НОВІ МЕТОДИ В СИСТЕМНОМУ АНАЛІЗІ, ІНФОРМАТИЦІ ТА ТЕОРІЇ ПРИЙНЯТТЯ РІШЕНЬ

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METHOD OF APPROXIMATION OF EVOLUTIONARY INCLUSIONS AND VARIATIONAL INEQUALITIES BY STATIONARY

P.O. KASYANOV, V.S. MEL'NIK, L. TOSCANO

The method of finite-difference approximations, advanced by C. Bardos and H. Brezis for the nonlinear evolutionary equations, is generalized on differential-operational inclusions which are tightly connected to evolutionary variational inequalities in Banach spaces.

INTRODUCTION

At studying of nonlinear evolutionary equations the some spread methods are used: Faedo-Galerkin, singular perturbations, difference approximations, nonlinear semigroups of operators and others [1, 2]. The dissemination of these approaches on evolutionary inclusions and variational inequalities encounters a series of basic difficulties. The method of nonlinear semigroups of operators in Banach spaces was developed for evolutionary inclusions in works of A.A. Tolstonogov [3], A.A. Tolstonogov and J.I. Umanskij [4], V. Barbu [2] and others. A method of singular perturbations H. Brezis [5] and Yu. Dubinskiy [6] on evolutionary inclusions have disseminated in A.N. Vakulenko's and V.S. Mel'nik works [7–9], a method of Galerkin's approximations in P.O. Kasyanov's works [10, 11].

In the present work the attempt to disseminate a method of difference approximations [1] on evolutionary inclusions and variational inequalities is undertaken for the first time.

PROBLEM FORMALIZATION

Let Φ be separable locally convex linear topological space; Φ' be the space identified to topologically conjugate to Φ space such, that $\Phi \subset \Phi'$; (f, φ) is the inner product (canonical pairing) of devices $f \in \Phi'$ and $\varphi \in \Phi$.

Let the three spaces V, H and V' are given, moreover

$$\Phi \subset V \subset \Phi', \quad \Phi \subset H \subset \Phi', \quad \Phi \subset V' \subset \Phi' \tag{1}$$

with continuous and dense embedding;

H is a Hilbert space (with inner product $(h_1, h_2)_H$ and corresponding norm $\| h \|_H$);
V be reflexive separable Banach space with norm $\| v \|_V$;
V' is the conjugate to V space with dual norm $\| f \|_{V'}$.

If $\varphi, \psi \in \Phi$, that $(\varphi, \psi) = (\varphi, \psi)_H$ is inner product of devices $\varphi \in V$ and $\psi \in V'$.

Let $V = V_1 \cap V_2$ and $\| \cdot \|_V = \| \cdot \|_{V_1'} + \| \cdot \|_{V_2'}$, where $(V_i, \| \cdot \|_{V_i})$, $i = \overline{1,2}$ is reflexive separable Banach spaces, embedding $\Phi \subset V_i \subset \Phi'$ and $\Phi \subset V_i' \subset \Phi'$ is dense and continuous. Spaces $(V_i', \| \cdot \|_{V_i'})$, $i = \overline{1,2}$ are topologically conjugate to $(V_i, \| \cdot \|_{V_i})$ concerning the bilinear form $(\cdot,\cdot)$. Then $V' = V_1' + V_2'$.

Let $A : V_1 \to V_1'$, $\varphi : V_2 \to R$ be a functional, Λ is non-bounded operator, which operates from V to V' with definitional domain $D(\Lambda; V, V')$. The following problem on searching of solutions by a method of finite differences is considered (see [1, chapter 2.7]):

$$u \in D(\Lambda; V, V'), \tag{2}$$

$$\Lambda u + A(u) + \partial\varphi(u) \ni f, \tag{3}$$

where $f \in V'$ fixed element; $\partial\varphi : V_2 \overset{\to}{\to} V_2'$ is subdifferential from the functional φ (see [13]).

THE BASIC GUESSES

Let us assume, that a set Φ is dense in space

$$(V \cap V', \| v \|_V + \| v \|_{V'}). \tag{4}$$

Remark 1. From (4) it follows, that

$$V \cap V' \subset H. \tag{5}$$

Really, if $v \in \Phi$, that $\| v \|_H^2 \le \| v \|_{V'} \| v \|_V$ whence, due to (4) it follows (5).

Remark 2. If $V \subset H$, it is possible to not introduce Φ and identifying H and H', at once receive the following line-up of embeddings:

$$V \subset H \subset V'. \tag{6}$$

Definition 1. The family of maps $\{G(s)\}_{s \ge 0}$ refers to as a *continuous semigroup* in a Banach space X, if $\forall s \ge 0 \quad G(s) \in L(X; X)$, $G(0) = Id$, $G(s+t) = G(s) \circ G(t) \quad \forall s, t \ge 0$, $G(t)x \overset{w}{\to} x$ as $t \to 0+ \quad \forall x \in X$.

Operator Λ**.** Let the family of maps $\{G(s)\}_{s \ge 0}$ be such that $\{G(s)\}_{s \ge 0}$ is continuous semigroup on V, H, V', that is there are three semigroups, defined in spaces V, H, and V' correspondingly, which coincide on Φ. Each of them we shall designate as $\{G(s)\}_{s \ge 0}$;

$$\{G(s)\}_{s\geq 0} \text{ is non-expanding semigroup in } H\,,$$

$$\text{that is } \|G(s)\|_{L(H;H)} \leq 1 \quad \forall s \geq 0\,. \tag{7}$$

Further let $-\Lambda$ be the infinitesimal generator of a semigroup $\{G(s)\}_{s\geq 0}$ with a definitional domain $D(\Lambda;V)$ (accordingly $D(\Lambda;H)$ or $D(\Lambda;V')$) in V (accordingly in H or in V'). In virtue of [14, theorem 13.35] such generator exists, moreover, it is densely defined closed linear operator in space V (accordingly in H or in V').

Let $\{G^*(s)\}_{s\geq 0}$ be the semigroup conjugated to $G(s)$, which operates accordingly in V, H, and V'. Let $-\Lambda^*$ is the infinitesimal generator of a semigroup $\{G^*(s)\}_{s\geq 0}$ with definitional domain $D(\Lambda^*;V)$ in V, $D(\Lambda^*;H)$ in H and $D(\Lambda^*;V')$ in V'. The operator Λ^* in H (accordingly in V or in V') is conjugated in sense of the theory of unlimited operators to the operator Λ in H (accordingly in V or in V'). It takes place the following.

Lemma 1. The sets $D(\Lambda;V')\bigcap V$ and $D(\Lambda^*;V')\bigcap V$ are dense in V.

Proof. Really, $\forall u \in V$ $\forall \varepsilon > 0$ $\exists \varphi \in \Phi:$ $\|u-\varphi\|_V < \varepsilon,$ $\varphi_n :=$ $= \left(I - \frac{1}{n}\Lambda \right)^{-1} \varphi \in D(\Lambda;V') \bigcap V$, $\varphi_n \to \varphi$ in V as $n \to \infty$.

The lemma is proved.

Now we define Λ as non-bounded operator, which operates from V to V' with definitional domain $D(\Lambda;V,V')$. Let us put

$$D(\Lambda;V,V') = \{v \in V \mid \text{the form } w \to (v, \Lambda^* w) \text{ is continuous on}$$

$$D(\Lambda^*;V')\bigcap V \text{ in topology, induced from space } V\}\,. \tag{8}$$

Then there is unique element $\xi_v \in V' : (v, \Lambda^* w) = (\xi_v, w)$. If $v \in D(\Lambda;V')\bigcap$ $\bigcap V$, that $\xi_v = \Lambda v$. Thus, generally we can put $\xi_v = \Lambda v$, whence

$$(v, \Lambda^* w) = (\Lambda v, w) \quad \forall w \in D(\Lambda^*;V') \bigcap V\,. \tag{9}$$

If we enter on $D(\Lambda;V,V')$ the norm $\|v\|_V + \|\Lambda v\|_{V'}$, we receive a Banach space. Let us similarly define space $D(\Lambda^*;V,V')$.

Remark 3. If $V \subset H$, then

$$D(\Lambda;V,V') = V \bigcap D(\Lambda;V') \quad \text{and} \quad D(\Lambda^*;V,V') = V \bigcap D(\Lambda^*;V').$$

In case when V does not include in H we assume that

$$V \bigcap D(\Lambda;V') \text{ dense in } D(\Lambda;V,V')\,,$$

$$V \bigcap D(\Lambda^*;V') \text{ dense in } D(\Lambda^*;V,V')\,. \tag{10}$$

Remark 4. ([1, chapter 2, remark 7.5., 7.6.]).

$$(\Lambda v, v) \geq 0 \quad \forall v \in D(\Lambda; V, V'), \quad (\Lambda^* v, v) \geq 0 \quad \forall v \in D(\Lambda^*; V, V'). \qquad (11)$$

Let us enter some new denotations. Let Y be some reflexive Banach space. As $C_v(Y)$ we designate the system of all nonempty convex closed bounded subsets from Y. For nonempty subset $B \subset Y$ we consider the closed convex hull of the given set $\overline{\mathrm{co}}\,(B) := \mathrm{cl}_Y(\mathrm{co}\,(B))$. With multi-valued map A it is comparable *upper* $[A(y), \omega]_+ = \sup\limits_{d \in A(y)} < d, w >_Y$ and *lower* $[A(y), \omega]_- = \inf\limits_{d \in A(y)} < d, w >_Y$ *function of support*, where $y, \omega \in Y$. Properties of the given maps are considered in works [15–17]. Later on $y_n \xrightarrow{w} y$ in Y will mean, that y_n weakly converges to y in space Y.

THE CLASSES OF MAPS

Let us consider the next classes of maps of pseudomonotone type:

Definition 2. Operator $A : V \to V'$ refers to *pseudomonotone*, if from $\{y_n\}_{n \geq 0} \subset V$, $y_n \xrightarrow{w} y_0$ in V, and $\overline{\lim\limits_{n \to \infty}} (A(y_n), y_n - y_0) \leq 0$ it follows, that $\exists \{y_{n_k}\}_{k \geq 1} \subset \{y_n\}_{n \geq 1}$:

$$\underline{\lim}_{k \to \infty} (A(y_{n_k}), y_{n_k} - w) \geq (A(y_0), y_0 - w) \quad \forall w \in V.$$

Definition 3. The next set:

$$\partial\varphi(v) = \{p \in V' \mid < p, u - v > \leq \varphi(u) - \varphi(v) \quad \forall\, u \in V\}$$

refers to *subdifferential map* form functional $\varphi : V \to \mathbf{R}$ in point $v \in V$.

Definition 4. Multi-valued map $A : V \rightrightarrows V^*$ refers to:

1) λ*-pseudomonotone*, if from $\{y_n\}_{n \geq 0} \subset V$, $y_n \xrightarrow{w} y_0$ in V and $\overline{\lim\limits_{n \to \infty}} (d_n, y_n - y_0) \leq 0$, where $d_n \in \overline{\mathrm{co}}\, A(y_n)$ $\forall n \geq 1$ it follows, that it is possible to choose such $\{y_{n_k}\}_{k \geq 0} \subset \{y_n\}_{n \geq 0}$, $\{d_{n_k}\}_{k \geq 0} \subset \{d_n\}_{n \geq 0}$ that

$$\forall\, w \in V \qquad \underline{\lim}_{k \to \infty} (d_{n_k}, y_{n_k} - w) \geq [A(y_0), y_0 - w]_-;$$

2) *bounded*, if A translates arbitrary bounded in V set in bounded in V^*;

3) *coercive*, if $\| v \|_V^{-1} [A(v), v]_+ \to +\infty$ as $\| v \|_V \to +\infty$;

4) satisfies *condition* (κ) if the map $V \ni v \to \| v \|_V^{-1} [A(v), v]_+ \in \mathrm{R}$ is bounded from below on bounded in $V \setminus \bar{0}$ sets, that is

$$\forall\, D \subset V \setminus \{\bar{0}\} - \text{bounded in } V \quad \exists\, c_1 \in \mathrm{R} : \ \frac{[A(v), v]_+}{\| v \|_V} \geq c_1 \quad \forall\, v \in D.$$

Remark, that the bounded multi-valued maps and monotone multi-valued operators, including subdifferential maps, are satisfying condition (κ).

Definition 5. Multivalued map $A:V \to C_v(V^*)$ satisfies *property* (M), if from $\{y_n\}_{n\geq 0} \subset V$, $d_n \in A(y_n)$ $\forall n \geq 1$: $y_n \overset{w}{\to} y_0$ in V, $d_n \overset{w}{\to} d_0$ in V', $\overline{\lim_{n\to\infty}} (d_n, y_n) \leq (d_0, y_0)$ it follows, that $d_0 \in A(y_0)$.

Definition 6. Operator $L: D(L) \subset V \to V^*$ refers to *maximally monotone*, if it is monotone and from $(w - L(u), v - u) \geq 0$ $\forall u \in D(L)$ it follows, that $v \in D(L)$ and $L(v) = w$.

Lemma 2. Let V, W be Banach spaces, densely and continuously embedded in locally convex linear topological space Y, $A: V \rightrightarrows V'$, $B: W \rightrightarrows W'$ — multi-valued λ-pseudomonotone maps and one of them is bound-valued. Then the multi-valued operator $A := A + B: V \bigcap W \rightrightarrows V' + W'$ is λ-pseudomonotone.

Proof. Let $y_n \overset{w}{\to} y$ in $X := V \bigcap W$ (that is $y_n \overset{w}{\to} y$ in V and $y_n \overset{w}{\to} y$ in W) and the next inequality is holds:

$$\overline{\lim_{n\to\infty}} < d_n, y_n - y >_X \leq 0, \tag{12}$$

where

$$d_n \in \overline{\text{co}}\, A(y_n) = \overline{\text{co}}\, A(y_n) + \overline{\text{co}}\, B(y_n). \tag{13}$$

Let us prove the last equality. It is obvious, that $\text{co}\, A(y_n) = \text{co}\, A(y_n) + {}$ ${} + \text{co}\, B(y_n)$ and, moreover, $\overline{\text{co}}\, A(y_n) \supset \overline{\text{co}}\, A(y_n) + \overline{\text{co}}\, B(y_n)$. Let us prove the inverse inclusion. Let x is a frontier point of $A(y_n)$. Then $\exists \{x_m\}_{m\geq 1} \subset \text{co}\, A(y_n) =$ $= \text{co}\, A(y_n) + \text{co}\, B(y_n)$: $x_m \overset{w}{\to} x$ in X as $m \to \infty$, because of Mazur theorem (see [14]), for an arbitrary convex set its weak and the strong closure is coincide. Hence, $\forall m \geq 1$ $\exists v_m \in A(y_n)$, $\exists w_m \in B(y_n)$: $v_m + w_m = x_m$ and, taking into account bound-valuededness of one of the maps and Banach-Alaoglu theorem, we obtain, within to a subsequence, $v_m \overset{w}{\to} v$ in V, $w_m \overset{w}{\to} w$ in W for some $v \in \overline{\text{co}}\, A(y_n)$, $w \in \overline{\text{co}}\, B(y_n)$. The statement (13) is proved. Consequently $d_n = d_n' + d_n''$, where $d_n' \in \overline{\text{co}}\, A(y_n)$, $d_n'' \in \overline{\text{co}}\, B(y_n)$. From here, within to a subsequence, we obtain one of two inequalities:

$$\overline{\lim_{n\to\infty}} < d_n', y_n - y >_V \leq 0, \qquad \overline{\lim_{n\to\infty}} < d_n'', y_n - y >_W \leq 0. \tag{14}$$

Without loss of generality, let us consider, that (within to a subsequence) $\overline{\lim_{n\to\infty}} < d_n', y_n - y >_V \leq 0$. Then, due to λ-pseudomonotony of A, $\exists \{y_m\}_m \subset \{y_n\}_{n\geq 1}$:

$$\underline{\lim_{m\to\infty}} < d_m', y_m - v >_V \geq [A(y), y - v]_- \quad \forall v \in V.$$

Let us put in last equality $v = y$, then

$$\varliminf_{m\to\infty} < d'_m, y_m - y >_V \ge [A(y), y-y]_- = 0 .$$

Hence, $\exists \lim_{m\to\infty} < d'_m, y_m - y >_V = 0$. Then, due to (12), $\varlimsup_{n\to\infty} < d'_m, y_m - y >_W \le 0$. Taking into account (14), λ-pseudomonotony of A and B, we have

$$\varliminf_{k\to\infty} < d'_{n_k}, y_{n_k} - v >_V \ge [A(y), y-v]_- \quad \forall v \in V,$$

$$\varliminf_{k\to\infty} < d''_{n_k}, y_{n_k} - w >_W \ge [B(y), y-w]_- \quad \forall w \in W.$$

Then from last two relations it follows

$$\varliminf_{k\to\infty} < d_{n_k}, y_{n_k} - x >_X \ge \varliminf_{k\to\infty} < d'_{n_k}, y_{n_k} - x >_V + \varliminf_{k\to\infty} < d''_{n_k}, y_{n_k} - x >_W \ge$$

$$\ge [A(y), y-x]_- + [B(y), y-x]_- = [A(y), y-x]_- \quad \forall x \in V \cap W.$$

The lemma is proved.

Lemma 3. Let V, W be Banach spaces, densely and continuously embedded in locally convex linear topological space Y, $A: V \rightrightarrows V'$, $B: W \rightrightarrows W'$ are multi-valued coercive maps, which satisfies condition (κ). Then the multi-valued operator $A := A + B : V \cap W \rightrightarrows V' + W'$ is coercive.

Proof. We obtain this statement arguing by contradiction. Let's assume, that $\exists \{x_n\}_{n\ge1} : \| x_n \|_X = \| x_n \|_V + \| x_n \|_W \to +\infty$ as $n \to \infty$, but $\sup_{n\ge1} \frac{[A(x_n), x_n]_+}{\| x_n \|_X} < +\infty$.

Case 1. $\| x_n \|_V \to +\infty$ as $n \to \infty$, $\| x_n \|_W \le c \ \ \forall n \ge 1$;

$$\gamma_A(r) := \inf_{\|v\|_V = \gamma} \frac{[A(v), v]_+}{\| v \|_V}, \quad \gamma_B(r) := \inf_{\|w\|_W = \gamma} \frac{[B(w), w]_+}{\| w \|_W}, \quad r > 0 .$$

Remark, that $\gamma_A(r) \to +\infty, \gamma_B(r) \to +\infty$ as $r \to +\infty$. Then $\forall n \ge 1$ $\| x_n \|_V^{-1} [A(x_n), x_n]_+ \ge \gamma_A(\| x_n \|_V) \| x_n \|_V$ and $\frac{[A(x_n), x_n]_+}{\| x_n \|_X} \ge \gamma_A(\| x_n \|_V) \times \times \frac{\| x_n \|_V}{\| x_n \|_X} \to +\infty$ as $\| x_n \|_V \to +\infty$ and $\| x_n \|_W \le c.$

In this case, due to condition (κ), $\forall n \ge 1$

$$\frac{[B(x_n), x_n]_+}{\| x_n \|_X} \ge \gamma_B(\| x_n \|_W) \frac{\| x_n \|_W}{\| x_n \|_X} \ge c_1 \frac{\| x_n \|_W}{\| x_n \|_X} \to 0 \quad \text{at} \quad n \to \infty ,$$

where $c_1 \in \mathrm{R}$ is the constant from condition (κ). It is clear, that

$$\frac{[A(x_n), x_n]_+}{\| x_n \|_X} = \frac{[A(x_n), x_n]_+}{\| x_n \|_X} + \frac{[B(x_n), x_n]_+}{\| x_n \|_X} \to +\infty \quad \text{as} \quad n \to \infty .$$

We have an inconsistency with boundedness of the left part of the given expression.

Case 2. The case $\| x_n \|_V \le c \quad \forall n \ge 1$ and $\| x_n \|_W \to \infty$ as $n \to \infty$ is investigated similarly.

Case 3. Let us consider the situation, when $\| x_n \|_V \to \infty$ and $\| x_n \|_W \to \infty$ as $n \to \infty$. Then,

$$+\infty > \sup_{n \ge 1} \frac{[A(x_n), x_n]_+}{\| x_n \|_X} \ge \gamma_A(\| x_n \|_V) \frac{\| x_n \|_V}{\| x_n \|_V + \| x_n \|_W} +$$

$$+ \gamma_B(\| x_n \|_W) \frac{\| x_n \|_W}{\| x_n \|_V + \| x_n \|_W} . \tag{15}$$

It is obvious, that $\forall n \ge 1 \ \frac{\| x_n \|_V}{\| x_n \|_X} > 0$ and $\frac{\| x_n \|_W}{\| x_n \|_X} > 0$. And, if even one of limits, for example $\frac{\| x_n \|_V}{\| x_n \|_X} \to 0$, that $\frac{\| x_n \|_W}{\| x_n \|_X} = 1 - \frac{\| x_n \|_V}{\| x_n \|_X} \to 1$. We have an inconsistency with (15).

The lemma is proved.

THE MAIN RESULT

Theorem. Let a) $A : V_1 \to V_1'$ be bounded pseudomonotone on V_1 operator, which satisfies the following coercive condition:

$$\frac{(A(u), u)}{\| u \|_{V_1}} \to +\infty \quad \text{as} \quad \| u \|_{V_1} \to +\infty ; \tag{16}$$

b) functional $\varphi : V_2 \to \mathrm{R}$ is convex, lower semicontinuous and the following takes place:

$$\frac{\varphi(v)}{\| v \|_{V_2}} \to +\infty \quad \text{as} \quad \| v \|_{V_2} \to +\infty ; \tag{17}$$

c) The operator Λ satisfies all listed above conditions, including conditions (7) and (10).

Then for every $f \in V'$ there exists such u, that satisfies (2) and (3).

Remark 5. If $V \subset H$, inclusion (2) implies, that $u \in V \bigcap D(\Lambda; V')$.

Proof. *The approximate solutions*. Natural approximation of inclusion (3) is inclusion

$$\frac{I - G(h)}{h} u_h + A(u_h) + \partial\varphi(u_h) \ni f \quad (h > 0) . \tag{18}$$

Though, if V does not include in H (18), generally speaking, has no solutions, and it is necessary to modify the given inclusion in appropriate way. We choose such sequence $\theta_h \in (0,1)$, that

$$\frac{1 - \theta_h}{h} \to 0 \quad \text{as} \quad h \to 0 . \tag{19}$$

Let us put $\theta_h = 1$ when $V \subset H$. Further, we take

$$\Lambda_h = \frac{I - \theta_h G(h)}{h} \tag{20}$$

and also replace (18) with the inclusion

$$\Lambda_h u_h + A(u_h) + \partial\varphi(u_h) \ni f . \tag{21}$$

Lemma 4. Inclusion (21) has a solution $u_h \in V \cap H$.

Proof. Let us enter the map

$$B = \Lambda_h + A : H \cap V_1 \to H + V_1' . \tag{22}$$

We consider the following variation inequality:

$$(B(u_h), v - u_h) + \varphi(v) - \varphi(u_h) \geq (f, v - u_h) \quad \forall v \in V \cap H . \tag{23}$$

Let us prove the existence of such $u_h \in V \cap H$, that is a solution of the given inequality. The given statement follows from [15, theorem 7], if to put $V = H \cap V_1$, $W = V_2$, $A = B$, $\varphi = \varphi$ and under condition of realization

Lemma 5. Operator B satisfies to the following conditions:

$$\text{i)}\ \frac{(B(u), u)}{\| u \|_{H \cap V_1}} \to +\infty \quad \text{as} \quad \| u \|_{H \cap V_1} \to \infty ; \tag{24}$$

ii) B *is* pseudomonotone on $H \cap V_1$; (25)

iii) B *is* bounded on $H \cap V_1$. (26)

Proof. i) As $G(s)$ is non-stretched on H, then $\forall v \in H$

$$(\Lambda_h v, v) = \frac{1}{h}(v - \theta_h G(h) v, v) \geq \frac{1}{h}\left(\| v \|_H^2 - \theta_h \| G(s) v \|_H \| v \|_H \right) \geq$$

$$\geq \frac{1 - \theta_h}{h} \| v \|_H^2 . \tag{27}$$

From here it follows the coercive condition and condition (κ) for Λ_h on H. Thus, due to (2), we can use lemma 3 for maps $A = \Lambda_h$ on $V = H$ and $B = A$ on $W = V_1$, whence it follows (24), if we prove, that A satisfies condition (κ). Really, if it is not true, then $\exists \{w_n\}_{n \geq 1} \subset V_1 \setminus \bar{0}$ such bounded in W, that $\| w_n \|_{V_1}^{-1} [A(w_n), w_n]_+ \to -\infty$ as $n \to \infty$, but in virtue of boundedness of A, we have

$$\| w_n \|_{V_1}^{-1} [A(w_n), w_n]_+ = \| w_n \|_{V_1}^{-1} (A(w_n), w_n) \geq -\sup_{n \geq 1} \| A(w_n) \|_{V_1} > -\infty .$$

iii) The boundedness of B on $H \cap V_1$ follows from the boundedness of Λ_h on H and A on V_1. The boundedness of Λ_h on H immediately follows from the definition of Λ_h and estimation (6).

ii). Let us prove the pseudomonotony of B on $H\bigcap V_1$. For this purpose we use lemma 2 with $A=\Lambda_h$ on $V=H$ and $B=A$ on $W=V_1$. From here, due to the pseudomonotony and to the property of bound-valuedness of A on V_1, it is enough to prove pseudomonotony of Λ_h on H. Let

$$y_n \to y \quad \text{in} \quad H, \quad \varlimsup_{n\to\infty}(\Lambda_h y_n, y_n - y) \le 0.$$

Then, from estimation (27) we have

$$\varliminf_{n\to\infty}(\Lambda_h y_n, y_n - y) \ge \varliminf_{n\to\infty}(\Lambda_h y_n - \Lambda_h y, y_n - y) + \varliminf_{n\to\infty}(\Lambda_h y, y_n - y) \ge 0 + 0 = 0 .$$

Hence $\exists \lim\limits_{n\to\infty}(\Lambda_h y_n, y_n - y) = 0$. Further, $\forall u \in H$, $\forall s > 0$ let $w := y +$ $+ s(u-y)$. Then

$$s(\Lambda_h y_n, y-u) \ge -(\Lambda_h y_n, y_n - y) + (\Lambda_h w, y_n - y) - s(\Lambda_h w, u-y) \quad \forall n \ge 1$$

and

$$s\varliminf_{n\to\infty}(\Lambda_h y_n, y-u) \ge -s(\Lambda_h w, u-y) \Leftrightarrow \varliminf_{n\to\infty}(\Lambda_h y_n, y-u) \ge -(\Lambda_h w, u-y) .$$

Let $s \to 0+$ then $\varliminf\limits_{n\to\infty}(\Lambda_h y_n, y-u) \ge -(\Lambda_h y, u-y) = (\Lambda_h y, y-u)$ and

$$\varliminf_{n\to\infty}(\Lambda_h y_n, y_h - u) \ge \varliminf_{n\to\infty}(\Lambda_h y_n, y_h - y) +$$
$$+ \varliminf_{n\to\infty}(\Lambda_h y_n, y-u) \ge (\Lambda_h y, y-u) \quad \forall u \in H .$$

Thus we have the required statement.

The lemma is proved.

To complete the proof of lemma 4 it is necessary to show, that for fixed $u_h \in H\bigcap V_1$ the variation inequality (23) is equivalent to inclusion (22). If $v \in H\bigcap V_1$ is arbitrary, then, by definition of subdifferential map, the inequality (23) is equivalent to $f - B(u_h) \in \partial\varphi(u_h)$, that in turn, by definition of B, it is equivalent to (22).

The lemma is proved.

The boundary transition on h. From lemma 4 for every $h > 0$ the existence of such $u_h \in H \cap V_1$ and $d_h \in \partial\varphi(u_h)$, that

$$\Lambda_h u_h + A(u_h) + d_h = f . \tag{28}$$

is follows. If we put in (23) $v = \bar{0}$, we obtain

$$(B(u_h), u_h) + \varphi(u_h) \le (f, u_h) + \varphi(\bar{0}). \tag{29}$$

Let us prove boundedness of $\{u_h\}_{h>0}$ in V as h close to zero. For this purpose we use advantage coercive conditions (16) and (24). Let us assume, that $\| u_h \|_V = \| u_h \|_{V_1} + \| u_h \|_{V_2} \to \infty$.

Case 1. $\| u_h \|_{V_1} \to \infty$, $\| u_h \|_{V_2} \le c$;

$$\gamma_B(r) := \inf_{\|u\|_{V_1}=r} \frac{(B(u),u)}{\|u\|_{V_1}}, \quad \gamma_\varphi(r) := \inf_{\|u\|_{V_2}=r} \frac{\varphi(u)}{\|u\|_{V_2}}, \quad r>0.$$

Remark, that $\gamma_B(r) \to +\infty$ and $\gamma_\varphi(r) \to +\infty$ as $r \to +\infty$. Then $\|u_h\|_{V_1}^{-1}(B(u_h),u_h) \ge \gamma_B(\|u\|_{V_1})\|u\|_{V_1}$ and

$$\|f\|_{V'} \leftarrow \|f\|_{V'} + \frac{\varphi(\bar{0})}{\|u_h\|_V} \ge \frac{(f,u_h)+\varphi(\bar{0})}{\|u_h\|_V} \ge \frac{(B(u_h),u_h)+\varphi(u_h)}{\|u_h\|_V} \ge$$

$$\ge \frac{\gamma_B(\|u_h\|_{V_1})\|u_h\|_{V_1}}{\|u_h\|_V} + \frac{\gamma_\varphi(\|u_h\|_{V_2})\|u_h\|_{V_2}}{\|u_h\|_V} \ge$$

$$\ge \frac{\gamma_B(\|u_h\|_{V_1})\|u_h\|_{V_1}}{\|u_h\|_{V_1}+c} + \frac{\gamma_\varphi(\|u_h\|_{V_2})\|u_h\|_{V_2}}{\|u_h\|_V} \to +\infty \quad as \quad \|u_h\|_V \to \infty.$$

We have an inconsistency with boundedness of the left part of the given inequality. It is necessary to notice, that last item in a right-side of last inequality tends to zero. It follows from boundedness from below of φ on the bounded sets (see [13]).

Case 2. The case $\|u_h\|_{V_1} \le c$, $\|u_h\|_{V_2} \to \infty$ is investigated similarly.

Case 3. Let us consider the situation, when $\|u_h\|_{V_1} \to \infty$, $\|u_h\|_{V_2} \to \infty$. Then,

$$\|f\|_{V'} \leftarrow \|f\|_{V'} + \frac{\varphi(\bar{0})}{\|u_h\|_V} \ge \frac{\gamma_B(\|u_h\|_{V_1})\|u_h\|_{V_1}}{\|u_h\|_{V_1}+\|u_h\|_{V_2}} + \frac{\gamma_\varphi(\|u_h\|_{V_2})\|u_h\|_{V_2}}{\|u_h\|_{V_1}+\|u_h\|_{V_2}}. \quad (30)$$

It is obvious, that $\frac{\|u\|_{V_1}}{\|u\|_V} > 0$ and $\frac{\|u\|_{V_2}}{\|u\|_V} > 0$. And, if even one of boundaries, for example, $\frac{\|u\|_{V_1}}{\|u\|_V} \to 0$, that $\frac{\|u\|_{V_2}}{\|u\|_V} = 1 - \frac{\|u\|_{V_1}}{\|u\|_V} \to 1$. We have an inconsistency in (30). Thus,

$$u_h \quad \text{are bounded in} \quad V \quad \text{as} \quad h \to 0. \quad (31)$$

Prove, that

$$d_h \quad \text{are bounded in} \quad V_2' \quad \text{as} \quad h \to 0. \quad (32)$$

First, from equality (28) we receive:

$$\sup_n (d_{h_n}, u_{h_n}) < \infty \quad \forall \{h_n\} \subset (0,+\infty): \; h_n \to 0 \quad \text{as} \quad n \to \infty. \quad (33)$$

Due to $u_h \in H$, from equality (28), estimation (31) and boundednesses of an operator A we have

$$\sup_n (d_{h_n}, u_{h_n}) = \sup_n (f, u_{h_n}) + \sup_n (-A(u_{h_n}), u_{h_n}) +$$

$$+ \sup_n (-\Lambda_{h_n} u_{h_n}, u_{h_n}) \le \|f\|_V' \sup_n \|u_{h_n}\|_V + \sup_n \|A(u_{h_n})\|_{V'} \sup_n \|u_{h_n}\|_V < +\infty.$$

Now, in virtue of (33), we prove (32). From $d_{h_n} \in \partial\varphi(y_{h_n})$ and from definition of subdifferential map, $\forall v \in V_2$

$$\sup_n (d_{h_n}, v) \le \sup_n (d_{h_n}, y_{h_n}) + \sup_n (d_{h_n}, v - y_{h_n}) \le \sup_n (d_{h_n}, y_{h_n}) + \varphi(v) - \varphi(y_{h_n}) \le$$

$$\le \sup_n (d_{h_n}, y_{h_n}) + \varphi(v) - \inf_n \varphi(y_{h_n}) < +\infty ,$$

as functional φ is bounded from below on bounded sets. From here, under Banach-Steingauss theorem (32) is follows.

From (31) and boundedness of an operator A on V_1 it follows, that

$$A(u_h) \quad \text{are} \quad \text{bounded} \quad \text{in} \quad V_1' \quad \text{as} \quad h \to 0 . \tag{34}$$

From equality (28), estimates (31), (32) and (34), under Banach-Alaoglu theorem, the existence of such subsequences $\{u_{h_n}\}_{n\ge1} \subset \{u_h\}_{h>0}$, $\{d_{h_n}\}_{n\ge1} \subset$ $\subset \{d_h\}_{h>0}$, $\{A(u_{h_n})\}_{n\ge1} \subset \{A(u_h)\}_{h>0}$ $(0 < h_n \to 0)$, which further we will designate simply as $\{u_h\}_{h>0}$, $\{d_h\}_{h>0}$, $\{A(u_h)\}_{h>0}$ accordingly, and elements $u \in V$, $\chi \in V_1$, $d \in V_2$ the next convergences

$$u_h \xrightarrow{w} u \quad \text{in} \quad V \quad A(u_h) \xrightarrow{w} \chi \quad \text{in} \quad V_1' \quad d_h \xrightarrow{w} d$$

$$\text{in} \quad V_2' \quad L_h u_h \xrightarrow{w} Lu \quad \text{in} \quad V' \tag{35}$$

are follows, in particular,

$$v_h := A(u_h) + d_h \xrightarrow{w} \chi + d =: w \quad \text{in} \quad V' . \tag{36}$$

Let us enter the following map: $C(v) = A(v) + \partial\varphi(v) : V \to C_v(V')$. Now prove, that the given map satisfies property (M). For this purpose it is enough to show λ-pseudomonotony of C on V. If C is λ-pseudomonotone on V and $\{y_n\}_{n\ge0} \subset V$, $d_n \in C(y_n)$ $\forall n \ge 1$:

$$y_n \xrightarrow{w} y_0 \quad \text{in} \quad V, \quad d_n \xrightarrow{w} d_0 \quad \text{in} \quad V' \quad \text{and} \quad \overline{\lim_{n\to\infty}} (d_n, y_n) \le (d_0, y_0),$$

then

$$\overline{\lim_{n\to\infty}} (d_n, y_n - y_0) \le \overline{\lim_{n\to\infty}} (d_n, y_n) + \overline{\lim_{n\to\infty}} (d_n, -y_0) \le (d_0, y_0) - (d_0, y_0) = 0 .$$

Hence, due to λ-pseudomonotony of C it follows, that $\exists \{y_{n_k}\}_{k\ge1} \subset$ $\subset \{y_n\}_{n\ge1}$, $\{d_{n_k}\}_{k\ge1} \subset \{d_n\}_{n\ge1}$:

$$\forall w \in V \quad \underline{\lim_{k\to\infty}} (d_{n_k}, y_{n_k} - w) \ge [C(y_0), y_0 - w]_- .$$

From here

$$[C(y_0), y_0 - w]_- \le \underline{\lim_{k\to\infty}} (d_{n_k}, y_{n_k} - w) \le \overline{\lim_{n\to\infty}} (d_n, y_n - w) \le$$

$$\le (d_0, y_0 - w) \quad \forall w \in V .$$

Hence $d_0 \in C(y_0)$. Thus C satisfies condition (M) on V.

In turn, lemma 2, pseudomonotony and bounded-valuedness of A on V_1 provides the last, if to prove λ-pseudomonotony of $\partial\varphi$ on V_2. As it is known, the last statement follows from [20.III, lemma 2, remark 2].

We use the fact, that C satisfies property (M) on V. Let us take v from $V \cap D(\Lambda^*;V')$. From (28) and (36) it follows, that

$$(u_h, \Lambda_h^* v) + (v_h, v) = (f, v) . \tag{37}$$

But

$$\Lambda_h^* v = \frac{I - G(h)^*}{h} v + \frac{I - \theta_h}{h} G(h)^* v \tag{38}$$

and due to (20), $\Lambda_h^* v \to \Lambda^* v$ in V'; and consequently, as h tends to zero in (37) we receive:

$$(u, \Lambda^* v) + (w, v) = (f, v) \qquad \forall v \in \mathrm{V} \cap D(\Lambda^*;V')$$

and (in virtue of (7), (8)) $u \in D(\Lambda, V, V')$

$$\Lambda u + w = f$$

and we prove the theorem, if we show that

$$w \in C(u) . \tag{39}$$

On the other hand, because of (28) and (36) for $v \in \mathrm{V} \cap D(\Lambda; V') \subset \mathrm{H}$, we have

$$(v_h, u_h - v) = (f, u_h - v) - (\Lambda_h v, u_h - v) - (\Lambda_h (u_h - v), u_h - v) \le$$

$$\le (f, u_h - v) - (\Lambda_h v, u_h - v) ,$$

as $\Lambda_h \ge 0$ in $\Lambda(H;H)$. From here

$$\limsup (v_h, u_h) \le (w, v) - (f, u - v) - (\Lambda v, u - v) \quad \forall v \in \mathrm{V} \cap D(\Lambda;V') .$$

But, due to (9), the same inequality is fulfilled $\forall v \in D(\Lambda;V,V')$, and when $v = u$ we obtain

$$\limsup (v_h, u_h) \le (w, u) ,$$

and also (39), because of C is the operator of type (M). The theorem is proved.

Example. Let Ω in $\mathbf{R}^n$ be a bounded region with regular boundary $\partial\Omega$, $S = [0,T]$ be finite time interval, $Q = \Omega \times (0;T)$, $\Gamma_T = \partial\Omega \times (0;T)$. As operator A we take $(Au)(t) = A(u(t))$, where

$$A(\varphi) = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(\left| \frac{\partial \varphi}{\partial x_i} \right|^{p-2} \frac{\partial \varphi}{\partial x_i} \right) + |\varphi|^{p-2} \varphi \tag{40}$$

(see [1, chapter 2.9.5]); V is closed subspace in Sobolev space $W^{1,p}(\Omega)$, $p > 1$ such, that

$$W_0^{1,p}(\Omega) \subset V \subset W^{1,p}(\Omega) \tag{41}$$

and

$$V_1 = L_p(0,T;V), \quad H = L_2(0,T;L_2(\Omega)), \quad V_2 = L_2(0,T;L_2(\Omega)).$$

We consider convex lower semicontinuous coercive functional $\psi : \mathbf{R} \to \mathbf{R}$ and its subdifferential $\Phi : \mathbf{R} \rightrightarrows \mathbf{R}$, that satisfies growth condition.

If we put $V = V_1 \cap V_2$ (from here $V' = L_q(0,T;V^*) + L_2(0,T;L_2(\Omega))$, where $\frac{1}{p} + \frac{1}{q} = 1$), we obtain the situation (6), if $p \geq 2$. At $1 < p < 2$ the common case takes place, if to take $\Phi = D(0,T;V)$ (see [1]).

As an operator Λ we take the derivation operator in sense of space of scalar distributions $D^*(0,T;V^*)$, $D(\Lambda;V,V') := W = \{y \in V \cap H \mid y' \in H + V'\}$

$$G(s)\varphi(t) := \{\varphi(t-s) \text{ at } t \geq s;\ 0 \text{ at } t \leq s\}.$$

Due to [1, chapter 2.9.5] and to the theorem, the next problem:

$$\frac{\partial y(x,t)}{\partial t} - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial y(x,t)}{\partial x_i}\right|^{p-2} \frac{\partial y(x,t)}{\partial x_i}\right) +$$

$$+ |y(x,t)|^{p-2} y(x,t) + \Phi(y(x,t)) \ni f(x,t) \quad \text{a.e. on } Q, \tag{42}$$

$$y(x,0) = 0 \quad \text{a.e. on } \Omega, \tag{43}$$

$$\frac{\partial y(x,t)}{\partial \nu_A} = g(x,t) \quad \text{a.e. on } \Gamma_T, \tag{44}$$

has a solution $y \in W$, obtained by finite differences method. Remark, that in (42)–(44): $f \in V'$, $y_0 \in L_2(\Omega)$ are fixed elements.

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