# Temperature conditions in a cone

*S.M.Ivakhnenko, M.A.Serga, S.N.Shevchuk, V.V.Yanovsky**

V.Bakul Institute for Superhard Materials, National Academy of Sciences of Ukraine, 2 Avtozavodskaya St., 04074 Kyiv, Ukraine
*Institute for Single Crystals, STC "Institute for Single Crystals", National Academy of Sciences of Ukraine, 60 Lenin Ave., 60001 Kharkiv, Ukraine

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The temperature field in a cone is described at certain boundary conditions. Such conditions arise often in technological systems at controlled heat removal. Accurate stationary solutions are obtained and anomalous properties are found to take place in that regime. The stage of transition to the stationary state is also considered analytically. The relationships obtained make it possible to control the heat flows and to optimize the required conditions.

Описано поле температур в конусе при определенных граничных условиях. Такие условия часто возникают в технологических системах для контролируемого отвода тепла. Получены точные стационарные решения и обнаружены аномальные свойства этого режима. Также аналитически изучена стадия выхода в стационарное состояние. Полученные зависимости позволяют управлять тепловыми потоками и оптимизировать требуемые режимы.

Thermal problems are of importance in numerous processes [1]. To study of temperature fields at phase transformations, such as the crystal growth, is particularly important. In this case, it is just the thermal problem that defines the growth conditions and the crystallization front movement speed [2, 3]. Rather complex situations arise sometimes in those situations. For example, the geometry selected in this work is associated with the growth optimization of diamond crystals. The selected conical shape facilitates the crystal nucleation and its subsequent growth. At a certain choice of the cone angle, the new nuclei formation is suppressed and thus the polycrystal growth is prevented. The optimization of that process requires the determination of temperature fields in a rather complex geometry. It is shown in this work that the thermal problem in such a region has some unusual properties, so that the solutions obtained differ qualitatively from those for cylindrical and other regions often being under study. This is due to the singularity of conical shape. Similar specific features were observed in hydrodynamics (at the flow about an angle) and in elasticity theory [4]. As will be shown below, this results in some physically unusual dependences of temperature in the stationary state and effects qualitatively the transition thereto. All the main characteristics of that thermal problem are obtained analytically.

Let the region where the temperature field will be considered to have a conical shape (Fig.1). A cone is defined by the base radius $r$ and the height $h$. The vertex angle is defined by those parameters as $\tan(\alpha) = \frac{r}{h}$. The temperature field in that region is defined by the heat conduction equation

$$\frac{\partial T}{\partial t} = \chi \Delta T$$

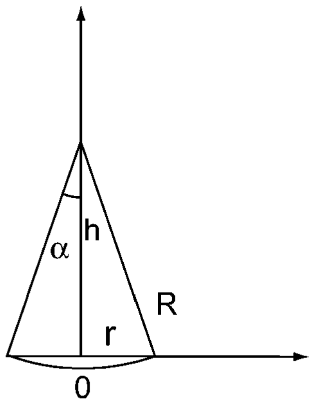


Fig. 1. A conical region: $h$, the cone height; $r$, the base radius; $\alpha$, the vertex angle.

and the boundary conditions. Here, $\chi$ is the temperature conduction coefficient. In what follows, the spherical coordinate system $((\rho, \theta, \varphi)$ will be used. The coordinates in that system vary within limits $0 \le \rho < \infty$, $0 \le \theta \le \pi$, $0 \le \varphi \le 2\pi$. As the problem is symmetrical, the solution and the boundary conditions are independent of $\varphi$ and thus the problem becomes an effectively two-dimensional one. To seek for the temperature field, the boundary conditions are to be met. From physical considerations, it is convenient to fix the total heat flow $q$ through the cone base. Further, it is convenient to guess that the angular dependence of the flow through the base is not very important and that it will be established in the course of the problem solution. Also, it is convenient to set the flow on a spherical segment of radius $R$ instead of the planar cone base. The heat flow through both those surfaces is obviously the same. Therefore, let the boundary condition at the spherical segment be set in the form

$$-\lambda \left.\frac{\partial T}{\partial \rho}\right|_{\rho=R} = q_0(\theta), \tag{1}$$

where $\lambda$ is the heat conduction coefficient. The function $q_0(\theta)$ should satisfy the integral restriction

$$q = \int_0^{\alpha} d\theta \sin\theta \int_0^{2\pi} d\varphi R^2 q_0(\theta) = 2\pi R^2 \int_0^{\alpha} d\theta \sin\theta q_0(\theta). \tag{2}$$

This equality fixes the heat flow through the lower cone base. Now let the boundary condition at the cone side surface be considered. Assuming the environmental temperature outside the cone to be $T_0$, let the boundary condition at the cone side surface be chosen according to the Newton law for heat exchange with the environment

$$\left.\frac{\partial T}{\partial \theta}\right|_{\theta=\alpha} = \delta \cdot (T|_{\theta=\alpha} - T_0). \tag{3}$$

Here, $\delta$ is the external heat emission coefficient; $\frac{\partial}{\partial \theta}$, the vector of normal to the cone side surface; $T_0$, the environmental temperature. In the stationary state, all those flows are coincident. In what follows, the temperature counted from the environmental one, $U = T - T_0$, will be used as a variable for convenience.

Let the heat conductivity equation be written in spherical coordinates:

$$\frac{\partial U}{\partial t} = \frac{\chi}{\rho^2 \sin(\theta)} \left( \sin(\theta) \frac{\partial}{\partial \rho} \left( \rho^2 \frac{\partial U}{\partial \rho} \right) + \frac{\partial}{\partial \theta} \left( \sin(\theta) \frac{\partial U}{\partial \theta} \right) \right). \tag{4}$$

Here, the symmetry of the problem is taken into account as the independence of the variable $\varphi$. The complete solution of the problem in a cone under the above boundary condition is an extremely difficult task. Therefore, let us start from the stationary solution.

When searching for the stationary temperature distribution, it is necessary to find the solution of equation

$$\frac{\chi}{\rho^2 \sin(\theta)}\left(\sin(\theta)\frac{\partial}{\partial\rho}\left(\rho^2\frac{\partial U}{\partial\rho}\right)+\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial U}{\partial\theta}\right)\right)=0$$

. Satisfying the boundary conditions

$$\left.\frac{\partial U}{\partial\theta}\right|_{\theta=\alpha}=\delta U|_{\theta=\alpha}, \tag{5}$$

$$-\lambda\left.\frac{\partial U}{\partial\rho}\right|_{\rho=R}=q_0(\theta). \tag{6}$$

To find the solution, the separation of variables can be used. However, it is convenient to seek the solution directly in the form

$$U=A\rho^{\nu}\Phi(\theta),$$

where $\nu$ is meanwhile an arbitrary number. Substituting it into the stationary heat conduction equation, we get for $\Phi$ the equation

$$\frac{1}{\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial\Phi}{\partial\theta}\right)+\nu(\nu+1)\Phi=0.$$

The solution of this equation depends on the parameter and is coincident with the Legendre function [5]

$$\Phi=P_\nu(\cos\theta).$$

Using the found solution, let us try to satisfy the boundary condition (5). Substituting the solution in to the boundary condition, we get an equation of the type

$$\left(\frac{\partial P_\nu(\cos\theta)}{\partial\theta}-\delta P_\nu(\cos\theta)\right)|_{\theta=\alpha}=0$$

or

$$\frac{\partial P_\nu(\cos\alpha)}{\partial\alpha}-\delta P_\nu(\cos\alpha)=0.$$

It is seen that the latter condition can be satisfied only at the certain value $\nu=\nu_*=\nu_*(\alpha,\delta)$. Actually, this boundary condition is an equation to obtain the $\nu$ value. Therefore, the usual method to search for a solution as an expansion into Legendre polynomials is not realizable in this case. It is difficult to find $\nu_*$ analytically, that is why it is more simply to find it numerically. However, using the properties of Legendre functions and the known recurrent relationships [5], the boundary condition can be transformed into a form convenient to obtain $\nu_*$:

$$\nu_*=\frac{\delta\sin\alpha P_{\nu_*}(\cos\alpha)}{\cos\alpha P_{\nu_*}(\cos\alpha)+P_{\nu_*-1}(\cos\alpha)}.$$

This equation can be used as the iteration equation to obtain explicit dependences of $\nu_*$ on physical parameters $\delta$ and $\alpha$:

$$\nu_*=\frac{\delta\sin\alpha}{\cos\alpha+P_{\nu_*-1}(\cos\alpha)/P_{\nu_*}(\cos\alpha)}.$$

Thus, the solution satisfying the boundary condition (5) has the form

$$U=A\rho^{\nu_*}P_{\nu_*}(\cos\theta)$$

It is important to note that $\nu_*=\nu_*(\alpha,\delta)$ depends explicitly on physical quantities, such as the cone opening and the characteristics of heat removal to environment. Such dependences are observed rarely in physical applications. Now let the boundary condition (6) be checked from which follows

$$-\lambda\nu_*R^{\nu_*-1}AP_{\nu_*}(\cos\theta)=q_0(\theta).$$

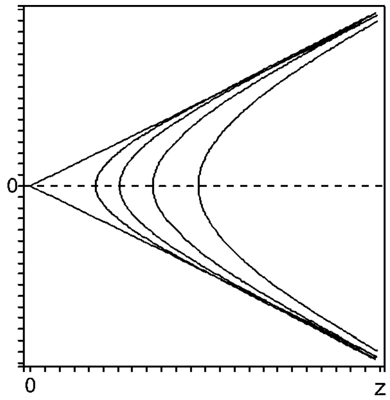


Fig. 2. Temperature distribution in a cone. Each line approaching the vertex corresponds to a temperature being one third of that for the preceding line. The temperature gradients at the cone surface increase as the distance from the vertex rises.

Taking into account the restriction (2), let the value of constant $A$ be established. To that end, let the obtained relation with respect to $\theta$

$$-A\lambda\nu_* R^{\nu_*-1}\int_0^{\alpha} P_{\nu_*}(\cos\theta)\sin\theta d\theta = \int_0^{\alpha} q_0(\theta)\sin\theta d\theta.$$

Thus, the constant $A$ is defined by the heat flow

$$A = -\frac{q}{2\pi R^{\nu_*+1}\lambda\nu_*.} \quad (7)$$

It is to mention that $q$ is the flow through the cone base surface. Thus, in the final form, we get the stationary solution

$$U = -\frac{q}{2\pi R^{\nu_*+1}\lambda\nu_*}\rho^{\nu_*}P_{\nu_*}(\cos\theta), \quad (8)$$

that satisfies the required boundary conditions, for example, at $q_0(\theta) \sim P_{\nu_*}(\cos\theta)$. The $q$ sign is negative, as the heat flow is directed oppositely to the $\vec{\rho}$ axis. The solution (8) belongs to unusual solution types where the power index includes physical parameters. Fig.2 shows an example of typical constant temperature lines for that stationary solution [6]. According to the Barenblatt's classification, it can be referred to auto-modeled solutions of second order. This solution defines completely the stationary distribution of temperature field in a cone and its transformations at varying physical conditions.

Now it is clear how the non-stationary solution is to search for. First of all, note that the boundary condition (6) is satisfied independently of the radial part. Therefore, it is reasonable to search for the solution using the separation of variables

$$U = F(\rho,t)\Phi(\theta).$$

Let that form be substituted into (4), and after obvious transformations we get

$$\frac{\rho^2}{F}\frac{\partial F}{\partial t} = \frac{\chi}{F}\frac{\partial}{\partial\varrho}\left(\rho^2\frac{\partial F}{\partial\varrho}\right) + \frac{\chi}{\Phi\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial\Phi}{\partial\theta}.\right)$$

Thus, this equation is satisfied if the functions $F$ and $\Phi$ satisfy the equations

$$\frac{\chi}{F}\frac{\partial}{\partial\varrho}\left(\rho^2\frac{\partial F}{\partial\varrho}\right) - \frac{\rho^2}{F}\frac{\partial F}{\partial t} = \mu,$$

$$\frac{\chi}{\Phi\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial\Phi}{\partial\theta}\right) = -\mu,$$

where $\mu$ is an arbitrary constant. Let those equation be rewritten in the equivalent form

$$\rho^2 \frac{\partial F}{\partial t} = \chi \frac{\partial}{\partial \varrho}\left(\rho^2 \frac{\partial F}{\partial \varrho}\right) - \mu F = 0, \tag{9}$$

$$\frac{\chi}{\sin(\theta)} \frac{\partial}{\partial \theta}\left(\sin(\theta) \frac{\partial \Phi}{\partial \theta}\right) + \mu \Phi = 0. \tag{10}$$

Comparing Eq. (10) with the equation for the angular part of the stationary solution, it is easy to see the coincidence of those equations as $\mu = \chi \nu_*(\nu_* + 1)$. Furthermore, it is clear that the boundary condition (6) will also be satisfied. In fact, it is just that circumstance that has defined the choice of the solution form. Thus, it remains to find the solution for the radial part $F$

$$\rho^2 \frac{\partial F}{\partial t} = \chi \frac{\partial}{\partial \varrho}\left(\rho^2 \frac{\partial F}{\partial \varrho}\right) - \chi \nu_*(\nu_* + 1) F = 0, \tag{11}$$

that satisfies the boundary condition (5) and the initial condition

$$F(\rho, t)|_{t=0} = 0.$$

Let us search for the solution of this equation in the form

$$F = e^{-\gamma^2 \chi t} f(\rho).$$

After substitution, we get an equation that is canonic and has known solutions:

$$-\gamma^2 \rho^2 \chi f = \chi \frac{\partial}{\partial \varrho}\left(\rho^2 \frac{\partial f}{\partial \varrho}\right) - \chi \nu_*(\nu_* + 1) f = 0.$$

In fact, let it be transformed to a more convenient form of Bessel equation:

$$\frac{\partial}{\partial \varrho}\left(\rho^2 \frac{\partial f}{\partial \varrho}\right) + (\gamma^2 \rho^2 - \nu_*(\nu_* + 1)) f = 0.$$

The solution of this equation is a linear combination of Bessel functions [7] $Z_{\nu_*+1/2}(\gamma\rho)/\sqrt{\rho}$. Thus, we get

$$f(\rho) = \frac{B J_{\nu_*+1/2}(\gamma\rho)}{\sqrt{\rho}} + \frac{C Y_{\nu_*+1/2}(\gamma\rho)}{\sqrt{\rho}}.$$

As the function $Y_{\nu_*+1/2}$ is divergent at $\rho = 0$, the constant $C = 0$ should be selected. Then the solution obtained must satisfy the boundary condition (5). Let that condition be checked:

$$-\lambda e^{-\gamma^2 \chi t} B \left.\frac{\partial}{\partial \rho} \frac{A J_{\nu_*+1/2}(\gamma\rho)}{\sqrt{\rho}}\right|_{\rho=R} P_{\nu_*}(\cos\theta) = q_0(\theta).$$

The right-hand part of that boundary condition is time-independent. The only possibility to attain the same for the left-hand part is to select $\gamma R = R_*$. Here, the dimensionless quantity $R_*$ is defined by the expression $J'_{\nu_*+1/2}(R_*) = 0$. Therefore,

$$\gamma = R_*/R.$$

In this case, the left-hand part losses the time dependence. It is of importance to note that $\gamma$ depends on $\nu_*$ and thus on the angle $\alpha$ and $\delta$, too. Now it is to take into account that the general solution can be selected as the sum of the stationary solution ane the found non-stationary one. Then, the boundary condition for the solution is

$$U = \left(A \rho^{\nu_*} + e^{-\gamma^2 \chi t} \frac{B J_{\nu_*+1/2}(\gamma\rho)}{\sqrt{\rho}}\right) P_{\nu_*}(\cos\theta).$$

Thus, the boundary conditions for that solution are satisfied. Let the initial condition for the solution be obtained:

$$U|_{t=0} = \left( A\rho^{\nu_*} + \frac{BJ_{\nu_*+1/2}(\gamma\rho)}{\sqrt{\rho}} \right) P_{\nu_*}(\cos\theta).$$

Taking into account the expansion of the Bessel function at small $\rho$, we get

$$U|_{t=0} = \left( A\rho^{\nu_*} + \frac{B\sqrt{\gamma}(\gamma\rho)^{\nu_*}}{\Gamma(\nu_* + 3/2)2^{\nu_*+1/2}} \right) P_{\nu_*}(\cos\theta).$$

Here $\Gamma(\nu_* + 3/2)$ is gamma function. It is easy to see that as the constant $B$

$$B = -A\Gamma\left(\nu_* + \frac{3}{2}\right)\left(\frac{2}{\gamma}\right)^{\nu_*+\frac{1}{2}}$$

in the region of small $\rho$, the necessary initial condition is realized. The solution takes the final form

$$U(t,\rho,\theta) = A\left(\rho^{\nu_*} - \Gamma\left(\nu_* + \frac{3}{2}\right)\left(\frac{2}{\gamma}\right)^{\nu_*+\frac{1}{2}} \frac{J_{\nu_*+1/2}(\gamma\rho)}{\sqrt{\rho}} e^{-\gamma^2\chi t}\right) P_{\nu_*}(\cos\theta)$$

and defines the relaxation to stationary temperature distribution. The $A$ value has been found during the search for the stationary one (see Eq. (7))). Further, it is easy to analyze the characteristic time of transition to the stationary state, the temperature distribution and its relaxation to stationary state.

The obtained temperature distribution in a cone is of a great importance for optimization of crystal nucleation and growth is that geometry. In such a system, it becomes possible to provide the conditions for the single crystal growth and to suppress simultaneously the formation of other nuclei. In such processes, of importance is the increment or the characteristic time of transition to the stationary state. The latter have been shown to depend in a non-trivial manner on the problem parameters such as the cone opening and the characteristics of heat exchange with the environment.

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## Температурні режими у конусі

*С.А.Івахненко, М.А.Серга, С.М.Шевчук, В.В.Яновський*

Описано поле температур у конусі при певних граничних умовах. Такі умови часто виникають у технологічних системах для контрольованого відведення тепла. Одержано точні стаціонарні розв'язання та виявлено аномальні властивості цього режиму. Аналітично досліджено також стадію виходу на стаціонарний стан. Одержані залежності дозволяють керувати тепловими потоками та оптимізувати потрібні режими.