ANOMALOUS TRANSPORT IN VELOCITY SPACE: EQUATION AND MODELS

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The problem of anomalous diffusion in the momentum space is considered on the basis of the appropriate probability transition function (PTF). A new general equation for the description of the diffusion of heavy particles in a gas of light particles is formulated on the basis of the new approach similar to one in the coordinate space [1]. The obtained results allow one to describe the various situations where the probability transition function has a long tail in the momentum space. The effective friction and diffusion coefficients are determined.

1. Introduction

Interest in anomalous diffusion is conditioned by a large variety of applications: semiconductors, polymers, some granular systems, plasmas under specific conditions, various objects in biological systems, physico-chemical systems, etc.

The deviation from a linear-in-time dependence $\langle r^2(t)\rangle \sim t$ of the mean square displacement has been experimentally observed, in particular, under essentially non-equilibrium conditions or for some disordered systems. The average square separation of a pair of particles passively moving in a turbulent flow grows, according to Richardson's law, with the third power of time [2]. For the diffusion typical of glasses and related complex systems [3], the observed time dependence is slower than a linear one. These two types of anomalous diffusion are obviously characterized as superdiffusion $\langle r^2(t)\rangle \sim t^\alpha$ ($\alpha > 1$) and subdiffusion ($\alpha < 1$) [4]. For the description of these two diffusion regimes, a number of effective models and methods have been suggested. The continuous time random walk (CTRW) model of Scher and Montroll [5], leading to the strongly subdiffusive behavior, provides a basis for understanding the photoconductivity in strongly disordered and glassy semiconductors. The Levy-flight model [6], leading to the superdiffusion, describes various phenomena as the self-diffusion in micelle systems [7] and reactions and transport in polymer systems [8] and is applicable even to the stochastic description of financial market indices [9]. For both cases, the so-called fractional differential equations in the coordinate and time spaces are applied as an effective approach [10].

However, a more general approach has been suggested recently in [1,11] which avoids the fractional differentiation, reproduces results of the standard fractional differentiation method, when it is applicable, and allows one to describe the more complicated cases of anomalous diffusion processes. In [12], these approach has been applied also to the diffusion in a time-dependent external field.

In this paper, the problem of anomalous diffusion in the momentum (velocity) space will be considered. In spite of a formal similarity, the diffusion in the momentum space is very different physically from that in the coordinate space. It is clear, because the momentum conservation, which takes place in the momentum space, has no analogy in the coordinate space.

Some aspects of the anomalous diffusion in the velocity space have been investigated for the last decade in a few papers [13–16]. On the whole, comparing with the anomalous diffusion in the coordinate space, the anomalous diffusion in the velocity space is poorly studied. The consequent way to describe the anomalous diffusion in the velocity space is, to the best of our knowledge, still absent.

In this paper, a new kinetic equation for the anomalous diffusion in the velocity space is derived (see also [17]) on the basis of the appropriate expansion of a PTF (in the spirit of the approach suggested in [1] for the diffusion in the coordinate space), and some particular problems are investigated on this basis.

The diffusion in the velocity space for the cases of the normal and anomalous behaviors of a PTF is presented in Section 2. Starting from the argumentation based on a PTF of the Boltzmann type, we derive a new kinetic equation which can be applied, in fact, to a wide class of PTFs. The particular cases of anomalous diffusion for collisions of hard spheres with the specific power-type prescribed distribution function of the light particles are analyzed in Section 3. The universal character of the anomalous diffusion in the velocity space is absent, by contrast with one in the coordinate space (when the anomalous diffusion exists for the PTFs power-dependent for all distances). The more general examples of the anomalous diffusion are considered as well. In Section 4, the short review of the anomalous diffusion in the coordinate space is presented.

2. Diffusion in the Velocity Space on the Basis of a Master-Type Equation

Let us consider now the main problem formulated in Introduction, namely, the diffusion in the velocity space (V-space) on the basis of a respective master equation which describes the balance of grains coming in and out a point $\mathbf{p}$ at the time moment t. The structure of this equation is formally similar to the master equation (41) in the coordinate space

$$\frac{df_g(\mathbf{p},t)}{dt} = \int d\mathbf{q}\times$$

$$\times \left\{W(\mathbf{q},\mathbf{p}+\mathbf{q})f_g(\mathbf{p}+\mathbf{q},\mathbf{t}) - W(\mathbf{q},\mathbf{p})f_g(\mathbf{p},t)\right\}. \quad (1)$$

Of course, there is no conservation law for the coordinate space, similar to that in the momentum space. The probability transition $W(\mathbf{p},\mathbf{p}')$ describes the probability for a grain with momentum $\mathbf{p}'$ (point $\mathbf{p}'$) to transfer from this point $\mathbf{p}'$ to the point $\mathbf{p}$ per unit time. The momentum transferred is equal to $\mathbf{q}=\mathbf{p}'-\mathbf{p}$. Assuming at the beginning that the characteristic momenta $\mathbf{q}$ are small, one may expand Eq. (1) and arrive at an equation of the Fokker–Planck type for the density distribution $f_g(\mathbf{p},t)$:

$$\frac{df_g(\mathbf{p},t)}{dt} = \frac{\partial}{\partial p_\alpha}\left[A_\alpha(\mathbf{p})f_g(\mathbf{p},t) + \frac{\partial}{\partial p_\beta}\left(B_{\alpha\beta}(\mathbf{p})f_g(\mathbf{p},t)\right)\right]. \quad (2)$$

$$A_\alpha(\mathbf{p}) = \int d^s q q_\alpha W(\mathbf{q},\mathbf{p});$$

$$B_{\alpha\beta}(\mathbf{p}) = \frac{1}{2}\int d^s q q_\alpha q_\beta W(\mathbf{q},\mathbf{p}). \quad (3)$$

The coefficients A_α and $B_{\alpha\beta}$ describe the friction force and the diffusion, respectively, and s is the dimension of the momentum space.

Because the velocity of heavy particles is small, the $\mathbf{p}$-dependence of the PTF can be neglected in the calculation of the diffusion, which is constant in this case, $B_{\alpha\beta} = \delta_{\alpha\beta}B$, where B is the integral

$$B = \frac{1}{2s}\int d^s q q^2 W(q). \quad (4)$$

By neglecting the $\mathbf{p}$-dependence of the PTF at all, we arrive to the coefficient $A_\alpha = 0$ (while the diffusion coefficient is constant). As is well known, this neglect is wrong, and the coefficient A_α for the Fokker–Planck equation can be determined by using the argument that the stationary distribution function is Maxwellian. On this way, we arrive at the standard form of the coefficient $MTA_\alpha(p) = p_\alpha B$ which is one of the forms of the Einstein relation. For the systems far from equilibrium, this argument is not acceptable.

To find the coefficients in the kinetic equation which are applicable also to slowly decreasing PTFs, let us use a more general way based on the difference of the velocities of light and heavy particles. To calculate the function A_α, we take the fact into account that the function $W(\mathbf{q},\mathbf{p})$ is scalar and depends on $q, \mathbf{q}\cdot\mathbf{p}, p$. Expanding $W(\mathbf{q},\mathbf{p})$ in $\mathbf{q}\cdot\mathbf{p}$, we obtain the approximate representation of the functions $W(\mathbf{q},\mathbf{p})$ and $W(\mathbf{q},\mathbf{p}+\mathbf{q})$:

$$W(\mathbf{q},\mathbf{p}) \simeq W(q) + \tilde{W}'(q)(\mathbf{q}\cdot\mathbf{p}) + \frac{1}{2}\tilde{W}''(q)(\mathbf{q}\cdot\mathbf{p})^2, \quad (5)$$

$$W(\mathbf{q},\mathbf{p}+\mathbf{q}) \simeq W(q) + \tilde{W}'(q)\,(\mathbf{q}\cdot\mathbf{p}) +$$

$$+\frac{1}{2}\tilde{W}''(q)(\mathbf{q}\cdot\mathbf{p})^2 + q^2\tilde{W}'(q), \quad (6)$$

where $\tilde{W}'(q) \equiv \partial W(q,\mathbf{q}\cdot\mathbf{p})/\partial(\mathbf{qp})\mid_{\mathbf{q}\cdot\mathbf{p}=0}$ and $\tilde{W}''(q) \equiv \partial^2 W(q,\mathbf{q}\cdot\mathbf{p})/\partial(\mathbf{qp})^2\mid_{\mathbf{q}\cdot\mathbf{p}=0}$.

Then, with the necessary accuracy, A_α equals

$$A_\alpha(\mathbf{p}) = \int d^s q q_\alpha q_\beta p_\beta \tilde{W}'(q) =$$

$$= p_\alpha \int d^s q q_\alpha q_\alpha \tilde{W}'(q) = \frac{p_\alpha}{s}\int d^s q q^2 \tilde{W}'(q). \quad (7)$$

If the equality $\tilde{W}'(q) = W(q)/2MT$ is fulfilled for the function $W(\mathbf{q},\mathbf{p})$, then we get the usual Einstein relation

$$MTA_\alpha(\mathbf{p}) = p_\alpha B. \quad (8)$$

Let us check this relation for Boltzmann collisions which are described by the PTF $W(\mathbf{q},\mathbf{p}) = w_{\rm B}(\mathbf{q},\mathbf{p})$ [11]:

$$w_{\rm B}(\mathbf{q},\mathbf{p}) = \frac{2\pi}{\mu^2 q}\int_{q/2\mu}^{\infty} du\, u\,\frac{d\sigma}{do}\times$$

$$\times \left[\arccos\left(1-\frac{q^2}{2\mu^2 u^2}\right), u\right] f_b(u^2+v^2-\mathbf{q}\cdot\mathbf{v}/\mu), \qquad (9)$$

where ($\mathbf{p}=M\mathbf{v}$) and $d\sigma/do$ and f_b are, respectively, the differential scattering cross-section and the distribution function for light particles. For the equilibrium Maxvellian distribution f_b^0, the equality $\tilde{W}'(q)=W(q)/2MT$ is evident, and we obtain the usual Fokker–Planck equation in the velocity space with the constant diffusion $D\equiv B/M^2$ and friction $\beta\equiv B/MT=DM/T$ coefficients which satisfy the Einstein relation.

The evident generalization of the PTF $w_{\mathrm{B}}^d(\mathbf{q},\mathbf{p})$ to the case of a driven (characterized by the drift velocity $\mathbf{u_d}$) distribution function ($\mathbf{p}=M\mathbf{v}$) reads

$$w_{\mathrm{B}}^d(\mathbf{q},\mathbf{p})=\frac{2\pi}{\mu^2 q}\int\limits_{q/2\mu}^{\infty} duu\cdot\frac{d\sigma}{do}\left[\arccos\left(1-\frac{q^2}{2\mu^2u^2}\right),u\right]\times$$

$$\times f_b(u^2+(\mathbf{v}-\mathbf{u_d})^2-\mathbf{q}\cdot(\mathbf{v}-\mathbf{u_d})/\mu). \qquad (10)$$

and will be considered in details separately.

For some non-equilibrium situations, the PTF can possess a long tail. In this case, we have derive a generalization of the Fokker–Planck equation in spirit of the above consideration for the coordinate case, because the diffusion and friction coefficients in the form Eqs. (4) and (7) diverge for large q if the functions have the asymptotic behavior $W(q)\sim 1/q^{\alpha}$ with $\alpha\leq s+2$ and (or) $\tilde{W}'(q)\sim 1/q^{\beta}$ with $\beta\leq s+2$.

Let us insert the expansions for W in Eq. (1) (as an example, we choose $s=3$; arbitrary s can be considered in a similar way). With the necessary accuracy, we find

$$\frac{df_g(\mathbf{p},t)}{dt}=\int d\mathbf{q}\{f_g(\mathbf{p}+\mathbf{q},t)[1+q_\alpha\partial/\partial p_\alpha]\times$$

$$\times[W(q)+\tilde{W}'(q)\,(\mathbf{q}\cdot\mathbf{p})+\frac{1}{2}\tilde{W}''(q)(\mathbf{q}\cdot\mathbf{p})^2]-f_g(\mathbf{p},t)\times$$

$$\times[W(q)+\tilde{W}'(q)\,(\mathbf{q}\cdot\mathbf{p})+\frac{1}{2}\tilde{W}''(q)(\mathbf{q}\cdot\mathbf{p})^2]\}. \qquad (11)$$

After the Fourier transformation $f(\mathbf{r})=\int\frac{d\mathbf{p}}{(2\pi)^3}\exp(i\mathbf{pr})f(\mathbf{p},t)$, Eq. (11) reads

$$\frac{df_g(\mathbf{r},t)}{dt}=\int d\mathbf{q}\{\exp(-i(\mathbf{qr})[W(q)-i\tilde{W}'(q)\,(\mathbf{q}\cdot\frac{\partial}{\partial\mathbf{r}})+$$

$$-\frac{1}{2}\tilde{W}''(q)(\mathbf{q}\cdot\frac{\partial}{\partial\mathbf{r}})^2]-[W(q)-i\tilde{W}'(q)\,(\mathbf{q}\cdot\frac{\partial}{\partial\mathbf{r}})-$$

$$-\frac{1}{2}\tilde{W}''(q)(\mathbf{q}\cdot\frac{\partial}{\partial\mathbf{r}})^2]\}f_g(\mathbf{r},t). \qquad (12)$$

We can rewrite this equation as

$$\frac{df_g(\mathbf{r},t)}{dt}=A(r)f(\mathbf{r})+$$

$$+B_\alpha(r)\frac{\partial}{\partial\mathbf{r}_\alpha}f(\mathbf{r},t)+C_{\alpha\beta}(r)\frac{\partial^2}{\partial\mathbf{r}_\alpha\partial\mathbf{r}_\beta}f(\mathbf{r},t), \qquad (13)$$

where

$$A(r)=\int d\mathbf{q}[\exp(-i(\mathbf{qr}))-1]W(q)=$$

$$=4\pi\int\limits_0^{\infty}dqq^2\left[\frac{\sin(qr)}{qr}-1\right]W(q), \qquad (14)$$

$$B_\alpha\equiv r_\alpha B(r);$$

$$B(r)=-\frac{i}{r^2}\int d\mathbf{q}\mathbf{q}\mathbf{r}[\exp(-i(\mathbf{qr}))-1]\tilde{W}'(q)=$$

$$=\frac{4\pi}{r^2}\int\limits_0^{\infty}dqq^2\left[\cos(qr)-\frac{\sin(qr)}{qr}\right]\tilde{W}'(q), \qquad (15)$$

$$C_{\alpha\beta}(r)\equiv r_\alpha r_\beta C(r)=$$

$$=-\frac{1}{2}\int d\mathbf{q}q_\alpha q_\beta[\exp(-i(\mathbf{qr}))-1]\tilde{W}''(q), \qquad (16)$$

$$C(r)=-\frac{1}{2r^4}\int d\mathbf{q}(\mathbf{qr})^{\mathbf{2}}[\exp(-i(\mathbf{qr}))-1]\tilde{W}''(q)=$$

$$=\frac{2\pi}{r^2}\int\limits_0^{\infty}dqq^4\left[\frac{2\sin(qr)}{q^3r^3}-\frac{2\cos(qr)}{q^2r^2}-\frac{\sin(qr)}{qr}+\frac{1}{3}\right]\tilde{W}''(q). \qquad (17)$$

For the isotropic function $f(\mathbf{r})=f(r)$, one can rewrite Eq. (13) in the form

$$\frac{df_g(r,t)}{dt}=A(r)f(r)+B(r)r\frac{\partial}{\partial r}f(r)+C(r)r^2\frac{\partial^2}{\partial r^2}f(r). \qquad (18)$$

In the case of a strongly decreasing PDF, the exponent under the integrals for the functions $A(r)$, $B(r)$, and $C(r)$ can be expanded as

$$A(r) \simeq= -\frac{r^2}{6}\int d\mathbf{q}\, q^2 W(q),$$

$$B(r) \simeq -\frac{1}{3}\int d\mathbf{q}\, q^2 \tilde{W}'(q);\ C(r) \simeq 0. \qquad (19)$$

Then the simplified kinetic equation in the case of a PTF which is short-range in the q variable (non-equilibrium, in general case) reads

$$\frac{df_g(r,t)}{dt} = A_0 r^2 f(r) + B_0 r \frac{\partial}{\partial r} f(r), \qquad (20)$$

where $A_0 \equiv -1/6\int d\mathbf{q}\, q^2 W(q)$ and $B_0 \equiv -1/3\int d\mathbf{q}\, q^2 \tilde{W}'(q)$.

The stationary solution of Eq. (18) for $C(r) = 0$ is as follows:

$$f_g(r,t) = C\exp\left[-\int_0^r dr' \frac{A(r')}{r'B(r')}\right] = C\exp\left[-\frac{A_0 r^2}{2B_0}\right]. \qquad (21)$$

The respective normalized stationary momentum distribution equals

$$f_g(p) = \frac{N_g B_0^{3/2}}{(2\pi A_0)^{3/2}}\exp\left[-\frac{B_0 p^2}{2A_0}\right]. \qquad (22)$$

Therefore, in Eq. (19), the constant $C = N_g$. Equation (20) and distribution (22) are generalizations of those in the Fokker–Planck case for the normal diffusion to a non-equilibrium situation, when the prescribed $W(\mathbf{q},\mathbf{p})$ is determined, e.g., by some non-Maxwellian distribution of small particles f_b. To show this in another way, let us make the Fourier transformation of (13) with $C = 0$ and the respective A and B_α:

$$\frac{df_g(\mathbf{p},t)}{dt} = -A_0\frac{\partial^2}{\partial p^2} f_g(\mathbf{p},t) - B_0\frac{\partial}{\partial p_\alpha} p_\alpha f_g(\mathbf{p},t)\ . \qquad (23)$$

Therefore, we arrive at the Fokker–Planck-type equation with the friction coefficient $\beta \equiv -B_0$ and the diffusion coefficient $D = -A_0/M^2$. In general, these coefficients (Eq. (19)) do not satisfy the Einstein relation.

In the case of the equilibrium W-function (e.g., $f_b = f_b^0$, see above), the equality $\tilde{W}'(q) = W(q)/2MT_b$ is fulfilled. Then $A(r)/rB(r) = MT_b r$ ($A_0 = MT_b B_0$). Only in this case, the Einstein relation between the diffusion and friction coefficients exists, and the standard Fokker–Planck equation is valid.

3. Models of Anomalous Diffusion in V-Space

Now, we can calculate the coefficients within the models of anomalous diffusion.

At first, we consider a simple model system of hard spheres with different masses m and $M \gg m$, $d\sigma/do = a^2/4$. Let us suppose that the small particles in the model under consideration are described by the prescribed stationary distribution $f_b = n_b\phi_b/u_0^3$ (where ϕ_b is a dimensionless distribution, and u_0 is the characteristic velocity for the distribution of small particles) and $\xi \equiv (u^2 + v^2 - \mathbf{q}\cdot\mathbf{v}/\mu)/u_0^2$. We have

$$W_a(\mathbf{q},\mathbf{p}) = \frac{n_b a^2\pi}{2\mu^2 u_0 q}\int_{(q^2/4\mu^2+v^2-\mathbf{q}\cdot\mathbf{v}\mu)/u_0^2}^{\infty} d\xi\cdot\phi_b(\xi). \qquad (24)$$

If the distribution $\phi_b(\xi) = 1/\xi^\gamma$ ($\gamma > 1$) possesses a long tail, we get

$$W_a(\mathbf{q},\mathbf{p}) = \frac{n_b a^2\pi}{2\mu^2 u_0 q}\frac{\xi^{1-\gamma}}{(1-\gamma)}\bigg|_{\xi_0}^{\infty} = \frac{n_b a^2\pi}{2\mu^2 u_0 q}\frac{\xi_0^{1-\gamma}}{(\gamma-1)}\ , \qquad (25)$$

where $\xi_0 \equiv (q^2/4\mu^2 + v^2 - \mathbf{q}\cdot\mathbf{v}/\mu)/u_0^2$.

In the case $p = 0$, the value $\xi_0 \to \tilde{\xi}_0 \equiv q^2/4\mu^2 u_0^2$, and we obtain the expression for the anomalous $W \equiv W_a$

$$W_a(\mathbf{q},\mathbf{p}=\mathbf{0}) = \frac{n_b a^2\pi}{2^{3-2\gamma}(\gamma-1)\mu^{4-2\gamma}u_0^{3-2\gamma}q^{2\gamma-1}} \equiv$$

$$\equiv \frac{C_a}{q^{2\gamma-1}}. \qquad (26)$$

The function $A(r)$, according to Eq. (14), is

$$A(r) \equiv 4\pi\int_0^\infty dq q^2\left[\frac{sin\,(qr)}{qr} - 1\right]W(q) =$$

$$= 4\pi C_a\int_0^\infty dq\frac{1}{q^{2\gamma-3}}\left[\frac{sin(qr)}{qr} - 1\right]. \qquad (27)$$

Comparing the reduced equation (see below) in the velocity space with that for the diffusion in the coordinate space ($2\gamma - 1 \leftrightarrow \alpha$ and $W(q) = C/q^{2\gamma-1}$), we can establish that the convergence of the integral on the right-hand side of Eq. (27) (3d case) is ensured if $3 < 2\gamma - 1 < 5$ or $2 < \gamma < 3$. The inequalities $\gamma < 3$ and $\gamma > 2$ ensure the convergence for small q ($q \to 0$) and $q \to \infty$, respectively.

To determine the structure of the transport process and the kinetic equation in the velocity space, we have to find the functions $\tilde{W}'(q)$ and $\tilde{W}''(q)$.

To find $\tilde{W}'(q)$ and $\tilde{W}''(q)$ for $p \neq 0$, we use the full value $\xi_0 \equiv (q^2/4\mu^2 + p^2/M^2 - \mathbf{q}\cdot\mathbf{p}/M\mu)/u_0^2$ and its derivatives on $\mathbf{q}\cdot\mathbf{p}$ at $p=0$, $\xi_0' = -1/M\mu u_0^2$ and $\xi_0'' = 0$. Then

$$\tilde{W}'(\mathbf{q},\mathbf{p}) \equiv \frac{n_b a^2 \pi}{2M\mu^3 u_0^3 q}\xi_0^{-\gamma},$$

$$\tilde{W}''(\mathbf{q},\mathbf{p}) \equiv \frac{n_b a^2 \pi\gamma}{2M^2\mu^4 u_0^5 q}\xi_0^{-\gamma-1}. \tag{28}$$

Therefore, for $p = 0$ ($\xi_0 \to \tilde{\xi}_0$), we obtain the functions

$$\tilde{W}'(q) \equiv \frac{(4\mu^2 u_0^2)^\gamma n_b a^2 \pi}{2M\mu^3 u_0^3 q^{2\gamma+1}},$$

$$\tilde{W}''(q) \equiv \frac{(4\mu^2 u_0^2)^{\gamma+1} n_b a^2 \pi\gamma}{2M^2\mu^4 u_0^5 q^{2\gamma+3}}. \tag{29}$$

We now establish the conditions of convergence of the integrals for $B(r)$ and $C(r)$. We have

$$B(r) = \frac{4\pi}{r^2}\int_0^\infty dq q^2 \left[\cos(qr) - \frac{\sin(qr)}{qr}\right]\tilde{W}'(q). \tag{30}$$

The convergence of $B(r)$ exists for small q if $\gamma < 2$ and for large $q \to \infty$ for $\gamma > 1/2$.

Finally, the convergence of $C(r)$ is determined by the equalities $\gamma < 2$ for small q and $\gamma > 1$ for large q:

$$C(r) = \frac{2\pi}{r^2}\int_0^\infty dq q^4 \times$$

$$\times\left[\frac{2\sin(qr)}{q^3r^3} - \frac{2\cos(qr)}{q^2r^2} - \frac{\sin(qr)}{qr} + \frac{1}{3}\right]\tilde{W}''(q). \tag{31}$$

Therefore, to ensure the convergence of A, B, and C at large q, we have to establish the convergence for A, which means $\gamma > 2$. To establish the convergence for small q, it is enough to establish the convergence for B and C, which means $\gamma < 2$. Therefore, for the purely power behavior of the function $f_b(\xi)$, the convergence is absent. However, for the anomalous diffusion in the momentum space, the convergence for small q is always ensured in reality, e.g., by a finite value of v or by a change of the small q-behavior of $W(q)$ (compare with the examples of anomalous diffusion in the coordinate space [1]). Therefore, in the model under consideration, the "anomalous diffusion in the velocity space" for the power behavior of $W(q)$, $\tilde{W}'(q)$, and $\tilde{W}''(q)$ at large q exists if, for large q, $W(q\to\infty) \sim 1/q^{2\gamma-1}$ asymptotically with $\gamma > 2$. At the same time, the expansion of the exponential function in Eqs. (14)–(17) in the integrands, which leads to the Fokker–Planck-type kinetic equation, is invalid for the power-type kernels $W(\mathbf{q},\mathbf{p})$.

Let us consider now the formal general model, for which we will not connect the functions $W(q)$, $\tilde{W}'(q)$, and $\tilde{W}''(q)$ with a specific form of $W(\mathbf{q},\mathbf{p})$. In this case, we can suggest that the functions possess the q-dependences independent of one another.

As an example, these dependences can be taken as those of the power type for three functions: $W(q) \equiv a/q^\alpha$, $\tilde{W}'(q) \equiv b/q^\beta$, and $\tilde{W}''(q) \equiv c/q^\eta$. Here, α, β, and η are independent and positive. As is follows from the consideration above, the convergence of the function W exists if $5 > \alpha > 3$ (for asymptotically small and large q, respectively). For the function $\tilde{W}'(q)$, the convergence condition is $5 > \beta > 2$ (for asymptotically small and large q, respectively). Finally, for the function $\tilde{W}''(q)$, the convergence condition is $7 > \eta > 5$ (for asymptotically small and large q, respectively).

For this example, the kinetic equation Eq. (13) reads

$$\frac{df_g(\mathbf{r},t)}{dt} = P_0 r^{\alpha-3} f(\mathbf{r},t) + r^{\beta-5}P_1 r_i \frac{\partial}{\partial r_i} f(\mathbf{r},t) +$$

$$+ r^{\eta-7}P_2 r_i r_j \frac{\partial^2}{\partial r_i \partial r_j} f(\mathbf{r},t), \tag{32}$$

where

$$P_0 \equiv 4\pi a \int_0^\infty d\zeta \zeta^{2-\alpha}\left[\frac{\sin\zeta}{\zeta} - 1\right], \tag{33}$$

$$P_1 = 4\pi b \int_0^\infty d\zeta q \zeta^{2-\beta}\left[\cos\zeta - \frac{\sin\zeta}{\zeta}\right], \tag{34}$$

$$P_2 = 4\pi c \int_0^\infty d\zeta \zeta^{4-\eta}\left[\frac{\sin\zeta}{\zeta^3} - \frac{\cos\zeta}{\zeta^2} - \frac{\sin\zeta}{2\zeta} + \frac{1}{6}\right]. \tag{35}$$

Taking the isotropy in the coordinate space into account, we can rewrite Eq. (32) in the form

$$\frac{df_g(r,t)}{dt} = P_0 r^{\alpha-3} f(r) +$$

$$+r^{\beta-4}P_1\frac{\partial}{\partial r}f(r,t)+r^{\eta-5}P_2\frac{\partial^2}{\partial r^2}f(r,t). \quad (36)$$

Naturally, Eqs. (32),(36) can be formally rewritten in the momentum space (or in the velocity one) via the fractional derivatives of various orders. As is easy to see, for the purely power behavior of the functions $W(q)$, $\tilde{W}'(q)$, and $\tilde{W}''(q)$, the solution with the convergent coefficients exists for the powers in the above-mentioned intervals. The universal character of anomalous diffusion in the velocity space in the case under consideration takes place in the interval of powers mentioned above.

Let us consider now the particular case of anomalous diffusion, when the specific structure of the PTF $W(\mathbf{q},\mathbf{p})$ provides a rapid (let us say, exponential) decrease of the functions $\tilde{W}'(q)$ and $\tilde{W}''(q)$. Therefore, the exponential function in the integrals for the coefficients $B(r)$ and $C(r)$ can be expanded, which gives $B(r)=B_0$ and $C(r)\simeq 0$, respectively. At the same time, the function $W(q)\equiv a/q^{\alpha}$ has a purely power dependence on q.

Then the kinetic equation (13) reads

$$\frac{df_g(\mathbf{r},t)}{dt}=P_0r^{\alpha-3}f(\mathbf{r},t)+B_0r_i\frac{\partial}{\partial r_i}f(\mathbf{r},t) \quad (37)$$

or, formally in the momentum space,

$$\frac{df_g(\mathbf{p},t)}{dt}=P_0D^{\nu}f(\mathbf{p},t)+B_0\frac{\partial}{\partial p_i}[p_if(\mathbf{p},t)], \quad (38)$$

where $\nu\equiv(\alpha-3)$ $(2>\nu>0)$, and we introduced the fractional differentiation operator $D^{\nu}\equiv\int d\mathbf{r}r^{\nu}\exp(-i\mathbf{pr})$ in the momentum space to compare this equation with a similar one in [13]. The stationary solution of Eq. (37) is

$$f_g(r,t)=C\exp\left[-\frac{P_0r^{\nu-1}}{B_0}\right]. \quad (39)$$

The similar consideration can be used for other types of anomalous diffusion in the velocity space. The physically important applications based on the physical models for the $W(\mathbf{q},\mathbf{p})$ function will be considered separately.

4. Appendix: Diffusion in the Coordinate Space on the Basis of a Master-Type Equation

Let us consider the diffusion in the coordinate space on the basis of a master equation which describes the balance of grains coming in and out a point r at the time moment t. The structure of this equation is formally similar to the master equation in the momentum space (see, e.g., [1, 11]). Of course, there is no conservation law for the coordinate space, similar to that in the momentum space:

$$\frac{df_g(\mathbf{r},t)}{dt}=\int d\mathbf{r}'\left\{W(\mathbf{r},\mathbf{r}')f_g(\mathbf{r}',\mathbf{t})-W(\mathbf{r}',\mathbf{r})f_g(\mathbf{r},t)\right\}. \quad (40)$$

The probability transition $W(\mathbf{r},\mathbf{r}')$ describes the probability for a grain to transfer from the point $\mathbf{r}'$ to the point $\mathbf{r}$ per unit time. We can rewrite this equation in the coordinates $\rho=\mathbf{r}'-\mathbf{r}$ and $\mathbf{r}$ as

$$\frac{df_g(\mathbf{r},t)}{dt}=\int d\rho\times$$

$$\times\left\{W(\rho,\mathbf{r}+\rho)f_g(\mathbf{r}+\rho,t)-W(\rho,\mathbf{r})f_g(\mathbf{r},t)\right\}. \quad (41)$$

Assuming that the characteristic displacements are small, one can expand Eq. (41) and get the Fokker–Planck-type equation for the density distribution $f_g(\mathbf{r},t)$

$$\frac{df_g(\mathbf{r},t)}{dt}=\frac{\partial}{\partial r_\alpha}\left[A_\alpha(\mathbf{r})f_g(\mathbf{r},t)+\frac{\partial}{\partial r_\beta}\left(B_{\alpha\beta}(\mathbf{r})f_g(\mathbf{r},t)\right)\right]. \quad (42)$$

The coefficients A_α and $B_{\alpha\beta}$, describing the acting force and the diffusion, respectively, can be written as functionals of the PTF in the coordinate space W (with dimension s) in the form

$$A_\alpha(\mathbf{r})=\int d^s\rho\rho_\alpha W(\rho,\mathbf{r}) \quad (43)$$

and

$$B_{\alpha\beta}(\mathbf{r})=\frac{1}{2}\int d^s\rho\rho_\alpha\rho_\beta W(\rho,\mathbf{r}). \quad (44)$$

In the isotropic case, the probability function depends on $\mathbf{r}$ and the modulus of ρ. For a homogeneous medium, when r-dependence of the PTF is absent, the coefficients $A_\alpha=0$, while the diffusion coefficient is constant with $B_{\alpha\beta}=\delta_{\alpha\beta}B$, where B is the integral

$$B=\frac{1}{2s}\int d^s\rho\rho^2W(\rho). \quad (45)$$

This consideration cannot be applied to specific situations, in which the integral in Eq. (45) is infinite. In that case, we have to examine the general transport equation (40). We will now consider the problem for the homogeneous and isotropic case where the PTF depends only on $|\rho|$. By the Fourier transformation, we arrive at the following form [11] of Eq. (40):

$$\frac{df_g(\mathbf{k},t)}{dt} = \int d^s\rho\,[\exp(i\mathbf{k}\rho) - 1] \times$$

$$\times W(|\rho|) f_g(\mathbf{k},t) \equiv X(\mathbf{k}) f_g(\mathbf{k},t), \tag{46}$$

where $X(\mathbf{k}) \equiv X(k)$. We assume a simple form of the PTF with a power dependence on the distance $W(\rho) = C/|\rho|^\alpha$, where C is a constant and $\alpha > 0$. Such a singular dependence is typical of the jump diffusion probability in heteropolymers in solution (see, e.g., [18], where the different applications of the anomalous diffusion are considered on the basis of the fractional differentiation method). In the one-dimensional case, we find

$$X(k) \equiv -4 \int_0^\infty du\, \sin^2\left(\frac{k\,u}{2}\right) W(u) =$$

$$= -2^{3-\alpha} C |k|^{\alpha-1} \int_0^\infty d\zeta \frac{\sin^2\zeta}{\zeta^\alpha}. \tag{47}$$

For the values $1 < \alpha < 3$, this function is finite and equal to

$$X(k) = -\frac{C\,\Gamma[(3-\alpha)/2]\,|k|^{\alpha-1}}{2^\alpha \sqrt{\pi}\,\Gamma(\alpha/2)(\alpha-1)}, \tag{48}$$

where Γ is the gamma-function. At the same time, the integral in Eq. (45) for such a PTF is infinite, because the usual diffusion is absent.

The procedure considered for the simplest cases of a power dependence of the PTF is equivalent to the equation with the fractional space differentiation [10, 18]

$$\frac{df_g(x,t)}{dt} = C \Delta^{\mu/2} f_g(x,t), \tag{49}$$

where $\Delta^{\mu/2}$ is a fractional Laplacian, a linear operator, whose action on the function $f(x)$ in the Fourier space is described by $\Delta^{\mu/2} f(x) = -(k^2)^{\mu/2} f(k) = -|k|^\mu f(k)$. In the case considered above, $\mu \equiv (\alpha - 1)$, where $0 < \mu < 2$. For more general PTFs, which (for arbitrary values of ρ) are not proportional to the α power of ρ, the method described above is also applicable, although the fractional derivative does not exist.

In the case of a purely power dependence of the PTF, the non-stationary solution for the density distribution describes the so-called superdiffusion (or Levy flights). The solution of Eq. (49) in the Fourier space reads

$$f_g(k,t) = \exp(-C|k|^\mu t), \tag{50}$$

which corresponds in the coordinate space to the so-called symmetric Levy stable distribution:

$$f_g(x,t) = \frac{1}{(kt)^{1/\mu}} L\left[\frac{x}{(kt)^{1/\mu}}; \mu, 0\right]. \tag{51}$$

In the general case, it follows from Eq. (46) that

$$f_g(k,t) = C_1 \exp[X(k)t] \tag{52}$$

with some constant C_1.

The consideration on the basis of a PTF given in this section allows us to avoid the fractional differentiation method and to consider more general physical situations of the non-power probability transitions. Let us consider that for a simple example. Taking (in the one-dimensional case) a PTF $W(u)$ in the form

$$W(u) = C\frac{1 - \exp[-\sigma u^p]}{u^\alpha}, \tag{53}$$

with $p > 0$, we get the function

$$X(k) = -2^{3-\alpha} C |k|^{\alpha-1} \int_0^\infty d\zeta \times$$

$$\times \frac{\{1 - \exp[-\sigma(2\zeta/|k|)^p\,]\}\, sin^2\zeta}{\zeta^\alpha} \equiv$$

$$\equiv -2^{3-\alpha} C |k|^{\alpha-1}\, T(\sigma/|k|^p, \alpha). \tag{54}$$

It is easy to see that the function $T(\sigma/|k|^p, \alpha)$ is finite for $1 < \alpha < p + 3$, because, for the small values of the distance for $p > 0$, the divergence is suppressed also for some powers $\alpha > 3$. A simple calculation for $\alpha = 2$ and $p = 1$ leads to the following result which cannot be found by the usual fractional differentiation method:

$$T(\sigma/|k|, 2) =$$

$$= \frac{\pi}{2} - \arctan(|k|/\sigma) + \frac{\sigma}{2|k|} \ln\left[1 + k^2/\sigma^2\right]. \tag{55}$$

The asymptotic behavior of the function $X(k)$ for $k \to 0$ (or $\sigma \to \infty$) is similar, as follows from Eq. (55), to the case $W(u) = C/u^\alpha$. In the case under consideration where $\alpha = 2$, the limit $X(k \to 0) \to -\pi C k$. For large values of k ($k \to \infty$ or $\sigma \to 0$), we find $X(k) \to (\sigma/k)\ln(k/\sigma)$.

In general case, the universal behavior of the function $X(k)$ is ensured by asymptotic properties of the PTF for large distances for $1 < \alpha < p + 3$.

5. Conclusions

In this paper, the problem of anomalous diffusion in the momentum (velocity) space is consequently considered. The new kinetic equation for anomalous diffusion in the velocity space is derived without suggestion about the equilibrium stationary distribution function. The model of anomalous diffusion in the velocity space is described on the basis of a respective expansion of the kernel in the master equation. The conditions of convergence for the coefficients of the kinetic equation are found in the particular cases. A wide variety of anomalous processes in the velocity space exists, because the three different coefficients in the general diffusion equation are present even in the isotropic case. The example of the Boltzmann kernel with the prescribed distribution function for light particles is studied, in particular for the interaction of hard spheres. In general, the Einstein relation for such a situation is not applicable, because the stationary state can be far from equilibrium. For the normal diffusion, the friction and diffusion coefficients are explicitly found in the non-equilibrium case. In the equilibrium case, the usual Fokker–Planck equation is reproduced as a particular case.

We also shortly reviewed the anomalous diffusion in the coordinate space.

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АНОМАЛЬНИЙ ТРАНСПОРТ У ПРОСТОРІ ШВИДКОСТЕЙ: РІВНЯННЯ ТА МОДЕЛІ

С.О. Триггер

Р е з ю м е

Проблему аномальної дифузії в імпульсному просторі розглянуто на основі відповідної функції імовірності переходів. Нове загальне рівняння для опису дифузії важких частинок у газі легких частинок одержано на основі нового наближення, яке використано автором раніше для дифузії в координатному просторі. Одержані результати дозволяють описати різні ситуації, коли функція імовірності переходів має довгий хвіст у імпульсному просторі. Одержано ефективні коефіцієнти тертя та дифузії.