CHAOTIC DYNAMICS AT CYCLOTRON RESONANCES

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Mechanisms and criteria for the transition to chaotic dynamics of particles and fields under conditions of electron cyclotron resonances (ECR) are considered. It is shown that the known conditions for the onset of dynamic chaos of charged particles in external electromagnetic fields require careful use. The mechanism of the appearance of regimes with dynamic chaos has been discovered and described, even under conditions of isolated cyclotron resonance. Anomalous sensitivity of particle dynamics to external fluctuations is described. It is shown that the higher moments of particle dynamics can play a significant role. In this case, the usual diffusion equations require a revision.

PACS: 41.75.-i; 05.45.-a

INTRODUCTION

It would seem that in the dynamics of particles and fields in ECR conditions everything is quite understandable and investigated. However, as will follow from the results of our work, many important questions of the dynamics of particles and fields in ECR have important features that were not previously studied. Really in section 2 it is shown, that the transition of regular dynamics to a regime of chaotic dynamics requires an additional analysis of transition conditions. In the 3rd section it is shown that the dynamics of particles at ECR is anomalously sensitive to external fluctuations. Ad-ditive fluctuations can give rise to superdiffusion. Multiplicative fluctuations give rise to a fluctuation instability. In the 4th section it is shown that a regime with dynamic chaos in the excitation of electromagnetic waves by a stream of charged particles can arise even under conditions of one isolated nonlinear cyclotron resonance.

In Section 5 it is shown that the higher moments can play a more significant role than the lower moments. This means that for the kinetic description of these regimes the known diffusion equations can not be used. Equations are needed in which these higher moments are taken into account

1. CONDITION OF ARISING REGIMES WITH DYNAMICAL CHAOS

Let us consider the motion of a charged particle in a constant magnetic field $\vec{H}_0$ and in the field of a plane electromagnetic wave of arbitrary polarization:

$$\mathbf{E} = \mathrm{Re}\,\mathbf{E}_0 e^{i(\mathbf{kr}-\omega t)},\ \mathbf{H} = \mathrm{Re}(c/\omega)[\mathbf{kE}],$$

$$\mathbf{E}_0 \equiv \left\{E_0\left(\alpha_x, i\alpha_y, \alpha_z\right)\right\}. \qquad (1)$$

Here E_0 is amplitude of electric field strength, $\boldsymbol{\alpha} \equiv \left\{\alpha_x, i\alpha_y, \alpha_z\right\}$ is vector polarization of wave.

The equations of motion of a charged particle in this case has the form

$$\frac{d\mathbf{p}}{dt} \equiv \frac{d}{dt}(\gamma m\mathbf{v}) = e\mathbf{E} + \frac{e}{mc}[\mathbf{vH}_0] + \frac{e}{mc}[\mathbf{vH}], \qquad (2)$$

$$d\mathbf{r}/dt = \mathbf{p}/\gamma .$$

For simplicity of writing the formulas, we shall consider the simplest structure of the electromagnetic-wave field

$$\mathbf{E} = \mathrm{Re}\left\{0, iE_y, 0\right\}, \mathbf{H} = \left\{0, 0, H_z\right\}, \mathbf{k} = \left\{k_x, 0, 0\right\}.$$

In this case, the criterion for the overlap of two neighboring nonlinear cyclotron resonances, which was obtained in [1, 2], can be written in the form

$$\varepsilon_0 \ge \omega_H^2 \Big/ \left(16|W_s| k_x^2\right), \qquad (3)$$

where $W_s \equiv -p_\perp J_s'(\mu)$, $\mu \equiv k_x p_\perp / \omega_H$, $p_x = p_\perp \cos\theta$, $p_y = p_\perp \sin\theta$ and dimensionless variables are introduced $\tau = \omega t$, $\mathbf{p} \to \mathbf{p}/mc$, $\mathbf{r} \to k\mathbf{r}$, $\mathbf{k} \to \mathbf{k}c/\omega$, $\varepsilon_0 = eE_0/mc\omega$, $\mathbf{h} = \mathbf{H}_0/|\mathbf{H}_0|$, $\omega_H = eH_0/mc\omega$.

From the condition (3) follows that at $W_s \to 0$ the wave amplitude necessary for the arising of regime with dynamic chaos tends to infinity. However, our numerical calculations show that this does not occur. We will remind that the criterion (3) has been obtained as a condition for overlapping two neighboring nonlinear cyclotron resonances (Chirikov criterion). Our analytical and numerical studies show that in this case ($\varepsilon_0 \to \infty$) for the formation of chaotic dynamics, the main role is played by other resonances that have not been taken into account in obtaining the criterion (3).

We note that the influence of a large number of nonlinear cyclotron resonances with which the particle interacts weakly can be modeled by the presence of an external noise influence. Indeed, as we shall see in the next section, the role of even small external fluctuations can radically change the dynamics of charged particles at cyclotron resonances. Qualitatively, these results can explain the arisen contradictions.

2. INFLUENCE FLUCTUATIONS ON PARTICLES DYNAMICS

It was shown in [3, 4] that under conditions close to autoresonance conditions, particle dynamics can be anomalously sensitive to external fluctuations. Below, we consider this question in more detail for the simplest structure of the field of an electromagnetic wave propagating along the direction of the field $\mathbf{H}_0 \parallel z$:

$$\mathbf{E} = \mathrm{Re}\left\{E_x, 0, 0\right\}, \mathbf{H} = \left\{0, H_y, 0\right\}, \mathbf{k} = \left\{0, 0, k_z \approx 1\right\}.$$

We will analyze the influence of additive and multiplicative fluctuations in the most interesting case, under conditions close to autoresonance:

$$R_s = k_z \upsilon_z + \left(s\omega_H/\gamma\right) - 1 \to 0 \cdot$$

2.1. ADDITIVE FLUCTUATION

At the beginning, we will estimate the role of additive fluctuations. For this, taking into account the small value of the field amplitude $\varepsilon_0 << 1$, the system of equations (2) can be linearized [4]:

$$\left(d\tilde{\gamma}/d\tau\right) = -B\tilde{\theta}, \quad \left(d\tilde{\theta}/d\tau\right) = \alpha\tilde{\gamma} + f, \qquad (4)$$

where $B \equiv \left(\varepsilon_0 W_1 / 2\gamma_0\right)\sin\theta_0$, $\theta_1 = \theta + \tilde{\theta}$, $\tilde{\theta} << 1$, $\tilde{\gamma} << 1$, $\alpha = \left(\partial R_{n0}/\partial\gamma\right)_{\gamma_0}$, $\gamma << \tilde{\gamma}$, $f = \tilde{\omega}_H$ is additive fluctuation force. At the analytical study, we assume that $f(\tau)$ – Gaussian, delta – correlated random process with zero mean:

$$\left\langle f(\tau) f(\tau')\right\rangle = 2D\delta(\tau - \tau'), \ \left\langle f \right\rangle = 0,$$

where D is diffusion coefficient.

In works [3, 4] it has been shown anomalous sensitivity of the particles dynamics to such fluctuations at approach to an autoresonance ($\alpha \to 0$). However, the energy gain of the particle remained diffusive:

$$\left\langle \gamma^2 \right\rangle = \left(DB/\alpha\right)\tau. \qquad (5)$$

It is of interest to find out under what conditions the law of ordinary diffusion is replaced by the law of superdiffusion:

$$\left\langle \gamma^2 \right\rangle = \left(2/3\right) D \cdot B \cdot \tau^3 \qquad (6)$$

Numerical studies were carried out for this purpose. For numerical calculations, the parameter value $B \sim 0.033$ was chosen. The value of the parameter α varied from $\alpha = 10^{-1}$ to $\alpha = 10^{-7}$. To find the mean values of the square of the energy increment, averaging over the ensemble of forty realizations was carried out. As fluctuations, a random variable with a uniform distribution law in the interval $-\Delta\omega_H, \Delta\omega_H$ was chosen. The value $\Delta\omega_H = 0.1$. Initial conditions for the addition of energy and phase: $\tilde{\gamma}(0) = 0$, $\tilde{\theta}(0) = \pi/60$.

Studies have shown that at change of the parameter α right up to $\alpha \geq 10^{-4}$ the dependence of the mean square of energy on time corresponds to the diffusion law (5) Figs. 1,a-b.

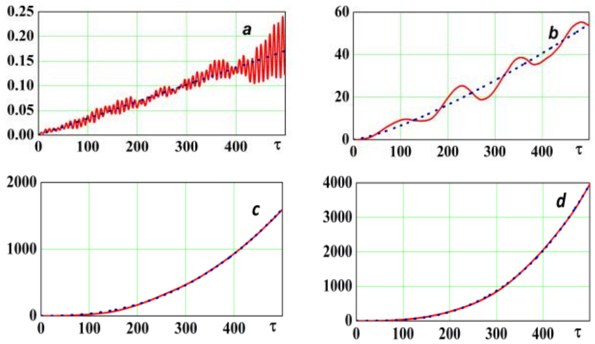


Fig. 1. The dependence of the mean square of the particle energy on time: a) $\alpha = 10^{-1}$; b) $\alpha = 10^{-4}$; c) $\alpha = 10^{-5}$; d) $\alpha = 10^{-7}$

Decrease the parameter α leads to a qualitative change in the dependence of the mean square of the particle energy on the time Figs. 1,c-d. The dependence of the mean square of energy on time instead of linear becomes quadratic with $\alpha = 10^{-5}$ and increases to cubic $\alpha = 10^{-7}$.

These results show that the presence of additive fluctuations, even of very small amplitudes, actually leads to the appearance of superdiffusion. However, this occurs only in an exceptionally small neighborhood of the exact fulfillment of the autoresonance conditions. Under conditions of real experiments, it is practically impossible to satisfy the conditions of autoresonance with the required accuracy. Therefore, it is necessary to focus on the formula (5), and not on the formula (6).

2.2. MULTIPLICATIVE FLUCTUATION

Let us now consider what the presence of multiplicative fluctuations will lead to. Such fluctuations arise, for example, in the presence of fluctuations in the amplitude of the wave in which the particle moves. In this case, the dynamics of a particle located not in the vicinity of a singular point of the "saddle" type, but in the neighborhood of the "center" is of the greatest interest. This is due to the fact that from the vicinity of the saddle point the particles exponentially move away from each other even under the action of regular forces. Equations for finding time dynamics $\tilde{\gamma}$ and $\tilde{\theta}$ particle, which are close to points of the "center" type of a mathematical pendulum, in this case it is convenient to represent in the form [4]:

$$\frac{du}{d\tau} = -(1 + f(\tau))\theta, \quad \frac{d\theta}{d\tau} = u. \qquad (7)$$

Here $\tau \equiv \omega \cdot t \cdot \sqrt{|\alpha \cdot B|}$. The relationship between particle energy and angle θ takes the form: $\tilde{\gamma} = \dot{\theta} \cdot \sqrt{|B/\alpha|}$.

The numerical analysis of equations (7) has been carried out for the initial conditions $u(0) = 0$, $\theta(0) = \pi/60$, amplitude of fluctuations $\Delta\omega_H = 0.1$. Fig. 2 shows the dependence of the mean square of the energy: the solid line is the result of numerical calculation, the dots indicate the approximation by the curve $F_{\exp}(\tau) = D_{mult} \cdot \exp(\delta\tau)$. The exponential dependence of the mean square of energy on time is clearly visible from the graphs of Fig. 2.

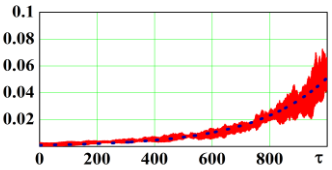


Fig. 2. The dependence of the mean square of the particle energy on time

3. EXCITATION OF ELECTROMAGNETIC WAVES BY A BUNDLE OF OSCILLATORS

The dynamics of charged particles in external prescribed electromagnetic fields becomes chaotic only in the presence of at least two nonlinear resonances. However, if an electromagnetic wave is excited by the particles themselves, then regimes with dynamic chaos can appear even in the presence of only one nonlinear resonance. Let us show this result. To this end, we consider the problem of excitation of an electromagnetic field by a monoenergetic beam of oscillators with a distribution function

$$f_0 = \frac{N_b}{2\pi p_\perp}\delta(p_\perp - p_{\perp 0})\delta(p_\parallel), \qquad (8)$$

where $p_\perp$, $p_\parallel$ perpendicular and parallel to the axis z impulse component, N_b – equilibrium beam density.

We shall consider the excitation of a wave propagating perpendicular to the external magnetic field. A complete nonlinear self-consistent system of equations that describes the dynamics of particles and fields consists of the Maxwell equations and the equations of particle motion. Such a system is given in [1, 2]. Below, we write out truncated system of equations describing the dynamics of particles and fields in an isolated cyclotron resonance with number s:

$$\frac{dp_\perp}{d\tau} = iJ_s'(\mu)e^{i\theta_s}\varepsilon,$$

$$\frac{d\theta_s}{d\tau} = \frac{s\omega_H}{\gamma} - 1 + \frac{1}{\omega_H}\left(1 - \frac{s^2}{\mu^2}\right)\mathrm{Re}\left(J_s(\mu)e^{i\theta_s}\varepsilon\right), \qquad (9)$$

$$\frac{d\varepsilon}{d\tau} = i\frac{\omega_b^2}{2\pi}\int_0^{2\pi} d\theta_{s0}\frac{p_\perp}{\gamma}J_s'(\mu)e^{-i\theta_s},$$

where: $p_\perp = p_\perp / mc$, $\mu = p_\perp/\omega_H$, $\gamma = \sqrt{1+\mu^2\omega_H^2}$, $\omega_H = eH_o/mc\omega$, $\omega_b^2 = 4\pi e^2 n_b / m_e\omega$, $\varepsilon = eE/mc\omega$.

From the results of numerical calculations shown in Fig. 3, we can mark out the following features of the dynamics of particles and fields:

with an increase in the density of active particles (within $0.002 < \omega_b^2 < 0.04$), the level of the excited field increases. The dynamics of particles and the excited field is regular;

with the beam density greater than $\omega_b^2 > 0.04$ a chaotic component appears in the dynamics of the excited field;

beginning approximately from the beam density of 0.5, the asymptotic value of the field does not exceed 0.15.

Thus, just as in overlapping cyclotron resonances (see, for example, [5]), the onset of local instability leads to a limitation of the level of the field excited by the beam (see Fig. 3).

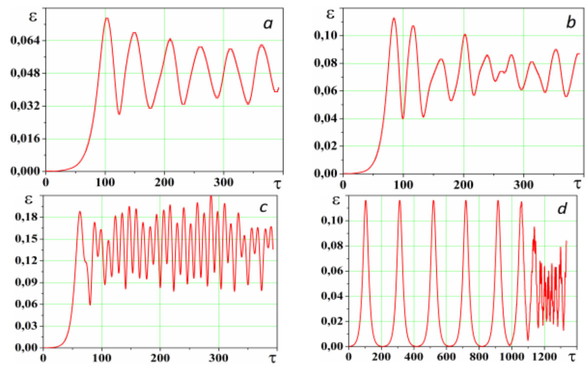


Fig. 3. The amplitude of the field versus time at a beam density: a) ω_b^2 =0.04; b) ω_b^2 =0.1; c) ω_b^2 =0.5; d) ω_b^2 =4

We note that the same stabilization process is also characteristic of plasma-beam instability [6]. Such dynamics of the field with increasing particle density remains fairly familiar as long as the particle density satisfies inequality $\omega_b^2 < 1$. With further increasing of the particle density, when $\omega_b^2 > 1$ it was possible to assume that excitation of oscillations at the selected frequencies ($\omega \approx \omega_H$) will be absent. Indeed, if inequality $\omega_b^2 > 1$ oscillations at frequencies $\omega \approx \omega_H$ are not eigenmodes in such medium. When excited, these oscillations are damped. Indeed, due to the nonequilibrium nature of the beam system at frequencies $\omega \approx \omega_H$, there is excitation of oscillations at these frequencies. However, these oscillations decay rapidly enough. The regime of relaxation oscillations appears in Fig. 3,d. It exists on a sufficiently large time interval. However, over time, this regime goes into a regime of chaotic oscillations, and the process of excitation of oscillations at these frequencies is stopped. With increasing particle density, the amplitudes of the excited oscillations decrease. As far as we know, the excitation of such relaxation oscillations has not yet been described. Such oscillations may, apparently, arise in the ionospheric plasma.

In the above model (see formulas (9)), one cyclotron resonance is isolated, and the dynamics of the interaction of particles and fields in the isolated cyclotron resonance model is studied. In this case, the chaotization mechanism due to the overlap of the cyclotron resonances is absent. An additional analysis was made of the dynamics of particles in an isolated cyclotron resonance. We assume that the amplitude of the wave is constant. In this case, the system of equations that describes the dynamics of a particle coincides with the system of equation (9), in which the third equation can be neglected. The dynamics of the particles is described by the first two equations. Such a system has the Hamiltonian:

$$H(\theta_s, I) = \frac{s}{\omega_H}\gamma - I + \frac{\varepsilon}{\omega_H}2I\frac{d}{dI}\left(J_s(\sqrt{2I})\right)\cos(\theta_s), \qquad (10)$$

where $I = \mu^2/2$.

It is easy to show that the phase portrait of the system with the Hamiltonian (10) is topologically similar to the phase portrait of the Duffing oscillator. For a small value of the parameter $G = \varepsilon / p_{\perp,0}^3$ ($p_{\perp,0}$ – initial particle momentum) ($G << 1$) on the phase plane there are three singular points. Two of them are points of the "center" type, one is the "saddle" type. If the amplitude is sufficiently large ($G >> 1$), then two singular points, namely the saddle point and the point of the "center" type merge and disappear. There remains only one singular point – a point of the "center" type. It is necessary to pay attention to that fact that oscillations of the Duffing oscillator are potential, and for the equations considered by us, it isn't possible to find potential. Typical types of phase portrait at small ($G << 1$) and at high ($G > 1$) field strengths of the external waves are presented in Fig. 4. There are selected regions for trapped particles and a region for transiting particles. As can be seen from Fig. 4,a, on the phase plane, in full accordance with the results given above, there are three singular points: two types of center and a saddle point. As amplitude of the wave increases, two points ("saddle" and "center" with $\theta_s = 0$) approach and disappear (see Fig. 4,b).

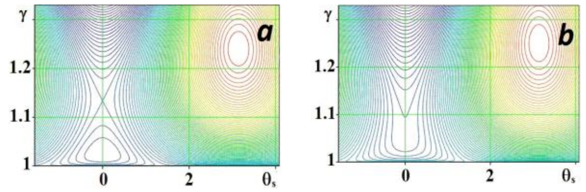
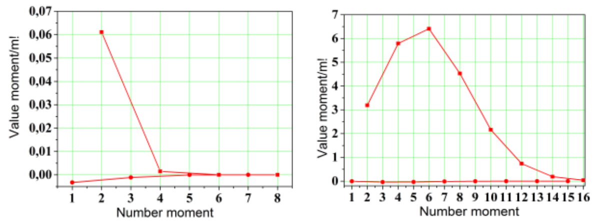


Fig. 4. Phase trajectories: a) ε =0.08; b) ε =0.105

Such a process of qualitative change in dynamics can cause a regime with dynamic chaos. Moreover, it can be seen that even the quantitative characteristics of the appearance of such a qualitative change in dynamics, given in Figs. 3,a-b, confirm this possibility. Indeed, it can be seen from this figure that as soon as the amplitude of the excited wave exceeds 0.105, the dynamics of the particles acquire an irregular character. With a further increase in the particle density, and, correspondingly, with increasing intensity of the excited wave, this irregularity becomes more noticeable. Already the intensity of the field being excited for short times may exceed 0.2. However, the dynamics of the particle turns out to be such that, irregularly oscillating, the field amplitude reaches a level of the order of 0.15. This value of the field strength agrees qualitatively with the intensity of the wave field at which a qualitative change in the phase dynamics of the particles occurs

4. ROLE OF THE MOMENTS IN DYNAMICS OF PARTICLES

Often, particle dynamics in regimes with dynamic chaos are described in the framework of a diffusion equation of the Einstein-Fokker-Planck type equations. In particular, this approach is used to describe the dynamics of particles in ECR. See, for example, [7] and the literature cited there. However, such diffusion equations are valid only when the higher moments rapidly decrease and it is sufficient to take into account only the second moment. Below we show that in the regimes with dynamic chaos in ECR, in most cases the higher moments can play a more significant role. They need to be taken into account. In this case, the kinetic equation must contain these moments. The results of numerical studies of the dependence of the magnitude of the moments on their number and on the field strength are presented in Fig. 5.



a *b*

Fig. 5. Dependences of the magnitudes of the moments divided by the factorial of their number $m!$ for the field amplitude: a) $\varepsilon_0 = 0.1$; b) $\varepsilon_0 = 0.19$

These figures show the dependence of the magnitude of the moments on their number and the magnitude of each moment is divided by the factorial of its number (on $m!$). It can be seen from these figures that, for a sufficiently low external field strength ($\varepsilon_0 = eE/mc\omega = 0.1$) the moments rapidly fall with increasing number (see Fig. 5,a).

However, for higher strengths (for $\varepsilon_0 = 0.19$) the higher moments turn out to be larger than the moments with smaller numbers. Fig. 5,b shows that the moments increase with the number up to the number $m = 6$.

This feature of the moments requires the modification of the equations for describing the particle kinetics. To do this, we write down the relationship between the particle density at the instant of time $\tau + \Delta\tau$ and the particle density at time τ:

$$n(p,\tau+\Delta\tau) = \int_{-\infty}^{\infty} \left[n(p-p',\tau) \right] f(p')dp' . \quad (11)$$

Expression (11) is a mathematical reflection of the fact that the density of particles that have a momentum p at a time $\tau + \Delta\tau$, will be determined by all other particles (with other energies). In this case, such particles with probability $f(p')$, after an interval of time $\Delta\tau$, acquire momentum p'. It is convenient to rewrite equation (11) in the form:

$$n(p,\tau+\Delta\tau) - n(p,\tau) = \int_{-\infty}^{\infty} \left[n(\nu-p',\tau) - n(p,\tau) \right] f(p')dp' . \quad (12)$$

If the moments are finite, then, decomposing the integrands (12) with respect to small displacements and limiting ourselves to the second moments, we obtain the usual diffusion equation for the particle density with the diffusion coefficient $D = \left\langle p^2 \right\rangle /2$. If the moments do not decrease, then a more general equation:

$$\frac{\partial n}{\partial \tau} = \sum_m \frac{\left\langle (p)^m \right\rangle}{m!} \frac{\partial^m n}{\partial p^m}, \quad m = 2j;\ j = \{1,2,3...\} . \quad (14)$$

For the case presented in Fig. 5,b, it is necessary to take into account 4-5 terms in the sum (14).

CONCLUSIONS

Thus, the results obtained above show that the chaotic dynamics of particles and fields in ECR is not fully understood at the present time. Note the most important result for the application. Great hopes were placed on using the autoresonance condition to accelerate charged particles and to excite high-frequency oscillations. 8 However, real attempts to construct such installations have shown their insignificant efficiency (see, for example, Ref. 9). Such insignificant efficiency of energy exchange between particles and waves can be related to the anomalous sensitivity of particle dynamics with respect to fluctuations (see Section 3 above).

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Article received 30.09.2017

ХАОТИЧЕСКАЯ ДИНАМИКА ПРИ ЦИКЛОТРОННЫХ РЕЗОНАНСАХ

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Рассмотрены механизмы и критерии перехода к хаотической динамике частиц и полей в условиях электронных циклотронных резонансов (ЭЦР). Показано, что известные условия возникновения динамического хаоса заряженных частиц во внешних электромагнитных полях требуют осторожного использования. Обнаружен и описан механизм возникновения режимов с динамическим хаосом даже в условиях изолированного циклотронного резонанса. Описана аномальная чувствительность динамики частиц на внешние флуктуации. Показано, что значительную роль могут играть высшие моменты динамики частиц. В этом случае привычные диффузионные уравнения требуют пересмотра.

ХАОТИЧНА ДИНАМІКА ПРИ ЦИКЛОТРОННИХ РЕЗОНАНСАХ

В.О. Буц, В.В. Кузьмін, О.П. Толстолужський

Розглянуто механізми та критерії переходу до хаотичної динаміки частинок і полів в умовах електронних циклотронних резонансів (ЕЦР). Показано, що відомі умови виникнення динамічного хаосу заряджених частинок у зовнішніх електромагнітних полях вимагають обережного використання. Виявлено та описано механізм виникнення режимів з динамічним хаосом навіть в умовах ізольованого циклотронного резонансу. Описана аномальна чутливість динаміки частинок на зовнішні флуктуації. Показано, що значну роль можуть грати вищі моменти динаміки частинок. В цьому випадку звичні дифузійні рівняння вимагають перегляду.