

Week fragmentation of round plates

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The simple model of binary week fragmentation of round plates is considered. It is shown, that distribution function of fragments by masses is depended on choice of fissures arise mechanism. The exact distribution function of fragments by masses is obtained with isotropic and anisotropic fissure spreading in plates and its asymptotic behaviour in small and large masses regions is studied.

Рассмотрена простая модель слабой бинарной фрагментации круглых пластин. Показано, что функция распределения осколков по массам зависит от выбора механизма зарождения трещин. Получена точная функция распределения осколков по массам при изотропном и анизотропном распространении трещины в пластинах, и изучены ее асимптотики в области малых и больших масс.

1. Introduction

Fragmentation is one of strongly not in equilibrium processes, the theory of which actively developing. There are many different variants of its construction. Discussion of some of them it can find in works [1–5]. Recently in [6] it proposed the simple model of random cut of rectangular plate. In that work it assumed arising of fissure on plate bound, uniformly on the bound, and linear and isotropic fissure spreading. With probability consideration for this case it obtained the distribution function of fragments by masses. In region of small masses fragment distribution by masses have power asymptotics $f(m) \sim m^{-1/2}$, and in region of large masses it try to $f(m) \approx const$.

There are some questions from physical point of view. At first, the degree of sensitivity of this result to plate form is not evident. Further, fissures arise on the bound of plate seems to be some synthetically. More natural is to assume the fissures arise on different material defects, that distribute homogeneously on all plate volume. In addition, the direction of fissure moving often is not isotropic. Simplest example are crystal materials, in which fissures spread in some special directions, depend on its crystal structure. Therefore, it is interesting to study anisotropy influence on distribution function of fragments by masses. Properly, this work is dedicated to answers on this questions.

In work it consider influence of plate form on the fragments distribution. It is shown, that with selecting round plate and with homogeneous fissure arise on bound, the fragments distribution function change. In range of small masses the power asymptotics $f(m) \sim m^{-2/3}$ has another exponent, in range of large masses the tending to $f(m) \approx const$ keeps.

Even more radical changing of distribution function with assumption of homogeneous fissure arising in all plate material. Both in case of isotropy of fissure spread direction and in anisotropic case, the distribution function by masses becomes universal $f(m) = const$.

It is interesting to note, that experimental destruction [7] of round sapphire plates by impact load points to distribution function tending in small-scale range to asymptotics $f(m) \approx const$. Of course, more careful experimental check of this observation needs move further to small-scale range, that bounded by technical complexity.

2. Fissure arise on the bound

Let's consider influence of plate form on distribution function by masses with assumption of fissure arise on the plate bound [6]. We shall assume, that points of fissure arise distributes on bound uniformly and its spread direction isotropic. For this let's consider some linear fissure, bringing to cut of round plate on two particles (see Fig.1). Every such fissure can be described by two parameters – position of arise point, that lie on bound and spread angle φ, as it shown on Fig.1. Of course, these parameters uniquely determine masses of two fragments, obtained after destruction. From simple geometric consideration it can easy obtain expression for mass of smaller segment through these parameters.

$$m(\varphi) = \rho R^2 \left(\varphi - \frac{1}{2}\sin 2\varphi\right), \qquad \varphi \in \left(0, \frac{\pi}{2}\right)$$

Here R – radius of round plate and ρ – «surface» density of plate material. The point of fissure arise is fixed and coincide with contact point of horizontal line and circle on Fig.1.

Since in each subdivision act of plate with mass M exactly two fragments form – first with mass $m \leq \frac{M}{2}$ and second with mass $M - m \geq \frac{M}{2}$, then fragments mass distribution for masses, large, than half of round mass M the same as fragments mass distribution for masses $m \leq \frac{M}{2}$. That is it is enough to investigate mass distribution of smaller fragments. In case of arise point is on the round bound, the mass of fragment depends only on cut angle φ, as it shown on Fig.1, but not on arise point. Therefore, expression for $m(\varphi)$ not contain the coordinate of arise point on plate bound.

So, in each subdivision act the microscopic density of mass distribution is determined by expression $\delta(m - m(\varphi))$ and desired distribution density function $f(m)$ – by averaging of microscopic density

$$f(m) = \langle \delta(m - m(\varphi)) \rangle, \tag{1}$$

where angle brackets designate the averaging on random values. The supposition about angle isotropy or equiprobability of all directions of fissure spreading brings to averaging on distribution density, equal to

$$p_{\angle}(\varphi) = \frac{2}{\pi}.$$

In this case, fragments distribution density by masses is equal to

$$f(m) = \int_0^{\pi/2} p_{\angle}(\varphi)\, \delta(m - m(\varphi))\, d\varphi.$$

Using known property of δ-function $\delta(m - m(\varphi)) = \sum_\alpha \frac{\delta(\varphi - \varphi_\alpha(m))}{|m'(\varphi_\alpha(m))|}$ (where $\varphi_\alpha(m)$ - roots of equation $m(\varphi) = m$), let's carry out the integration

$$f(m) = \int_0^{\pi/2} p_{\angle}(\varphi)\, \frac{\delta(\varphi - \varphi(m))}{m'(\varphi)} d\varphi = \int_0^{\pi/2} \frac{2}{\pi} \frac{\delta(\varphi - \varphi(m))}{2R^2 \rho \sin^2\varphi} d\varphi = \frac{1}{\pi R^2 \rho \sin^2 \varphi(m)}.$$

Taking into account monotonicity of mass increase with angle φ, only one root dives contribution in integral. Thus, it obtains the exact distribution density in implicit form. Further let's consider asymptotics of this distribution density in range «small» and «large» masses.

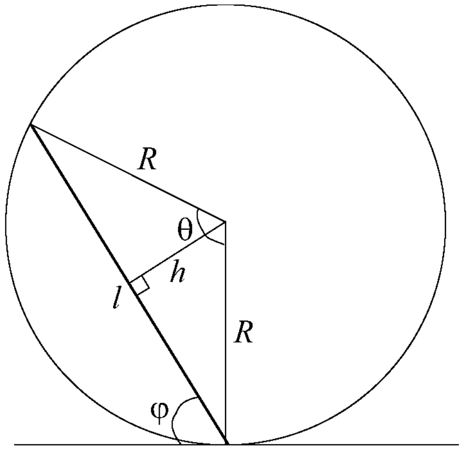

Fig. 1. Destruction of round plate by line fissure on two fragments is shown. Fissure direction determining by angle φ.

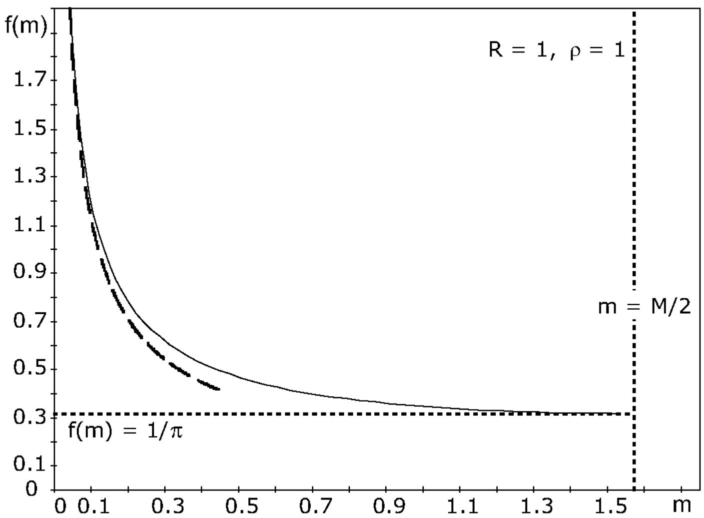

Fig. 2. The exact fragments distribution density by masses shown by solid curve. Asymptotics (2) represents by pecked line and asymptotics (3) – by dotted line. It can easy see coincidence of asymptotics in correspond regions with exact distribution density. In construction $f\,(m)$ parameters ρ and R chosen equal to one.

Let's start from region of small masses. In this region it can easy fing roots of equation $m\,(\varphi) = m$, expanding $m\,(\varphi)$ at small $\varphi \ll \pi/2$. Then $m\,(\varphi) \approx \rho R^2 \cdot \left(\frac{2}{3}\varphi^3\right)$ and respectively, asymptotics of distribution density by masses

$$f\,(m) \sim m^{-2/3}. \tag{2}$$

Now let's consider region of large masses $m \approx M/2$, that corresponds to angles $\varphi \approx \frac{\pi}{2}$. Expanding $m\,(\varphi)$ on small deviation $\varepsilon = \left(\frac{\pi}{2} - \varphi\right) \ll 1$ we obtain $m\,(\varphi) \approx \rho R^2 \left(\frac{\pi}{2} - 2\varepsilon\right)$. Using this relation it can easy obtain the asymptotic form of distribution density in large-scale region

$$f\,(m) \approx \frac{1}{\pi \rho R^2} = \frac{1}{M}. \tag{3}$$

Thus fragments distribution density by masses demonstrates simple asymptotics behaviour in regions of small and large masses. Comparison of exact distribution density with obtained asymptoticses given on Fig.2

In [6] it obtained asymptotics for rectangular plates fragmentation with probabilistic consideration. For distribution density in range of small masses $f(m) \sim m^{-1/2}$, and in range of large masses $f(m) \approx \frac{4}{M}\left[\frac{\gamma arctg\gamma + arcctg\gamma}{\pi(1+\gamma)}\right]$ (where γ – relation of length of rectangle sides $\gamma = a/b$). By comparison of these results with obtained above, it can conclude, that dependency of distribution density by masses on plate form presents. This dependency especially strong for behaviour of distribution density in range of small masses.

Now let's consider influence of destruction mechanism on distribution density by masses.

3. Isotropic case of thin disc fragmentation

General mechanisms of material destruction under load connected with presence of different defects, that brings to strains concentration, that gives beginning and spreading of fissures (see for example [8]). In crystal materials it can also observe mechanism of fissure appearance at concentration of strains on defect under effect of boundary dislocation. Such mechanisms of fissure appearance and material destruction depends on distribution of defects in material. As s rule, defects uniformly distribute in material volume. It means, that from physical point of view process of fissure arise distributed uniformly in material volume and not concentrated on its bound. So we need consider the case of uniformly distribution of fissures in material bound, in our case – on the surface of round plate. In this connection it needs to distinguish two cases. Case of isotropic fissure spreading and anisotropic. Last case easy been observed at crystal fragmentation, where destruction happens in several special directions. These directions defines by crystal structure. In this section we consider only isotropic case.

Let's start from introducing of coordinates, that uniquely determine arise point and spread direction of idealised linear fissure. Of course, arise point position can be determined in polar coordinates. For comparison with previous case ease, let's choose next coordinates, that fully determine fissure. Let position of fissure arise point is determined by distance r to round center and spread direction – by angle φ between this direction and line, perpendicular to radius r (see Fig.3). Knowledge of coordinates r and φ let us uniquely determine mass of fragment, that was formed at disk subdivision by such linear fissure.

$$m = \rho R^2 \left[\arccos\left(\frac{r}{R}\cos\varphi\right) - \frac{r}{R}\cos\varphi\sqrt{1-\left(\frac{r}{R}\cos\varphi\right)^2}\right].$$

It can easy note, that fragment mass is determined by automodelling combination $h = r\cos\varphi$, and not separately r and φ. Physical sense of value $h = r\cos\varphi$ determines, as it easy can prove, by next – h is the length of perpendicular, dropped from center of plate on linear fissure (see Fig.3). Therefore, before obtaining of distribution function of fragments by sizes, let's discuss statistical properties of value h. For considered case, distribution of fissure arise points positions is uniform and direction of fissure spreading – isotropic. It means, that

$$p_r(r)\,dr = \frac{2\pi r dr}{\pi R^2} = \frac{2r}{R^2}dr, \tag{4}$$

$$p_\angle(\varphi) = \frac{2}{\pi}, \quad \varphi \in \left[0, \frac{\pi}{2}\right].$$

Using this distribution functions it can determine the distribution function of value h, averaging its microscopic distribution on corresponding distributions r and φ.

$$p_h(h) = \langle\delta(h - r\cos\varphi)\rangle = \int_0^{\pi/2}\frac{2}{\pi}d\varphi\int_0^R\frac{2rdr}{R^2}\cdot\delta(h - r\cos\varphi) = \frac{4}{\pi R}\sqrt{1-(h/R)^2}.$$

Thus, averaging microscopic distribution function on h, it obtains fragments distribution function

$$f(m) = \langle\delta(m - m(h))\rangle = \int_0^R \delta(m - m(h))p_h(h)\,dh. \tag{5}$$

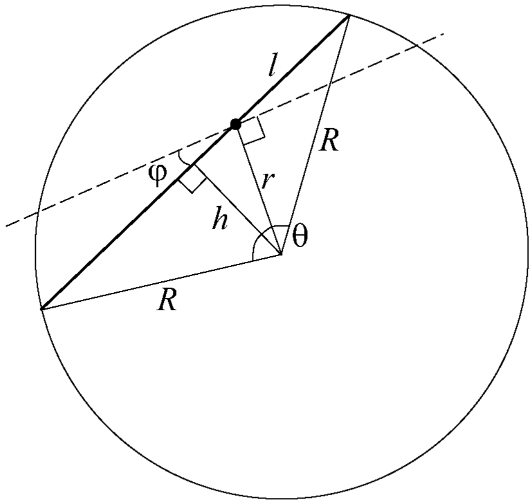

Fig.3. Arise point and spread direction of linear fissure in round are shown. Also designations for corresponding coordinates and values introduced.

Here fragment's mass $m(h) = \rho R^2 \left(\arccos\left(\frac{h}{R}\right) - \frac{h}{R}\sqrt{1-\left(\frac{h}{R}\right)^2}\right)$, and averaging is making on distribution function $p_h(h)$. Taking into account monotonicity of $m(h)$ decreasing, it can easy calculate corresponding integral and find the fragments distribution function

$$f(m) = -\left.\frac{p_h(h)}{m'(h)}\right|_{h=h(m)} = \frac{2}{\pi\rho R^2} = \frac{2}{M}. \tag{6}$$

Note, that by integrating of expression $-\frac{\pi\rho R^2}{2} \cdot p_h(h) = m'(h)$ from some h_0 to R, it deduced, that probability $h \in [h_0, R]$ is equal to $\frac{m(h)}{M/2}$, i.e. proportional to mass part, cut with $h = h_0$ or plate area part, that is same. Thus, the reason of disappearance of depending on m related to homogeneity of arise points distribution. Let's consider the integral distribution function

$$F(m) = \int_0^m f(m)dm,$$

that corresponds to probability of appearance of fragment with mass less, then m. Such distribution function should be monotonically increasing and reach 1 at $m = \frac{M}{2}$. In homogeneous and isotropic case this probability is determining only by mass ratio

$$F(m) = \frac{2m}{M},$$

coefficient 2 takes into account, that this is distribution function for smaller fragment, mass of which can not be more, than $\frac{M}{2}$.

It is interesting to note, that the case of homogeneous cutting of round in integral geometry [9], when cut lines homogeneously fill the round region, essentially different from case, shown above. It supposes, that cut lines have rotate, scale and translation symmetry. Then, according to [10], distribution function $p(h) = \frac{1}{R}$ for this case, and therefore, averaging with this distribution function brings to nontrivial depending of distribution function $f(m)$ on masses. In this case the analytical expression for $f(m)$ more complicate. Therefore, let's show obtained numerically explicit form of distribution function $f(m)$ on Fig.4. In small mass region asymptotic law takes form

$$f(m) \approx \frac{1}{(2\sqrt{3}\rho R^2)^{2/3}} \cdot m^{-1/3}, \qquad m \ll 1, \tag{7}$$

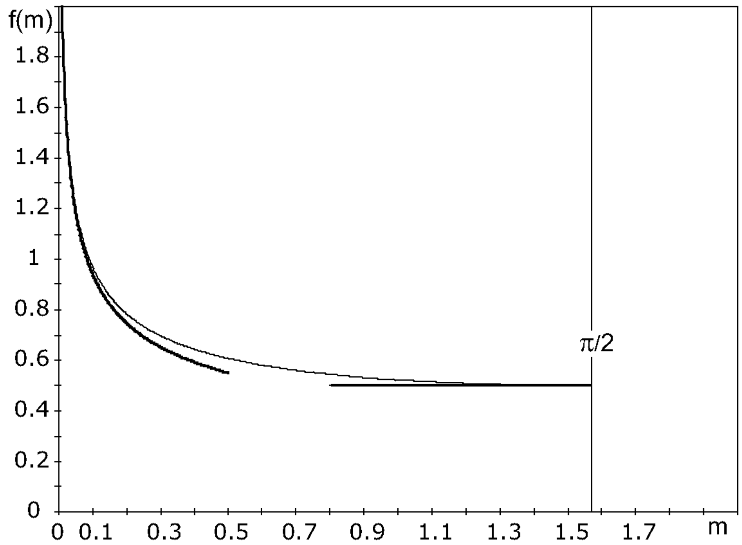

Fig. 4. Fragments distribution density by masses in case of homogeneous arrangement of fissures in round, obtained numerically. Asymptoticses (7), (8) in small-scale and large-scale region shown by bolder lines.

In large-scale region, as it was before, constant value is reaching

$$f(m) \approx \frac{1}{2\rho R^2}. \tag{8}$$

Comparing obtained asymptoticses, it can see sensitivity of asymptotic behaviour of fragments distribution function to mechanisms of fissures arise and material destruction. Thus, type of fragments distribution in small-scale region gives possibility to determine the realised fissure arise mechanism and round disc destruction.

4. Anisotropic fragmentation

In this section we shall consider the case of anisotropic fissure spreading. Such anisotropy is characteristic, foe example, for plates, cut from crystals. As in previous section, we shall determine the position of fissure arise point by pair of numbers (r, α). The supposition of arise points distribution homogeneity preserves.

Let's choose the direction of horizontal axe for anisotropic material along some fixed direction in sample, defined by form of anisotropy of this material. In this case, the probability of fissure spread direction will depend only on angle $\gamma \in [0, \pi]$ (Fig.5). Thus, random values are fissure arise point position (r, α) and spread direction γ. Directions distribution function $p_\gamma(\gamma)$ determines by anisotropic material structure and will be discuss follow. Arise points are distributed uniformly on disc, as in isotropic case (see relation 4)),

$$p_\alpha(\alpha) = \frac{1}{2\pi}, \text{where } \alpha \in [0, 2\pi]$$

Writing fragment mass for this case, it can obtain

$$m = \rho R^2 \left[\arccos\left(\frac{r}{R}|\sin(\gamma-\alpha)|\right) - \frac{r}{R}|\sin(\gamma-\alpha)|\sqrt{1-\left(\frac{r}{R}|\sin(\gamma-\alpha)|\right)^2}\right] =$$

$$= \rho R^2 \left[\arccos\left(\frac{z}{R}\right) - \frac{z}{R}\sqrt{1-\left(\frac{z}{R}\right)^2}\right]$$

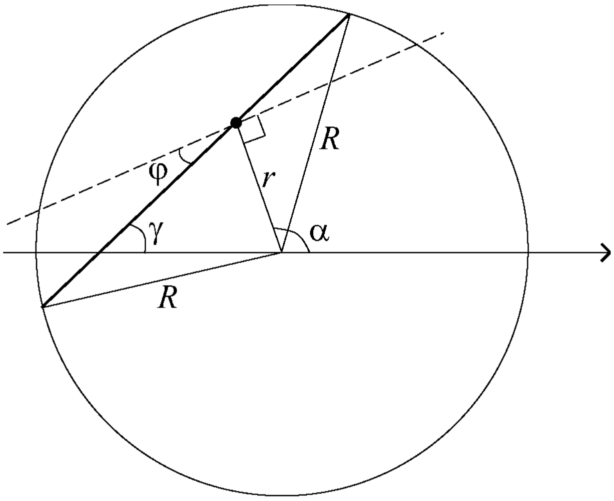

Fig. 5. Coordinates (r, α, γ), useful for description of line fissure location in anisotropic materials are shown.

It can easy note, that fragment mass is determining by only one value $z = r|\sin(\gamma - \alpha)|$. It means that it need average precisely on this value for obtaining fragments distribution function by masses. Let's find distribution function of value $z = r|\sin(\gamma - \alpha)|$ by averaging microscopic distribution function at first on arise point position and at second on directions γ with angle distribution function $p_\gamma(\gamma)$.

$$p_z(z) = \langle \delta(z - r|\sin(\gamma - \alpha)|) \rangle_{r,\alpha,\gamma} = \frac{z}{\pi R^2} \int_0^{\pi} \int_0^{2\pi} \frac{p_\gamma(\gamma)}{\sin^2(\gamma - \alpha)} \times$$
$$\times \left\{ \Theta \left(|\sin(\gamma - \alpha)| - \frac{z}{R} \right) \right\} d\alpha d\gamma.$$

Here indexes point variables, on which it makes averaging. It is easy to prove, that result of integration on α not dependes on value γ, and only on variable z.

$$\int_0^{2\pi} \frac{1}{\sin^2(\gamma - \alpha)} \left\{ \Theta \left(|\sin(\gamma - \alpha)| - \frac{z}{R} \right) \right\} d\alpha = \frac{4R}{z} \sqrt{1 - \left(\frac{z}{R} \right)^2}.$$

Then remaining integration on γ can be easy done and gives only norm factor, equal to 1. These calculations can be done at any anisotropy form, i.e. any form of function $p_\gamma(\gamma)$. In result we obtain

$$p_z(z) = \frac{4}{\pi R} \sqrt{1 - \left(\frac{z}{R} \right)^2}.$$

Consequently, in anisotropic case fragment distribution function by masses insensitive to choice of function $p_\gamma(\gamma)$. It means, that in case of isotropy violation the universal behaviour of fragments masses distribution function realises, independent of distribution $p_\gamma(\gamma)$ of fissures by spread directions. For obtain this universal distribution function, let's average microscopic mass distribution on found distribution function $p_z(z)$

$$f(m) = \langle \delta(m - m(z)) \rangle = \int_0^R \delta(m - m(z)) p_z(z) dz.$$

Substitution of found function $p_z(z)$ bringes to simple integrals

$$f(m) = \langle \delta(m - m(z)) \rangle = \int_0^R \delta(m - m(z)) \cdot \frac{4R}{z} \sqrt{1 - \left(\frac{z}{R} \right)^2} dz.$$

Comparing this expression with expression (5), it can easy note their identity. Thus, we obtain

$$f(m) = \frac{2}{\pi R^2 \rho} = \frac{2}{M} \tag{9}$$

universal distribution density at anisotropic case. It is important to note, that form of this function not depends on anisotropy form, i.e. on form $p_\gamma(\gamma)$. Thus, homogeneous distribution of fissure arise points brings to universal distribution of fragments by masses (9) as at isotropic distribution of spread directions, and at any anisotropy of these directions distribution. One must note, that behaviour of fragments distribution function at small-scale region can be criterion for destruction mechanisms determination.

References

1. V.V.Yanovsky, A.V.Tur, O.V.Kuklina, *Zh.Exper.Teor.Fiz.*, **137**, 986 (2010).
2. Fielder G., *Mon.Not.R.Astr.Soc.*, **129**, 351 (1965).
3. R.F.Machado, J.K.Leal da Silva, *Phys.Rev.E*, **52**, 6037 (1995).
4. A.N.Kolmogorov, *Dokl. AN USSR*, **31**, 99 (1941).
5. J.Mark, R.Simha, *Trans. Faraday. Soc.*, **35**, 611 (1940).
6. F.Parisio, L.Dias, *Phys.Rev.*, **E84**, 035101(R) (2011).
7. R.Ye.Brodskii, P.V.Konevskiy, R.I.Safronov, *Functional Materials*, **18**, 200 (2011).
8. A.N.Orlov, Introduction in Theory of Defects in Crystals, Moscow (1983) [in Russian].
9. L.A.Santalo, Integral Geometry and Geometric Probabilities, Nauka, Moscow (1983) [in Russian].
10. E.T.Jaynes, *Foundations of Physics*, **3**, 477 (1973).

Слабка фрагментація круглих пластин

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Розглянуто просту модель слабкої бінарної фрагментації круглих пластин. Показано, що функція розподілу уламків за масами залежить від вибору механізму зародження тріщин. Отримано точну функцію розподілу уламків за масами при ізотропному й анізотропному поширенні тріщини у пластинах, і вивчено її асимптотики в області малих і великих мас.