Radiative conduction problem in the thin layer of semi-transparent medium. The weak absorption case

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Heat conduction problem in the thin layer of semi-transparent media is studied. Together with mechanism of ordinary thermal conductance, the energy transfer by electromagnetic irradiation is taken into account. The Stefan-Boltzmann irradiation law is used. The case of small energy absorption has been studied. It corresponds to small value of the optical length λ. First terms of the asymptotic expansion on λ powers of the stationary temperature distribution in the layer are obtained.

Решается задача о теплообмене в тонком слое полупрозрачной среды. Наряду с механизмом теплопроводности, учитывается перенос энергии электромагнитным излучением. Используется закон излучения Стефана-Больцмана. Рассмотрен случай слабого поглощения излучения средой, т.е. малой оптической длины λ. Получены первые члены асимптотического разложения по степеням λ для стационарного распределения температуры в слое.

The problems connected with the temperature distribution $T(\mathbf{x}, t)$ in semi-transparent media when the radiative conduction influence on the formation of it, are much more complicated in comparison with ordinary heat conduction problems when the thermal conductivity mechanism is taken into account only (see, for example, [1]). These problems are not ordinary on its mathematical setting and some complicated mathematical problems arise as the sequence of this unusual behavior when we study them. The complexity is connected with non-local nature of the evolution equation for the temperature distribution. Besides, this equation is nonlinear. The nonlinearity of the equation is the sequence of the nonlinear dependence on the temperature of the energy flux value irradiated by each volume unit of the medium sample. The non-local form of the equation is connected with the fact that the physical mechanism of the heat transfer by the irradiation has non-local character. In the evolution equation for the distribution $T(\mathbf{x}, t)$, the total irradiation getting into the point $\mathbf{x}$ at the time moment t should be taken into account. And this radiation is represented by the sum of contributions in the total energy flux of rays having been undertaken the different number of reflections from different sample boundary points. From here, it follows that the evolution equation for the temperature distribution should not be local, i.e. it is not formed only by the values $T(\mathbf{x}, t)$ and its derivatives at each space-time point ($\mathbf{x}$, t), but, otherwise, it should be the integral and differential ones where integral terms describe the balance of the heat flux in each space point $\mathbf{x}$ at each time moment t when the energy is transferred by the irradiation. The form of these integral terms is closely connected with boundary conditions of irradiation. There is yet physical reason of the appearance of non-local structure of the evolution equation. It is connected with the finiteness of the irradiation wave length. However, as a rule, this sircumstance is neglected when radiative conductance problems are set using the so-called geometric optics approximation [2]. But even at the geometric optics approximation, due to the large variety of the irradiation and absorption conditions for the electromagnetic radiation of the unit medium volume, it is not succeeded to formulate the evolution equation for the distribution $T(\mathbf{x},t)$ by the universal way[3]. The simplification of the heat radiative conductance problem arises in those cases when the solution of non-local equation, in frameworks of the perturbation theory, it is succeeded to reduce to the solving of the local equations when the irradiation absorption coefficient α is large for the medium under consideration [4]. It is connected with the fact that light rays may influence appreciably on the temperature distribution only in the medium domains being at small distance from the irradiation point. Then, in the frameworks of the perturbation theory on the α^{-1} powers, we obtain some differential equations [4]. Such a simplification of the problem arises at the small α value. But, in this case, the coefficient of the ray reflection from the boundary should be small, $r \ll 1$. In this case, firstly, it may be possible to calculate the energy flux of the light radiation in each point $\mathbf{x}$ of the sample by its representation in the form of series on the number of reflection from the boundary. Then, each series term being proportional to a power of the reflection coefficient r is represented by the integral over the sample taking into account the contributions to the total irradiation flux from all points. However, it may be possible to expand each of these summands on α powers. The zero order approximation terms represent itself the constant values which are depended self-consistently on the distribution $T(\mathbf{x},t)$. Since these constants do not influence to the divergence of the irradiation energy flux, then it is arisen the possibility to solve the problem of radiative conduction both the evolution one and the stationary one. It is done by the expansion of the solution into double series on powers of values r and α where, at the zero approximation on α, the solution is determined by the ordinary thermal conductivity equation. In this work we study the solution of the stationary one-dimensional heat radiative conduction problem in the thin layer of semi-transparent medium. In this case, it is succeeded удаётся to avoid the expansion on powers of the reflection coefficient.

Let us consider the evolution equation for one-dimensional temperature distribution $T(x,t)$ which is constructed on the basis of the energy balance in each small material volume. It is putt that the decreasing of the energy flux $P(x)$ (it is positive if it is directed to right in accordance with the choice of the coordinate axe x) in each fixed point of the material sample, is spent on the local increasing of the internal energy. At the same time, the heat is taken out from the small volume under consideration by the thermal conductivity mechanism into the domains neighboring with this volume. The algebraic sum of these two changes of the internal energy in the volume under consideration is equal to the heat variation $N\rho\dot{T}(x,t)$ in it during the same time. Consequently, the evolution equation has the following form

$$N\rho\frac{\partial T}{\partial t} = \varkappa\frac{\partial^2 T}{\partial x^2} - \frac{\partial P}{\partial x} \tag{1}$$

where N is the material specific heat, ρ and $\varkappa$ are its density and the thermal conductivity, correspondingly. In this model, we neglect the dependencies on the temperature of the values N and $\varkappa$ supposing that the temperature distribution $T(x,t)$ is varied not very strong that it is necessary to take into account these dependencies. Second summand in right-side of Eq.(1) is the divergence of the energy flux. To obtain the closed evolution equation for the distribution $T(x,t)$, it is necessary to have the explicit expression of $P(x,t)$. We use the following formula which has been obtained in [4],

$$P(x,t) = \alpha\sigma\int_0^L Q(x,y)T^4(y,t)dy\,. \tag{2}$$

Such an expression of the radiation energy flux takes place at the one-dimensional irradiation propagation in the sample having the length L, when the Kirchhoff law is fulfilled which asserts that absorption and irradiation intensities of the radiation energy connected with the unit area element are equal to each other. The last value in Eq.(2) is designated by the letter α. Further, it is put in Eq.(2) that the Stefan-Boltzmann irradiation law takes place in each point of the medium. This law connects the energy flux $P_0(x,t)$ irradiated by the unit area element in the point x at the time moment t with the temperature distribution. According to this law, we put that $P_0(x,t) = \sigma T^4(x,t)$ where $T(x,t)$ is the medium temperature in the point x and σ is the Stefan-Boltzmann constant. On the work [4], the transition function $Q(x,y)$ is calculated also for the case of the one-dimensional case of the ray propagation in the thin layer of the semi-transparent medium. It has the following form

$$Q(x,y) = \frac{1}{2}\mathrm{sgn}(x-y)e^{-\alpha|x-y|} + \\ + \frac{re^{-\alpha L}}{1-r^2e^{-2\alpha L}}\left[\mathrm{sh}\,\alpha(L-x-y) + re^{-\alpha L}\mathrm{sh}\,\alpha(x-y)\right]. \quad (3)$$

After the substitution of this expression in Eq.(2), the equation (1) has the self-consistent form

$$N\rho\frac{\partial T}{\partial t} = \varkappa\frac{\partial^2 T}{\partial x^2} - \alpha\sigma\frac{\partial}{\partial x}\int_0^L Q(x,y)T^4(y,t)dy\,. \quad (4)$$

For this equation, some different formulations of initial boundary problems with fixed conditions for the distribution $T(x,t)$ at the sample boundaries have sense. Particularly, it may be possible to fix the temperature values on the opposite boundary plates of medium layer as such conditions. We designate them by $T_- = T(0,t)$, $T_+ = T(L,t)$. From the physical point of view, the availability of thermostats with the temperature values pointed out at the sample ends.

In this work we study the static solution $T(x)$ of the equation (4) when the left-side is equal to zero

$$\varkappa\frac{\partial^2 T}{\partial x^2} = \alpha\sigma\frac{\partial}{\partial x}\int_0^L Q(x,y)T^4(y,t)dy\,. \quad (5)$$

This equation possesses the first integral

$$\frac{\partial T(x)}{\partial x} = \frac{\alpha\sigma}{\varkappa}\int_0^L Q(x,y)T^4(y)dy + A \quad (6)$$

where the constant A should be determined on the basis of boundary conditions. From Eq.(6), it follows

$$T(x) = T_- + Ax + \frac{\alpha\sigma}{\varkappa}\int_0^L \bar{Q}(x,y)T^4(y)dy\,, \quad (7)$$

$$\bar{Q}(x,y) = \int_0^x Q(z,y)dz \quad (8)$$

taking into account the boundary condition at the left end of the sample. From here, we obtain the expression for the constant C in terms of the unknown solution $T(x)$

$$A = \frac{T_+ - T_-}{L} - \frac{\alpha\sigma}{\varkappa L}\int_0^L \bar{Q}(L,y)T^4(y)dy\,. \quad (9)$$

The common solution of the integral equations (6) and (9) brings to the desired temperature distribution in the sample. For the analysis of these equations, it is convenient to pass to the physically dimensionless variables. Let us introduce the dimensionless coordinates $\xi = x/L$, $y/L = \eta$ and the dimensionless temperature distribution $T(x)/T_+ = \Theta(\xi)$ (in the case $T_+ \geq T_-$). We introduce also the dimensionless parameters: the relative temperature $T_-/T_+ = \theta$, the reduced optical length $\lambda = \alpha L$ and the reduced irradiation intensity $\varepsilon = \sigma T_+^3 L/\varkappa$. Then, we find from Eq.(3) that

$$Q(x,y) \equiv q(\xi,\eta) = \frac{1}{2}\mathrm{sgn}(\xi-\eta)e^{-\lambda|\xi-\eta|} +$$

$$+\frac{re^{-\lambda}}{1-r^2e^{-2\lambda}}\left[\operatorname{sh}\lambda(1-\xi-\eta)+re^{-\lambda}\operatorname{sh}\lambda(\xi-\eta)\right]\,, \tag{10}$$

$$\frac{1}{L}\bar{Q}(x,y)\equiv\bar{q}(\xi,\eta)=\int\limits_0^{\xi}q(\zeta,\eta)d\zeta\,. \tag{11}$$

At these designations, Eqs.(6) and (9) are represented in the form

$$\Theta(\xi)=\theta+C\xi+\lambda\varepsilon\int\limits_0^1\bar{q}(\xi,\eta)\Theta^4(\eta)d\eta\,, \tag{12}$$

$$C=1-\theta-\lambda\varepsilon\int\limits_0^1\bar{q}(1,\eta)\Theta^4(\eta)d\eta \tag{13}$$

where $C=AL/T_+$.

It is not obvious that the problem setting in the previous item has the solution at any values of parameters λ and ε. Let us find the solution of Eqs.(12),(13) using the smallness of the parameter λ. We build it as the expansion on the λ parameter powers

$$\Theta(\xi)=\Theta^{(0)}(\xi)+\Theta^{(1)}(\xi)+\Theta^{(2)}(\xi)+...\,, \tag{14}$$

$$C=C^{(0)}+C^{(1)}+C^{(2)}+... \tag{15}$$

being generally speaking asymptotical. With this aim, we represent the kernel $q(\xi,\eta)$ in the form of the expansion on λ powers

$$q(\xi,\eta)=\frac{1}{2}\operatorname{sgn}(\xi-\eta)+\lambda(h-1/2)(\xi-\eta)+\lambda g(1-\xi-\eta)+O(\lambda^2)\,, \tag{16}$$

где $g=re^{-\lambda}(1-r^2e^{-2\lambda})^{-1}$, $h=gre^{-\lambda}$.

At zero approximation, we find from Eqs.(12),(13)

$$C^{(0)}=1-\theta\,,\quad \Theta^{(0)}(\xi)=(1-\theta)\xi+\theta\,. \tag{17}$$

The reflections from the boundary are not taken into account in this solution. Let us find the next approximation

$$C^{(1)}=-\frac{1}{2}\lambda\varepsilon\int\limits_0^1(1-2\eta)[\Theta^{(0)}(\eta)]^4d\eta\,,$$

$$\frac{d\Theta^{(1)}}{d\xi}=C^{(1)}+\frac{1}{2}\lambda\varepsilon\int\limits_0^1\operatorname{sgn}(\xi-\eta)[\Theta^{(0)}(\eta)]^4d\eta\,. \tag{18}$$

Putting $z=\Theta^{(0)}(\xi)$, we find

$$\int\limits_0^1\operatorname{sgn}(\xi-\eta)[\Theta^{(0)}(\eta)]^4d\eta=(1-\theta)^{-1}\int\limits_\theta^1\operatorname{sgn}\left(\Theta^{(0)}(\xi)-z\right)z^4dz=$$

$$=\frac{1}{5(1-\theta)}\left(2\left[\Theta^{(0)}(\xi)\right]^5-\theta^5-1\right)$$

taking into account the explicit form of the solution (17). Therefore, we obtain from Eq.(18) the equation

$$\Theta^{(1)}(\xi)=C^{(1)}\xi+\frac{\lambda\varepsilon}{10(1-\theta)}\int\limits_0^{\xi}\left(2\left[\Theta^{(0)}(\eta)\right]^5-\theta^5-1\right)d\eta=$$

$$= C^{(1)}\xi + \frac{\lambda\varepsilon}{30(1-\theta)^2}\left(\left[\Theta^{(0)}(\xi)\right]^6 - \theta^6\right) - \frac{\lambda\varepsilon}{10(1-\theta)}\left(1+\theta^5\right)\xi\,,$$

since $\Theta^{(1)}(0) = 0$ on the basis of Eq.(12). Due to this reason, $\Theta^{(1)}(1) = 0$ that permits to find the constant $C^{(1)}$ without the calculation the integral. In a result, we obtain

$$\Theta^{(1)}(\xi) = \frac{\lambda\varepsilon}{30(1-\theta)^2}\left(\left[\Theta^{(0)}(\xi)\right]^6 - \theta^6 - \xi(1-\theta^6)\right)\,. \tag{19}$$

The second approximation is calculated by the same way on the basis of the expression

$$\frac{d\Theta^{(2)}}{d\xi} = C^{(2)} + 2\lambda\varepsilon\int_0^1 \mathrm{sgn}(\xi-\eta)[\Theta^{(0)}(\eta)]^3[\Theta^{(1)}(\eta)]d\eta +$$

$$+\lambda^2\varepsilon\int_0^1 [(h-1/2)(\xi-\eta) + g(1-\xi-\eta)][\Theta^{(0)}(\eta)]^4 d\eta\,. \tag{20}$$

The account of the ray reflections from the sample boundary is shown in this approximation. The evaluation of integrals on $z = \Theta^{(0)}(\eta)$ brings us to the next expression

$$\frac{d\Theta^{(2)}}{d\xi} = C^{(2)} + \frac{(\lambda\varepsilon)^2\theta(1-\theta^5)}{15(1-\theta)^4}\int_\theta^1 \mathrm{sgn}(\Theta^{(0)}(\xi)-z)(1-z)z^3 dz +$$

$$+\frac{\lambda^2\varepsilon}{(1-\theta)^2}\int_\theta^1 [(h-g-1/2)\Theta^{(0)}(\xi) + g(1+\theta) - z(h+g-1/2)]z^4 dz =$$

$$= C^{(2)} + \frac{(\lambda\varepsilon)^2\theta(1-\theta^5)}{30(1-\theta)^4}\left[[\Theta^{(0)}(\xi)]^4\left(1-\frac{4}{5}\Theta^{(0)}(\xi)\right) - \frac{1}{2}\theta^4\left(1-\frac{4}{5}\theta\right) - \frac{1}{10}\right] +$$

$$+\frac{\lambda^2\varepsilon}{5(1-\theta)^2}\left[\left(\left(h-g-\frac{1}{2}\right)\Theta^{(0)}(\xi) + g(1+\theta)\right)(1-\theta^5) - \right.$$

$$\left. - \frac{5}{6}\left(h+g-\frac{1}{2}\right)(1-\theta^6)\right]\,. \tag{20}$$

Taking into account the conditions $\Theta^{(2)}(0) = \Theta^{(2)}(1) = 0$, the integration on ξ of the expression in the right-side from zero to arbitrary point gives the final result for the second approximation

$$\Theta^{(2)}(\xi) = \frac{(\lambda\varepsilon)^2\theta(1-\theta^5)}{150(1-\theta)^5}\times$$

$$\times\left[[\Theta^{(0)}(\xi)]^5\left(1-\frac{1}{3}\Theta^{(0)}(\xi)\right) - \frac{2}{3}\xi - \theta^5\left(1-\frac{1}{3}\theta\right)(1-\xi)\right] -$$

$$-\frac{\lambda^2\varepsilon}{20(1-\theta)^3}\,\frac{1+3re^{-\lambda}}{1+re^{-\lambda}}\left[[\Theta^{(0)}(\xi)]^2 - \xi - \theta^2(1-\xi)\right]$$

where explicit expressions for g, h coefficients are used.

We remark that the solution obtained in the work which is represented in the form of the power λ expansion, is only suitable at small value $\lambda\varepsilon$. It is due to the presence the infinite set of summands being proportional to $(\lambda\varepsilon)^n$, $n \in \mathbb{N}$ in the expansion obtained such that they have no the supplement multipliers λ^m. Thus, it is necessary to consider the value $\lambda\varepsilon$ as the effective parameter in the expansion. This sircumstance narrows down strongly the domain where our expansion is applicable, since the dimensionless parameter ε is larger as a rule than the unity in the real experiment. We point out also that the using the iteration procedure for nonlinear equation (12) is possible only at very small values λ. This domain of the

λ values is essentially more narrow in comparison with the domain where the expansion on λ powers is only asymptotical. But from the numerical point of view, the smallness of the value λ provides the good accordance of some first terms of the expansion with the strict solution. Such a situation takes place in the general case (see., for example, [5], [6]). In our case, it is the consequence of the fact that the energy flux is strongly depends on the temperature in the Stefan-Boltzmann low. At these conditions, The map in the right-side of Eq.(12) is contractive only at very small value of the parameter λ [7].

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Radiative conduction problem in the thin layer of semi-transparent media. The weak absorption case

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Досліджується задача про теплообмін у тонкому шарі напівпрозорого середовища. Разом з механізмом теплопровідності, облічується перенесення енергії електромагнітного випромінювання. Використовується закон випромінювання Стефана-Больцмана. Розглянуто випадок малого поглинання випромінювання середовищем, тобто малої оптичної довжини λ. Одержані перші члени асимптотичного розкладу за степенями λ для стаціонарного розподілу температури у шарі.