

New vortex structures in the two-dimensional hydrodynamic

A.V.Tur, K.N.Kulik*, V.V.Yanovsky*

Universite de Toulouse [UPS], CNRS, Center d'Etude Spatiale des Rayonnements, 9 avenue du Colonel Roche, BP 4346, 31028 Toulouse Cedex 4, France
*Institute for Single Crystals, STC "Institute for Single Crystals" National Academy of Sciences of Ukraine, 60 Lenina Ave., Kharkiv 31001, Ukraine

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In this work we found the new class of exact stationary solutions for 2D-Euler equations. The obtained solutions are expressed in terms of elementary functions. These solutions represent complex singularity point surrounded by vortex satellites structure.

В работе найден новый класс точных стационарных решений двумерных уравнений Эйлера. Полученные решения выражаются через элементарные функции. Найденные решения представляют собой сложную особую точку, окруженную симметричной структурой вихревых сателлитов.

1. *Introduction*

The importance of exact solutions for 2D-Euler equations is well known. Today, the list of exact solutions is quite impressive. Without pretending to be exhaustive we will mention only some of them. First of all, this are classical solutions with smooth vorticity, such as Runkine and Kirchof type vortices (see, for example, the standard references [1-3], elliptical Moor and Suffman vortices [2], Lamb dipole [1] and Stuart vortex pattern [4]. It is interesting, that these classical solutions are still topical (see for example recent works [5, 6]). The generalization of these solutions are the models of different coherent structures, vortex patches, and vortex crystals (see, for example, [7-13] and references therein), which are well observed in numerical and laboratory experiments (see, for example, [14-21]).

But in this work we show that 2D-Euler equation has a new class of exact stationary solutions with complex singular point, which index is equal to three. These solutions are found in explicit form and expressed in terms of elementary functions. Obtained solutions describe localized vortex structures, in which complex singular point is surrounded by vortices satellites. In addition we discuss the equation of motion for singularities and we give the sufficient conditions of immobility for singular points; without this equation one can not guarantee the stationarity of solution.

2. *Vortex structures*

We use 2D-Euler equation in form of Poisson brackets for the vorticity ω and the stream function Ψ [2]

$$\frac{\partial \Delta\Psi}{\partial t} + \{\Delta\Psi, \Psi\} = 0, \tag{1}$$

Poisson brackets has the usual form

$$\{A,B\}=\varepsilon_{ik}\frac{\partial A}{\partial x_i}\frac{\partial B}{\partial x_k}=\frac{\partial A}{\partial x}\frac{\partial B}{\partial y}-\frac{\partial A}{\partial y}\frac{\partial B}{\partial x}$$

when $\varepsilon_{ik}-$is single anti symmetrical tensor. Now we consider the problem of exact stationary solutions of equation (1). Further, it is easy to consider the Poisson brackets

$$\frac{\partial \Psi}{\partial x}\frac{\partial \Delta\Psi}{\partial y}-\frac{\partial \Psi}{\partial y}\frac{\partial \Delta\Psi}{\partial x}=0 \tag{2}$$

as dimensionless. For the simplest case exact stationary solutions were found independently and in a different way in papers [22, 23]. In this work we study a more general *anzatz*. Let us suppose that the vorticity can be presented in the form

$$\Delta\Psi=\exp\left(-\frac{\Psi}{\Gamma_0}\right)+4\pi n\Gamma_0\delta(\vec{x}-\vec{x}_0)-4\pi\vec{D}\frac{\partial}{\partial\vec{x}}\delta(\vec{x}-\vec{x}_0) \tag{3}$$

when function $f(\Psi)$ is chosen in the same way as in the Stuart work [4]; we suppose that coefficients Γ and $\vec{D}$ are constant, and the coordinate $\vec{x}_0$ is not depending on time. (For the sake of simplicity we can choose $\vec{x}_0=0$). n- is positive integer number. By means of evident reskaling

$$\frac{\Psi}{\Gamma_0}\to\Psi',\quad x\to\Gamma_0^{\frac{1}{2}}x',\quad y\to\Gamma_0^{\frac{1}{2}}y',\quad \frac{\vec{D}}{\Gamma_0^{\frac{3}{2}}}\to\vec{D}' \tag{4}$$

equation (3) is reduced to the more simple equation (the primes were omitted):

$$\Delta\Psi=\exp(-\Psi)+4\pi n\delta(\vec{x})-4\pi\vec{D}\frac{\partial}{\partial\vec{x}}\delta(\vec{x}). \tag{5}$$

First of all, let us find exact solutions for the equation (5), and then we prove, that they are exact stationary solutions of 2D-Euler equation (2). Now we can look for the solutions of equation (5) in the Liouville form

$$\Psi=-\ln 8\frac{\left|u'(z)\right|^2}{\left(1+|u(z)|^2\right)^2}, \tag{6}$$

when $u'(z)=\frac{du(z)}{dz}$, is the unknown for the moment function of complex variable $z=x+iy$ and $u(z)-$ is primitive function.

Direct calculation of $\Delta\Psi$ (6) gives

$$\Delta\Psi=8\frac{\left|u'(z)\right|^2}{\left(1+|u(z)|^2\right)^2}-\Delta\ln\left|u'(z)\right|^2, \tag{7}$$

it is important to note that the formula (7) is valid for arbitrary analytical function $u'(z)$ independently of its singularities structure.

We substitute the formula (7) into equation (5) and obtain the equation for the function $\left|u'(z)\right|^2$

$$-\Delta\ln\left|u'(z)\right|^2=4\pi n\delta(\vec{x})-4\pi\vec{D}\frac{\partial}{\partial\vec{x}}\delta(\vec{x}). \tag{8}$$

It is easy to see that the equation (8) is satisfied, if we choose the function $\left|u'(z)\right|^2$ in the form

$$\left|u'(z)\right|^2=\frac{1}{|z|^{2n}}\exp\vec{D}\frac{\partial}{\partial\vec{x}}\ln|z|^2. \tag{9}$$

Indeed

$$\ln\left|u'(z)\right|^2 = -n\ln|z|^2 + \vec{D}\frac{\partial}{\partial\vec{x}}\ln|z|^2. \tag{10}$$

The first term in (10) gives the Green function of Laplace equation

$$\Delta\ln|z|^2 = 4\pi\delta(\vec{x}) \tag{11}$$

and describes the point vortex. The second term in (10) is a result of application of the operator $\vec{D}\frac{\partial}{\partial\vec{x}}$ to the equation (11) and describes the point dipole. Let us introduce the complex dipole moment

$$D = D_1 + iD_2. \tag{12}$$

Then the dipole operator $\vec{D}\frac{\partial}{\partial\vec{x}}$ can be written in the complex form

$$\vec{D}\frac{\partial}{\partial\vec{x}} = D\frac{\partial}{\partial z} + \overline{D}\frac{\partial}{\partial\overline{z}}. \tag{13}$$

(When $\overline{D}, \overline{z}$- denote complex conjugated values).

The function (9) can be written down in the form

$$\left|u'(z)\right|^2 = \frac{1}{|z|^{2n}}\exp\left(D\frac{\partial}{\partial z} + \overline{D}\frac{\partial}{\partial\overline{z}}\right)(\ln z + \ln\overline{z}) = \frac{1}{z^n}\exp\left(\frac{D}{z}\right)\frac{1}{\overline{z}^n}\exp\left(\frac{\overline{D}}{\overline{z}}\right). \tag{14}$$

From formula (14) follows, that functions $u'(z)$ can be chosen in the form

$$u'(z) = \frac{1}{z^n}\exp\left(\frac{D}{z}\right). \tag{15}$$

In the point $z = 0$, the function $u'(z)$ (15) has essential singular point, which joins the pole of order n. Now let us find the primitive function $u_n(z)$

$$u_n(z) = \int \exp\left(\frac{D}{z}\right)\frac{dz}{z^n}. \tag{16}$$

Using the new variable $w = \frac{D}{z}$, we obtain

$$u_n(w) = -\frac{1}{D^{n-1}}\int W^{(n-2)}\exp W dW. \tag{17}$$

From the formula (17) we can see that the primitive function is elementary function only with $n \geq 2$. This particular case is examined in this work. (Others cases will be considered separately).

Integration by parts in the formula (17) with $n \geq 2$, gives

$$u_n(z) = -\frac{1}{Dz^{n-2}}\exp\left(\frac{D}{z}\right)P_{n-2}(z); \tag{18}$$

where the polynomial $P_{n-2}(z)$ has the form

$$\begin{aligned} P_{n-2}(z) = 1 - (n-2)\left(\frac{z}{D}\right) + (n-2)(n-3)\left(\frac{z}{D}\right)^2 + \cdots \\ +(-1)^{n-3}(n-2)!\left(\frac{z}{D}\right)^{n-3} + (-1)^{n-2}(n-2)!\left(\frac{z}{D}\right)^{n-2}. \end{aligned} \tag{19}$$

As a result, $|u_n(z)|^2$ has the form:

$$|u_n(z)|^2 = \frac{1}{|D|^2|z|^{2(n-2)}}\exp\left(2\frac{D_1x + D_2y}{x^2+y^2}\right)|P_{n-2}(z)|^2. \tag{20}$$

The primitive function $u_n(z)$ (18), like the function $u^{'}(z)$ (15) has in $z=0$ the essential singular point, which joins the pole of order $(n-2)$. In the real form $\left|u^{'}(z)\right|^2$, has obviously the following form

$$\left|u^{'}(z)\right|^2 = \frac{1}{(x^2+y^2)^n}\exp\left(2\frac{D_1x+D_2y}{x^2+y^2}\right). \tag{21}$$

Consequently, the essential singular point describes in the complex form the singularities of point dipole kind, while the pole describes the point vortex, since the expression (21) generates following terms in stream function (6)

$$-\ln\left|u^{'}(z)\right|^2 = n\ln(x^2+y^2) - 2\frac{D_1x+D_2y}{x^2+y^2}.$$

Hence, the exact solution of the equation (5) is given by the formula (6), where $u(z)$ is defined by the expression (18), while $u^{'}(z)-$ by the formula (15). Now we can prove, that the obtained solution turns into zero the Poisson brackets (2). At first, let us calculate the velocity field. Derivatives $\frac{\partial\Psi}{\partial x}$ and $\frac{\partial\Psi}{\partial y}$ have the form

$$\frac{\partial\Psi}{\partial x} = \frac{2}{1+|u(z)|^2}\frac{\partial}{\partial x}u(z)\overline{u(z)},$$

$$\frac{\partial\Psi}{\partial y} = \frac{2}{1+|u(z)|^2}\frac{\partial}{\partial y}u(z)\overline{u(z)}.$$

Using the formulae

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial z} + \frac{\partial}{\partial \overline{z}},\ \frac{\partial}{\partial y} = i\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \overline{z}}\right),$$

we obtain a more convenient formula for derivatives

$$\frac{\partial\Psi}{\partial x} = \frac{2}{1+|u(z)|^2}\left(\overline{u}\frac{du}{dz} + u\overline{\frac{du}{dz}}\right),$$

$$\frac{\partial\Psi}{\partial y} = \frac{2i}{1+|u(z)|^2}\left(\overline{u}\frac{du}{dz} - u\overline{\frac{du}{dz}}\right).$$

Taking into account the formula (15), after simple algebraic transformations we obtain the expression for components of velocity field

$$\frac{\partial\Psi}{\partial x} = -\frac{2\exp\left(\frac{D\overline{z}+\overline{D}z}{|z|^2}\right)}{(1+|u(z)|^2)\,|D|^2\,|z|^{2n}}(D\overline{z}^2\overline{P}_{n-2} + \overline{D}z^2P_{n-2}), \tag{22}$$

$$\frac{\partial\Psi}{\partial y} = -\frac{2i\exp\left(\frac{D\overline{z}+\overline{D}z}{|z|^2}\right)}{(1+|u(z)|^2)\,|D|^2\,|z|^{2n}}(D\overline{z}^2\overline{P}_{n-2} - \overline{D}z^2P_{n-2}). \tag{23}$$

(Let us remind, that $n \geq 2$). Now we show that function (6), (18) is an exact solution of Poisson brackets (2). For that we substitute the expression for vorticity (3) and derivatives (22), (23) into Poisson brackets (2). First of all we examine the simplest case $n=2$. In this case the polynomial $P_{n-2}=1$ and derivatives (22), (23) take the simple form

$$\frac{\partial\Psi}{\partial x} = -\frac{4\exp\left(\frac{D\overline{z}+\overline{D}z}{|z|^2}\right)}{(1+|u|^2)\,|D|^2\,|z|^4}(D_1x^2+2D_2xy-D_1y^2), \tag{24}$$

$$\frac{\partial\Psi}{\partial y} = \frac{4\exp\left(\frac{D\overline{z}+\overline{D}z}{|z|^2}\right)}{(1+|u|^2)\,|D|^2\,|z|^4}(D_2x^2-2D_1xy-D_2y^2), \tag{25}$$

$$|u(z)|^2 = \frac{1}{|D|^2}\exp\left(\frac{D\overline{z}+\overline{D}z}{|z|^2}\right). \tag{26}$$

We write the Poisson brackets (2) in the explicit form

$$\{\Psi, \Delta\Psi\} = 4\pi n\left[\frac{\partial\Psi}{\partial x}\delta(x)\delta^{'}(y) - \frac{\partial\Psi}{\partial y}\delta(y)\delta^{'}(x)\right] - \tag{27}$$

$$-4\pi\left[D_1\frac{\partial\Psi}{\partial x} - D_2\frac{\partial\Psi}{\partial y}\right]\delta^{'}(x)\delta^{'}(y) -$$

$$-4\pi\left[D_2\frac{\partial\Psi}{\partial x}\delta(x)\delta^{''}(y) - D_1\frac{\partial\Psi}{\partial y}\delta(y)\delta^{''}(x)\right].$$

It is obvious, that all the terms in the first brackets (27) are equal to zero because they contain this kind of zeros

$$x^2\delta(x),\ x\delta(x),\ x^2\delta^{'}(x);\ \ y^2\delta(y),\ y\delta(y),\ y^2\delta^{'}(y).$$

In the second brackets one part of terms is also equal to zero, but there are dangerous terms of this type: $xy\delta^{'}(x)\delta^{'}(y)$. However, these terms are part of second brackets in the following combination

$$\left(D_1\frac{\partial\Psi}{\partial x} - D_2\frac{\partial\Psi}{\partial y}\right)\delta^{'}(x)\delta^{'}(y) = [\cdots](-D_1D_2xy + D_2D_1xy)\delta^{'}(x)\delta^{'}(y) = 0,$$

i.e. are reciprocally cancelled. (Here brackets $[\cdots]$ denote the common factor). Now we consider the last brackets in (27). In this brackets also, one part of terms turns into zero at once, but there are dangerous terms of this kind: $y^2\delta(x)\delta^{''}(y)$ and $x^2\delta(y)\delta^{''}(x)$. These dangerous terms are part of the brackets (27) in the following combination

$$[\cdots][D_2D_1y^2\delta(x)\delta^{''}(y) - D_1D_2x^2\delta(y)\delta^{''}(x)] \tag{28}$$

(Here brackets $[\cdots]$ denote the common factor). Now we use the formula

$$a(x)\delta^{''}(x) = a^{''}(0)\delta(x) - 2a^{'}(0)\delta^{'}(x) + a(0)\delta^{''}(x), \tag{29}$$

From this formula one can see, that dangerous terms have the form

$$D_1D_2\delta(x)\delta(y)\frac{d^2}{dy^2}y^2 - D_1D_2\delta(x)\delta(y)\frac{d^2}{dx^2}x^2 = 0 \tag{30}$$

and are cancelled in commutator (27). Others terms are obviously zero. Consequently, the Poisson brackets turn into zero for all singularities. According to the results of chapter 3, this guarantees that singularities do not move and that the dipole moment $\vec{D}$ is conserved. It is proved that the obtained solution of equation (5), is exact, stationary solution of 2D-Euler equation (2) with $n = 2$. Let us consider now the general case $n > 2$. In this case velocities (22), (23) contain the polynomials $P_{n-2}(z)$ and $\overline{P_{n-2}}(z)$ (19). It is clear now that the additional powers z or $\overline{z}$ in these polynomials generate in Poisson brackets zero terms only. The dangerous terms coincide only with the first term in the polynomial $P_{n-2}(z)$, i.e. unit. But these terms correspond to the case $n = 2$, and are already been considered. Hence, we prove that formulae in (6), together with the function $u_n(z)$ (18), give exact stationary solution of 2D-Euler equation with $n \geq 2$. In explicit form this solution has the form

$$\Psi = -\ln 8 - \ln\left|u^{'}(z)\right|^2 + 2\ln(1 + |u(z)|^2) = \tag{31}$$

$$= -\ln 8 + n\ln(x^2 + y^2) - 2\frac{D_1x + D_2y}{x^2 + y^2} +$$

$$+2\ln\left[1 + \frac{|P_{n-2}(z)|^2}{|D|^2(x^2 + y^2)^{(n-2)}}\exp\left(2\frac{D_1x + D_2y}{x^2 + y^2}\right)\right].$$

3. *Conclusions*

Exact localized solutions obtained in this work describe vortex structure of the complex form, where the singular point is surrounded be vortices satellites. With increasing of the number n the vortices satellites have tendency to form symmetrical necklaces. The existence of exact solutions with complex singularities itself is an important fact that is why in this work we contented ourself with consideration of simplest class of exact solutions expressed in elementary functions. We did not deal with questions of the construction of more complex solutions expressed by special functions and with important questions of stability of vortex configurations with complex singular points. All these questions must be examined separately and some of them will be studied in next works.

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Нові вихорові структури у двовимірній гідродинаміці

А.В.Тур, К.М.Кулик, В.В.Яновський

У роботі знайдено новий клас точних стаціонарних рішень двовимірних рівнянь Ейлера. Отримані розв'язки записуються через елементарні функції. Знайдені розв'язки мають вигляд складної особливої точки, яку оточує симетрична структура вихорових сателітів.