

Analytic approach to the heat radiative conduction problem in semi-transparent media. The large optical length approximation

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The one dimensional stationary problem of heat radiative conduction is solved used Stefan-Boltzmann approximation in the case of optical semi-transparent media. The case of the specimen large optical length λ is investigated. In frameworks of the perturbation theory on the parameter λ^{-1}, the asymptotically exact formulas are obtained in main approximation. The nonlinear integral and differential equation for the temperature distribution in the specimen is derived. It is done on the basis of strict account of the radiation transfer in the geometric optics approximation.

Решается одномерная стационарная задача радиационно-кондуктивного теплообмена в т.н. сером приближении в случае оптически полупрозрачной среды. Рассмотрен случай большой оптической длины λ образца. В рамках теории возмущений по параметру λ^{-1} получены асимптотически точные формулы в главных приближениях. Выведено нелинейное интегро-дифференциальное уравнение для распределения температуры в образце на основе точного учёта переноса излучения в приближении геометрической оптики.

1. Introduction. The problems of the heat radiative conduction calculation are mainly solved by numerical methods in the physical literature (see, for example, [1, 2]). The main difficulty of the numerical analysis of such problems that distinguished their from standard boundary and initial boundary problems which are set in mathematical physics consists of the radiation transfer account. It is connected with the fact that the heat radiative conduction problem is decomposed by natural way into two different problems according with it mathematical formulation (see, Sec.2). First of them consists of the calculation of the radiative energy flux that is transferred in optical semi-transparent specimen at an arbitrary temperature distribution in it. In one-dimensional case, this problem is reduced to the calculation of the function $P(x)$, (x is the moving coordinate of the point in the specimen). It is equal to the energy that is flowed through the point x during the time unit with the account of the flow direction. If the specimen is optically uniform so the dependence on x in the flux $P(x)$ takes place only by the functional dependence of the energy flux on the temperature distribution $T(x)$ in the specimen. The calculation of the function $P(x)$ is realized on the basis of the kinetic equation of the radiation transfer in the geometric optics approximation with the account of boundary conditions for rays at the specimen boundary. This equation has the radiation source that depends on the temperature distribution. The formal solution of the radiation transfer equation with the account of boundary conditions leads to the system of integral equations for the function $P(x)$ determining it at an arbitrary temperature distribution (see, for example, monographs [3, 4] where these equations are given in the most general form). The evaluation of the function $P(x)$ on the basis of these equations leads to the second problem when the thermal conductivity equation with the source is solved. Such a source is performed by the divergence of the energy flux (it is the derivative $dP(x)/dx$ in one-dimensional case). This equation represents itself the evolution nonlinear integral and differential equation in general case. However, its solving is the standard problem of mathematical physics at the given initial and boundary conditions. This problem is sufficiently complicated and it have no the strict analytic solution with the exception of the trivial situation when the temperature distribution is constant at constant and uniform boundary conditions. At the same time, the described problem has the nonstandard form since the integral equations for the energy flux contain the unknown functional variable, i.e. the temperature distribution. In one-dimensional case, their solution is represented by a functional on $T(x)$ and some difficulties arises at the realization of their analytic solving. Moreover, it is desirable to solve the first problem by explicit way in frameworks of the analytic approach. In other case, some supplementary obstructions arise when the thermal conductivity equation is solved. The pointed out circumstances complicate essentially the solving of the heat radiative conduction problem in the framework analytic approach. Just due to this reason, such a problem is usually solved numerically. It is done by the algorithm construction of numerical procedure for the simultaneous solving of the radiation transfer equation and the thermal conductivity one (see, [3, 4, 5]).

In this work, in one-dimensional variant of the heat radiative conduction problem, it is succeeded to solve by relatively simple way the first problem of the above-described ones. Taking into account the radiation energy transfer, we determine the energy flux $P(x)$ as the functional $P[T(x)]$ on the temperature distribution $T(x)$ in the specimen and calculate the flux divergence of the radiation energy in the explicit form at an arbitrary temperature distribution. This gives the possibility to set and to solve in one-dimensional case some problems with different sources of the radiation energy.

In Sec.2 the function $P(x)$ is calculated in the one-dimensional specimen at an arbitrary temperature distribution. In Sec.3 the nonlinear thermal conductivity equation for the function $T(x,t)$ is derived.

2. The radiation transfer problem in the one-dimensional case.

Let there exists the distributed radiation source on the segment $[0, L]$. It has the intensity $P_0(y)$ in each point $y \in [0, L]$ where the function $P_0(y)/2$ represents physically the electromagnetic energy value relating to the unit cross-section area element being perpendicular to the segment. It may flow in both possible directions (on the left and on the right) during the unit time. The multiplier $1/2$ arises in the connection with the fact that, in the stationary case under consideration, the source irradiates uniformly in time with the identical intensity in both directions. We put that the distribution density of such irradiation sources on the unit length is equal α, i.e. the distribution measure on the segment with the length dy is equal αdy. Each ray moves uniformly into the segment with the light rate and, after the boundary attaining, it reflects with the probability (the reflection coefficient) r. Further, it continues the moving in the opposite direction. The ray motion, i.e. its part which remains into the specimen at each reflection of all subsequent reflections from boundaries is continued unboundedly.

We denote by $Q_\pm(s|y) \leq 1$ the contributions of the parts of electromagnetic energy having arisen in the point y to the common flux, i.e. the parts relative to the initial flux $P_0(y)$, which are taken place after rays went the distance s. Signs + (-) denote the contribution to the energy flux of the ray that is irradiated to the right (to the left) from the source in the point y. Thus, the rays irradiated out the point y in the directions $\pm$ which, have intensities $Q_\pm(s|y)$, after they pass the distance s, correspondingly. The energy value losing by them after the passage of the segment having the length ds, is equal $\alpha Q_\pm(s|y)ds$. It is done due the radiation absorption by the medium. The absorption coefficient α coincides with the irradiation coefficient which defines the distribution of sources. These coincidence is justified by the Kirchhoff law. It states that the radiation absorption intensity in each space point coincides with the intensity of its irradiation.

Functions $Q_\pm(s|y)$ satisfy the kinetic equation of the heat transfer

$$\frac{\partial Q_\pm}{\partial s} = -\alpha Q_\pm$$

at the initial condition $Q_\pm(0|y) = 1/2$ where the time is measured by units of distance having passed by the ray with the constant light rate. Therefore, $Q_\pm(s|y) = \exp(-\alpha s)/2$, if the ray have no reflections from segment boundaries. In general case, at the account of boundary reflections, we have

$$Q_\pm(s|y) = r^{n_\pm(s|y)} \exp(-\alpha s)/2$$

where $n_\pm(s|y)$ is the number of ray reflections from specimen boundaries. This ray is irradiated from the point y in the direction + (-) and passes the distance s. It takes place since the ray loses the part of intensity which is determined by the reflection coefficient $r \le 1$ at each boundary reflection.

Further, we denote by $P_\pm(x)$ the irradiation energy flux in the point x passing to the right and to the left, correspondingly. We consider the pair of these functions as the two-component vector $(P_+(x), P_-(x))$. At this time, the total energy flux $P(x)$ in the point x is equal to the difference of introduced fluxes

$$P(x) = P_+(x) - P_-(x)\,. \tag{1}$$

In stationary state, each of fluxes $P_\pm(x)$ represents the sum of fluxes $Q_\pm(s|y)$ of all rays irradiated from all points y in different directions with different number of reflections from specimen boundaries during time s. Then, the following conditions should be fulfilled for all rays when the point y is fixed: 1) rays come to the point x from the left/from the right; 2) the total distance length of each ray is determined on the basis of the conditions of its outcome from the point y and its income to the point x. In connection with this fact, we denote by $Q_{\mu\nu}(x, y)$ those parts of the irradiation energy flux which are transferred by rays outcoming from the point y in the direction ν and incoming to the point x moving in the direction μ. It is done for each fixed pair of points (x, y) and for each pair of signs (μ, ν) pointing out the directions $\mu, \nu = \pm$. The collection of introduced functions composes the 2×2-matrix with indexes μ, ν. We call it *the transfer matrix.*

Since, according to the definition of the matrix $Q_{\mu\nu}(x, y)$ (its elements describe the parts of the energy flux), it is necessary to integrate over all parts of the total energy flux from all points y and to sum over both irradiation directions $\nu = \pm$ in order to obtain the vector $(P_+(x), P_-(x))$, then it is connected with the irradiation intensity function $P_0(y)$ by the relation

$$P_\mu(x) = \frac{1}{2} \sum_{\nu=\pm} \alpha \int_0^L Q_{\mu\nu}(x, y) P_0(y) dy\,. \tag{2}$$

Therefore, on the basis of Eq.(1), the irradiation energy flux $P(x)$ in the point x is expressed by the following way

$$P(x) = \frac{1}{2}\alpha \sum_{\mu,\nu=\pm} \mu \int_0^L Q_{\mu\nu}(x, y) P_0(y) dy\,. \tag{3}$$

We calculate the matrix $Q_{\mu\nu}(x, y)$ by the direct recount of contribution parts of each ray but we does not solve the integral equations of the radiation transfer which are mentioned above in the introduction. It may be done due to the one-dimensional geometry of the problem under consideration.

Let $X_{\mu\nu}(s|y)$ be the moving point coordinate attained by the ray which has passed the distance s outcoming from the point y in the direction ν and incoming to the moving point going in the direction μ. This ray comes to the point x after the passing of distances equal to $s^{(1)}_{\mu\nu}, s^{(2)}_{\mu\nu}, \ldots$. They are determined as solutions of the equation

$$X_{\mu\nu}(s^{(i)}_{\mu\nu}|y) = x\,, \quad i = 1, 2, \ldots \tag{4}$$

and, therefore, they are the functions on x and y. But further, we do not denote explicitly this dependence.

Then, the functions $Q_{\mu,\nu}(x, y)$ are represented by the formula

$$Q_{\mu\nu}(x, y) = \sum_{i=0}^{\infty} Q_\nu(s^{(i)}_{\mu\nu}|y)\,. \tag{5}$$

Thus, for the calculation of the matrix $Q_{\mu\nu}(x, y)$, it is necessary to find the trajectories $X_{\mu\nu}(s|y)$.

Let us calculate the number $n_\mu(s|y)$. If the ray has been irradiated to the right, then, after $n_+(s|y)$ boundary reflections, we have

$$(L-y)+n_+(s|y)L < s < (L-y)+(n_+(s|y)+1)L\,.$$

Therefore, taking the integral part, we obtain

$$\left[\frac{1}{L}\,(s+y-L)\right] = n_+(s|y)\,. \tag{6}$$

Strictly by the same way, after $n_-(s|y)$ reflections from boundaries, we have for the ray which is irradiated to the left

$$y+(n_-(s|y)-1)L < s < y+n_-(s|y)L\,.$$

Therefore,

$$\left[\frac{1}{L}\,(s-y+L)\right] = n_-(s|y)\,. \tag{7}$$

Now, let us calculate the distances $s^{(i)}_{\mu\nu}$, $i=1,2,...$. The trajectories $X_{\mu\nu}(s|y)$ are periodical on s with the period $2L$. Due to this reason, each trajectory $X_{\mu\nu}(s|y)$ is built at $s<2L$ and, after that, it is continued periodically. We note that, for the calculation of values $s^{(i)}_{\mu\nu}$, it is important to know not the trajectories but the equations connected them with the distance s.

The function $X_{++}(s|y)$ is defined by the equality $X_{++}(s|y)=s+y$ if $s<L-y$. Further, the function $X_{++}(s|y)$ have no sense at $2L-y>s>L-y$. The periodical continuation of the function $X_{++}(s|y)$ from the region $s<L-y$ for s satisfying the condition $2nL-y<s<(2n+1)L-y$, $n=1,2,...$ gives the equation

$$X_{++}(s|y)+(2n-1)L+(L-y)=s\,. \tag{8}$$

The function $X_{++}(s|y)$ is not defined for s satisfying the condition $(2n+1)L-y<s<2(n+1)L-y$, $n=0,1,2,...$.

The function $X_{-+}(s|y)$ does not exist at $s<L-y$ and, therefore, it have no sense at arbitrary shifts of this region which are equal to $2Ln$, $n=1,2,...$, i.e. at $2nL-y<s<(2n+1)L-y$. Otherwise, it has the sense at $2L-y>s>L-y$ and it is defined by the equation

$$(L-X_{-+}(s|y))+(L-y)=s\,,\quad X_{-+}(s|y)=2L-y\,.$$

In this case, due to the periodicity, we have the equation

$$(L-X_{-+}(s|y))+(L-y)+2nL=s \tag{9}$$

at $L-y+(2n+1)L>s>(2n+1)L-y$, $n=0,1,2,...$.

The functions $X_{--}(s|y)$, $X_{+-}(s|y)$ are calculated by the analogous way. If $s<y$, then

$$X_{--}(s|y)=y-s$$

and the function $X_{+-}(s|y)$ have no sense.

At the shift on $2nL$, we have the equation for the function $X_{--}(s|y)$,

$$(L-X_{--}(s|y))+y+(2n-1)L=s \tag{10}$$

for all s satisfying the inequality $y+(2n-1)L<s<2nL+y$, $n=1,2,...$.

Vice versa, the function $X_{+-}(s|y)$ has the sense at the condition $y+2nL<s<y+(2n+1)L$, $n=0,1,2,...$ and it satisfies the equation

$$y+2nL+X_{+-}(s|y)=s\,. \tag{11}$$

It have no sense at $y+(2n-1)L<s<y+2nL$, $n=1,2,...$.

Thus, it follows from the fulfilled analysis that Eq.(4) has the following solutions. For the simplicity, we do not point out explicitly those variables from which the reflection numbers n_ν depend on. Putting that $X_{++}(s|y)$ is equal to x in Eq.(8), we have the expression

$$s_{++}^{(n+1)} = 2nL + x - y\,, \quad n_+ = 2n,\ n = 0,1,2,...\,, \tag{12}$$

for values $s_{++}^{(i)}$, $i = 1,2,...$ at $x > y$ and at $x < y$ $s_{++}^{(n)} = 2nL + x - y\,,\quad n_+ = 2n,\ n = 1,2,...\,,$ correspondingly. By the analogous way, we obtain

$$s_{-+}^{(n+1)} = 2(n+1)L - x - y\,, \quad n_- = 2n+1,\ n = 0,1,2,... \tag{13}$$

from Eq.(9) at any relation between x and y. From Eq.(10), we find

$$s_{--}^{(n+1)} = 2nL + y - x\,, \quad n_- = 2n,\ n = 0,1,2,... \tag{14}$$

at $x < y$ and $s_{--}^{(n)} = 2nL + y - x\,,\quad n_- = 2n,\ n = 0,1,2,...$ at $x > y$, correspondingly. From Eq.(11), we obtain the expression

$$s_{+-}^{(n+1)} = y + 2nL + x\,, \quad n_+ = (2n+1)\,,\ n = 0,1,2,... \tag{15}$$

for the function $s_{+-}^{(n)}$ at any relation between x and y.

Now, we may to calculate the transfer matrix $Q_{\mu\nu}(x,y)$. According to the definition (5), we have

$$Q_{\mu\nu}(x,y) = \sum_{i=1}^{\infty} Q_\nu(s_{\mu\nu}^{(i)}|y) = \sum_i r^{n_{\mu\nu}^{(i)}} e^{-\alpha s_{\mu\nu}^{(i)}}\,,$$

where values $n_{\mu\nu}^{(i)} = n_\nu(s_{\mu\nu}^{(i)}|y)$, $i = 1,2,...$ are given by formulas (12)-(15). Substituting the corresponding expressions and producing summations, we obtain

$$Q_{++}(x,y) = \theta(x-y)e^{-\alpha(x-y)} + \sum_{m=1}^{\infty} r^{2m} e^{-\alpha(2mL+x-y)} =$$

$$= e^{-\alpha(x-y)}\left[\theta(x-y) + r^2 e^{-2\alpha L}(1 - r^2 e^{-2\alpha L})^{-1}\right]\,, \tag{16}$$

$$Q_{-+}(x,y) = \sum_{m=0}^{\infty} r^{2m+1} e^{-\alpha(2(m+1)L-x-y)} =$$

$$= re^{-\alpha(2L-x-y)}(1 - r^2 e^{-2\alpha L})^{-1}\,, \tag{17}$$

$$Q_{--}(x,y) = \theta(y-x)e^{-\alpha(y-x)} + \sum_{m=1}^{\infty} r^{2m} e^{-\alpha(2mL+y-x)} =$$

$$= e^{-\alpha(y-x)}\left[\theta(y-x) + r^2 e^{-2\alpha L}(1 - r^2 e^{-2\alpha L})^{-1}\right]\,, \tag{18}$$

$$Q_{+-}(x,y) = \sum_{m=0}^{\infty} r^{2m+1} e^{-\alpha(2mL+x+y)} = re^{-\alpha(x+y)}(1 - r^2 e^{-2\alpha L})^{-1}\,, \tag{19}$$

where $\theta(\cdot)$ is the Heaviside function.

On the basis of the calculated matrix elements, we count the kernel of the integral transformation

$$Q(x,y) = \frac{1}{2}\sum_{\mu,\nu=\pm} \mu Q_{\mu\nu}(x,y) = \frac{1}{2}\left(Q_{++} - Q_{--} + Q_{+-} - Q_{-+}\right)(x,y) =$$

$$= \frac{1}{2}\mathrm{sgn}(x-y)e^{-\alpha|x-y|} + \frac{re^{-\alpha L}}{1 - r^2 e^{-2\alpha L}}\left[\mathrm{sh}\alpha(L-x-y) + re^{-\alpha L}\mathrm{sh}\alpha(x-y)\right]\,. \tag{20}$$

This kernel defines according to Eq.(3) the expression of the desired irradiation energy flux in each space point x in dependence on the temperature distribution $T(y)$ in the specimen,

$$P(x) = \alpha \int_0^L Q(x, y) P_0(y) dy \,, \tag{21}$$

where $P_0(y)$ is the functional on $T(y)$, $P_0(y) = P_0[T(y)]$.

3. The problem setting of the heat radiative conduction. After the solving of the radiation transfer problem, to set the problem of the self-consistent determination of the temperature distribution in the specimen which is taken place due to thermal conductivity and radiation transfer, it is necessary to reformulate the evolution equation of the temperature distribution at the given radiation energy flux. We suppose that such an equation for the instant temperature distribution $T(x,t)$ is the thermal conductivity equation with the source having the form of the energy flux divergence [3]. In the one-dimensional case under consideration, it has the form

$$\rho c \frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2} - \frac{\partial P(x)}{\partial x} \tag{22}$$

where c is the specific heat of the mass unit, ρ is the density of the material, κ is the thermal conductivity coefficient. On the basis of Eq.(22), the balance principle has been found. It is supposed that the outcome of the energy flux in each space point is spent on the local increase of the internal energy in this point and the thermal conductivity process sends away the heat from it.

To turn the equation into the self-consistent one, it is necessary to define the functional $P_0[T]$. The form of this functional is determined by the optical properties of the specimen material. Further, we analyze the simplest model, i.e. such an approximation, when it is put that $P_0[T] = \sigma T^4$, where σ is the Stefan-Boltzmann constant.

In the one-dimensional case, the heat radiative conductivity problem consists of the solving of the initial boundary problem of the Eq.(22) at the fixed temperature boundary conditions at specimen endpoints (but not at the fixed energy flux when the energy transfer is found). In this work the boundary conditions are confined in the form of the temperature constancy at the endpoints, i.e. those values $T_- = T(0)$, $T_+ = T(L)$ are fixed.

At the equilibrium state, Eq.(22) has the form

$$\kappa \frac{d^2 T}{dx^2} = \frac{dP(x)}{dx}$$

or, taking into account our irradiation model,

$$\kappa \frac{d^2 T}{dx^2} = \alpha \sigma \frac{d}{dx} \int_0^L Q(x, y) T^4(y) dy \,. \tag{23}$$

This equation has the first integral

$$\kappa \frac{dT}{dx} = \alpha \sigma \int_0^L Q(x, y) T^4(y) dy + C \tag{24}$$

with an indefinite constant $C = \text{const}$. But one may calculate this constant on the basis of the boundary conditions only after the construction of the general solution of Eq.(24). This fact is inconvenient when the formulated boundary problem is solved. However, in the frameworks of a perturbation theory, one may find this constant together with the functional dependence $T(y)$.

4. The large absorption approximation. In this work we use the large absorption approximation for the equilibrium state calculation. However, the subsequent solving of this problem in the form of the asymptotic series on inverse α powers or, that is equivalent, on inverse powers of the optical length

$\lambda = \alpha L$, reduces to some tedious calculations. Therefore, we simplify the problem in this work and we have calculated first terms of the decomposition pointed out up to α^{-2} inclusively using the supposition of the reflection coefficient smallness. This supposition permits to simplify the kernel $Q(x,y)$. Namely, one may be restricted only by one term

$$P(x) = \frac{\alpha\sigma}{2}\int_0^L \mathrm{sgn}(x-y)e^{-\alpha|x-y|}T^4(y)dy$$

in the flux expression at $r \ll 1$. As a result, we obtain the integral and differential equation

$$\kappa\frac{d^2T}{dx^2} = \frac{\alpha\sigma}{2}\frac{d}{dx}\int_0^L \mathrm{sgn}(x-y)e^{-\alpha|x-y|}T^4(y)dy\,.$$

We differentiate explicitly the right-hand side of it,

$$\frac{d}{dx}\int_0^L \mathrm{sgn}(x-y)e^{-\alpha|x-y|}T^4(y)dy = 2T^4(x) - \alpha\int_0^L e^{-\alpha|x-y|}T^4(y)dy\,.$$

Then, we obtain the equation for the determination of the equilibrium temperature distribution in the form

$$\kappa\frac{d^2T}{dx^2} = \alpha\sigma T^4(x) - \frac{1}{2}\alpha^2\sigma\int_0^L e^{-\alpha|x-y|}T^4(y)dy\,. \quad (25)$$

Now, we build the solution of Eq.(25) in the form of asymptotic decomposition on the inverse absorption coefficient. From the formal mathematical point of view, the building of such a decomposition corresponds to the study of the temperature distribution in the limit $\alpha \to \infty$.

We decompose the following integral on the powers α^{-1},

$$\int_0^L e^{-\alpha|x-y|}T^4(y)dy = \alpha^{-1}\int_{-\alpha x}^{\alpha(L-x)} e^{-|y|}T^4(\alpha^{-1}y + x)dy\,.$$

For this, we substitute the following decomposition

$$T^4(\alpha^{-1}y + x) = T^4(x) + \frac{y}{\alpha}\left(T^4(x)\right)' + \frac{y^2}{2\alpha^2}\left(T^4(x)\right)'' + \frac{y^3}{6\alpha^3}\left(T^4(x)\right)''' + O(\alpha^{-4})\,.$$

As a result, we obtain

$$\int_0^L e^{-\alpha|x-y|}T^4(y)dy == \frac{2}{\alpha}T^4(x) + \frac{2}{\alpha^3}\left(T^4(x)\right)'' + O(\alpha^{-5})\,.$$

Here, the exponentially small terms which are connected with the continuation of the integration region to the total axe, are thrown off. Such a neglecting is not justified, generally speaking, at the neighborhoods of segment endpoints. Besides, we have taken into account that integrals with odd powers are equal to zero. The substitution of this decomposition into the equation gives

$$\kappa\frac{d^2T}{dx^2} = -\frac{\sigma}{\alpha}\frac{d^2}{dx^2}T^4(x) + O(\alpha^{-3})\,. \quad (26)$$

We get this equation from Eq.(25) up to α^{-3}. Then, it may be used for the finding of stationary temperature distribution only up to α^{-2} inclusively. We have

$$\kappa T(x) = -\frac{\sigma}{\alpha}T^4(x) + Cx + D \quad (27)$$

after the integration of Eq.(26) where C and D are constant. Substitution of the boundary conditions gives the equations for the determination of these constants,

$$\kappa T_- + \frac{\sigma}{\alpha} T_-^4 = D\,, \quad \kappa T_+ + \frac{\sigma}{\alpha} T_+^4 = CL + D\,.$$

Therefore,

$$C = \frac{\kappa}{L}(T_+ - T_-) + \frac{\sigma}{\alpha L}\left(T_+^4 - T_-^4\right) > 0\,.$$

Thus, we determine Eq.(27) for the stationary temperature distribution completely. It represents the algebraic equation of fourth degree. Since it is obtained up to $O(\alpha^{-2})$ then it is sufficiently to solve the equation with the same accuracy. In zero approximation, we obtain

$$T^{(0)}(x) = \frac{x}{L}(T_+ - T_-) + T_-\,. \tag{28}$$

since the constants C and D have both zero order terms and terms being proportional α^{-1}.

For the obtaining of the next approximations, we substitute the decomposition

$$T(x) = T^{(0)}(x) + \alpha^{-1}T^{(1)}(x) + \alpha^{-2}T^{(2)}(x) + O(\alpha^{-3})\,. \tag{29}$$

in Eq.(27). Selecting the equal order terms on α^{-1} in the obtained expression, we find

$$T^{(1)}(x) = \frac{\sigma}{\kappa}\left[T_-^4 + \frac{x}{L}\left(T_+^4 - T_-^4\right) - [T^{(0)}(x)]^4\right]\,. \tag{30}$$

since C and D contain terms $\sim \alpha^{-1}$. In the next approximation, the constants C and D have no terms $\sim \alpha^{-2}$ and, therefore, we obtain from Eq.(27) the expression

$$T^{(2)}(x) = -\left(\frac{2\sigma}{\kappa}\right)^2 [T^{(0)}(x)]^3 \left[T_-^4 + \frac{x}{L}\left(T_+^4 - T_-^4\right) - [T^{(0)}(x)]^4\right]\,. \tag{31}$$

We must give one remark relative to the qualitative property of the distribution $T(x)$ in used approximation. From Eq.(26), differentiating of the right-hand side, we find that

$$\left(\kappa + \frac{\sigma}{\alpha}\right)\frac{d^2T}{dx^2} = -12\frac{\sigma}{\alpha}T^2\left(\frac{dT}{dx}\right)^2 < 0\,.$$

Thus, the temperature distribution in the specimen is concave it is and increasing since $dT^{(0)}(x)/dx > 0$.

5. Conclusion. In this work we have investigated the heat radiative conduction problem in the unbounded layer of semi-transparent medium. Such a problem is reduced effectively to the problem in the one-dimensional specimen. Solving the last problem, we obtain on the basis of the strict solution of the radiation transfer problem the integral and differential equation for the equilibrium temperature distribution determination. We restrict our calculation by large absorption approximation at the evaluation of this distribution. For the simplicity, we use the smallness of the reflection coefficient. In order to refuse this assumption when the decomposition on powers of the inverse value λ^{-1} of the optical length is built and when the decomposition on powers of small λ is built, it is necessary the reformulation of boundary problem for the reconstruction of integral and differential equation to the equivalent integral equation. It is supposed to do in the next publication.

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Аналітичний підхід до проблеми радіаційно-кондуктивного теплообміну у напівпрозорих середовищах. Наближення великої оптичної довжини

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Розв'язується одновимірна стаціонарна проблема радіаційно-кондуктивного теплообміну у т.з. сірому наближенні у випадку оптично напівпрозорого середовища. Розлянуто випадок великої оптичної довжини λ зразка. В рамках теорії збурень за параметром λ^{-1}, одержано асимптотичні формули у головному наближенні. Виведено нелінійне інтегро-диференціальне рівняння для розподілу температури у зразку на основі точного обліку переносу випромінювання у наближенні геометрічної оптики.