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Commensurate—incommensurate phase transitions for multichain quantum spin models: exact results

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The behavior in an external magnetic field is studied exactly for a wide class of multichain quantum spin models. It is shown that the magnetic field together with the interchain couplings cause commensurate—incommensurate phase transitions between the gapless phases in the ground state. The conformal limit of these models is studied and it is shown that the low-lying excitations for the incommensurate phases are not independent, because they are governed by the same magnetic field (chemical potential for excitations). A scenario for the transition from one to two space dimensions for the exactly integrable multichain quantum spin models is proposed, and it is shown that the incommensurate phases in an external magnetic field disappear in the limit of an infinite number of coupled spin chains. The similarities in the external field behavior for the quantum multichain spin models and a wide class of quantum field theories are discussed. The scaling exponents for the appearence of the gap in the spectrum of low-lying excitations of the quantum multichain models due to the relevant perturbations of the integrable theories are calculated.

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1. Introduction

There has recently been considerable interest on low-dimensional quantum-correlated spin and electron systems. These systems, especially one-dimensional (1*D*), manifest the specific features of, e.g., magnetic behavior at low temperatures, which are absent for the standard, conventional 3*D* magnetic systems. Spin systems usually manifest 1*D* behavior at temperatures higher than the temperature of the 3*D* magnetic ordering, but lower than the maximum characteristic energy of the interaction between spins, i.e., in our case the intrachain spin-spin coupling. The origin of such specific features is the enhancement of the quantum fluctuations of the 1*D* systems due to the peculiarities of the 1*D* density of states together with the quantum nature of spins.

Moreover, during the last decade a large number of new quasi-1*D* spin compounds have been created and studied experimentally. These compounds manifest at low temperatures the properties of a single or several quantum spin chains weakly coupled to each other [1,2]. It is strongly believed that this class of compounds will provide new information on the transition from 1*D* to 2*D* in quantum many-body physics. It is very important, because the 2*D* quantum many-body physics has been a challenge for both theorists and experimentalists since the beginning of the study of low dimensional quantum systems. On the other hand, the advantage of the 1*D* theoretical studies is the possibility of obtaining exact solutions by using non-perturbative methods, which are difficult to apply for the higher-dimensional quantum many-body models. The results of the exact calculations of the 1*D* models can serve as testing grounds for the use of perturbative and numerical methods in more realistic situations.

Recently several exactly solvable models [3–5] have been introduced, in which the zigzag-like interaction between *two* quantum spin chains was studied exactly using the Bethe *ansatz* technique [6]. This method is widely known by now, see, e.g., the recent monography [7] and references therein. The Bethe ansatz method permits exact calculation of the static characteristics of quantum many-body systems, such as the ground state behavior, the influence of an external magnetic field, and the thermodynamic features of the temperature dependence of the specific heat, magnetic susceptibility, etc. These results should apply to more-realistic systems, but it is not obvious how the interactions between the chains modify the answers. The mean-field-like approximations for the interchain couplings are not sufficient, because the mean field approach in any version already implies the existence of the (sometimes hidden) order parameter. It is, unfortunately, also unclear whether the numerical calculations, which can be directly applied for the quantum many-body systems of very small sizes, by now (say, at most several tens of sites) describe well the properties of the real systems, in which, even in quasi-$1D$ ones, the number of sites is at least of order 10^8. On the other hand, it must be admitted that some features of the exactly solvable $1D$ models are far from what is observed experimentally, but these unrealistic features of the $1D$ models are known and simple to recognize.

The behavior of the multichain spin systems in an external magnetic field is especially interesting see, e.g., [5,8–10] because of (i) the possibility of experimental observations due to recent progress in high-magnetic-field measurements and (ii) very interesting theoretically predictable effects which are possible to recognize in experiments, such as phase transitions in the external magnetic field. However, several important issues are far from being resolved in the quantum two-chain spin models. For example, there are three questions that need to be answered: (1) Are the properties of those exactly solvable two-chain spin models unique or is it possible to say something about the more general class of two-chain quantum spin models? (2) How are the multichain quantum models connected to the $2D$ many-body systems, i.e., what is the scenario of the transition from $1D$ to $2D$ when one increases the number of coupled chains while keeping the conditions of integrability? (3) What will happen with the behavior of the nonintegrable multichain spin models if one goes beyond the framework of integrability, i.e., adding some perturbations to the exactly solvable model? (For example, Ref. 10 implies that namely the spin chirality, which separately breaks the time-reversal and parity symmetries in the two-chain integrable model [11], is the reason for the emergence of the additional phase transitions in an external magnetic field for the two-chain spin-$1/2$ model as compared to the single-chainsystem.)

The goal of this paper is to answer these questions. First, we revisit the exactly integrable two-chain spin-$1/2$ model and show that the inclusion of the magnetic anisotropy of the «easy-plane» type, with which the system stays in the quantum critical region, will not drastically change the behavior in an external magnetic field but will shift the critical values of the magnetic fields and intrachain couplings at which the phase transitions occur and will affect the critical exponents. We will show that these two-chain spin models share the most important features of the behavior in an external field with the wide class of (1+1) quantum field theories. Next, we will introduce the higher-spin versions of the two-chain spin models, e.g., investigating the important class of $1D$ two-chain quantum ferrimagnets with different spin values at the sites of each chain. We will also investigate the behavior of the multichain exactly solvable spin models in an external magnetic field and show how the additional phase transitions arising due to the increasing number of chains vanish in the quasi-$2D$ limit. Finally, we will show how the relevant deviations from integrability, e.g., the absence of terms in the Hamiltonian which separately break the parity and time-reversal symmetries give rise to gaps in the spectra of low-lying excitations of the multichain quantum spin systems, and we will calculate the scaling exponents for the gaps.

The paper is organized as follows. In Section 2 we revisit the exactly solvable two-chain uniaxial spin model [4] to remind the reader of the main steps of the Bethe ansatz. The investigations [9,10] of isotropic spin-$1/2$ two-chain models are generalized in this section for the case of uniaxial magnetic anisotropy. The calculations in this section are rather simple, but we will write them in detail because they provide the basis for the more nontrivial generalizations of this class of models, and will be used in the following Sections. In Section 3 we point out the similarities between the behavior of the uniaxial two-chain quantum spin models and a class of quantum field theories (QFT) in an external magnetic field, predicting new phases for the QFT. In Section 4 we introduce the $SU(2)$ generalization of the integrable two-chain model for higher values of the site spins (possibly different) in each chain, i.e., a quantum ferrimagnet. We point out the similarities of the quantum ferrimagnet with QFT with a nonzero Wess—Zumino term and predict new phases for the latter in an external magnetic field. We derive integral equations for the critical exponents. In Section 5 we consider the multichain quantum spin model and discuss how the external field behavior of the integrable multichain models is changed when the number of chains is increased while preserving the exact solvability. In Section 6 we briefly sketch how the deviations from integrability change the magnetic and low-temperature properties of this class of multichain quantum spin systems. The paper is closed with a discussion of the main results and some conclusions.

2. Two-chain uniaxial quantum spin model

A common property of some of the Bethe ansatz solutions is the presence of shifts θ_j of the spectral parameter λ for the associated transfer matrix of an algebraic version of the Bethe ansatz (the Quantum Inverse Scattering Method (QISM) [7]). Those shifts also appear in the Bethe ansatz equations (BAE) for the quantum numbers called rapidities, which parametrize the eigenfunctions and eigenvalues of the Hamiltonians. Hence, the distributions of the rapidities are also affected by the shifts. An interesting property is connected with those shifts: depending on their values and the external magnetic field, even for (quasi)particles of the same type, additional minima may appear in distributions of the rapidities. These additional minima also result in nonmonotonic behavior of the dispersion laws of the low-lying excitations. Also, they provide additional Dirac seas for low-lying excitations, changing the structures of the physical ground states of the models. These additional minima determine the special behavior of the models in an external magnetic field [3,5,9,10]. In particular, the appearance of the new phases and new phase transitions is due to the emergence of these new minima in the distributions of the quantum numbers.

To set the stage, let us first remind the reader about the main steps of the QISM. The common feature of the Bethe-ansatz-solvable models is the factorization of the monodromy matrix (the ordered product of all two-particle scattering matrices, which depend on some spectral parameter) [7]. Exact (Bethe ansatz) integrability requires exclusively elastic scattering between (quasi)particles. For such theories the two-particle scattering matrices and L operators satisfy the Yang—Baxter relation [7,12]. In turn, the factorization of the monodromy matrices guarantees that they satisfy the Yang—Baxter equations, too. The transfer matrices of the associated statistical problem are traces over some additional, auxiliary subspace of monodromy matrices [7]. The most important feature of transfer matrices with different spectral parameters is their commutativity. The necessary and sufficient condition for this is the validity of the Yang—Baxter equations for two-particle scattering matrices and hence for monodromy matrices. The commutativity of the transfer matrices implies that one can construct an infinite number of integrals of motion, which commute with one another and with the transfer matrix. Therefore the exact integrability is proved. Usually the structure of these integrals of motion is determined by their locality. For instance, the best-known of series of integrals of motion is the series of derivatives with respect to the spectral parameter of the logarithm of a transfer matrix taken at some special value of the former [7]. Locality means that for the first derivative of the logarithm of the transfer matrix (usually called the Hamiltonian of the lattice system) only short-range particle—particle interactions contribute.

In this paper we will see that namely the aforementioned shifts of the spectral parameters yield new phases in the ground state behavior in an external magnetic field of a wide class of exactly solvable models, quantum spin multichain models and QFT. We will show that in the conformal limit these phases of the lattice models correspond to one Wess—Zumino—Witten (WZW) model or to several of them with dressed charges (proportional to the compactification radii) of scalar or matrix types for each of the phases, respectively.

Let us start with the form of the Bethe ansatz equations (BAE) for the set of rapidities $\{u_\alpha\}_{\alpha=1}^{M}$. In this paper we will concentrate only on the critical, «easy-plane» type of the magnetic anisotropy for the antiferromagnetic spin multichain models, $0 \le \gamma \le \pi/2$ ($\gamma = \pi/q$, where q is an integer, parametrizing the magnetic anisotropy), and the repulsive interactions in QFT. This corresponds to hyperbolic or rational solutions of the Yang—Baxter equations for the two-particle scattering matrices, or to $U(1)$ and $SU(2)$ symmetries of the scattering processes, respectively. For the simplest case of one shift θ, which pertains to the two-chain quantum spin models and most a QFT, the BAE have the form (here we use the more general hyperbolic parametrization first; for the rational limit see below) [4]

$$\prod e_1^{N_\pm}(u_\alpha \pm \theta) = e^{i\pi M} \prod_{\beta=1,\ \beta\neq\alpha}^{M} e_2(u_\alpha - u_\beta)\ , \tag{1}$$

where $N_\pm$ are the numbers of sites in each of the spin chains; $e_n(x) = \sinh(x + i\gamma n/2) \times \sinh(x - i\gamma n/2)^{-1}$; and M is the number of down spins. The shift θ determines the interchain coupling constant for two-chain quantum spin-1/2 models [4,11,13]. Please note that the Bethe ansatz equations are just the quantization conditions for the rapidities, which parametrize the eigenwaves and eigenvalues of the many-body quantum model. The Hamiltonian is the first derivative of the logarithm of the transfer matrix (note that the transfer matrix of the two coupled spin chains in this integrable model is the product of two «standard» transfer matrices of each chain with the spectral parameters $\lambda \pm \theta$ [11]:

$$\hat{H}_{1/2} = \frac{1}{\sinh^2\theta + \sin^2\gamma} \times$$

$$\times \sum_n \Big(\cos\gamma \sinh^2\theta \, (\mathbf{S}_{n,1}\,\mathbf{S}_{n+1,1} + \mathbf{S}_{n,2}\,\mathbf{S}_{n+1,2}) +$$

$$+ 2\sin^2\gamma \hat{I}\mathbf{S}_{n,1}\,(\mathbf{S}_{n,2} + \mathbf{S}_{n+1,2}) +$$

$$+ 2\sin\gamma\sinh\theta\,(\hat{J}\mathbf{S}_{n+1,2} - \hat{J}\mathbf{S}_{n,1})\,[\mathbf{S}_{n+1,1}\times\mathbf{S}_{n,2}]\Big)\,, \quad (2)$$

where $\hat{I} = \mathrm{diag}\,(\cosh\theta, \cosh\theta, \cos\gamma)$ and $\hat{J} = \mathrm{diag}\,(\cos\gamma, \cos\gamma, \cosh\theta)$, diag (a, b, c) is 3×3 diagonal matrix, and $[\times]$ denotes the vector product. Please note that the sum runs over n to N_+ for the chain with spins $S_{n,1}$ and to N_- for the chain with spins $S_{n,2}$. The parameter θ determines the intrachain coupling in our two-chain spin model. For $\theta = 0$ the Hamiltonian and BAE coincide with the ones for the single «easy-plane» antiferromagnetic spin-1/2 chain of length $N_+ + N_-$ with only nearest-neighbor interactions in it. The eigenvalue of the Hamiltonian (energy) is parametrized as the function of the rapidities as follows:

$$E = \sin\gamma \sum_{\pm}\sum_{\alpha=1}^{M} N_{\pm}\,[e_1(u_\alpha \pm \theta) + e_1^{-1}(u_\alpha \pm \theta)] + E_0\,, \quad (3)$$

where E_0 is the energy of the vacuum (ferromagnetic) state (with $M = 0$). The isotropic $SU(2)$-symmetric antiferromagnetic quantum spin two-chain model [9,10,11,13] can be obtained from the uniaxial ($U(1)$-symmetric) one of Eqs. (1)–(3) by the simple change of variables in the limit: $u_\alpha \to \gamma u_\alpha$, $\lambda \to \gamma\lambda$, $\theta \to \gamma\theta$, $\gamma \to 0$. (The last limit corresponds to the rational, $SU(2)$-symmetric solution of the Yang—Baxter equations for two-particle scattering matrices.) The two-chain isotropic ($SU(2)$-symmetric) spin-1/2 Hamiltonian obtained in this limit from Eq. (2) takes the form [4,9,10,11,13]

$$\hat{H}_{\text{is}} = \frac{1}{1+\theta^2} \times$$

$$\times \sum_n \Big(\theta^2(\mathbf{S}_{n,1}\mathbf{S}_{n+1,1} + \mathbf{S}_{n,2}\mathbf{S}_{n+1,2}) + 2\mathbf{S}_{n,1}(\mathbf{S}_{n,2} + \mathbf{S}_{n+1,2}) +$$

$$+ 2\theta(\mathbf{S}_{n+1,2} - \mathbf{S}_{n,1})[\mathbf{S}_{n+1,1}\times\mathbf{S}_{n,2}]\Big)\,. \quad (4)$$

The summations over n run to $N_\pm$ for each kind of spins, respectively. Note that for $\theta \to \infty$ Eqs. (4) and the BAE recover the Hamiltonian and BAE of two decoupled spin-1/2 chains of lengths $N_\pm$ with the only nearest-neighbor interactions in each of the chains.

The solution to the BAE (1) is usually obtained in the thermodynamic limit ($N_\pm$, $M \to \infty$, with the ratio $M/(N_+ + N_-)$ fixed). Here instead of the discret set of rapidities one introduces the distribution of a continuous density of rapidities. The ground state corresponds to the solutions of the BAE with negative energies, i.e., it is connected with the filling up the Dirac sea(s) for the model. For the «easy-plane» antiferromagnetic two-chain spin-1/2 model the ground state corresponds to the filling of the Dirac sea for real rapidities, i.e., no spin bound states have negative energies. In the thermodynamic limit the real roots of Eqs. (1) are distributed continuously over some intervals, which determine the Dirac seas of the model. The set of integral equations for the dressed densities of the rapidities u_α ($\rho(u)$) and dressed energies of the low-lying quasiparticles ($\varepsilon(u)$) are (see, e.g., Ref. 7 for the standard procedure of deriving these integral equations from the BAE and Refs. 11,13 for the isotropic two-chain spin-1/2 model)

$$\rho(u) + \int_{(Q)} dv\; K(u-v)\,\rho(v) = \sum_{\pm} \frac{N_\pm}{N}\,\rho_\pm^0 \quad (5)$$

and

$$\varepsilon(u) + \int_{(Q)} dv\; K(u-v)\,\varepsilon(v) = h - \sum_{\pm} \frac{N_\pm}{N}\,\varepsilon_\pm^0\,, \quad (6)$$

where the kernels of the integral equations are

$$K(u) = \frac{\partial \ln e_2(u)}{\partial u} = \frac{\sin(2\gamma)}{2\pi[\cosh(u) - \cos(2\gamma)]}\,, \quad (7)$$

and h is an external magnetic field. The values

$$\rho_\pm^0(u) = \frac{\partial \ln e_1(u\pm\theta)}{\partial u} \equiv$$

$$\equiv \frac{\partial p_\pm^0(u)}{\partial u} = \frac{\sin\gamma}{2\pi[\cosh(u\pm\theta) - \cos\gamma]} \quad (8)$$

are bare densities of the rapidities, and

$$\varepsilon_\pm^0(u) = h - \frac{\sin^2\gamma}{\cosh(u\pm\theta) - \cos\gamma} \quad (9)$$

are bare energies (here «bare» corresponds to non-interacting particles, and the interaction «dresses» them as usual [7]). The integrations are performed over the domain (Q), determined in such a way that the dressed energies inside these intervals are negative. The limits of integration are determined by the zeros of the dressed energies and are the Fermi points for each sea. The analysis of the integral equations (5) and (6) in an external magnetic field shows that in general, for some values of θ and h, there can be one Dirac sea (it corresponds to one minimum of the bare densities of the rapidities and, hence, to one minimum of the bare energy). On the other hand, for higher values of θ and for some domain of h, two Dirac seas of the same type (gapless, see below) of excitations are possible (for two minima of the bare energies of the rapidities and thus two minima of the bare density). Note that for $\theta \to \infty$ at fixed $N_{\pm}$ all the roots of the integral BAE separate into two sets of «right-» and «left-moving» seas, centered at $\pm\theta$, respectively.

Here we briefly revisit the analysis of Refs. 9,10, but for the case of the uniaxial two-chain model. Analytical solutions to Eqs. (5) and (6) can be easily obtained in closed form in the limit of zero field and equal lengths of the chains $N_+ = N_-$. The simplest nontrivial exited quasiparticle (spinon) is a hole in the Dirac sea for real rapidities, with the quasimomentum

$$p(u_0) = 2 \arctan \left(\frac{\sinh (\pi u_0/\gamma)}{\cosh (\pi\theta/\gamma)} \right), \qquad (10)$$

where u_0 is the spinon's rapidity. Note that for topological reasons such particles have to exist in pairs for the $SU(2)$-symmetric case, etc. [14,15]. The energy of this spinon is given by

$$^{o}(u_0) = -\sin\gamma \frac{\partial p(u_0)}{\partial u_0} . \qquad (11)$$

It can be rewritten as a function of the quasimomentum, i.e., in the form of the commonly used dispersion law

$$^{o}(p) = \frac{\pi}{\gamma} \sin\gamma \tanh \frac{\pi\theta}{\gamma} \sin \frac{p}{2} \left[\cos^2 \frac{p}{2} + \sinh^{-2} \frac{\pi\theta}{\gamma} \right]^{1/2} . \qquad (12)$$

A spinon corresponds in the usual Bethe ansatz classification of BAE solutions to a string of length 1 [7]. Naturally Eqs. (1) have string solutions of higher lengths too. Other spin excitations can be obtained as combinations of spinon quasiparticles and higher-length strings with different rapidities. However, spinons here are picked out because only their dressed energies may be negative, i.e., only spinons may form Dirac seas of the ground state of the model.

One can see that the dispersion law Eq. (12) of the low-lying excitation of the «easy-plane» two-chain spin-1/2 antiferromagnetic model is factorized into two parts: the gapless part at $p = 0$, π and the gapped one at $p = \pi/2$, cf. [9,10]. The former corresponds to the oscillations of the magnetization, while the latter is connected with the oscillations of the staggered magnetization [9]. The analysis, similar to the analysis of the solutions of Eqs. (5) and (6) for nonzero magnetic field $h \neq 0$ (here we point out that according to the very accurate analysis [16] the solution of the integral BAE in the first-order approximation reproduces correctly both the low- and high-coupling asymptotic behavior), shows that: (i) the dressed energy of a spinon as a function of the dressed quasimomentum has only one extremum, a maximum at $p = \pi/2$ for $\theta < \theta_c$, and (ii) for $\theta > \theta_c$ there are two maxima and one minimum (situated at $p = \pi/2$). At the (tri)critical point θ_c , the minimum disappears and the two maxima join into a flatter one (at $p = \pi/2$). In the limit $\theta \to \infty$ the mimimum is transformed into a cusp. It reveals that the gap of the staggered magnetization vanishes in this limit of two independent spin chains. This simple picture helps us to understand what happens if one switches on an external magnetic field h. Besides the usual phase transition to the ferromagnetic (spin-polarized) phase at

$$h_s = \sum_{\pm} \frac{N_{\pm}}{N} \varepsilon^0_{\pm}(0) , \qquad (13)$$

there is an additional transition between two phases. One of these corresponds to one Dirac sea of spinons (at small θ), while the other one is connected with two Dirac seas for the same kind of spinons (at large θ). It can also be seen from the right-hand side of Eqs. (5) and (6) for the densities and dressed energies that the bare density and bare energy (corresponding to terms which do not depend on $\rho(u)$ and $\varepsilon(u)$) have either one or two minima, respectively. Hence, they reproduce the same property in the dressed characteristics: the interaction simply «dresses» the (quasi)particles, as usual, but the «dressing» does not affect the picture qualitatively. The new critical field value can be approximated by $h_c \approx (\pi/\gamma) \sin\gamma \cosh^{-1} (\pi\theta/\gamma)$ in the first-order approximation [9]. In this approximation the tricritical point is the root of the equation $1 \approx \sinh (\pi\theta_c /\gamma)$. At this point two second-order phase transition lines h_s and h_c join. Hence, the «easy-plane» magnetic anisotropy in the antiferromagnetic two-chain model does not change qualitatively the ground state behavior in the external magnetic field; cf. [9,10]. However it changes the critical values of the magnetic field and the intra-chain coupling. The difference between the two (gapless) phases is obvious: the first phase corresponds to a Néel-like antiferromagnetic ground state for spins in both chains (along the zigzag line), while the second phase is connected with Néel-like antiferromagnetic ground states in each of the chains, i.e., effectively to two magnetic sublattices in the two-chain model.

That is why our simple model explains in which domains of parameters the two-chain spin system behaves like a one-sublattice quantum «easy-plane» antiferromagnet, and where it behaves like a two-sublattice one. Note also that the phase transitions we study here are manifestations of the commensurate—incommensurate phase transitions for spin systems. One can obviously see this, because the intra-chain coupling for two spin chains can be interpreted as the next-nearest-neighbor spin interactions for a single spin chain of higher length $N_+ + N_-$. Here the magnetic couplings are spin-frustrated, and so the emergence of the incommensurate magnetic states is understandable.

As a consequence of the conformal invariance of (1+1)-dimensional quantum systems, the classification of universality classes is simple in terms of the central charge (conformal anomaly C) of the underlying Virasoro algebra [17]. The critical exponents in a conformally invariant theory are scaling dimensions of the operators within the quantum model. They can be calculated considering the finite-size (mesoscopic) corrections for the energies and quasimomenta of the ground state and low-lying excited states. Conformal invariance formally requires all gapless excitations to have the same velocity (Lorentz invariance). The complete critical theory for systems with several gapless excitations with different Fermi velocities is usually given as a semidirect product of these independent Virasoro algebras [18]. Here we briefly sketch the procedure and write the results for the finite-size corrections to the energy, following the standard procedure; see, e.g., Ref. 18. One can see that for $\theta < \theta_c$ and for $\theta > \theta_c$, $h < h_c$, the conformal limit of our uniaxial two chain spin-1/2 model corresponds to one level-1 Kac—Moody algebra (one WZW model of level 1 with the conformal anomaly $C = 1$). The finite-size correction to the energy is rather standard (cf. [18]):

$$E_{fs}(N_+ + N_-) = -\frac{\pi}{6} v_F + 2\pi v_F (\Delta_l + \Delta_r) , \quad (14)$$

where v_F is the Fermi velocity of the spinon, and the conformal dimensions of the primary operators are (please note that the lower indices denote the conformal dimensions for right- and left-moving quasiparticles, at the right and left Fermi point, respectively)

$$2\Delta_{l,r} = \left(\frac{\Delta M}{2z} \pm z\Delta D\right)^2 + 2n_{l,r} , \quad (15)$$

where ΔM is an integer denoting the change of the number of particles induced by the primary operator; ΔD is an integer (half-integer) denoting the number of transferred particles from the right to the left Fermi point (back scattering processes); and $n_{l,r}$ are the numbers of particle—hole excitations of right- and left-movers. The values of the quantum numbers are restricted by $\Delta D = \Delta M/2 \pmod 1$. The dressed charge $z = \xi(Q)$ is the solution of the (standard) integral equation [18]

$$\xi(u) + \int_{(Q)} dv\, K(u - v)\xi(v) = 1 \quad (16)$$

taken at the limits of integration (these are the Fermi points, symmetric with respect to zero). In this phase there is only one region of integration over v. The dressed charge is a scalar. The behavior of our class of models in this phase in the conformal limit is rather standard [18]. The correlation functions decay asymptotically $\propto (x - v_F t)^{-\Delta_l} \times (x + v_F t)^{-\Delta_r}$. The choice of the appropriate quantum numbers of excitations ΔM, ΔD, and $n_{l,r}$ is determined for the leading asymptotic terms of the correlators by taking the possible numbers with smallest exponents.

But for $\theta > \theta_c$, $h > h_c$, the conformal limit of the «easy-plane» two-chain spin-1/2 model corresponds to the semidirect product of two level-1 Kac—Moody algebras, both with conformal anomalies $C = 1$, i.e., to two WZW models both of level 1 [9,10]. The Dirac seas (i.e., the possible spinons with negative energies) are in the intervals $[-Q^+, -Q^-]$ and $[Q^-, Q^+]$ (minima in the distributions of rapidities at $\mp\theta$). This can be interpreted as symmetrically distributed (around zero) Dirac seas of «particles» for $[-Q^+, Q^+]$ and the Dirac sea of «holes» for $[-Q^-, Q^-]$. In fact the valley in the density distribution for «particles» and the maximum for «holes» are in one-to-one correspondence with the maxima and minimum of the dispersion law for spinons. The second critical field h_c in this language corresponds to the van Hove singularity of the empty band of «holes». Naturally, the Fermi velocities of «particles» are positive, $v_F^+ = (2\pi\rho(Q^+))^{-1}\varepsilon'(u)|_{u=Q^+}$, while the Fermi velocities of «holes» are negative $v_F^- = -(2\pi\rho(Q^-))^{-1} \times \varepsilon'(u)|_{u=Q^-}$. The finite-size corrections to the energy for this case are

$$E_{fs}(N_+ + N_-) = -\frac{\pi}{6}(v_F^+ + v_F^-) + + 2\pi\left(v_F^+(\Delta_l^+ + \Delta_r^+) + v_F^-(\Delta_l^- + \Delta_r^-)\right), \qquad (17)$$

where the dispersion laws of «particles» and «holes» are linearized about the Fermi points for each Dirac sea. The conformal dimensions of the primary operators are (the upper indices denote Dirac seas; the lower indices denote right and left Fermi points of each of these two Dirac seas, cf. [10] for the isotropic spin-1/2 two-chain model):

$$2\Delta_{l,r}^{\mp} = = \left[\frac{(x_{-\pm}\Delta M^+ - x_{+\pm}\Delta M^-)}{2\det \hat{x}} \mp \frac{(z_{-\pm}\Delta D^+ - z_{+\pm}\Delta D^-)}{2\det \hat{z}}\right]^2 + 2n_{l,r}^{\mp}, \qquad (18)$$

where the minus sign between the terms in square brackets corresponds to the right-movers and the plus sign to the left-movers. Here $\Delta M^\pm$ denote the differences between the numbers of particles excited in the Dirac seas of «particles» and «holes» labeled by the upper indices. $\Delta D^\pm$ denote the numbers of backward scattering excitations, and $n_{l,r}^\pm$ are the numbers of particle—hole excitations for right- and left-movers of each of the Dirac seas (for «particles» and «holes»). Please note that $\Delta M^\pm$ and $\Delta D^\pm$ are not independent. Their values are restricted by the following relations: $\Delta M^+ - \Delta M^- = \Delta M$ and $\Delta D^+ - \Delta D^- = \Delta D$, where ΔM and ΔD determine in a standard way the changes of the total magnetization and the total momentum of the system, respectively, due to excitations. Please note that in Refs. 10, 19 these restrictions were missing; this resulted in, for example, the invalid statement that four independent backscattering low-lying excitations are possible. However one can see that only two of them are really independent. The same is true for excitations that change the total magnetization of the system: there are only two independent of four possible such excitations. This is a direct consequence of the fact that only one magnetic field determines the filling of the two Dirac seas for «particles» and «holes» or, in other words, two the two Dirac seas for spinons at $\pm\theta$.

The dressed charges $x_{ik}(Q^k)$ and $z_{ik}(Q^k)$ (i, $k = +, -$) are matrices in this phase. They can be expressed by using the solution of the integral equation [18,20]

$$f(u|Q^\pm) = \left(\int_{-Q^+}^{Q^+} - \int_{-Q^-}^{Q^-}\right) K(u-v) f(v|Q^\pm) = K(u - Q^\pm) \qquad (19)$$

,

with [18]

$$z_{ik}(Q^k) = \delta_{i,k} + (-)^k \frac{1}{2}\left(\int_{Q^i}^{\infty} - \int_{-\infty}^{-Q^i}\right) dv\ f(v|Q^k)\ ,$$

$$x_{ik}(Q^k) = \delta_{i,k} - (-)^k \int_{-Q^i}^{Q^i} dv\ f(v|Q^k)\ . \qquad (20)$$

Note that the dressed charges depend on the value of the magnetic anisotropy γ via the kernels, while they depend indirectly on the value of the intrachain coupling constant θ, only via the limits of integration. In the first-order approximation one can write the solutions as

$$x_{ik}(Q^k) \approx \delta_{i,k} - (-)^k \int_{-Q^i}^{Q^i} dv\ K(v - Q^k) + \ldots$$

and

$$z_{ik}(Q^k) \approx \delta_{i,k} + (-)^k(1/2)\left(\int_{Q^i}^{\infty} - \int_{-\infty}^{-Q^i}\right) dv\ K(u - Q^k) + \ldots$$

The Dirac sea for «holes» disappears, naturally for $h \to h_c$, $\theta \to \theta_c$. The slopes of the dressed energies of «particles» and «holes» at the Fermi points of the Dirac seas (Fermi velocities) differ in general from each other. Therefore we have a semidirect product of two algebras. Hence, in this region the dressed charges are 2×2 matrices. This means that the conformal limit of the «easy-plane» two-chain spin-1/2 model corresponds to one or two WZW theories, depending on the values of the intrachain coupling, magnetic anisotropy, and magnetic field. At the critical line h_c the Dirac sea of «holes» disappears as well as the components of the dressed charge matrix $\hat{\chi}$ (with square root singularities of the critical exponents for the correlation functions). Note that the dressed charge z becomes $z = (2x)^{-1}$ at the phase transition line h_c. This corresponds to the disappearence of one of the WZW CFTs. Unfortunately, it is impossible to obtain an analytical solution to Eqs. (19) in closed form for a finite interchain coupling θ. Naturally, in the limiting cases of two independent chains of lengths $N_{\pm}$, $\theta \to \infty$, and a single chain of length $N_+ + N_-$, $\theta = 0$, the solutions of Eqs. (16), (19), (20) coincide with the well-known solutions (see Ref. 18). The correlation functions of the uniaxial two-chain spin-1/2 model decay algebraically in this phase $\propto (x - v_F^+ t)^{-\Delta_l^+}(x - v_F^- t)^{-\Delta_l^-}(x + v_F^+ t)^{-\Delta_r^+}(x + v_F^- t)^{-\Delta_r^-}$ with the minimal exponents of the possible quantum numbers of excitations $\Delta M^{\pm}$, $\Delta D^{\pm}$, and $n_{l,r}^{\pm}$. We point out once more that the same magnetic field plays the role of a chemical potential for the «particles» and «holes», or for the spinons of both Dirac seas in the second phase, and hence this choice of «minimal quantum numbers» is constrained.

We must point out here that there is a crucial difference between our situation and the case of dressed charge matrices appearing for systems with the internal structure of bare particles [18]. There the two Dirac seas of the ground states are connected with different kinds of excitations, e.g., holons and spinons for the repulsive Hubbard model, or Cooper-like singlet pairs and spinons for the supersymmetric t–J model. They correspond to two different kinds of Lagrange multipliers, chemical potentials, and magnetic fields. Thus the low-lying excitations of the conformal theories in the spin and charge sectors of these correlated electron models are practically independent of each other (spin—charge separation). Note that the spin and charge sectors are connected via the off-diagonal elements of the dressed charge matrix, though. This is a consequence of the fact that, say, holons or unbound electrons carry both charge and spin. On the other hand, two Dirac seas appear for the same kinds of particles for the models studied in this paper, which are also connected with the same magnetic field governing the filling of both Dirac seas. These seas appear due to two minima in the bare energy distribution and correspond to nonzero shift θ in the Bethe ansatz equations. In other words, the two Dirac seas are determined by the interchain coupling and appear if the values of the coupling and external magnetic field are higher than the threshold values θ_c and h_c, respectively. We believe that such a threshold behavior does not depend on the integrability of the model and is a generic feature for any multi-chain quantum spin models.

The low-temperature Sommerfeld approximation shows that, as usual, the low-temperature specific heat off the critical lines is proportional to T. On the critical lines the van Hove singularities produce $\sqrt{T}$ low-temperature behavior of the specific heat, while at the tricritical point we have $T^{1/4}$ behavior.

What are the changes due to the different lengths of the chains $N_+ \neq N_-$? One can see obviously that the values of the spinon momentum, energy, and velocity (which was $v = (\pi/\gamma) \sin\gamma \tanh(\pi\theta/\gamma)$) become functions of $N_+ - N_-$. For example, the velocity renormalizes as $v \to v[1 + (N_+ - N_-)^2 \tanh^2(\pi\theta/2\gamma)/N^2]^{-1}$. This introduces dependences of the critical values θ_c and h_c (as well as of the saturation field h_s) on the difference $N_+ - N_-$. Also, the Fermi velocities and Fermi points for finite-size corrections become dependent on this difference. One can in principle consider different coupling constants $J_{\pm}$ for each of the chains (overall multipliers [21]). This produces renormalizations similar to the effect of $N_+ \neq N_-$, i.e., the velocity, for example, renormalizes as $v \to J_+ v[1 + (J_- / J_+)^2 \tanh^2(\pi\theta/2\gamma)]^{-1}$.

3. Connections to the quantum field theories

The studies presented in the previous section, being rather standard (note, though, some important new features, which were absent in the previous studies [4,9,10,11,13,19], such as the dependence of the critical values of the interchain coupling and external magnetic field on the parameter of magnetic anisotropy and on the difference in the lengths of the chains; also the important restrictions on the quantum numbers of low-lying conformal excitations). However, we will use the results of that Section for novel studies for a wider classes of exactly solvable models in Sections 3–5. For instance, in this Section we point out the important similarities in the behaviors of the two-chain quantum spin model considered in the previous Section and several models of QFT.

Really, when examinating Eqs. (1), one can see that these Bethe ansatz equations coincide with the equations which describe the behavior of the spin (color) sector of some QFT. $N_{\pm}$ corresponds to the numbers of (bare) particles with positive and negative chiralities. For example, for the chiral-invariant Gross—Neveu model [14,22] we have to put $\gamma \to 0$ (i.e., the $SU(2)$-symmetric case, equivalent to the $SU(2)$-symmetric Thirring model), and $\theta = (1 - g^2)/2g$, where g is the coupling constant of the chiral invariant Gross—Neveu QFT [14]. As to the Lagrange multiplier h, it can play the roles of either an external magnetic field or the chemical potential, or an external topological field dual to the topological Noether current in QFT. Here we point out that in fact in QFT the theorists are interested in physical particles, which have a finite mass (gap). In the chiral-invariant Gross—Neveu model the gap of the staggered oscillations of the two-chain quantum spin model plays the role of the physical mass of the particle (spinor) [13,14]. As to the (gapless) oscillations of the magnetization of the two-chain spin model, we point out that they are consequences of the lattice and play the role of the massless fermion doublers of the lattice QFT [23]. The results of the previous section mean that the behavior of the chiral-invariant Gross—Neveu model (or $SU(2)$-symmetric Thirring model) in an external magnetic field depends strongly on the coupling constant θ (or equivalently on g). For $\theta < \theta_c$ the conformal limit of the QFT corresponds to one level-1 WZW model with the conformal dimension $C = 1$. However, for $\theta > \theta_c$ $(-\theta_c - \sqrt{\theta_c^2 + 4} < 2g < -\theta_c + \sqrt{\theta_c^2 + 4})$ the conformal limit of this QFT in an external magnetic field corresponds to the semidirect product of two level-1 WZW models with the conformal dimensions $C = 1$. Two kinds of conformal points for this QFT have been mentioned already [24] in a slightly different context. They were connected with one WZW theory or two WZW theories, coupled via a current—current interaction. This is related to right—left symmetry of the chiral invariant Gross—Neveu QFT (see also Refs. 32,33 for the case of the QFT for the principal chiral field).

Note that the condition $h > h_c$ in the QFT means that the magnetic field is larger than the mass of the physical particle (color spinor). In this sense, in the region of magnetic field values $h < h_c$ the results of the QFT (see, e.g., [22]) predict zero magnetization; however, a different lattice regularization, similar to the lattice scheme used in the previous Section, predicts a nonzero magnetization of the chiral-invariant Gross—Neveu model in this region. This is an indirect effect of the fermion doublers. In other words, it is connected with the well-known mapping of the lattice (e.g., Thirring) model under regularization onto two continuum QFTs either both bosonic (the free bosonic and sine-Gordon QFT [25]), or both fermionic ones (a free one and the continuum massive Thirring model). There are necessarily two such theories because of the Nielsen—Ninomiya fermion doublers: remember that we have started from a lattice [23].

For other models of QFT the lattice regularization procedure [26–28] has been used. Here θ plays the role of the cutoff for keeping the mass of the physical particle finite. For example, for the $U(1)$-symmetric Thirring QFT [23,29] one can use the results of the previuos section with the limit $\theta \to \infty$ taken after the thermodynamic limit (L, $N_\pm$, $M \to \infty$ with their ratios fixed, L is the size of the box). In this case one can obviously obtain the conformal limit of the theory with nonzero physical masses of the particles. Naturally, in the limit $\theta \to \infty$ we always exist, in an external magnetic field, in the phase with two Dirac seas. Here the latters correspond to the right- and left-moving particles (with positive and negative chiralities). Actually here our point of view coincides with that of the field theorists. Recently it was shown [30] that for the(1+1)-dimensional sine-Gordon model the lattice regularization scheme in the «light-cone» approach gives results similar to ours for the conformal limit of the model. It was shown there that at the UV fixed point the conformal dimensions of the sine-Gordon model are determined by two $U(1)$ charges of excitations (the usual one and the chiral charge). The chiral charge corresponds to the number of excitations transferred from one Dirac sea to the other, similar to our results (note that the above-mentioned lattice-regularized sine-Gordon case corresponds in our notation to $\theta \to \infty$, where the integral equations for the particles with the positive and negative chiralities are totally decoupled). We point out here, that such behavior is not unexpected, because the sine-Gordon QFT belongs to the same class of models studied in our paper, i.e., its Bethe ansatz description features a shift of rapidities in the Bethe ansatz equations in the lattice-regularized theory [30].

4. Higher spin (chirality) generalizations

For the higher spin generalizations of the Bethe ansatz theory presented in Section 2 we can write the BAE in the form

$$\prod_{\pm} e_{n_\pm}^{N_\pm}(u_\alpha \pm \theta) = \mathrm{e}^{i\pi M} \prod_{\beta=1,\beta\neq\alpha}^{M} e_2(u_\alpha - u_\beta) \ , \quad (21)$$

where $n_\pm = 2S_\pm$ are the values of the spins in each chain or the colors of the bare particles in QFT. The eigenvalue of the transfer matrix can be written as

$$\Lambda(\lambda) = \prod_{\alpha=1}^{M} \frac{\sinh(\lambda - u_\alpha + i\gamma/2)}{\sinh(u_\alpha - \lambda + i\gamma/2)} +$$

$$+ e^{i\pi M} \prod_{\pm} \left(\frac{\sinh(\lambda \pm \theta)}{\sinh(i\gamma n_\pm/2 - \lambda \mp \theta)} \right)^{N_\pm} \times$$

$$\times \prod_{\alpha=1}^{M} \frac{\sinh(u_\alpha - \lambda + 3i\gamma/2)}{\sinh(\lambda - u_\alpha - i\gamma/2)} . \quad (22)$$

Similar new phases with one or two kinds of Dirac seas for similar kinds of low-lying excitations also exist for a number of models in which $n_\pm \neq 1$, e.g., for the higher-spin ($S > 1/2$) two-chain models with equal spins in each chain, $SU(n+1)$ CIGN QFT [31]: there $n_+ = n_- = n \neq 1$; for the principal chiral field models (nonlinear σ model) for CP-symmetric [32] (there $n_+ = n_- \to \infty$) and CP-asymmetric cases [33] (there $n_+ \neq n_-$, $(n_+ + n_-) \to \infty$, $(n_+ - n_-)$ fixed, i.e., the symmetry $SU(2) \times SU(2) \propto O(4)$); and for the $O(3)$-symmetric nonlinear σ model [34] as well as for spin-($S_+ \equiv 2n_+$) — spin-($S_- \equiv 2n_-$) two-chain models (quantum two-chain ferrimagnet). Note that for spins $S \neq 1/2$ the procedure of the construction of the Hamiltonian is more complicated, because it corresponds to the two-chain uniaxial generalization of the Takhtajan-Babujian model; see, e.g., Ref. 35. For the simplest case of isotropic exchange interactions between the spins and between the chains the Hamiltonian has the form

$$H = \sum_n \Big\{ \theta^2 (H_{S_+,S_+,n_1,n_1+1} + H_{S_-,S_-,n_2,n_2+1}) + 2(H_{S_+,S_-,n_1,n_2} + H_{S_-,S_+,n_1,n_2+1}) +$$
$$+ \left[(H_{S_+,S_+,n_1,n_1+1} + H_{S_-,S_-,n_2,n_2+1}), (H_{S_+,S_-,n_1,n_2} + H_{S_-,S_+,n_1,n_2+1}) \right] +$$
$$+ 2i\theta \left[(H_{S_+,S_+,n_1,n_1+1} + H_{S_-,S_-,n_2,n_2+1}), (H_{S_+,S_-,n_1,n_2} + H_{S_-,S_+,n_1,n_2+1}) \right] \Big\} , \quad (23)$$

where [...]{[...]} denote the (anti)commutator,

$$H_{S_1,S_2,n,n+1} =$$
$$= \sum_{j=|S_1-S_2|+1}^{S_1+S_2} \sum_{k=|S_1-S_2|+1}^{j} \frac{k}{k^2+\theta^2} \prod_{l=|S_1-S_2|}^{S_1+S_2} \frac{x - x_l}{x_j - x_l}, \quad (24)$$

$x = \mathbf{S}_{1,n} \mathbf{S}_{2,n+1}$ and $2x_j = j(j+1) - S_1(S_1+1) - S_2(S_2+1)$. The summation over n runs to $N_\pm$ in each chain. One can obviously see that for $S_\pm = 1/2$ the Hamiltonian (23) recovers the isotropic antiferromagnetic spin-1/2 Hamiltonian (4) investigated in Section 2. For a single spin chain, $\theta = 0$, $N_+ = N_-$ the Hamiltonian coincides with the known Hamiltonian of alternating spin chains [36–38]. The Bethe-ansatz studies of the model for $n_\pm$ can be performed in complete analogy with the above-mentioned case $n_\pm = 1$, keeping in mind, of course, the main difference: for the $SU(2)$-symmetric or uniaxial higher-spin models the ground state corresponds to the filling up of the Dirac seas for spin strings of lengths $n_\pm$ [35]. The well-known fusion scheme can be used for the case of a flavor-degenerate situation of the chiral invariant Gross—Neveu QFT, in the absence of flavor fields [39]. Note that, except for the $O(3)$-symmetric case, $\gamma = 0$ everywhere in the above-mentioned models of QFT. This corresponds to rational solutions of the Yang—Baxter equation for the two-particle scattering matrices. For the two spin chains the two-chain quantum ferrimagnet model corresponds to two Takhtajian—Babujian chains with different values of the site spins, coupled due to nonzero θ. The total quasimomentum and the energy of the system in the framework of the lattice (local) regularization scheme for some QFT can be written as [23]

$$-2a_t E = \sum_\pm \sum_{\alpha=1}^{M} \frac{\partial}{\partial u_\alpha} N_\pm \ln e_{n_\pm}(u_\alpha \pm \theta) ,$$
$$iaP = \sum_\pm \sum_{\alpha=1}^{M} N_\pm \ln e_{n_\pm}(u_\alpha \pm \theta) , \quad (25)$$

where a and a_t denote the space and time lattice constants, respectively, and their ratio fixes the velocity of light («light-cone» approach). The CP-symmetric (chiral invariant) case corresponds to the situation in which $n_+ = n_- = n$. The Dirac seas are related to the dressed (quasi)particles with negative energies (strings of length $n_\pm$). The behavior of the dispersion law for excited particles in the CP-symmetric case ($n_+ = n_- = n$ and $N_+ = N_-$) is similar to Eq. (12): for instance, for the chiral-invariant Gross—Neveu QFT and principal chiral field model the right-hand side of Eq. (12) must be simply multiplied by $\sin(\pi r/n{+}1)/\sin(\pi/n+1)$, and the parameter θ in Eq. (12) has to be replaced by $(n+1)\theta/2$, $r = 1, \ldots, n$ is the rank of a fundamental representation of the $SU(n+1)$ algebra. All the previously mentioned characteristic features from the case $n_\pm = 1$ persist. The differences are in the levels of the Kac—Moody algebras in the conformal limit: the conformal anomalies are $C = 3n/(n+2)$. Now the conformal field theory is a semidirect product of a Gaussian ($C = 1$) [40] and a $Z(n)$ parafermion models [41]: the operators identified from the scaling behavior of states consisting of Dirac sea strings only (found from finite-size corrections) are found to be composite operators formed by the product of a Gaussian-type operator and the operator in the parafermionic sector. To find the nonzero contributions from parafermions (constant shifts) one can consider the states with strings of other lengths than the Dirac sea present [42]. For the scaling dimensions these shifts are $(2r - r^2)/(2n+4)$, $r = 1, 2, \ldots$

From now on we concentrate on the $n_+ \neq n_-$ situation. For the two-chain spin system the situation corresponds to the quantum ferrimagnet. Here we point out that due to the zigzag-like interactions in the system and spin frustration the ferrimagnets of this class are in the singlet ground state (compensated phase) for $h = 0$, unlike the standard classical ferrimagnets in uncompensated phases. The integral equations that determine the physical vacuum of the systems are similar to Eqs. (5) and (6). They reveal one or several minima of the corresponding distributions of dressed energies and densities with possible negative energy states, i.e., one or several Dirac seas:

$$\varepsilon_\tau(u) + \int dv\ K_{\tau\tau'}(u-v)\varepsilon_{\tau'}(v) = h\,\frac{N_\tau}{N}\,n_\tau + \sum_\pm \frac{N_\pm}{N}\,\varepsilon^0_{\tau,\pm}\,,$$

$$\rho_\tau(u) + \int dv\ K_{\tau\tau'}(u-v)\rho_{\tau'}(v) = \sum_\pm \frac{N_\pm}{N}\,\rho^0_{\tau,\pm}\,. \qquad (26)$$

The index τ enumerates two possible Dirac seas and appears because $n_+ \neq n_-$, and the $\pm$ enumerate two possible minima due to the nonzero shift θ. The index τ was naturally absent for the CP-symmetric case $n_+ = n_-$. Note that for quantum two-chain ferrimagnets the investigated gapless phases in the ground state in an external magnetic field are similar to the spin-compensated and uncompensated phases. Thus the phase transition between those phases is similar in nature to the well-known spin-flop phase transition in the classical theory of magnetism. Note, though, that the spin-flop transition is of the first order («easy-axis» magnetic anisotropy), while the transition under study is a second-order one («easy-plane» anisotropy). The Fourier transform of the kernel is given by

$$2\coth(\omega/2)\times$$

$$\times[\mathrm{diag}(e^{-n_+|\omega/2|}\cosh(n_+\omega/2), e^{-n_-|\omega/2|}\cosh(n_-\,\omega/2)) -$$

$$\hat{\sigma}_x\,(e^{-(n_+-n_-)|\omega/2|} - e^{-(n_++n_-)|\omega/2|})]\,, \qquad (27)$$

where diag (a, b) is 2×2 diagonal matrix and $\hat{\sigma}_x$ is the usual Pauli matrix. Note that after taking the limit $(n_+ + n_-) \to \infty$, which is the case of the CP-asymmetric case of the QFT for the principal chiral field, i.e., with the Wess—Zumino term [33], the inverse kernel coincides formally (up to a constant multiplier) with the one for the case $n_+ = n_- = 1$. This indicates a global $O(4)$ ($O(3)$) symmetry of the principal chiral field [33]. There may also be two different behaviors, corresponding to one or several Dirac seas for $n_+ \neq 1$ or $n_- \neq 1$. Naturally, in the conformal limit the associated WZW CFTs have different conformal anomalies determined by $n_\pm$: $C_\pm = 3n_\pm/(n_\pm + 2)$. For the determination of the Gaussian parts of the conformal dimensions of primary operators, Eqs. (18) can be used. One has to add the input from the parafermionic sectors too [41,42]. The elements of the dressed charge matrices are the solutions of the following system of integral equations:

$$\xi_{\tau,\tau'}(u) + \sum_\pm \int dv\,K_{\tau'}(u-v)\xi_{\tau,\pm}(v) = \delta_{\tau,\tau'}\,, \qquad (28)$$

in which the summation over $\pm$ is due to the two possible Dirac seas (two minima in the distribution of rapidities) at $\pm\theta$. For different values of the spins, $n_+ \neq n_-$, a transition between two different phases is induced by increasing an external magnetic field to some critical value, even in the absence of the shift θ [37,38]. This differs from the CP-symmetric case $n_+ = n_-$, where the phase transition is only connected with the nonzero value of the intrachain coupling parameter θ. For the CP-symmetric case, one or two Dirac seas of the same type of excitations exist due to the nonzero θ. But in the CP-asymmetric case the existence of two Dirac seas can be related to two kinds of different low-lying excitations (particles). They are strings of lengths n_+ and n_-, respectively. In this situation the dispersion laws may be independent (not only factorized as for the previous CP-symmetric cases). The (new) phase transition at h_c reveals the van Hove singularity of the empty Dirac sea for the longer strings. The spin saturation field h_s is connected with the empty Dirac sea of strings of the smaller length.

5. Multi-chain quantum spin models

It is worthwhile to mention that phase transitions in an external magnetic field, similar to the ones studied in this paper for uniaxial spin chains and QFT, have been already studied in $1D$ quantum alternating single spin chains [37,38], spin-1/2 isotropic two-chain models [9,10], and correlated electron models with the finite concentration of magnetic impurities [43]. The Bethe ansatz equations for those models are similar to the ones studied in the present paper, Eqs. (1), (21). Note that the energies for spin models are defined (as usual for the lattice models) as the first logarithmic derivatives of the transfer matrices. The factorization of the dispersion law for the lowest excitations (spinon) reveals essentially two kinds of magnetic oscillations: excitations of the magnetization and oscillations of the staggered magnetization, i.e., the manifestation of essentially two magnetic sublattices. Naturally, the existence of the latters persists in the continuum limit of such systems, too (cf., for instance, the standard theory of antiferromagnetism). Two nonferromagnetic phases also reveal themselves in finite-size corrections to the energies of these quantum spin models. There, instead of a scalar dressed charge for the phase with one Dirac sea for spinons, 2×2 dressed charge matrices appear in the second phase, with two Dirac seas for spin strings of different lengths in an alternating spin chain [37,38] or for spinons of the same kind in zigzag-like coupled spin chains (see [9,10] for the isotropic two-chain spin-1/2 model).

The symmetry-breaking terms [the difference $(n_+ - n_-) = 2(S_1 - S_2)$, or nonzero θ] in BAE are actually the reason for the emergence of several gapless phases (or two Dirac seas) in the ground state in an external magnetic field. It is also interesting to note that a homogenuous shift of rapidities can be removed for one Dirac sea in the case of periodic boundary conditions by an appropriate unitary (gauge) transformation (shift of variables), e.g., $u_\alpha \to u_\alpha \pm \theta$. But in the case of open boundary conditions, the BAE take the form (for reasons of simplicity we write the free boundary situation only, without the external boundary potential):

$$\prod_{\pm} e^{2N}_{n_\pm}(u_\alpha \pm \theta) = \prod_{\pm}\prod_{\beta} e_2(u_\alpha \pm u_\beta) \ . \qquad (29)$$

It is clear that for the open chain one cannot remove the shift θ of the rapidities u_α from one Dirac sea by a special choice of the gauge. From this point of view the latter case is close to the CP-asymmetric situation in QFT.

One can see from the structure of the Hamiltonians that for the two-chain spin models the parameter θ characterizes the intrachain coupling for each chain (or the next-nearest-neighbor interaction in a single spin chain picture). It is obvious to introduce the series of $\{\theta_j\}_{j=1}^J$ (for each chain) and to construct the Hamiltonian of the exactly integrable multichain (J is the number of chains) spin model. For the simplest case of all $S = 1/2$ isotropic antiferromagnetic chains the Hamiltonian reads [4]:

$$\hat{H}_J = A \sum_n \left\{ \left[\prod_{i,k} (\theta_i - \theta_k) \right] \hat{P}_{S_{n,r} S_{n+1,r}} + \right.$$

$$\left. + \sum_{p<q} \frac{\Pi_{i,k}(\theta_i - \theta_k)}{(\theta_p - \theta_q)} \left[\hat{P}_{S_{n,q} S_{n+1,p}} , \hat{P}_{S_{n,q} S_{n+1,q}} + \hat{P}_{S_{n,p} S_{n+1,p}} \right] + \ldots + \left(\sum_{j=1}^{J} \hat{P}_{S_{n,j} S_{n,j+1}} - \hat{P}_{S_{n,J} S_{n,J+1}} + \hat{P}_{S_{n,J} S_{n+1,1}} \right) \right\} , \qquad (30)$$

where A is the normalization constant (which depends on θ_j); $\hat{P}_{S_a S_b} \propto (1/2)\hat{I} \otimes \hat{I} + 2\mathbf{S}_a \otimes \mathbf{S}_b$ is the permutation operator; and [.,.] denotes a commutator. Note that in the case of $J \neq 2$ the integrable model corresponds to the pair couplings not only between the nearest-neighbor spins but also to the next-nearest three-spin, etc., couplings. All those terms are only essential in quantum mechanics, because in classical physics they are total time derivatives [11] and do not change the equations of motion. The Bethe ansatz equations have the form

$$\prod_{j=1}^{J} e_1^{N_j}(u_m + \theta_j - \theta_1) = \mathrm{e}^{i\pi M} \prod_{k}^{M} e_2(u_m - u_k) \ , \quad (31)$$

where M is the total number of down spins and N_j is the number of sites in the jth chain. The previously studied situation $J = 2$ corresponds to the shift of the variables $u_m \to u_m + \theta$ with $\theta_2 - \theta_1 = -2\theta$. Now $\theta_j - \theta_1$ determines the values of the intrachain couplings in chain j.

The analysis of the low-temperature thermodynamics of the multichain spin system is analogous to the situation of $J = 2$ studied in Sections 2–4. From the structure of the Bethe ansatz equations in the thermodynamic limit N_j , $M \to \infty$, with their ratios fixed, one can see that for the J-chain model (for different θ_j) there can exist, generally speaking, J phase transitions of the second order in the ground state in an external magnetic field. These are none other than the commensurate—incommensurate phase transitions for the quantum multichain spin model with different couplings between the chains. The values of the critical fields h_{c_1} , ..., $h_{c_{J-1}}$ and the value of the magnetic field of the transition to the ferromagnetic state h_s depend on the set of θ_j , i.e., on the intrachain couplings (and also on the values of the magnetic anisotropy constants, which can be taken different for each chain; this does not destroy the integrability). The ferromagnetic state is gapped, while all other phases are gapless in the integrable multichain spin quantum model. There are also $J - 1$ tricritical points at which the lines of the phase transitions h_{c_j} join the line of the spin-saturation phase transition. Naturally, the phase that corresponds to the lowest value of the magnetic field, say $h < h_{c_1}$ for special values of θ_j (the condition is similar to $\theta < \theta_c$ for $J = 2$), has in the conformal limit one scalar dressed charge. Hence, in the conformal limit our multichain spin model behaves as the level-1 WZW CFT. In the next phase the multichain quantum spin model behaves as the semidirect product of two WZW CFTs, hence their dressed charges are 2×2 matrices, and so on, until the last gapless phase, which corresponds to the semidirect product of J WZW CFTs with $J \times J$ dressed charge matrices. Note that J in this approach also denotes the number of possible Dirac seas (each of them is connected with the same magnetic field, so the excitations in each of them are not independent), and thus, with one-half of the number of Fermi points. In the limit $J \to \infty$ (i.e., quasi-2D spin system) one obtains the (2D) Fermi surface instead of the set of 1D Fermi points (the latter become distributed more closely to each other with the growth of J). In this limit the differences between θ_j tend to zero, and that is why the differences between θ_{c_j} , h_{c_j} , and also between h_{c_j} and h_s disappear, too. Therefore in this limit only h_s survives. This means that for the quasi-2D limit of such an integrable model of J coupled quantum spin chains for $J \to \infty$ we expect only two phases in the ground state in an external magnetic field: the ferromagnetic gapped one and the gapless phase, which in the conformal limit corresponds to one WZW CFT (with a single scalar dressed charge). The phase transition between these two phases in the ground state in an external magnetic field is of the second order.

6. Behavior of the nonintegrable multichain spin systems

So far we have studied only integrable multichain quantum spin models. We have shown that the commensurate—incommensurate phase transitions of the second order have to reveal themselves in an external magnetic on account of the intrachain interactions (or the next-nearest interactions in a single quantum spin chain picture). We have shown that the emergence of these phase transitions does not depend on the value of the site spins; they emerge in the presence of «easy-plane» magnetic anisotropy, which keeps the system in the critical (gapless) region. It is not clear, however, which features of the behavior of the integrable models with the «fine-tuned» parameters have to exist for more realistic multichain models, and what are the qualitative differences that are expected to exist between the integrable multichain models and real multichain spin systems.

We have to add one more thing to clarify the situation: we study (quasi)-1D spin quantum models, for which one can use the Lieb—Schultz—Mattis theorem (and its generalizations) [8,44]. However, it is obvious that due to the frustration of the interactions between neighboring spins and the presence of additional terms in the Hamiltonians which violate the time-reversal and parity symmetries in the systems (spin chiralities or spin currents), for all of the spin models studied in this paper one cannot satisfy the conditions of the theorem. Hence it cannot be applied (at least not directly). That is why for all the models studied there are no spin gaps (except for the trivial one for the spin-polarized ground state). (Here we are not talking about the gaps connected with the magnetic anisotropy but rather about the Haldane-like spin gaps [45] which appear even for the isotropic spin—spin interaction, and about fractional magnetization plateaus [8]). As we argued before [11], it is the presence of the chiral spin terms (or the operators of the nonzero spin currents) in the Hamiltonian (which are total time derivatives and do not change the classical equations of motion but rather affect the topological properties, like the choice of the θ-vacuum in Haldane's approach) is the reason why the low-lying spin excitations (and particles for lattice QFT) for our class of models are gapless and our low-energy theories are conformal. It has to be mentioned that recent results of the perturbative RG analysis of the zigzag spin-1/2 chain without three-spin terms shows the tendency of the RG currents to flow to the state with the parity and time-reversal violation [46]. By the way, one can obviously see that the XY limit of the two-chain spin model does not correspond to the free fermion point of the exactly solvable model, and this coincides with the results of Ref. 46. Note, though, that in the latter it was erroneously concluded that the time-reversal and parity symmetries were violated by the two-chain zigzag spin Hamiltonian with only two-spin couplings (i.e., the nearest- and next-nearest-neighbor interactions in the single chain picture), without spin current terms in the Hamiltonian. Hence the symmetry of the state considered was lower than the symmetry of the Hamiltonian there.

Naturally, the relevant perturbations to our integrable models will immedeately produce spin gaps. As usual, the algebraic (power-law) decay of the correlation functions in the ground state of the models considered in this paper determines the quantum criticality. This means that, starting from the (conformal) exact solutions obtained in this paper, one can argue that the response of the more realistic spin systems to perturbations can be evaluated by using perturbative methods, e.g., in a renormalization group framework. For example, let us study the effect of relevant perturbations to the Hamiltonians considered, $\hat{H}_r = \hat{H} + \delta\hat{H}_1$, where one can choose as $\hat{H}_r$, e.g., the standard Heisenberg or uniaxial Hamiltonians for several coupled quantum spin chains, and as $\hat{H}$ the Hamiltonians of spin chains considered exactly in this paper for some values of θ, where the three-spin terms are relevant. The correction to the ground state energy and the excitation gap (mass of the particle in QFT) for the quantum critical system are $\Delta E \propto -\delta^{(d+z)/y}$ and $m \propto \delta^{1/y}$, respectively, where d is the dimension of the system, and z is the dynamical critical exponent. For the conformally invariant systems studied here one has $d = z = 1$. The application of the standard scaling relations yields $y + x = 2(= z + d)$, where x is the scaling dimension, i.e., $x = 2\Delta_l + 2\Delta_r$, found in the previous sections (for the phases with the dressed charge matrices the summation over upper indices is meant). Hence the gap for the low-lying excitations (the mass of the physical particles in QFT) for the perturbed systems will be $m \propto \delta^{1/2(1-\Delta_l-\Delta_r)}$. Note that because of scaling, the behavior of the critical exponents (which are related to the exponents we introduced for the integrable multichain spin models) in the vicinities of the lines of phase transitions has to be universal, and this can be checked experimentally. We expect that the spin gap has to exist for values of the isotropic zigzag interchain coupling higher than or of the order of 0.5 for the two-chain spin-1/2 system [9], where the three-spin couplings are relevant and the emergence of the spin gap is known exactly [47].

Very recently, density matrix renormalization group numerical studies of the two-chain zigzag spin-1/2 model (without chiral three-spin terms in the Hamiltonian) were performed in Ref. 48. These numerical studies strongly support the picture proposed here (see also Ref. 9): the magnetization as a function of the magnetic field in the ground state reveals (i) one second-order phase transition (to the spin-saturation phase) for weak intrachain coupling; (ii) one more second-order phase transition between the magnetic (gapless) phases in the intermediate region of intrachain coupling, and (iii) in addition to those second-order phase transitions, one to the gapped phase with zero magnetization (plateau) at an intrachain coupling value of 0.5.

We should also mention that it is not the chiral spin terms (as implied in Ref. 10) but the intrachain coupling that is responsible for the commensurate-incommensurate phase transitions between the gapless phases in this class of models. As to the aforementioned spin currents, their «fine-tuned» values produce the cancellation of the spin gap for zero magnetic field [9]. We should also note that to our mind some features of the phase diagram obtained in Ref. 19 are artefacts of the small number of sites involved into the numerical calculations. In Fig. 5 of Ref. 19 in the regions $0.52 < \kappa < 0.6$ (corresponding to intrachain couplings, normalized to the value of the interchain interaction, in the range [0.54–0.75]) we can obviously see that when increasing the value of the magnetic field one goes from the gapped phase with zero magnetization into the gapless one with two Dirac seas of low-lying excitations, then reaches the gapless phase with one Dirac sea, then returns to the gapless phase with two Dirac seas, and finally reaches the spin-saturated phase. To our mind this return to the already passed phase is nonphysical. One can clearly see that the region of values of the intrachain couplings in which these strange returns happen is reduced in size when going from 16 sites in the numerical calculations to 20 sites. This confirms that presently achieved sizes of the quantum systems for numerical calculations can produce even qualitatively invalid results, and analytical calculations are necessary, too.

We point out that despite the fact that the relevant perturbations in general produce a gap for the low-lying excitations, one can apply the results of this paper to real gapless multichain spin systems, too. For example, it was recently observed that even for the two-leg ladder system $SrCa_{12}Cu_{24}O_{41}$ the spin gap collapses under pressure [49].

7. Concluding remarks

In this paper, motivated by recent progress in the experimental measurements for multichain spin systems, we have theoretically studied the behavior in an external field of a wide class of multichain quantum spin models and quantum field theories. First, we have investigated the external field behavior of the exactly integrable two-chain spin-1/2 model and have shown that the inclusion of the magnetic anisotropy of the «easy-plane» type, for which the system stays in the quantum critical region, does not qualitatively change the behavior in an external magnetic field. However, we have shown that the magnetic anisotropy changes the critical values of the magnetic fields and intrachain couplings at which the phase transitions occur, and affects the critical exponents. We have shown that the external-field-induced phase transitions we discussed are the commensurate—incommensurate phase transitions due to the next-nearest-neighbor two-spin interactions, which are present in these multichain models with zigzag-like couplings. We have pointed out that the low-lying excitations of the conformal limit of our class of multichain spin models are not independent in the incommensurate phase, because they are governed by the same magnetic field. We have shown that these two-chain quantum spin models share the most important features of the behavior in an external field with the wide class of (1+1) quantum field theories.

We have introduced higher-spin versions of the two-chain exactly solvable spin models, e.g., we have investigated the important class of 1*D* two-chain quantum ferrimagnets with different spin values at the sites of each chain. Here we have shown that the phase transitions in an external magnetic field in this exactly solvable two-chain quantum ferrimagnet are similar in nature to the phase transitions between the spin-compensated and uncompensated phases in ordinary classical 3*D* ferrimagnets.

We have also studied the behavior of the multichain exactly solvable spin models in an external magnetic field and shown how the additional phase transitions arising due to the increasing number of chains vanish in the quasi-2*D* limit. Hence, to the best of our knowledge, we have proposed the first exact scenario of the transition from 1*D* to 2*D* quantum spin models in the presence of an external magnetic field. We have argued that the commensurate—incommensurate phase transitions in the multichain quantum spin models have to disappear in the limit of an infinite number of chains.

Finally, we have shown how the relevant deviations from integrability, i. e., the presence of three-spin (spin chiral) terms in the Hamiltonians which separately break the parity and time-reversal symmetries, give rise to gaps in spectra of the low-lying excitations of the multichain quantum spin systems, and we have calculated the critical scaling exponents for these gaps. We pointed out the qualitative agreement of our exact analytical calculations with recent numerical simulations for zigzag spin models.

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