 

# Evolution of sine-Gordon equation kinks in the presence of spatial perturbations

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Using numerical methods, the influence of damping, external power and spatial modulations of parameters on Kinks of the modified sine-Gordon equation is investigated. Changes of the kink structure, velocity and width at the moment of intersection of localization field of the spatial modulations of parameters have been studied.

С помощью численных методов исследуется влияние затухания, внешней силы и пространственной модуляции параметров на кинки модифицированного уравнения синус-Гордона. Изучено изменение структуры, скорости и ширины кинка при пересечении области локализации пространственной модуляции параметров.

In the recent years, the dynamic of typological solitons (for example, kinks) attracts more and more the attention of researchers [1–4]. This is connected with the fact that, though solitons initially were appeared at the integrated system study, they began soon to be used also for the non-integrated systems that describe many physical applications [5]. For example, equation of sine-Gordon solitons describes in solid-state physics the domain wall (DW) in magnetics, dislocations in crystals, fluxions in Josephson contacts and transitions, etc. [3, 5–7]. In many cases, the soliton behavior can be described in the frame of point particle model, and then their temporal evolution will conform to simple differential equations. However, an account for perturbation influence results often in considerable change of soliton structure, and so they must be described as deformable particles.

Excitation of the soliton inner freedom degrees may be of great importance in some physical processes [8]. Such inner modes may include the translation and internal modes. The latter are believed to be associated with long-living oscillations of the soliton width [9]. It is known that the unperturbed sine-Gordon equation does not have inner modes. Today, the question — what perturbations can stimulate inner mode of sine-Gordon equation solitons — attracts a great attention. For instance, many works are devoted to the influence of time-depending heterogeneous outer force [9–11].

The spatial modulation (heterogeneity) of the system parameters also is a case of great interest [12]. In weakly heterogeneous case, those perturbations can be believed to do not change considerably the form of modified sine-Gordon equation (MSGE) solitons, influencing mainly their dynamics [12]. In strongly heterogeneous case, the MSGE solitons form must undergo strong changes, excitation of soliton modes and emission of excitations disengaging as free waves should be expected. The most interesting case is when the size of kink and the size characterizing the heterogeneity of the parameters are of the same order of magnitude, so the kink form must undergo strong changes when crossing the heterogeneous area [1]. The large perturbation influence on MSGE solution in general case can be studied as a rule only using numerical method. In this work, influence of damping, external force and space modulation of parameters on the MSGE kinks is investigated.

Let the kinks of the following MSGE be considered:

$$\frac{\partial^2\theta}{\partial\tilde{x}^2} - \frac{\partial^2\theta}{\partial\tilde{t}^2} - \frac{\tilde{K}}{2}\sin 2\theta = h\sin\theta + \alpha\frac{\partial\theta}{\partial\tilde{t}}, \quad (1)$$

where $\theta$ is a function of the dimensionless coordinate $\tilde{x}$ and dimensionless time $\tilde{t}$; $\tilde{K} = \tilde{K}(\tilde{x})$ is a certain function characterizing a local heterogeneity of system parameters, $h$ is a dimensionless parameter characterizing the external force magnitude; $\alpha$, a parameter characterizing dissipation in the system. It is to note that there are real physical systems that satisfy the selected equation (1). For example, domain walls in real ferro- and antiferromagnetics, that move under the influence of external magnetic field [7, 13]. Then, $\tilde{K}$ has a physical sense of uniaxial magnetic anisotropy and can take both positive and negative values.

Let the function $\tilde{K}$ be modeled by a step having the height or depth $\tilde{K}$, when constructing the perturbation theory for this equation [12]. For the cases of large $\tilde{K}$ and $\tilde{W}$ values considered in this work, the numerical method of iterations for the explicit scheme [14] will be used. The scheme of the realized numerical experiment is as follows. At the initial time moment, there is a static kink that is a solution of non-perturbated equation (1):

$$\theta_0(\tilde{x}) = 2\mathrm{arctg}(e^{\tilde{x}}) \qquad (2)$$

for which boundary conditions are $\theta(\pm\infty) = 0, \pi$. The presence of an external force results in the acceleration of the kink during some finite time to the stationary speed confirming at a high accuracy to the well-known formula [13]:

$$\tilde{v}_{limit} = \frac{\chi}{\sqrt{1+\chi^2}}, \qquad (3)$$

where $\chi = h/\alpha$. Then the kink accelerated to the stationary speed crosses the area of parameter $\tilde{K}$ heterogeneity and its evolution can be observed. The evolution study of the kink moving in a coasting manner at a constant speed in non-dissipative environment is also possible. All the results presented in this work were investigated for the cases of small dissipation ($\alpha = 10^{-2}$) and outer force.

Let us specify the function $\tilde{K}(\tilde{x})$ in the form of a step:

$$\tilde{K} = \begin{cases} 1, \tilde{x} \le \tilde{x}_0 \\ \tilde{K}, \tilde{x} > \tilde{x}_0 \end{cases}. \qquad (4)$$

The case $|1 - \tilde{K}| << 1$, $\alpha = h = 0$ is investigated using perturbation theory for the solitons in [12]. Let us consider from here on that $\tilde{K}$ can take arbitrary values both more and less than one.

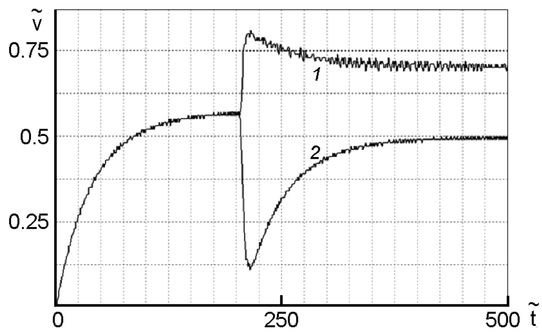


Fig. 1. Time dependence of the kink center speed ($1 - \tilde{K} = 0.5$; $2 - \tilde{K} = 1.5$), at the passage of the step region.

The calculation results show that at the moment when kink passes the step region, low amplitude waves appear spreading to the left and to the right therefrom. At the moment of passing the step localization area, the kink speed drops or rises abruptly, and then the speed goes to some other stationary speed (Fig. 1) corresponding the equation (3) ($\tilde{v}$ ($\tilde{K} = 0.5$) = 0.71, $\tilde{v}$ ($\tilde{K} = 1.5$) = 0.49). As the initial crossing speed of the step area increases, the kink speed change decreases. An increase of $|1 - \tilde{K}|$, in contract, causes an increased change of the speed.

In Fig. 2, the time dependences of the kink width are presented. The analytical kink width was calculated using the well-known formula:

$$\tilde{\delta} = \sqrt{1 - \tilde{v}^2}. \qquad (5)$$

It is seen in the Figure that at the stationary motion up to the step area, numerical and analytically calculated values coincide, but after the crossing of this area boundary, the difference is considerable. This is connected with the change of not only the motion speed but also of the kink structure. The presence of oscillations in kink width after the step area crossing indicates the excitation of inner oscillation modes, which can be referred to the "internal shape mode" in terms of modern non-integrated MSGE [9].

Comparison of Figs. 2a and 2b shows that in case $\tilde{K} = 0.5$, the internal shape mode amplitude is larger and the frequency is lower than in case $\tilde{K} = 1.5$. It is to note also that when passing the step localization area, $\delta_0(\tilde{K} \ne 1)$ in equation (5), generally speaking, differs from $\delta_0(\tilde{K} = 1)$. We do not take account of this fact while plotting the curve 2 in Fig. 2.

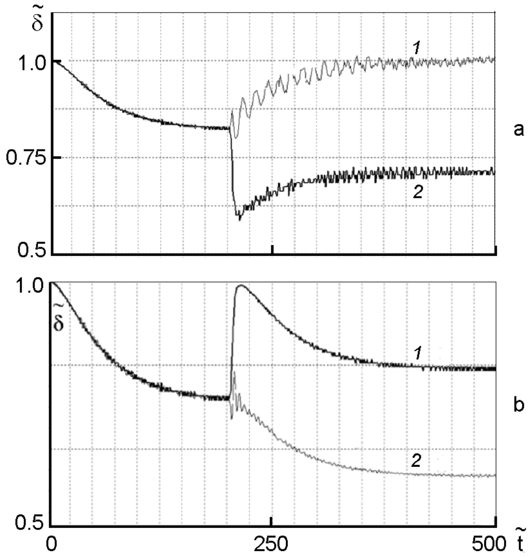


Fig. 2. Time dependences of kink width at the passage of the step region determined numerically (*1*) and analytically from equation (5) (*2*) at $\tilde{K}$ = 0.5 (a) and $\tilde{K}$ = 1.5 (b).

The coasting passing of the kink through the step localization area was also investigated. In this case, the numerical experiment scheme is as follows: when the kink was accelerated to the stationary speed, the outer force action was over and the damping was zeroed, then, after time $\Delta\tilde{t} = 20$, the step area appeared on the considerable distance from it. Comparing the results, a conclusion can be drawn that there is a qualitative difference from the case when the outer force and damping are present. In the case $\tilde{K} = 0.5$, one can state that in the presence of damping and outer force, there is a local maximum of kink center speed at the crossing of the step localization area $\tilde{v} = 0.83$, and then, the speed decreases and goes to the stationary value. In the case of coasting kink motion, the kink center speed increases sharply at the crossing moment of the step area and changes no more because of damping absence. For the case $\tilde{K} = 1.5$, the same is observed, but now the kink speed decreases after crossing the step localization area. It is to note that at the coasting DW motion, after the crossing the NCMA area, the speeds of the DW center take values $\tilde{v}$, $(\tilde{K} = 0.5) = 0.83$ and $\tilde{v}$, $(\tilde{K} = 1.5) = 0.22$ close to those defined by the analytical formula

$$\tilde{v}_{after} = \tilde{v}_{before} - \frac{1}{2}(\tilde{K} - 1)\left(\frac{1}{\tilde{v}_{before}} - \tilde{v}_{before}\right),$$

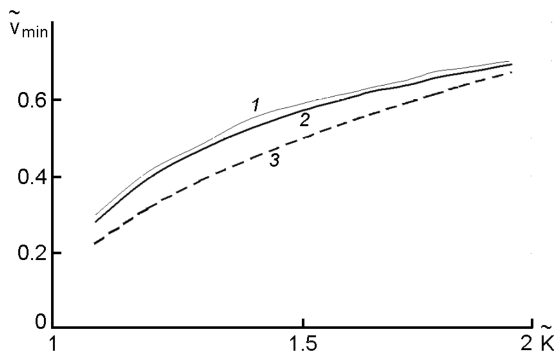


Fig. 3. Dependences of the minimum kink speed $\tilde{v}_{min}$ required to overcome the step region determined numerically for coasting motion (*1*) and for $h \neq 0$ (*2*) and analytically from equation (6) (*3*).

found in [12] using perturbation theory.

In this case, there is a minimal speed $\tilde{v}_{min}$ for $\tilde{K} > 1$ which is necessary to overcome the step area. Physically, this is connected with the fact that at the "barrier" crossing, a fraction of the kink kinetic energy should be spent for the potential energy increase, because of parameter changes. This minimal speed for the cases of motion under the influence of outer force and coasting was calculated (Fig. 4) compared with the analytical equation [12] correct for small values $(\tilde{K} - 1)$:

$$\tilde{v}_{\min} = \left(\frac{\tilde{K} - 1}{2}\right)^{1/2}. \quad (6)$$

In is seen from the Figure that the speeds (curves *1* and *2*) necessary to overcome the "barrier" area, differ very little in both cases; this evidences a weak dependence on external force and damping in our case.

In case $\tilde{v} < \tilde{v}_{min}$, $\tilde{K} > 1$ at coasting motion of the kink, it is reflected elastically from the step area and, as a result, it changes its motion direction to the opposite. At $h \neq 0$, $\alpha \neq 0$, the damping oscillations in the step area (or kink pinning) are possible. In addition, an emission in the form of low-amplitude waves is observed.

From the time dependences of kink center coordinate and speed for the investigated case, we got that the kink center reaches the step area border only at the initial time moment, all the following oscillations appear close to that area. The observed oscillations can be regarded as near to harmonic only after time $\Delta\tilde{t} \sim 200$.

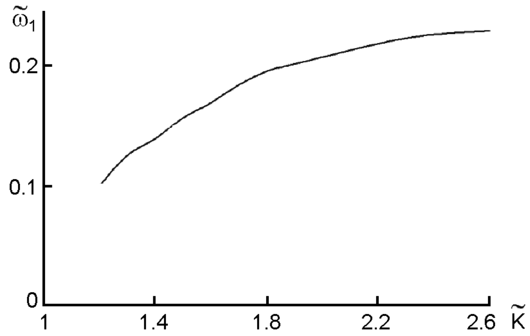


Fig. 4. Dependence of translation mode frequency of the kink oscillation on the step height $\tilde{K}$.

In Fig. 4, presented is the translation mode frequency $\tilde{\omega}_1$ of the kink fluctuation as a function of the step height $\tilde{K}$, calculated from the dependence $\tilde{x}(\tilde{t})$. It is seen from the Figure, first, that the dependence is close to $\omega_1(\tilde{K}) \sim \tilde{K}^{1/2}$, second, with the increase of $\tilde{K}$, the oscillation frequency increases and in the limiting value $\tilde{K} \to \infty$ it approaches $\omega \cong 0.25$. After the stop, the kinks with a structure "fixed" to the defect are observed, what is similar to the results of work.

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# Еволюція кінків рівняння синус-Гордона у присутності просторового збурення

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За допомогою числових методів досліджується вплив затухання, зовнішньої сили та просторової модуляції параметрів на кінки модифікованого рівняння синус-Гордона. Досліджено зміни структури, швидкості та ширини кінка при перетинанні області локалізації просторової модуляції параметрів.