(h^4_{12})^2 = -\varepsilon_4$, which implies that $\varepsilon_4 = -1$, and $h^4_{12} = \mp 1$. That is, M lies in $\mathbb{H}^3_1(-1) \subset \mathbb{H}^{m-1}_s(-1)$. Without loss of generality, we may take $h^4_{12} = 1$. Therefore, for the maximal surface M we obtain

$$A_4 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

By choosing a new orthonormal tangent frame $\{\bar{e}_1, \bar{e}_2\}$, we can have the shape operator A_4 as

$$A_4 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Since M lies in a totally geodesic anti-de Sitter 3-space in $\mathbb{H}^3_1(-1) \subset \mathbb{H}^{m-1}_s(-1) \subset \mathbb{E}^m_{s+1}$, we can assume that M is immersed in $\mathbb{H}^3_1(-1) \subset \mathbb{E}^4_2$ without loss of generality. Let $\{e_1, e_2, e_3, e_4 = \mathbf{x}\}$ be a local orthonormal frame on M in $\mathbb{E}^4_2$ such that e_1, e_2 are tangent to M, and e_3, e_4 are normal to M. Since the normal connection of M is flat, we have $\omega_{34} = 0$. In addition, as M is flat, we can take local coordinates (u, v) on M with $\omega_1 = du$ and $\omega_2 = dv$. So we have

$$\omega_{12} = \omega_{34} = 0,\ \omega_{13} = -\omega_1,\ \omega_{23} = \omega_2,\ \omega_{14} = -\omega_1,\ \omega_{24} = -\omega_2. \tag{4.4}$$

Therefore, the connection forms ω_{AB} of M coincide with the connection forms of $\mathbb{H}^1(-a^{-2}) \times \mathbb{H}^1(-b^{-2}) \subset \mathbb{H}^3_1(-1) \subset \mathbb{E}^4_2$ for $a = b = \frac{1}{\sqrt{2}}$ given by (4.1). As a consequence of the fundamental theorem of submanifolds, M is congruent to an open part of $\mathbb{H}^1(-2) \times \mathbb{H}^1(-2) \subset \mathbb{H}^3_1(-1) \subset \mathbb{H}^{m-1}_s(-1) \subset \mathbb{E}^m_{s+1}$.

Case (a.2): $h^4_{12} = 0$ and $\omega_{12}(e_j) \neq 0$ at least for one $j = 1, 2$. Thus, we have $A_3 = A_4 = \cdots = A_{m-1} = 0$ and $A_m = -I$, and the Gaussian curvature $K = -1$. So, M is an open part of hyperbolic space $\mathbb{H}^2(-1)$ in $\mathbb{H}^{m-1}_s(-1) \subset \mathbb{E}^m_{s+1}$.

Case (b): $h^4_{12} = 0$. By a similar argument given in Case (a), it can be easily seen that M is an open part of $\mathbb{H}^1(-2) \times \mathbb{H}^1(-2) \subset \mathbb{H}^3_1(-1) \subset \mathbb{H}^{m-1}_s(-1) \subset \mathbb{E}^m_{s+1}$ or an open part of the hyperbolic space $\mathbb{H}^2(-1)$ in $\mathbb{H}^{m-1}_s(-1) \subset \mathbb{E}^m_{s+1}$.

The converse follows from Corollary 4.2 and Example 4.7. ∎

We have stated before that a map may have a nonzero constant component in its spectral decomposition. We will investigate submanifolds of a pseudo-hyperbolic space $\mathbb{H}^{m-1}_s(-1) \subset \mathbb{E}^m_{s+1}$ with 1-type pseudo-hyperbolic Gauss map having a nonzero constant component in its spectral decomposition.

Now we provide the example to be used in the proof of the next theorem.

E x a m p l e 4.9. (*Space-like surface with flat normal bundle and zero mean curvature vector $\hat{H}$ in $\mathbb{H}^4_1(-1) \subset \mathbb{E}^5_2$*)

Let $\mathbf{x} : M \longrightarrow \mathbb{H}^4_1(-1) \subset \mathbb{E}^5_2$ be an oriented space-like isometric immersion from a surface M into an anti-de Sitter space $\mathbb{H}^4_1(-1) \subset \mathbb{E}^5_2$ defined in [10] by

$$\mathbf{x}(u, v) = (1, \cosh u \sinh v, \sinh u, \cosh u \cosh v, 1). \tag{4.5}$$

If we put $e_1 = \frac{\partial}{\partial u},\ e_2 = \frac{1}{\cosh u}\frac{\partial}{\partial v}$,

$$e_3 = (\frac{3}{2}, \cosh u \sinh v, \sinh u, \cosh u \cosh v, \frac{1}{2})$$

and

$$e_4 = (\frac{1}{2}, \cosh u \sinh v, \sinh u, \cosh u \cosh v, -\frac{1}{2}),\ e_5 = \mathbf{x},$$

then $\{e_i\}$ for $i = 1, \ldots, 5$ form an orthonormal frame field on M. A straightforward computation gives

$$\begin{aligned} h^3_{11} = h^3_{22} = h^4_{11} = h^4_{22} = -1, \quad h^3_{12} = h^4_{12} = 0, \\ \omega_{12}(e_1) = 0, \quad \omega_{12}(e_2) = \tanh u, \quad \omega_{34} = 0, \\ \|\hat{h}\|^2 = 0, \quad \hat{H} = e_4 - e_3 = (-1, 0, 0, 0, -1). \end{aligned} \tag{4.6}$$

If we use (4.6), then equation (3.6) reduces to

$$\triangle\tilde{\nu} = 2\hat{H} \wedge e_1 \wedge e_2 = -2e_3 \wedge e_1 \wedge e_2 + 2e_4 \wedge e_1 \wedge e_2. \tag{4.7}$$

If we put

$$\tilde{c} = \tilde{\nu} - e_3 \wedge e_1 \wedge e_2 + e_4 \wedge e_1 \wedge e_2 \tag{4.8}$$

and

$$\tilde{\nu}_p = e_3 \wedge e_1 \wedge e_2 - e_4 \wedge e_1 \wedge e_2, \tag{4.9}$$

then we have $\tilde{\nu} = \tilde{c} + \tilde{\nu}_p$. It can be shown that $e_i(\tilde{c}) = 0$ for $i = 1, 2$, i.e., $\tilde{c}$ is a constant vector. Using (4.7), (4.8) and (4.9), we arrive at $\Delta\tilde{\nu}_p = -2\tilde{\nu}_p$. Thus, M has a 1-type pseudo-hyperbolic Gauss map with nonzero constant component in its spectral decomposition.

Theorem 4.10. *Let M be a space-like oriented surface in $\mathbb{H}^4_1(-1) \subset \mathbb{E}^5_2$ with zero mean curvature vector $\hat{H}$ in an anti-de Sitter space $\mathbb{H}^4_1(-1)$. Then M has a 1-type pseudo-hyperbolic Gauss map with nonzero constant component in its spectral decomposition if and only if M is an open part of the surface defined by* (4.5) *which is of curvature -1 and totally umbilical with constant zero mean curvature vector.*

P r o o f. Assume that $\mathbf{x} : M \longrightarrow \mathbb{H}^4_1(-1) \subset \mathbb{E}^5_2$ is an oriented isometric immersion from a space-like surface M into $\mathbb{H}^4_1(-1)$, and the pseudo-hyperbolic Gauss map $\tilde{\nu}$ of $\mathbf{x}$ is of 1-type with nonzero constant component in its spectral decomposition. Then we have $\triangle\tilde{\nu} = \lambda_p(\tilde{\nu} - \tilde{c})$ for a real number $\lambda_p \neq 0$ and for some constant vector $\tilde{c}$, from which we get

$$(\Delta\tilde{\nu})_i = \lambda_p(\tilde{\nu})_i, \tag{4.10}$$

where $(.)_i = e_i(.)$ and

$$e_i(\tilde{\nu}) = \sum_{r=3}^{4}\sum_{k=1}^{2}\varepsilon_r h_{ik}^r \mathbf{x} \wedge \underbrace{e_r}_{k-th} \wedge e_2. \tag{4.11}$$

By a long computation we obtain that

$$\begin{aligned}
e_i(\Delta\tilde{\nu}) &= (\|\hat{h}\|^2)_i\tilde{\nu} + \|\hat{h}\|^2\sum_{r=3}^{4}\sum_{k=1}^{2}\varepsilon_r h_{ik}^r \mathbf{x} \wedge \underbrace{e_r}_{k-th} \wedge e_2 + 4D_{e_i}\hat{H} \wedge e_1 \wedge e_2 \\
&+ 2\sum_{r=3}^{4}\sum_{k=1}^{2}\varepsilon_r h_{ik}^r \hat{H} \wedge \underbrace{e_r}_{k-th} \wedge e_2 + 2\sum_{k=1}^{2}\delta_{ik}\hat{H} \wedge \underbrace{\mathbf{x}}_{k-th} \wedge e_2 \\
&- 2\sum_{\substack{j,k=1\\ j\neq k}}^{2}\omega_{jk}(e_i)\mathbf{x} \wedge \underbrace{e_k}_{j-th} \wedge \underbrace{D_{e_k}\hat{H}}_{k-th} - 2\sum_{\substack{j,k=1\\ j\neq k}}^{2}\sum_{r=3}^{4}\varepsilon_r h_{ij}^r \mathbf{x} \wedge \underbrace{e_r}_{j-th} \wedge \underbrace{D_{e_k}\hat{H}}_{k-th} \\
&+ 2\sum_{k=1}^{2}\Big\langle A_{D_{e_k}\hat{H}}(e_i), e_k\Big\rangle\tilde{\nu} - 2\sum_{k=1}^{2}\mathbf{x} \wedge \underbrace{D_{e_i}D_{e_k}}_{k-th}\hat{H} \wedge e_2 \\
&+ 2\varepsilon_3\varepsilon_4\big(e_i(R_{321}^4)\mathbf{x} + R_{321}^4 e_i\big) \wedge e_3 \wedge e_4 - 2\varepsilon_3\varepsilon_4 R_{321}^4\sum_{k=1}^{2}h_{ik}^3\mathbf{x} \wedge e_k \wedge e_4 \\
&- 2\varepsilon_3\varepsilon_4 R_{321}^4\sum_{k=1}^{2}h_{ik}^4\mathbf{x} \wedge e_3 \wedge e_k.
\end{aligned} \tag{4.12}$$

Since M has a zero mean curvature vector $\hat{H}$ in $\mathbb{H}_1^4(-1)$, then $\langle\hat{H},\hat{H}\rangle = 0$ and $\hat{H} \neq 0$. Considering (4.10) and (4.11), the term $D_{e_i}\hat{H} \wedge e_1 \wedge e_2$ appears only in $e_i(\Delta\tilde{\nu})$, not in $e_i(\tilde{\nu})$, and thus we have $D_{e_i}\hat{H} = 0$. Since the co-dimension of M in $\mathbb{H}_1^4(-1)$ is two, and $\hat{H}$ is parallel, then $R^D = 0$, i.e., the normal bundle is flat. So we can choose $\{e_1, e_2\}$ such that the shape operators A_3, A_4 are both diagonal. As $D\hat{H} = 0$ and $R^D = 0$, equation (4.12) reduces to

$$\begin{aligned}
e_i(\Delta\tilde{\nu}) &= (\|\hat{h}\|^2)_i\tilde{\nu} + \|\hat{h}\|^2\sum_{r=3}^{4}\sum_{k=1}^{2}\varepsilon_r h_{ik}^r \mathbf{x} \wedge \underbrace{e_r}_{k-th} \wedge e_2 \\
&+ 2\sum_{r=3}^{4}\sum_{k=1}^{2}\varepsilon_r h_{ik}^r \hat{H} \wedge \underbrace{e_r}_{k-th} \wedge e_2 + 2\hat{H} \wedge \underbrace{\mathbf{x}}_{i-th} \wedge e_2.
\end{aligned} \tag{4.13}$$

Now, using (4.11) and (4.13), from (4.10) we obtain that $\|\hat{h}\|_i^2 = 0$, i.e., $\|\hat{h}\|^2$ is constant, which implies that the scalar curvature is constant because of (2.9).

On the other hand, we have

$$\|\hat{h}\|^2 \sum_{r=3}^{4}\sum_{k=1}^{2} \varepsilon_r h_{ik}^r \mathbf{x} \wedge \underbrace{e_r}_{k-th} \wedge e_2 + 2\hat{H} \wedge \underbrace{\mathbf{x}}_{i-th} \wedge e_2 = \lambda_p \sum_{r=3}^{4}\sum_{k=1}^{2} \varepsilon_r h_{ik}^r \mathbf{x} \wedge \underbrace{e_r}_{k-th} \wedge e_2 \tag{4.14}$$

and

$$\sum_{k=1}^{2}\sum_{r=3}^{4} \varepsilon_r h_{ik}^r \hat{H} \wedge \underbrace{e_r}_{k-th} \wedge e_2 = 0. \tag{4.15}$$

Since $h_{12}^3 = h_{12}^4 = 0$, from (4.15) we get

$$\mathrm{tr}A_4 h_{ii}^3 - \mathrm{tr}A_3 h_{ii}^4 = 0 \tag{4.16}$$

for $i = 1, 2$. Considering $\hat{H} = 0$, we can take $\hat{H} = \varepsilon_3 a e_3 + \varepsilon_4 b e_4$, where $a = \frac{1}{2}\mathrm{tr}A_3$, $b = \frac{1}{2}\mathrm{tr}A_4$ and $a^2 = b^2$. That is, $b = \varepsilon^* a$ with $\varepsilon^* = \pm 1$. Therefore, $\hat{H} = a(\varepsilon_3 e_3 + \varepsilon^* \varepsilon_4 e_4)$. Without loss of generality, if we take $\varepsilon_3 = 1$, $\varepsilon_4 = -1$, then $\hat{H} = a(e_3 - \varepsilon^* e_4)$. As $\hat{H}$ is parallel, $D\hat{H} = 0$ implies that a is a nonzero constant.

Now, from (4.16) we have $a(h_{ii}^3 - \varepsilon^* h_{ii}^4) = 0$, $i = 1, 2$, i.e., $h_{ii}^3 = \varepsilon^* h_{ii}^4$ for $i = 1, 2$. These give us $\|\hat{h}\|^2 = 0$. So the scalar curvature $S = -2$ and the Gaussian curvature $K = -1$. Hence, from (4.14) we find $\lambda_p h_{ii}^3 = -2a$ and $\lambda_p h_{ii}^4 = -2a$ for $i = 1, 2$, which imply that h_{ii}^r's are constant and e_3, e_4 are umbilical. Thus, M is a totally umbilical surface in $\mathbb{H}_1^4(-1) \subset \mathbb{E}_2^5$. Taking the sum of $\lambda_p h_{11}^3 = -2a$ and $\lambda_p h_{22}^3 = -2a$, we obtain that $a(\lambda_p + 2) = 0$ that gives $\lambda_p = -2$. So, we have $h_{11}^3 = h_{22}^3 = \varepsilon^* h_{11}^4 = \varepsilon^* h_{22}^4$, and hence $A_3 = aI$, $A_4 = \varepsilon^* aI$. Now it is easy to see that $\widetilde{\nabla}_{e_i} \hat{H} = 0$, that is, $\hat{H} = a(e_3 - \varepsilon^* e_4)$ is a constant vector. It follows from the proof of Theorem 8.1 given in [10] that M is congruent to the surface defined by (4.5) which is totally umbilical with constant zero mean curvature vector and of curvature -1.

The converse of the proof follows from Example 4.9. ∎

Lemma 4.11. *Let M_t^n be a pseudo-Riemannian hypersurface with index t in $\mathbb{H}_s^{n+1}(-1) \subset \mathbb{E}_{s+1}^{n+2}$. Then we have*

$$\Delta(e_{n+1} \wedge e_1 \wedge \cdots \wedge e_n) = -n(\hat{\alpha}\tilde{\nu} + e_{n+1} \wedge e_1 \wedge \cdots \wedge e_n), \tag{4.17}$$

where $\hat{\alpha}$ is the mean curvature of M_t^n in $\mathbb{H}_s^{n+1}(-1)$.

P r o o f. Let M_t^n be a pseudo-Riemannian hypersurface with index t in $\mathbb{H}_s^{n+1}(-1) \subset \mathbb{E}_{s+1}^{n+2}$. Let $e_1, e_2, \ldots, e_{n+1}, e_{n+2}$ be a local orthonormal frame on M_t^n in $\mathbb{E}_{s+1}^{n+2}$ with signatures $\varepsilon_A = \langle e_A, e_A \rangle = \mp 1$ for $A = 1, 2, \ldots, n+2$ such that

$e_1, e_2, \ldots, e_n$ are tangent to M_t^n, and $e_{n+1},\ e_{n+2} = \mathbf{x}$ are normal to M_t^n, where $\mathbf{x}$ is the position vector of M_t^n. As M_t^n is a hypersurface in $\mathbb{H}_s^{n+1}(-1) \subset \mathbb{E}_{s+1}^{n+2}$, the normal vector e_{n+1} in $\mathbb{H}_s^{n+1}(-1)$ is parallel, i.e., $De_{n+1} = 0$.

Let us put $\bar{\nu} = e_{n+1} \wedge e_1 \wedge \cdots \wedge e_n$. By differentiating $\bar{\nu}$, we obtain

$$e_i\bar{\nu} = \varepsilon_i e_{n+1} \wedge e_1 \wedge \cdots \wedge e_{i-1} \wedge \mathbf{x} \wedge e_{i+1} \wedge \cdots \wedge e_n. \tag{4.18}$$

Since $\nabla_{e_i} e_i = \sum_j \varepsilon_j \omega_{ij}(e_i) e_j$ and $De_{n+1} = 0$, we have

$$(\nabla_{e_i} e_i)\bar{\nu} \;=\; \sum_{j=1}^{n} \omega_{ij}(e_i) e_{n+1} \wedge e_1 \wedge \cdots \wedge \underbrace{\mathbf{x}}_{j-th} \wedge \cdots \wedge e_n. \tag{4.19}$$

Differentiating $e_i\bar{\nu}$ in (4.18), we get

$$e_i e_i(\bar{\nu}) = \varepsilon_i\bar{\nu} + h_{ii}^{n+1}\tilde{\nu} - \sum_{\substack{j=1 \\ i\neq j}}^{n} \omega_{ji}(e_i) e_{n+1} \wedge e_1 \wedge \cdots \wedge \underbrace{\mathbf{x}}_{j-th} \wedge \cdots \wedge e_n. \tag{4.20}$$

Using $n\hat{\alpha} = \mathrm{tr}A_{n+1} = \sum_i \varepsilon_i h_{ii}^{n+1}$, we obtain that

$$\begin{aligned} \Delta\bar{\nu} &= -\sum_i \varepsilon_i (e_i e_i - \nabla_{e_i} e_i)\bar{\nu} \\ &= -n\hat{\alpha}\tilde{\nu} - n\bar{\nu} + \sum_{i,j=1}^{n} (\omega_{ij}(e_i) + \omega_{ji}(e_i)) e_{n+1} \wedge e_1 \wedge \cdots \wedge \underbrace{\mathbf{x}}_{j-th} \wedge \cdots \wedge e_n, \end{aligned} \tag{4.21}$$

which gives (4.17) as $\omega_{ji}(e_i) + \omega_{ij}(e_i) = 0$. ∎

Theorem 4.12. *An n-dimensional pseudo-Riemannian oriented submanifold M_t^n with index t and non-zero mean curvature vector $\hat{H}$ of a pseudo-hyperbolic space $\mathbb{H}_s^{m-1}(-1) \subset \mathbb{E}_{s+1}^m$ has a 1-type pseudo-hyperbolic Gauss map with nonzero constant component in its spectral decomposition if and only if M_t^n is an open part of a non-flat, non-totally geodesic and totally umbilical pseudo-Riemannian hypersurface of the totally geodesic pseudo-hyperbolic space $\mathbb{H}_{s^*}^{n+1}(-1) \subset \mathbb{H}_s^{m-1}(-1) \subset \mathbb{E}_{s+1}^m$ for $s^* = t \leq s$ or $s^* = t+1 \leq s$, that is, M_t^n is an open part of $\mathbb{H}_t^n(-c) \subset \mathbb{H}_{t+1}^{n+1}(-1)$ of curvature $-c$ for $c > 1$ or $\mathbb{H}_t^n(-c) \subset \mathbb{H}_t^{n+1}(-1)$ of curvature $-c$ for $0 < c < 1$ or $\mathbb{S}_t^n(c) \subset \mathbb{H}_t^{n+1}(-1)$ of curvature $c > 0$.*

P r o o f. Let $\mathbf{x} : M_t^n \longrightarrow \mathbb{H}_s^{m-1}(-1) \subset \mathbb{E}_{s+1}^m$ be an oriented isometric immersion from a pseudo-Riemannian manifold M_t^n into a pseudo-hyperbolic space $\mathbb{H}_s^{m-1}(-1) \subset \mathbb{E}_{s+1}^m$. Assume that M_t^n has a non-zero mean curvature vector $\hat{H}$ in

$\mathbb{H}_s^{m-1}(-1)$ and a 1-type pseudo-hyperbolic Gauss map with nonzero component in its spectral decomposition. Then we have $\Delta\tilde{\nu} = \lambda_p(\tilde{\nu} - \tilde{c})$ for a real number $\lambda_p \neq 0$ and some constant vector $\tilde{c} \in \mathbb{E}_q^N$. So, we have

$$(\Delta\tilde{\nu})_i = \lambda_p(\tilde{\nu})_i, \tag{4.22}$$

where $(\cdot)_i = e_i(\cdot)$. By a direct long computation, from (4.22), we obtain that

$$\begin{aligned}
e_i(\Delta\tilde{\nu}) &= (\|\hat{h}\|^2)_i\tilde{\nu} + \|\hat{h}\|^2 \sum_{r=n+1}^{m-1}\sum_{k=1}^{n} \varepsilon_r h_{ik}^r \ \mathbf{x} \wedge e_1 \wedge \cdots \wedge \underbrace{e_r}_{k-th} \wedge \cdots \wedge e_n \\
&+2nD_{e_i}\hat{H} \wedge e_1 \wedge \cdots \wedge e_n + n\sum_{k=1}^{n}\sum_{r=n+1}^{m-1} \varepsilon_r h_{ik}^r \hat{H} \wedge e_1 \wedge \cdots \wedge \underbrace{e_r}_{k-th} \wedge \cdots \wedge e_n \\
&+ n\sum_{k=1}^{n} \varepsilon_i\delta_{ik}\hat{H} \wedge e_1 \wedge \cdots \wedge \underbrace{\mathbf{x}}_{k-th} \wedge \cdots \wedge e_n \\
&- n\sum_{\substack{j,k=1\\ j\neq k}}^{n} \varepsilon_k\omega_{jk}(e_i)\mathbf{x} \wedge e_1 \wedge \cdots \wedge \underbrace{e_k}_{j-th} \wedge \cdots \wedge \underbrace{D_{e_k}\hat{H}}_{k-th} \wedge \cdots \wedge e_n \\
&- n\sum_{\substack{j,k=1\\ j\neq k}}^{n}\sum_{r=n+1}^{m-1} \varepsilon_r h_{ij}^r \ \mathbf{x} \wedge e_1 \wedge \cdots \wedge \underbrace{e_r}_{j-th} \cdots \wedge \underbrace{D_{e_k}\hat{H}}_{k-th} \wedge \cdots \wedge e_n \\
&+ n\sum_{k=1}^{n} \varepsilon_k\Big\langle A_{D_{e_k}\hat{H}}(e_i), e_k\Big\rangle\tilde{\nu} - n\sum_{k=1}^{n} \mathbf{x} \wedge e_1 \wedge \cdots \wedge \underbrace{D_{e_i}D_{e_k}\hat{H}}_{k-th} \wedge \cdots \wedge e_n \\
&+ \sum_{\substack{r,s=n+1\\ s<r}}^{m-1}\sum_{\substack{j,k=1\\ j\neq k}}^{n} \varepsilon_r\varepsilon_s\{e_i(R_{sjk}^r)\mathbf{x} + R_{sjk}^r e_i\} \wedge e_1 \wedge \cdots \wedge \underbrace{e_s}_{k-th} \wedge \cdots \wedge \underbrace{e_r}_{j-th} \wedge \cdots \wedge e_n \\
&+ \sum_{\substack{r,s=n+1\\ s<r}}^{m-1}\sum_{\substack{j,k,\ell=1\\ j,k,\ell\neq}}^{n} \varepsilon_r\varepsilon_s R_{sjk}^r\{\sum_{h=1}^{n} \varepsilon_h\omega_{\ell h}(e_i)\mathbf{x} \wedge e_1 \wedge \cdots \wedge \underbrace{e_h}_{\ell-th} \wedge \cdots \wedge \underbrace{e_s}_{k-th} \wedge \cdots \wedge \\
&\underbrace{e_r}_{j-th} \wedge \cdots \wedge e_n + \sum_{t=n+1}^{m-1} \varepsilon_t h_{i\ell}^t\mathbf{x} \wedge e_1 \wedge \cdots \wedge \underbrace{e_t}_{\ell-th} \wedge \cdots \wedge \underbrace{e_s}_{k-th} \wedge \cdots \wedge \underbrace{e_r}_{j-th} \wedge \cdots \wedge e_n\} \\
&- \sum_{r,s=n+1}^{m-1}\sum_{\substack{j,k,\ell=1\\ j\neq k}}^{n} \varepsilon_r\varepsilon_s\varepsilon_\ell R_{sjk}^r h_{i\ell}^s\mathbf{x} \wedge e_1 \wedge \cdots \wedge \underbrace{e_\ell}_{k-th} \wedge \cdots \wedge \underbrace{e_r}_{j-th} \wedge \cdots \wedge e_n
\end{aligned}$$

$$+\sum_{r,s,t=n+1}^{m-1}\sum_{\substack{j,k=1\\ j\neq k}}^{n}\varepsilon_r\varepsilon_s\varepsilon_t R^r_{sjk}\omega_{st}(e_i)\mathbf{x}\wedge e_1\wedge\cdots\wedge\underbrace{e_t}_{k-th}\wedge\cdots\wedge\underbrace{e_r}_{j-th}\wedge\cdots\wedge e_n. \tag{4.23}$$

Case (a): $\hat{H}=0$. Then equation (4.23) becomes

$$\begin{aligned}
e_i(\Delta\tilde{\nu}) &= (\|\hat{h}\|^2)_i\tilde{\nu}+\|\hat{h}\|^2\sum_{r=n+1}^{m-1}\sum_{k=1}^{n}\varepsilon_r h^r_{ik}\ \mathbf{x}\wedge e_1\wedge\cdots\wedge\underbrace{e_r}_{k-th}\wedge\cdots\wedge e_n\\
&+\sum_{\substack{r,s=n+1\\ s<r}}^{m-1}\sum_{\substack{j,k=1\\ j\neq k}}^{n}\varepsilon_r\varepsilon_s\{e_i(R^r_{sjk})\mathbf{x}+R^r_{sjk}e_i\}\wedge e_1\wedge\cdots\wedge\underbrace{e_s}_{k-th}\wedge\cdots\wedge\underbrace{e_r}_{j-th}\wedge\cdots\wedge e_n\\
&+\sum_{\substack{r,s=n+1\\ s<r}}^{m-1}\sum_{\substack{j,k,\ell=1\\ j,k,\ell\neq}}^{n}\varepsilon_r\varepsilon_s R^r_{sjk}\{\sum_{h=1}^{n}\varepsilon_h\omega_{\ell h}(e_i)\mathbf{x}\wedge e_1\wedge\cdots\wedge\underbrace{e_h}_{\ell-th}\wedge\cdots\wedge\underbrace{e_s}_{k-th}\wedge\cdots\wedge\underbrace{e_r}_{j-th}\\
&\wedge\cdots\wedge e_n+\sum_{t=n+1}^{m-1}\varepsilon_t h^t_{i\ell}\mathbf{x}\wedge e_1\wedge\cdots\wedge\underbrace{e_t}_{\ell-th}\wedge\cdots\wedge\underbrace{e_s}_{k-th}\wedge\cdots\wedge\underbrace{e_r}_{j-th}\wedge\cdots\wedge e_n\}\\
&-\sum_{r,s=n+1}^{m-1}\sum_{\substack{j,k,\ell=1\\ j\neq k}}^{n}\varepsilon_r\varepsilon_s\varepsilon_\ell R^r_{sjk}h^s_{i\ell}\mathbf{x}\wedge e_1\wedge\cdots\wedge\underbrace{e_\ell}_{k-th}\wedge\cdots\wedge\underbrace{e_r}_{j-th}\wedge\cdots\wedge e_n\\
&+\sum_{r,s,t=n+1}^{m-1}\sum_{\substack{j,k=1\\ j\neq k}}^{n}\varepsilon_r\varepsilon_s\varepsilon_t R^r_{sjk}\omega_{st}(e_i)\mathbf{x}\wedge e_1\wedge\cdots\wedge\underbrace{e_t}_{k-th}\wedge\cdots\wedge\underbrace{e_r}_{j-th}\wedge\cdots\wedge e_n.
\end{aligned} \tag{4.24}$$

By comparing (3.3), (4.22) and (4.24), we get $\|\hat{h}\|^2_i = R^r_{sjk}=0$. So, M^n_t has a flat normal bundle, and $\|\hat{h}\|^2$ is constant. On the other hand, the scalar curvature is constant by (2.9). Thus, Theorem 4.1 implies that M^n_t has the 1-type pseudo-hyperbolic Gauss map $\tilde{\nu}$ with $\tilde{c}=0$. This is a contradiction, and thus $\hat{H}\neq 0$.

Case (b): $\hat{H}\neq 0$. We observe that the term $D_{e_i}\hat{H}\wedge e_1\wedge\cdots\wedge e_n$ appears only in $(\Delta\tilde{\nu})_i$, not in $e_i(\tilde{\nu})$. Hence, considering (3.3), (4.22) and (4.23), we obtain that $D\hat{H}=0$. Then M^n_t has a nonzero parallel mean curvature vector $\hat{H}$ in

$\mathbb{H}_2^{m-1}(-1)$. In this case, (4.23) becomes

$$
\begin{aligned}
e_i(\Delta\tilde{\nu}) &= (\|\hat{h}\|^2)_i\tilde{\nu} + \|\hat{h}\|^2 \sum_{r=n+1}^{m-1}\sum_{k=1}^{n}\varepsilon_r h_{ik}^r\ \mathbf{x}\wedge e_1\wedge\cdots\wedge\underbrace{e_r}_{k-th}\wedge\cdots\wedge e_n \\
&+ n\sum_{k=1}^{n}\varepsilon_i\delta_{ik}\hat{H}\wedge e_1\wedge\cdots\wedge\underbrace{\mathbf{x}}_{k-th}\wedge\cdots\wedge e_n \\
&+ n\sum_{k=1}^{n}\sum_{r=n+1}^{m-1}\varepsilon_r h_{ik}^r\hat{H}\wedge e_1\wedge\cdots\wedge\underbrace{e_r}_{k-th}\wedge\cdots\wedge e_n \\
&+ \sum_{\substack{r,s=n+1\\ s<r}}^{m-1}\sum_{\substack{j,k=1\\ j\neq k}}^{n}\varepsilon_r\varepsilon_s\{e_i(R_{sjk}^r)\mathbf{x}+R_{sjk}^r e_i\}\wedge e_1\wedge\cdots\wedge\underbrace{e_s}_{k-th}\wedge\cdots\wedge\underbrace{e_r}_{j-th}\wedge\cdots\wedge e_n \\
&+ \sum_{\substack{r,s=n+1\\ s<r}}^{m-1}\sum_{\substack{j,k,\ell=1\\ j,k,\ell\neq}}^{n}\varepsilon_r\varepsilon_s R_{sjk}^r\{\sum_{h=1}^{n}\varepsilon_h\omega_{\ell h}(e_i)\mathbf{x}\wedge e_1\wedge\cdots\wedge\underbrace{e_h}_{\ell-th}\wedge\cdots\wedge\underbrace{e_s}_{k-th}\wedge\cdots\wedge\underbrace{e_r}_{j-th} \\
&\wedge\cdots\wedge e_n + \sum_{t=n+1}^{m-1}\varepsilon_t h_{i\ell}^t\mathbf{x}\wedge e_1\wedge\cdots\wedge\underbrace{e_t}_{\ell-th}\wedge\cdots\wedge\underbrace{e_s}_{k-th}\wedge\cdots\wedge\underbrace{e_r}_{j-th}\wedge\cdots\wedge e_n\} \\
&- \sum_{r,s=n+1}^{m-1}\sum_{\substack{j,k,\ell=1\\ j\neq k}}^{n}\varepsilon_r\varepsilon_s\varepsilon_\ell R_{sjk}^r h_{i\ell}^s\mathbf{x}\wedge e_1\wedge\cdots\wedge\underbrace{e_\ell}_{k-th}\wedge\cdots\wedge\underbrace{e_r}_{j-th}\wedge\cdots\wedge e_n \\
&+ \sum_{r,s,t=n+1}^{m-1}\sum_{\substack{j,k=1\\ j\neq k}}^{n}\varepsilon_r\varepsilon_s\varepsilon_t R_{sjk}^r\omega_{st}(e_i)\mathbf{x}\wedge e_1\wedge\cdots\wedge\underbrace{e_t}_{k-th}\wedge\cdots\wedge\underbrace{e_r}_{j-th}\wedge\cdots\wedge e_n.
\end{aligned}
\tag{4.25}
$$

From (3.3), (4.22) and (4.25), we get $\|\hat{h}\|_i^2 = 0$, that is, $\|\hat{h}\|^2$ is constant, and also from (2.9), the scalar curvature of M_t^n is constant. On the other hand, considering (3.3), (4.22) and (4.25), we have

$$
\begin{aligned}
&\|\hat{h}\|^2\sum_{r=n+1}^{m-1}\sum_{k=1}^{n}\varepsilon_r h_{ik}^r\mathbf{x}\wedge e_1\wedge\cdots\wedge\underbrace{e_r}_{k-th}\wedge\cdots\wedge e_n \\
&+ n\sum_{k=1}^{n}\varepsilon_i\delta_{ik}\hat{H}\wedge e_1\wedge\cdots\wedge\underbrace{\mathbf{x}}_{k-th}\wedge\cdots\wedge e_n
\end{aligned}
$$

$$
\begin{aligned}
&-\sum_{r,s=n+1}^{m-1}\sum_{\substack{j,k,\ell=1\\ j\neq k}}^{n}\varepsilon_r\varepsilon_s\varepsilon_\ell R^r_{sjk}h^s_{i\ell}\mathbf{x}\wedge e_1\wedge\cdots\wedge\underbrace{e_\ell}_{k-th}\wedge\cdots\wedge\underbrace{e_r}_{j-th}\wedge\cdots\wedge e_n\\
&=\lambda_p\sum_{r=n+1}^{m-1}\sum_{k=1}^{n}\varepsilon_r h^r_{ik}\mathbf{x}\wedge e_1\wedge\cdots\wedge\underbrace{e_r}_{k-th}\wedge\cdots\wedge e_n
\end{aligned}
\tag{4.26}
$$

and

$$
\begin{aligned}
&n\sum_{k=1}^{n}\sum_{r=n+1}^{m-1}\varepsilon_r h^r_{ik}\hat{H}\wedge e_1\wedge\cdots\wedge\underbrace{e_r}_{k-th}\wedge\cdots\wedge e_n\\
&+\sum_{\substack{r,s=n+1\\ s<r}}^{m-1}\sum_{\substack{j,k=1\\ j\neq k}}^{n}\varepsilon_r\varepsilon_s R^r_{sjk}e_i\wedge e_1\wedge\cdots\wedge\underbrace{e_s}_{k-th}\wedge\cdots\wedge\underbrace{e_r}_{j-th}\wedge\cdots\wedge e_n=0.
\end{aligned}
\tag{4.27}
$$

As $\hat{H}$ is non-zero, we may put $\hat{H}=\varepsilon_{n+1}\hat{\alpha}e_{n+1}$, where $n\hat{\alpha}=\sum_{i=1}^{n}\varepsilon_i h^{n+1}_{ii}$. From (4.27) we have $R^r_{sjk}=0$ for $r,s\geq n+2$ and $j,k=1,\ldots,n$. Also, as $D\hat{H}=0$, it is seen that $R^{n+1}_{sjk}=0$. Thus the normal bundle of M^n_t is flat. Therefore, equation (4.27) is reduced to

$$
n\sum_{k=1}^{n}\sum_{r=n+2}^{m-1}\varepsilon_r\varepsilon_{n+1}\hat{\alpha}h^r_{ik}\,e_{n+1}\wedge e_1\wedge\cdots\wedge\underbrace{e_r}_{k-th}\wedge\cdots\wedge e_n=0. \tag{4.28}
$$

This equation implies that $h^r_{ik}=0$ for $r\geq n+2$ and $i,k=1,\ldots,n$. Thus, the first normal space Imh is spanned by e_{n+1}, i.e., from Erbacher's Reduction Theorem, M^n_t lies in a totally geodesic pseudo-hyperbolic space $\mathbb{H}^{n+1}_{s^*}(-1)\subset\mathbb{H}^{m-1}_s(-1)\subset\mathbb{E}^m_{s+1}$ for $s^*=t\leq s$ or $s^*=t+1\leq s$.

Now, using equation (4.26), we obtain that

$$
\left(\|\hat{h}\|^2-\lambda_p\right)h^{n+1}_{ik}=n\hat{\alpha}\varepsilon_i\delta_{ik} \tag{4.29}
$$

for $i,k=1,\ldots,n$. It is seen that $\lambda_p\neq\|\hat{h}\|^2$ as $\hat{\alpha}\neq 0$. If we take the sum of (4.29) for $i=k$ and i from 1 to n, then we get $n\hat{\alpha}(\|\hat{h}\|^2-n-\lambda_p)=0$, that is, $0\neq\lambda_p=\|\hat{h}\|^2-n$. Hence $h^{n+1}_{ik}=\hat{\alpha}\varepsilon_i\delta_{ik}$ from (4.29), i.e., the shape operator of M^n_t is diagonal. Moreover, $\lambda_p=\|\hat{h}\|^2-n=n(\varepsilon_{n+1}\hat{\alpha}^2-1)\neq 0$, and from (2.9) we have $S=n(n-1)(\varepsilon_{n+1}\hat{\alpha}^2-1)=(n-1)\lambda_p$, that is, M^n_t is non-flat.

Consequently, M^n_t is an open part of a non-flat, non-totally geodesic and totally umbilical pseudo-Riemannian hypersurface of a totally geodesic pseudo-hyperbolic space $\mathbb{H}^{n+1}_{s^*}(-1)\subset\mathbb{H}^{m-1}_s(-1)\subset\mathbb{E}^m_{s+1}$ for $s^*=t\leq s$ or $s^*=t+1\leq s$, that is, following [1], M^n_t is an open part of $\mathbb{H}^n_t(-c)\subset\mathbb{H}^{n+1}_{t+1}(-1)$ of curvature

$-c$ for $c > 1$ or $\mathbb{H}_t^n(-c) \subset \mathbb{H}_t^{n+1}(-1)$ of curvature $-c$ for $0 < c < 1$ or $\mathbb{S}_t^n(c) \subset \mathbb{H}_t^{n+1}(-1)$ of the curvature $c > 0$.

Conversely, assume that M_t^n is an open part of a non-flat, non-totally geodesic and totally umbilical pseudo-Riemannian hypersurface of a totally geodesic pseudo-hyperbolic space $\mathbb{H}_{s^*}^{n+1}(-1) \subset \mathbb{H}_s^{m-1}(-1) \subset \mathbb{E}_{s+1}^m$ for $s^* = t \leq s$ or $s^* = t + 1 \leq s$. Now we suppose that M_t^n is immersed in $\mathbb{H}_{s^*}^{n+1}(-1) \subset \mathbb{E}_{s^*+1}^{n+2}$. Let $e_1, \ldots, e_{n+1}, e_{n+2} = \mathbf{x}$ be a local orthonormal frame on M_t^n in $\mathbb{E}_{s^*+1}^{n+2}$ such that $e_1, \ldots, e_n$ are tangent to M_t^n, and $e_{n+1}, e_{n+2} = \mathbf{x}$ are normal to M_t^n, where $\mathbf{x}$ is the position vector of M_t^n. Since M_t^n is a pseudo-Riemannian hypersurface of $\mathbb{H}_{s^*}^{n+1}(-1)$, the normal bundle of M_t^n in $\mathbb{E}_{s^*+1}^{n+2}$ is flat, and the mean curvature vector $\hat{H} = \varepsilon_{n+1}\hat{\alpha}e_{n+1}$ is parallel in $\mathbb{E}_{s^*+1}^{n+2}$ because M_t^n has the nonzero constant mean curvature $\hat{\alpha}$ in $\mathbb{H}_{s^*}^{n+1}(-1)$. Also, as M_t^n is totally umbilical, we get $\|\hat{h}\|^2 = \varepsilon_{n+1}n\hat{\alpha}^2$, and hence, from (3.6) we have

$$\Delta\tilde{\nu} = \varepsilon_{n+1}n\hat{\alpha}(\hat{\alpha}\tilde{\nu} + e_{n+1} \wedge e_1 \wedge e_2 \wedge \cdots \wedge e_n). \tag{4.30}$$

We put

$$\tilde{c} = \frac{-1}{\varepsilon_{n+1}\hat{\alpha}^2 - 1}(\tilde{\nu} + \varepsilon_{n+1}\hat{\alpha}e_{n+1} \wedge e_1 \wedge e_2 \wedge \cdots \wedge e_n)$$

and

$$\tilde{\nu}_p = \frac{\varepsilon_{n+1}\hat{\alpha}}{\varepsilon_{n+1}\hat{\alpha}^2 - 1}(\hat{\alpha}\tilde{\nu} + e_{n+1} \wedge e_1 \wedge e_2 \wedge \cdots \wedge e_n),$$

where $\varepsilon_{n+1}\hat{\alpha}^2 - 1 \neq 0$ because M_t^n is a non-flat hypersurface in $\mathbb{H}_{s^*}^{n+1}(-1)$ (note that for a flat totally umbilical hypersurface in $\mathbb{H}_{s^*}^{n+1}(-1)$, $\varepsilon_{n+1} = 1$ and $\hat{\alpha}^2 = 1$), then we have $\tilde{\nu} = \tilde{c} + \tilde{\nu}_p$. As $\hat{\alpha}$ is a constant, it is easily seen that $e_i(\tilde{c}) = 0$, $i = 1, \ldots, n$, i.e., $\tilde{c}$ is a constant vector. Using (4.17) and (4.30), from a direct computation we obtain that $\Delta\tilde{\nu}_p = n(\varepsilon_{n+1}\hat{\alpha}^2 - 1)\tilde{\nu}_p$. Therefore, the pseudo-hyperbolic Gauss map $\tilde{\nu}$ is of 1-type with nonzero constant component in its spectral decomposition. ∎

We have the following corollaries.

Corollary 4.13. *A hyperbolic space $\mathbb{H}^n(-c)$ of curvature $-c$ for $c > 1$ in the anti-de Sitter space $\mathbb{H}_1^{n+1}(-1) \subset \mathbb{E}_2^{n+2}$ is the only space-like hypersurface with 1-type pseudo-hyperbolic Gauss map having a nonzero constant component in its spectral decomposition.*

Corollary 4.14. *An anti-de Sitter space $\mathbb{H}_1^n(-c)$ of curvature $-c$ for $c > 1$ in the pseudo-hyperbolic space $\mathbb{H}_2^{n+1}(-1) \subset \mathbb{E}_3^{n+2}$ is the only Lorentzian hypersurface with 1-type pseudo-hyperbolic Gauss map having a nonzero constant component in its spectral decomposition.*

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