RESUMMATION IN QCD FRACTIONAL ANALYTIC PERTURBATION THEORY

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We describe the generalization of Analytic Perturbation Theory (APT) for QCD observables, initiated by Radyushkin, Krasnikov, Pivovarov, Shirkov, and Solovtsov, to fractional powers of coupling — Fractional APT (FAPT). The basic aspects of FAPT are shortly summarized. We describe how to treat heavy-quark thresholds in FAPT and then show how to resum perturbative series in both the one-loop APT and FAPT. As an application, we consider the FAPT description of the Higgs boson decay $H^0 \to b\bar{b}$. The main conclusion is: To achieve an accuracy of the order of 1%, it is enough to consider up to the third correction.

1. APT and FAPT in QCD

In the standard QCD Perturbation Theory (PT), we know that the Renormalization Group (RG) equation $da_s[L]/dL = -a_s^2 - \ldots$ for the effective coupling $\alpha_s(Q^2) = a_s[L]/\beta_f$ with $L = \ln(Q^2/\Lambda^2)$, $\beta_f = b_0(N_f)/(4\pi) = (11 - 2N_f/3)/(4\pi)$[1]. Then the one-loop solution generates the Landau pole singularity, $a_s[L] = 1/L$.

Strictly speaking, the QCD Analytic Perturbation Theory (APT) was initiated by the paper by N. N. Bogoliubov et al. [1], where the ghost-free effective coupling for QED has been constructed. Then in 1982, Radyushkin [2] and Krasnikov and Pivovarov [3] suggested, by using the same dispersion technique, the regular (for $s \geq \Lambda^2$) QCD running coupling in a Minkowskian region, the famous $\dfrac{1}{\pi}\arctan\dfrac{\pi}{L}$. After that in 1995, Jones and Solovtsov discovered the coupling which appears to be finite for all s and coincides with the Radyushkin one for $s \geq \Lambda^2$, namely $\mathfrak{A}_1[L]$ in Eq. (2b). Just in the same time, Beneke et al. [4, 5] within the renormalization-based approach and Shirkov and Solovtsov [6] within the same dispersion approach of [1] discovered the ghost-free coupling $\mathcal{A}_1[L]$, Eq. (2a), in a Euclidean region.

[1] We use notations $f(Q^2)$ and $f[L]$ in order to specify the arguments we mean — squared momentum Q^2 or its logarithm $L = \ln(Q^2/\Lambda^2)$, that is $f[L] = f(\Lambda^2 \cdot e^L)$ and Λ^2 is usually referred to $N_f = 3$ region.

But the Shirkov–Solovtsov approach, named APT, was more powerful: in the Euclidean domain, $-q^2 = Q^2$, $L = \ln Q^2/\Lambda^2$, it generates the following set of images for the effective coupling and its n-th powers: $\{\mathcal{A}_n[L]\}_{n\in\mathbb{N}}$; whereas, in the Minkowskian domain, $q^2 = s$, $L_s = \ln s/\Lambda^2$, it generates another set, $\{\mathfrak{A}_n[L_s]\}_{n\in\mathbb{N}}$ (see also in [7]). APT is based on the RG and causality, which guarantees the standard perturbative UV asymptotics and spectral properties. The power series $\sum_m d_m a_s^m[L]$ is transformed into a non-power series $\sum_m d_m \mathcal{A}_m[L]$ in APT.

By the analytization in APT for an observable $f(Q^2)$, we mean the "Källén–Lehmann" representation

$$\left[f(Q^2)\right]_{\text{an}} = \int_0^\infty \frac{\rho_f(\sigma)}{\sigma + Q^2 - i\epsilon}\, d\sigma \tag{1}$$

with $\rho_f(\sigma) = \dfrac{1}{\pi}\,\text{Im}\left[f(-\sigma)\right]$. Then, in the one-loop approximation, $\rho_1(\sigma) = 1/\sqrt{L_\sigma^2 + \pi^2}$ and

$$\mathcal{A}_1[L] = \int_0^\infty \frac{\rho_1(\sigma)}{\sigma + Q^2} d\sigma = \frac{1}{L} - \frac{1}{e^L - 1}\,, \tag{2a}$$

$$\mathfrak{A}_1[L_s] = \int_s^\infty \frac{\rho_1(\sigma)}{\sigma} d\sigma = \frac{1}{\pi}\, \arccos\, \frac{L_s}{\sqrt{\pi^2 + L_s^2}}\,, \tag{2b}$$

whereas the analytic images of higher powers ($n \geq 2, n \in \mathbb{N}$) are

$$\begin{pmatrix}\mathcal{A}_n[L]\\ \mathfrak{A}_n[L_s]\end{pmatrix} = \frac{1}{(n-1)}\left(-\frac{d}{dL}\right)^{n-1}\begin{pmatrix}\mathcal{A}_1[L]\\ \mathfrak{A}_1[L_s]\end{pmatrix}. \tag{3}$$

At first glance, the APT is a complete theory providing tools to produce an analytic answer for any perturbative series in QCD. But Karanikas and Stefanis [8] suggested the principle of analytization "as a whole" in the Q^2 plane for hadronic observables calculated perturbatively. More precisely, they proposed the analytization recipe for terms like $\int_0^1 dx \int_0^1 dy\, \alpha_s\left(Q^2 xy\right) f(x) f(y)$,

which can be treated as an effective account for the logarithmic terms in the next-to-leading-order approximation of the perturbative QCD. This actually generalizes the analytic approach suggested in [9]. Indeed, in the standard QCD PT, one has also:
(i) the factorization procedure in QCD that gives rise to the appearance of logarithmic factors of the type $a_s^{\nu}[L]\,L$;
(ii) the RG evolution that generates evolution factors of the type $B(Q^2) = \left[Z(Q^2)/Z(\mu^2)\right]\, B(\mu^2)$ which are reduced in the one-loop approximation to $Z(Q^2) \sim a_s^{\nu}[L]$ with $\nu = \gamma_0/(2b_0)$ being a fractional number.
All that means that, in order to generalize APT in the "analytization as a whole" direction, one needs to construct analytic images of new functions $a_s^{\nu},\ a_s^{\nu}\,L^m, \ldots$. This task has been performed in the frames of the so-called FAPT suggested in [10, 11]. Now we briefly describe this approach.

In the one-loop approximation using recursive relation (3), we can obtain explicit expressions for $\mathcal{A}_\nu[L]$ and $\mathfrak{A}_\nu[L]$:

$$\mathcal{A}_\nu[L] = \frac{1}{L^\nu} - \frac{F(e^{-L}, 1-\nu)}{\Gamma(\nu)}\,, \tag{4a}$$

$$\mathfrak{A}_\nu[L] = \frac{\sin\left[(\nu-1)\arccos\left(\frac{L}{\sqrt{\pi^2+L^2}}\right)\right]}{\pi(\nu-1)\left(\pi^2+L^2\right)^{(\nu-1)/2}}\,. \tag{4b}$$

Here, $F(z,\nu)$ is the reduced Lerch transcendental function which is an analytic function in ν. They have very interesting properties which were discussed extensively in our previous papers [10–13].

The construction of FAPT with a fixed number of quark flavors, N_f, is a two-step procedure: we start with the perturbative result $\left[a_s(Q^2)\right]^{\nu}$, generate the spectral density $\rho_\nu(\sigma)$ using Eq. (1), and then obtain analytic couplings $\mathcal{A}_\nu[L]$ and $\mathfrak{A}_\nu[L]$ via Eqs. (2b). Here, N_f is fixed and factorized out. We can proceed in the same manner for N_f-dependent quantities $\left[\alpha_s(Q^2;N_f)\right]^{\nu} \Rightarrow \bar{\rho}_\nu(\sigma;N_f) = \bar{\rho}_\nu[L_\sigma;N_f] \equiv \rho_\nu(\sigma)/\beta_f^{\nu} \Rightarrow \bar{\mathcal{A}}_\nu[L;N_f]$ and $\bar{\mathfrak{A}}_\nu[L;N_f]$; here, N_f is fixed, but not factorized out.

The global version of FAPT [12] which takes heavy-quark thresholds into account is constructed along the same lines but starting from the global perturbative coupling $\left[\alpha_s^{\text{glob}}(Q^2)\right]^{\nu}$, being a continuous function of Q^2 due to choosing different values of QCD scales Λ_f corresponding to different values of N_f. Here, we illustrate the case of only one heavy-quark threshold at $s = m_4^2$, corresponding to the transition $N_f = 3 \to N_f = 4$. Then we obtain the discontinuous spectral density

$$\rho_n^{\text{glob}}(\sigma) = \theta\left(L_\sigma < L_4\right)\,\bar{\rho}_n\left[L_\sigma;3\right] + $$

$$+\theta\left(L_4 \le L_\sigma\right)\,\bar{\rho}_n\left[L_\sigma+\lambda_4;4\right]\,, \tag{5}$$

with $L_\sigma \equiv \ln\left(\sigma/\Lambda_3^2\right)$, $L_f \equiv \ln\left(m_f^2/\Lambda_3^2\right)$ and $\lambda_f \equiv \ln\left(\Lambda_3^2/\Lambda_f^2\right)$ for $f = 4$ which is expressed in terms of fixed-flavor spectral densities with 3 and 4 flavors, $\bar{\rho}_n[L;3]$ and $\bar{\rho}_n[L+\lambda_4;4]$. However, it generates the continuous Minkowskian coupling

$$\mathfrak{A}_\nu^{\text{glob}}[L] = \theta\left(L < L_4\right)\left(\bar{\mathfrak{A}}_\nu[L;3] + \Delta_{43}\bar{\mathfrak{A}}_\nu\right) +$$

$$+\theta\left(L_4 \le L\right)\,\bar{\mathfrak{A}}_\nu[L+\lambda_4;4] \tag{6a}$$

with $\Delta_{43}\bar{\mathfrak{A}}_\nu = \bar{\mathfrak{A}}_\nu[L_4+\lambda_4;4] - \bar{\mathfrak{A}}_\nu[L_4;3]$ and the analytic Euclidean coupling $\mathcal{A}_\nu^{\text{glob}}[L]$

$$\mathcal{A}_\nu^{\text{glob}}[L] = \bar{\mathcal{A}}_\nu[L+\lambda_4;4] +$$

$$+\int\limits_{-\infty}^{L_4} \frac{\bar{\rho}_\nu[L_\sigma;3] - \bar{\rho}_\nu[L_\sigma+\lambda_4;4]}{1+e^{L-L_\sigma}}\,dL_\sigma \tag{6b}$$

(for more details, see [12]).

2. Resummation in the One-Loop APT and FAPT

We consider now the perturbative expansion of a typical physical quantity, like the Adler function and the ratio R, in the one-loop APT. Due to a limited space of our presentation, we provide all formulas only for quantities in the Minkowski region:

$$\mathcal{R}[L] = \sum_{n=1}^{\infty} d_n\,\mathfrak{A}_n[L]\,. \tag{7}$$

We suggest that there exists the generating function $P(t)$ for coefficients $\tilde{d}_n = d_n/d_1$:

$$\tilde{d}_n = \int\limits_0^\infty P(t)\,t^{n-1}dt \quad \text{with} \quad \int\limits_0^\infty P(t)\,dt = 1\,. \tag{8}$$

To shorten our formulae, we use the following notation for the integral $\int_0^\infty f(t)P(t)dt$: $\langle\langle f(t)\rangle\rangle_{P(t)}$. Then the coefficients $d_n = d_1\,\langle\langle t^{n-1}\rangle\rangle_{P(t)}$, and we have, as has been shown in [14], the exact result for the sum in (7)

$$\mathcal{R}[L] = d_1\,\langle\langle \mathfrak{A}_1[L-t]\rangle\rangle_{P(t)}\,. \tag{9}$$

The integral with respect to the variable t here has a rigorous meaning ensured by the finiteness of the coupling

$\mathfrak{A}_1[t] \leq 1$ and the fast fall-off of the generating function $P(t)$.

In our previous publications [12, 15], we constructed generalizations of these results, first, to the case of the global APT, when heavy-quark thresholds are taken into account. Then one starts with a series of the type (7), where $\mathfrak{A}_n[L]$ are substituted by their global analogs $\mathfrak{A}_n^{\text{glob}}[L]$ (note that, due to different normalizations of global couplings, $\mathfrak{A}_n^{\text{glob}}[L] \simeq \mathfrak{A}_n[L]/\beta_f$, the coefficients d_n should be also changed). Then

$$\mathcal{R}^{\text{glob}}[L] = d_1\theta(L<L_4)\Big\langle\Big\langle \Delta_4\bar{\mathfrak{A}}_1[t] + \bar{\mathfrak{A}}_1\Big[L-\frac{t}{\beta_3};3\Big]\Big\rangle\Big\rangle_{P(t)} +$$

$$+d_1\theta(L\geq L_4)\Big\langle\Big\langle \bar{\mathfrak{A}}_1\Big[L+\lambda_4-\frac{t}{\beta_4};4\Big]\Big\rangle\Big\rangle_{P(t)}, \qquad (10)$$

where $\Delta_4\bar{\mathfrak{A}}_\nu[t] \equiv \bar{\mathfrak{A}}_\nu\Big[L_4+\lambda_4-t/\beta_4;4\Big]-\bar{\mathfrak{A}}_\nu\Big[L_3-t/\beta_3;3\Big]$.

The second generalization has been obtained for the case of the global FAPT. Then the starting point is a series of the type $\sum_{n=0}^{\infty} d_n\,\mathfrak{A}_{n+\nu}^{\text{glob}}[L]$, and the result of summation is a complete analog of Eq. (10) with substitutions

$$P(t) \Rightarrow P_\nu(t) = \int_0^1 P\left(\frac{t}{1-x}\right)\frac{\nu\, x^{\nu-1}dx}{1-x}, \qquad (11)$$

$d_0 \Rightarrow d_0\,\bar{\mathfrak{A}}_\nu[L]$, $\bar{\mathfrak{A}}_1[L-t] \Rightarrow \bar{\mathfrak{A}}_{1+\nu}[L-t]$, and $\Delta_4\bar{\mathfrak{A}}_1[t] \Rightarrow \Delta_4\bar{\mathfrak{A}}_{1+\nu}[t]$. All needed formulas have been also obtained in parallel for the Euclidean case.

3. Applications to Higgs Boson Decay

Here, we analyze the Higgs boson decay to a $\bar{b}b$ pair. For its width, we have

$$\Gamma(\text{H} \to b\bar{b}) = \frac{G_F}{4\sqrt{2}\pi}\,M_H\,\widetilde{R}_{\text{S}}(M_H^2) \qquad (12)$$

with $\widetilde{R}_{\text{S}}(M_H^2) \equiv m_b^2(M_H^2)\,R_{\text{S}}(M_H^2)$, and $R_{\text{S}}(s)$ is the R-ratio for the scalar correlator (see [10, 16] for details). In the one-loop FAPT, this generates the following non-power expansion[2]:

$$\widetilde{\mathcal{R}}_{\text{S}}[L] = 3\hat{m}_{(1)}^2\times$$

$$\times\left\{\mathfrak{A}_{\nu_0}^{\text{glob}}[L] + d_1^S\sum_{n\geq1}\frac{\tilde{d}_n^{\text{S}}}{\pi^n}\mathfrak{A}_{n+\nu_0}^{\text{glob}}[L]\right\}, \qquad (13)$$

[2] Appearance of denominators π^n in association with the coefficients $\tilde{d}_n$ is due to the d_n normalization.

where $\hat{m}_{(1)}^2 = 9.05 \pm 0.09$ GeV^2 is the RG-invariant of the one-loop $m_b^2(\mu^2)$ evolution $m_b^2(Q^2) = \hat{m}_{(1)}^2\,\alpha_s^{\nu_0}(Q^2)$ with $\nu_0 = 2\gamma_0/b_0(5) = 1.04$, and γ_0 is the quark-mass anomalous dimension. This value of $\hat{m}_{(1)}^2$ has been obtained using the one-loop relation [17] between the pole b-quark mass of [18] and the mass $m_b(m_b)$.

For the generating function $P(t)$, we take the Lipatov-like model of [15] with $\{c = 2.4,\ \beta = -0.52\}$

$$\tilde{d}_n^{\text{S}} = c^{n-1}\frac{\Gamma(n+1)+\beta\,\Gamma(n)}{1+\beta}, \qquad (14a)$$

$$P_{\text{S}}(t) = \frac{(t/c)+\beta}{c\,(1+\beta)}\,e^{-t/c}. \qquad (14b)$$

It gives a very good prediction for $\tilde{d}_n^{\text{S}}$ with $n = 2, 3, 4$, calculated in the QCD PT [16]: 7.50, 61.1, and 625 in comparison with 7.42, 62.3, and 620. Then we apply the FAPT resummation technique to estimate how good is FAPT in approximating the whole sum $\widetilde{\mathcal{R}}_{\text{S}}[L]$ in the range $L \in [11.5, 13.7]$ which corresponds to the range $M_H \in [60, 180]$ GeV^2 with $\Lambda_{\text{QCD}}^{N_f=3} = 189$ MeV and $\mathfrak{A}_1^{\text{glob}}(m_Z^2) = 0.122$. In this range, we have ($L_6 = \ln(m_t^2/\Lambda_3^2)$)

$$\frac{\widetilde{\mathcal{R}}_{\text{S}}[L]}{3\,\hat{m}_{(1)}^2} = \mathfrak{A}_{\nu_0}^{\text{glob}}[L] + \frac{d_1^{\text{S}}}{\pi}\,\langle\langle\bar{\mathfrak{A}}_{1+\nu_0}\Big[L+\lambda_5-\frac{t}{\pi\beta_5};5\Big]\rangle\rangle_{P_{\nu_0}^{\text{S}}} +$$

$$+\frac{d_1^{\text{S}}}{\pi}\,\langle\langle\Delta_6\bar{\mathfrak{A}}_{1+\nu_0}\left[\frac{t}{\pi}\right]\rangle\rangle_{P_{\nu_0}^{\text{S}}} \qquad (15)$$

with $P_{\nu_0}^{\text{S}}(t)$ defined via Eqs. (14) and (11).

Now we analyze the accuracy of the truncated FAPT expressions

$$\widetilde{\mathcal{R}}_{\text{S}}[L;N] = 3\hat{m}_{(1)}^2\left[\mathfrak{A}_{\nu_0}^{\text{glob}}[L] + d_1^{\text{S}}\sum_{n=1}^{N}\frac{\tilde{d}_n^{\text{S}}}{\pi^n}\mathfrak{A}_{n+\nu_0}^{\text{glob}}[L]\right] \qquad (16)$$

and compare them with the total sum $\widetilde{\mathcal{R}}_{\text{S}}[L]$ in Eq. (15) using the relative errors $\Delta_N[L] = 1 - \widetilde{\mathcal{R}}_{\text{S}}[L;N]/\widetilde{\mathcal{R}}_{\text{S}}[L]$. In Fig. 1, we show these errors for $N = 2$, $N = 3$, and $N = 4$ in the analyzed range of $L \in [11, 13.8]$. We see that already $\widetilde{\mathcal{R}}_{\text{S}}[L;2]$ gives accuracy of the order of 2.5%, whereas $\widetilde{\mathcal{R}}_{\text{S}}[L;3]$ of the order of 1%.

Looking at Fig. 1, we understand that, only in order to have the accuracy better than 0.5%, we need to take the 4-th correction into account. We verified also that the uncertainty due to $P(t)$-modelling is small $\lesssim 0.6\%$, while the on-shell mass uncertainty is of the order of 2%. The overall uncertainty then is of the order of 3% (see Fig. 2).

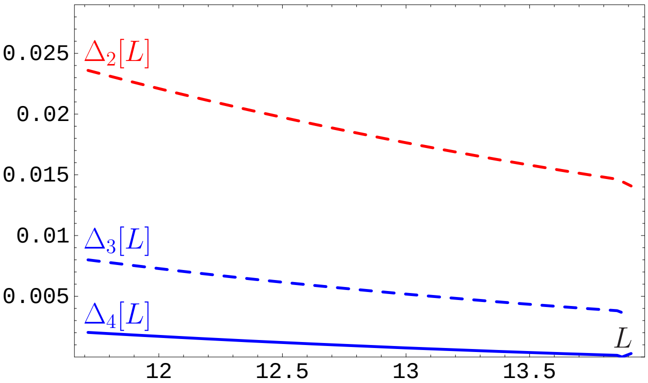

Fig. 1. Relative errors $\Delta_N[L]$, $N = 2, 3$, and 4, of the truncated FAPT in comparison with the exact summation result, Eq. (15)

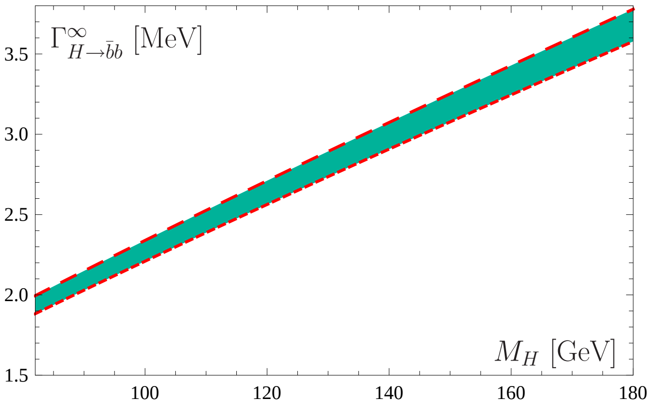

Fig. 2. Width $\Gamma_{H\to b\bar{b}}$ as a function of the Higgs boson mass M_H in the resummed FAPT. The width of the shaded strip is due to the overall uncertainties induced by the uncertainties of the resummation procedure and the pole mass error-bars

4. Conclusions

In this report, we have described the resummation approach in the global versions of the one-loop APT and FAPT and argued that it produces finite answers, provided the generating function $P(t)$ of perturbative coefficients d_n is known. The main conclusion is: To achieve an accuracy of the order of 1% it is enough to consider up to the third correction—in complete agreement with Kataev–Kim [17]. The d_4 coefficient is needed only to estimate the corresponding generating functions $P(t)$.

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ПЕРЕПІДСУМОВУВАННЯ У ДРОБОВО-АНАЛІТИЧНІЙ КХД-ТЕОРІЇ ЗБУРЕНЬ

О.П. Бакулев

Р е з ю м е

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