J. H. Dshalalow[1], **R. Robinson**
Department of Mathematical Sciences, College of Science
(Florida Institute of Technology, Melbourne, FL 32901, USA)
E-mail's: eugene@fit.edu and robinsor@my.fit.edu

On Fluctuations of a Nonmonotone Marked Point Process[2]

The present article investigates a bivariate recurrent process, which can describe the behavior of a nonmonotone financial instrument observed at random times. We are able to find explicitly the joint distribution of the highest value of the instrument prior to its first drop using a game-theoretic approach.

Исследован бивариантный рекуррентный процесс, с помощью которого можно описать поведение немонотонного финансового инструмента, наблюдаемое в случайные моменты времени. С использованием теоретико-игрового подхода явно определено объединенное распределение наибольшей величины инструмента, предшествующей его первому падению.

K e y w o r d s: random walk analysis, stock market, stochastic games, antagonistic games, fluctuation theory, marked point process, compound Poisson process, ruin time, exit time, first passage time, stochastic finance.

1. Introduction. In various applications to economics it is of interest to investigate a financial instrument observed over some random times. The key reference points of interest are: the time of the first drop of the stock or hedge fund if it increases, its highest value prior to the first drop and other data.

In our recent article [1], we assumed that the stochastic process which modeled a financial instrument we studied was monotone. In the present setting, the stochastic process under investigation is not monotone.

Consider a financial instrument observed at random moments of time $\tau_0, \tau_1, \tau_2 ...$ with respective values $B_0, B_1, B_2, ...$ so that

$$(\mathcal{B}, \mathcal{T}) = \sum_{k \geq 0} Y_k \varepsilon_{\tau_k}$$

(Y_k are the increments $B_k - B_{k-1}, B_0 = Y_0$ and ε_a is the Dirac measure) is a delayed marked renewal process. Unlike our earlier assumption [1] on Y_k's, here we assume

[1]The corresponding author.

[2] This research is supported by the US Army Grant No. W911NF-07-1-0121.

that they need not be positive. Consequently, the marginal process $\mathcal{B}$ is recurrent and not renewal as in [1] where we were interested in the behavior of $\mathcal{B}$ around some critical threshold L.

In the present paper we are concerned with the time when $\mathcal{B}$ drops for the first time while increasing, or opposite, when $\mathcal{B}$ raises up for the first time while decreasing. For now, we introduce the auxiliary process $\mathcal{A}=(A_n)$ with increments $X_k = A_k - A_{k-1}$ defined as

$$X_k = \begin{cases} 0, & Y_k \geq 0, \\ 1, & Y_k < 0 \end{cases} \tag{1}$$

and «attach» it to $(\mathcal{B},\mathcal{T})$ upon $\mathcal{T}$ to have the bivariate marked point process

$$(\mathcal{A},\mathcal{B},\mathcal{T}) = \sum_{k\geq 0}(X_k, Y_k)\,\varepsilon_{\tau_k}.$$

In this form, the first drop of $\mathcal{B}$ will coincide with the time τ_k when the auxiliary marginal process $\mathcal{A}$ will once hit 1. We therefore consider $(\mathcal{A},\mathcal{B},\mathcal{T})$ as the thri-variate generalized random walk process with exactly one «active» entry $\mathcal{A}$. The other two entries $\mathcal{B}$ and $\mathcal{T}$ will be referred to as «passive». Associate with the random walk is the r.v.

$$\nu = \min\{n : A_n = X_0 + \ldots + X_n = N\}$$

(*exit index*) for some positive integer N. For example, if $N=1$, then the value of the exit index is the νth observation of the random walk when $\mathcal{B}$ drops for the first time. In the general case of N, the exit index ν will stop the observation process when $\mathcal{B}$ drops for the Nth time. The corresponding time τ_ν is referred to as the *exit time* (or *first passage time* or *hitting time*). The following functional

$$\Psi(w,z,u,\upsilon,\vartheta,\theta) = E\left[w^{A_{\nu-1}} z^{A_\nu} e^{uB_{\nu-1}} e^{\upsilon B_\nu} e^{-\vartheta\tau_{\nu-1}} e^{-\theta\tau_\nu}\right]$$

gives all needed information upon the exit time τ_ν and pre-exit time $\tau_{\nu-1}$. The latter is of our particular interest, since at this moment, the process $\mathcal{B}$ assumes the largest value while appreciating before dropping for the first time (restricted to the observation process $\mathcal{T}$).

We observe that $(\mathcal{A},\mathcal{B},\mathcal{T})$ can be treated as a one-sided antagonistic game, in which player A sustains damages from another player B at random times. Positive values of increments Y_k's can be attributed to strikes that hit A, while negative values of Y_k's can be interpreted as restoration of wealth of A. Alternatively, the random walk $(\mathcal{A},\mathcal{B},\mathcal{T})$ can also be regarded as a two-sided game, in which positive values of Y_k's can be attributed to the consecutive strikes sustained by player A, while negative strikes can be regarded as losses to player B. It resembles a gambler's ruin problem, in which a game consists of a series of contests and each successful contest for player A means a loss for player B and visa versa. In this case, the first passage time τ_ν represents the first successful contest for player A, along with the total capital $C - B_\nu$ of player A at time τ_ν and the capital B_ν of player B. Thus, $B_{\nu-1}$ is the highest capital value of player B prior to his first loss. We are interested exactly in $B_{\nu-1}$.

For an arbitrary value of N, τ_ν will give the observed time when player B suffers his Nth loss and B_ν will be his capital. Note that B_ν can also be negative (the case of a debt). It would be of interest to investigate at what time player A or player B gets bankrupted, given some nonpositive threshold values, but this is not the aim of our paper. We are interested in an analytically tractable form for the joint pdf (probability density function) and PDF (probability distribution function of r. v.'s (random variables) $\tau_{\nu-1}$ and $B_{\nu-1}$. In a special case of the instrument with position independent marking, we manage to find it as the double inverse of Laplace—Carson transform and we obtained an explicit formula in terms of a modified Bessel function. Marginal pdf 's and PDF's are also given and they are very tame. For a numerical illustration, we plotted the joint PDF.

We apply the tools of fluctuation analysis specifically designed for this class of games (café. [1]). A more general literature on fluctuation theory can be referred to Bimgham [2], Kyprianou and Pistorius [3], Redner [4], and Takács [5], of which [3] contains applications to economics and [4] — applications to physics. Other applications of fluctuation theory to finance can be found in Dshalalow [6, 7] and Dshalalow and Liew [8]. For the game-theoretic aspect of our paper in connection with its antagonistic nature we mention papers by Fishburn [9], Konstantinov and Polovinkin [10], and Shashikin [11]. The two latter papers deal with applications to economics.

2. The Formalism. We begin with more general assumptions on random walk $(\mathcal{A}, \mathcal{B}, \mathcal{T})$. For a moment we go over its generic version assuming that it is marked point process with position dependent marking, i.e. (X_k, Y_k) depend on τ_k through

$$\Delta_k = \tau_k - \tau_{k-1} \tag{2}$$

only. Suppose $(X_k, Y_k, \Delta_k): \Omega \to \mathbb{N}_0 \times \mathbb{R} \times \mathbb{R}_+$, $k = 0, 1, \ldots$ is a sequence of independent random vectors on a probability space $(\Omega, \mathcal{F}, P)$ and that for $k = 1, 2, \ldots$, they are also identically distributed. In other words,

$$(X_k, Y_k, \Delta_k) \in [(X, Y, \Delta)], \tag{3}$$

where the latter is the equivalence class of random vectors with the common joint transform

$$\gamma(z, \upsilon, \theta) := E[z^X e^{\upsilon Y - \theta\Delta}], \ |z| \le 1, \ \mathrm{Re}(\theta) \ge 0. \tag{4}$$

Also, let

$$\gamma_0(z, \upsilon, \theta) := E[z^{X_0} e^{\upsilon Y_0 - \theta\Delta_0}], \ |z| \le 1, \ \mathrm{Re}(\theta) \ge 0. \tag{5}$$

Under the above assumptions, the following major formula for the functional $\Psi(w,z,u,\upsilon,\vartheta,\theta)$ holds true:

Theorem 1 [7]. Let $(\mathcal{A},\mathcal{B},\mathcal{T})$ be a bivariate delayed marked renewal process with positron dependent marking defined in (2)—(5). Let $\mathcal{A}$ be its discrete component valued in $\mathbb{N}_0$, which is *active* in the sense of section 1. For a positive integer N, let

$$v:=\inf\{n: A_n = X_0+\ldots+X_n \geq N\}$$

be the exit index of $(\mathcal{A},\mathcal{B},\mathcal{T})$. Then it holds true that

$$\Psi(w,z,u,\upsilon,\vartheta,\theta)=E[w^{A_{v-1}} z^{A_v} e^{uB_{v-1}} e^{\upsilon B_v} e^{-\vartheta\tau_{v-1}} e^{-\theta\tau_v}]=\mathcal{D}_x^{N-1}\{\gamma_0(z,\upsilon,\theta)-$$

$$-\gamma_0(xz,\upsilon,\theta)+\frac{\gamma_0(xwz,u+\upsilon,\vartheta+\theta)}{1-\gamma(xwz,u+\upsilon,\vartheta+\theta)}[\gamma(z,\upsilon,\theta)-\gamma(xz,\upsilon,\theta)]\}$$

where $\mathcal{D}^{L-1}$ is the inverse of the operator

$$D_p\{f(p)\}(x):=\sum_{p=0}^{\infty} x^p f(p)(1-x),\ \|x\|<1 \tag{6}$$

(applied to an integrable function f on set $\mathbb{N}_0$). The inverse $\mathcal{D}^k$ is

$$k\mapsto \mathcal{D}_x^k\varphi(x,y)=\begin{cases}\lim\limits_{x\to 0}\dfrac{1}{k!}\dfrac{\partial^k}{\partial x^k}\left[\dfrac{1}{1-x}\varphi(x,y)\right], & k\geq 0,\\ 0, & k<0\end{cases} \tag{7}$$

if applied to a function $\varphi(x,y)$ analytic at zero in the first variable.

Consider a special case with $N=1$ and the initial values $\tau_0=0$ a.s. and $Y_0\geq 0$ is a constant a.s. As mentioned in the Introduction, considering (1), it will correspond to the first observed drop of the process on those paths which are monotone nondecreasing. In this case (i.e with $N=1$), the operator $\mathcal{D}^0$ will be in its simplest form. Furthermore, we would like to look into the marginal functional $E[e^{uB_{v-1}} e^{-\vartheta\tau_{v-1}}]$ of the pre-exit elements, i.e. of the process attaining it highest value B_{v-1} before the first drop on one of the observation epochs, along with the associated observation time τ_{v-1}. Considering (7) we easily arrive at the marginal joint transform

$$\Psi_{v-1}(u,\vartheta)=E[e^{-uB_{v-1}} e^{-\vartheta\tau_{v-1}}]=\frac{\gamma_0(0,-u,\vartheta)[\gamma(1,0,0)-\gamma(0,0,0)]}{1-\gamma(0,-u,\vartheta)}. \tag{8}$$

In (8), for convenience, we changed the sign of u to its opposite (especially considering that $B_{v-1}\geq 0$). The components of (8) are as follows:

$$\gamma(0,-u,\vartheta)=E[0^X e^{-uY} e^{-\theta\Delta}]. \tag{9}$$

Note that

$$E[0^X] = P\{X=0\} = P\{Y \geq 0\} = E[\mathbf{1}_{\mathbb{R}_+}(Y)]. \tag{10}$$

We go on with more assumptions. We assume that the marginal process $(\mathcal{B}, \mathcal{T})$ is with position independent marking, Δ is exponentially distributed with parameter δ, and Y has a Laplace symmetric pdf (probability density function) with parameter λ:

$$f_Y(x) = \frac{\lambda}{2} e^{-\lambda|x|}, \ x \in \mathbb{R}. \tag{11}$$

In other words, the marginal Laplace—Stieltjes transform of Δ and moment generating function of Y, respectively, are

$$E\, e^{-\vartheta\Delta} = \frac{\delta}{\delta + \vartheta}, \tag{12}$$

$$m_Y(u) = E\, e^{uY} = \frac{\lambda^2}{\lambda^2 - u^2}. \tag{13}$$

It can be readily shown that

$$m_Y^+(u) = E[e^{uY} \mathbf{1}_{\mathbb{R}_+}(Y)] = \frac{\lambda}{2(\lambda - u)} \tag{14}$$

and thus

$$m_Y^+(-u) = E[e^{-uY} \mathbf{1}_{\mathbb{R}_+}(Y)] = \frac{\lambda}{2(\lambda + u)}. \tag{15}$$

Consequently, from (9)—(15) we have

$$\gamma(0, -u, \vartheta) = \frac{\delta}{\delta + \vartheta} E[e^{-uY} \mathbf{1}_{\mathbb{R}_+}(Y)] = \frac{\delta}{\delta + \vartheta} \frac{\lambda}{2(\lambda + u)}. \tag{16}$$

Furthermore, from (10) and (11),

$$\gamma(1,0,0) = 1 \text{ and } \gamma(0,0,0) = P\{Y \geq 0\} = \frac{1}{2}.$$

Therefore, the functional $\Psi_{\nu-1}$ of (8) turns

$$\Psi_{\nu-1}(u, \vartheta) = e^{-uY_0} \frac{1}{2} \frac{1}{1 - \frac{\delta}{\delta + \vartheta} \frac{\lambda}{2(\lambda + u)}}. \tag{17}$$

After a straightforward algebra, we have (17) reduce to

$$\Psi_{v-1}(u,\vartheta)=\frac{1}{2}e^{-uY_0}\left[1+\frac{1}{2}\frac{\delta\lambda}{u+\lambda}\frac{1}{\vartheta+\delta\left(1-\dfrac{\lambda}{2(u+\lambda)}\right)}\right]. \tag{18}$$

Our next goal is to find the joint pdf of the r.v.'s B_{v-1} and τ_{v-1}. We will use the Fubini's theorem to invert the double Laplace transform, first applying it w.r.t. variable ϑ and then — variable u.

Let us denote

$$\psi(u,\vartheta)=\frac{1}{2}\frac{\delta\lambda}{u+\lambda}\frac{1}{\vartheta+\delta\left(1-\dfrac{\lambda}{2(u+\lambda)}\right)}. \tag{19}$$

Then, applying the inverse Laplace transform to $\psi(u,\vartheta)$ w.r.t. ϑ we have from standard tables of Laplace transform:

$$\mathcal{L}_{\vartheta}^{-1}\{\psi(u,\vartheta)\}(y)=\frac{\delta\lambda}{2}\frac{1}{\lambda+u}e^{-\delta y}e^{\frac{\lambda}{2(u+\lambda)}y}.$$

The next step will require some auxiliary result not available in the tables we know.

Lemma 1. For three real numbers: $a>0$, b and c, it holds true that

$$\mathcal{L}_y^{-1}\left\{\frac{\exp\left(\dfrac{a}{y+b}\right)}{y+c}\right\}(q)=e^{-bq}I_0(2\sqrt{aq})+(b-c)\,e^{-cq}\int_{u=0}^{q}e^{-(b-c)u}I_0(2\sqrt{au})\,du,$$

where $I_0(x)$ is the modified Bessel function of order zero.

P r o o f. Using the representation

$$\frac{1}{y+c}=\frac{y+b}{(y+c)(y+b)}=\frac{y+c+b-c}{(y+c)(y+b)}=\frac{b-c}{(y+c)(y+b)}+\frac{1}{y+b}$$

and by linearity of $\mathcal{L}^{-1}$ we obtain

$$\mathcal{L}_y^{-1}\left\{\frac{\exp\left(\dfrac{a}{y+b}\right)}{y+c}\right\}(q)=\mathcal{L}_y^{-1}\left\{\frac{\exp\left(\dfrac{a}{y+b}\right)}{y+b}\right\}+(b-c)\,\mathcal{L}_y^{-1}\left\{\frac{\exp\left(\dfrac{a}{y+b}\right)}{y+b}\frac{1}{y+c}\right\}.$$

The above transforms are now easy to invert. First, due to Bateman and Erdélyi [12],

$$\mathcal{L}_y^{-1}\left\{\frac{\exp\left(\frac{a}{y+b}\right)}{y+b}\right\}=e^{-bq}I_0(2\sqrt{aq})\,.$$

In the second term, the expression $\frac{\exp\left(\frac{a}{y+b}\right)}{y+b}\frac{1}{y+c}$ can be regarded as a product of two Laplace transforms: $\frac{\exp\left(\frac{a}{y+b}\right)}{y+b}$ and $\frac{1}{y+c}$. Consequently, the inverse of $\frac{\exp\left(\frac{a}{y+b}\right)}{y+b}\frac{1}{y+c}$ will be the convolution of their respective inverses and thus it yields

$$(b-c)\,\mathcal{L}_y^{-1}\left\{\frac{\exp\left(\frac{a}{y+b}\right)}{y+b}\frac{1}{y+c}\right\}=(b-c)\,e^{-cq}\int\limits_{u=0}^{q}e^{c(q-u)}e^{-bu}I_0(2\sqrt{au})\,du=$$

$$=(b-c)\,e^{-cq}\int\limits_{u=0}^{q}e^{-(b-c)u}I_0(2\sqrt{au})\,du\,.$$

So, we are done with the proof of the lemma.

Applying the Laplace inverse transform w.r.t. variable u to $\mathcal{L}_\vartheta^{-1}\{\psi(u,\vartheta)\}(y)$ (ψ is defined in (19)) and using Lemma 1 we arrive at

$$g(x,y):=\mathcal{L}_{\vartheta,u}^{-1}\{\psi(u,\vartheta)\}(x,y)=\frac{\delta\lambda}{2}e^{-\delta y}e^{-\lambda x}I_0\left(2\sqrt{\frac{\delta\lambda xy}{2}}\right). \qquad (20)$$

Therefore, using the well-known property of the Laplace inverse we have

$$\mathcal{L}_{\vartheta,\,u}^{-1}\{e^{-uY_0}\psi(u,\vartheta)\}(x,y)=g(x-Y_0,y)\mathbf{1}_{(Y_0,\infty)}(x)\,. \qquad (21)$$

The remaining part of the expression $\frac{1}{2}e^{-uY_0}\left[1+\frac{1}{2}\frac{\delta\lambda}{u+\lambda}\frac{1}{\vartheta+\delta\left(1-\frac{\lambda}{2(u+\lambda)}\right)}\right]$ to be dealt with is constant **1** to which we need to apply the double inverse:

$$\mathcal{L}^{-1}_{\vartheta,\,u}\{1\}(x,\,y)=\delta_D(x)\,\delta_D(y)\,, \tag{22}$$

where δ_D is the Dirac delta. Consequently, from (22),

$$\mathcal{L}^{-1}_{\vartheta,\,u}\{e^{-uY_0}\}(x,\,y)=\delta_D(x-Y_0)\,\delta_D(y)\,\mathbf{1}_{(Y_0,\infty)}(x)\,. \tag{23}$$

Finally, compiling all formulas (20)—(23) we have the joint pdf of $B_{\nu-1}$ and $\tau_{\nu-1}$ in the form

$$f(x,y)=\frac{1}{2}\mathbf{1}_{(Y_0,\infty)}(x)\left\{\delta_D(x-Y_0)\,\delta_D(y)+\right.$$
$$\left.+\frac{\delta\lambda}{2}e^{-\delta y}e^{-\lambda(x-Y_0)}I_0\left(2\sqrt{\frac{\delta\lambda y\,(x-Y_0)}{2}}\right)\right\}. \tag{24}$$

From (24), using the definition of Dirac delta, we get the joint PDF (probability distribution function) of $B_{\nu-1}$ and $\tau_{\nu-1}$

$$F(x,y)=\int_{-\infty}^{x}\int_{-\infty}^{y}f(u,v)\,dvdu=$$
$$=\frac{\delta\lambda}{4}\int_{u=0}^{x}\mathbf{1}_{(Y_0,\infty)}(u)\,e^{-\lambda(u-Y_0)}\int_{v=0}^{y}e^{-\delta v}I_0\left(2\sqrt{\frac{\delta\lambda v(u-Y_0)}{2}}\right)dvdu+$$
$$+\frac{1}{2}\mathbf{1}_{(Y_0,\infty)}(x)\,\mathbf{1}_{(0,\infty)}(y). \tag{25}$$

Let us summarize the main result of this section as.

Theorem 2. Let $(\mathcal{B},\mathcal{T})=\sum_{k\geq 0}Y_k\varepsilon_{\tau_k}$ be a marked point process on probability space $(\Omega,\mathcal{F},P)$ with position independent marking, exponentially distributed time increments $\Delta_k=\tau_k-\tau_{k-1}$, symmetrical Laplace distribution of marks Y_k, all specified by (11)—(13), and the initial condition of $(\mathcal{B},\mathcal{T})$, $Y_0>0$ (being a.s. a constant) and $\tau_0=0$, a.s. Then, the pdf and PDF of the highest value of the process $B_{\nu-1}$ before the first drop jointly with the time $\tau_{\nu-1}$ of the first drop satisfies formulas (24) and (25), respectively. Here $I_0(z)$ is the modified Bessel function of order zero.

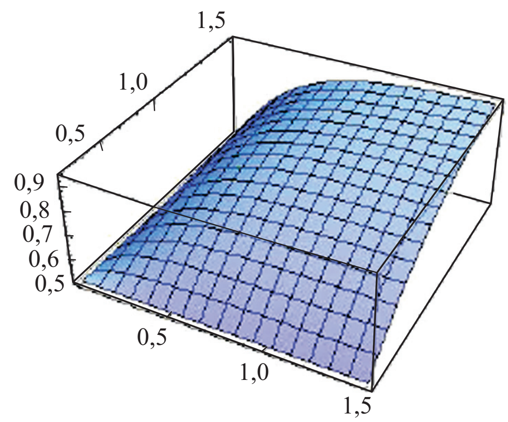

Fig. 1

Figures 1—3 are three different views of plots of the PDF F given in (25) for a special case (by using *Mathematica*) with $\delta = 5$, $\lambda = 4$ and $Y_0 = 0{,}5$ on the intervals $0,5 \le x \le 1,5$ and $0 \le y \le 1,5$. For convenience, the graph is plotted beginning from the rectangle $[0{,}5, \infty) \times [0, \infty)$ where F is positive. We omitted plotting for $x \le 0,5$ where $F(x, y) = 0$.

3. The Marginal Distribution of $B_{\nu-1}$. To find the marginal pdf of $B_{\nu-1}$ we integrate $f(x, y)$ with respect to y:

$$f_{B_{\nu-1}}(x) = \int_{y=0}^{\infty} f(x, y)\, dy = \frac{1}{2} \mathbf{1}_{(Y_0, \infty)}(x)\, \delta_D(x - Y_0) +$$

$$+ \frac{\delta\lambda}{4} \mathbf{1}_{(Y_0, \infty)}(x)\, e^{-\lambda(x - Y_0)} \int_{y=0}^{\infty} e^{-\delta y} I_0\left(2\sqrt{\frac{\delta\lambda y\,(x - Y_0)}{2}}\right) dy. \qquad (26)$$

To simplify (26) we need the following lemma.

Lemma 2. For two positive real numbers A and B the following exponential integral of the modified Bessel function is equal to

$$\int_{u=0}^{\infty} e^{-Bu} I_0(2\sqrt{Au})\, du = \frac{1}{B} e^{\frac{A}{B}}.$$

P r o o f. We use the formula [13],

$$\int_{t=0}^{\infty} I_\mu(\alpha t)\, e^{-\sigma^2 t^2} t^{\mu+1} dt = \frac{\alpha^\mu}{(2\sigma^2)^{\mu+1}} e^{-\frac{\alpha^2}{4\sigma^2}}, \qquad (27)$$

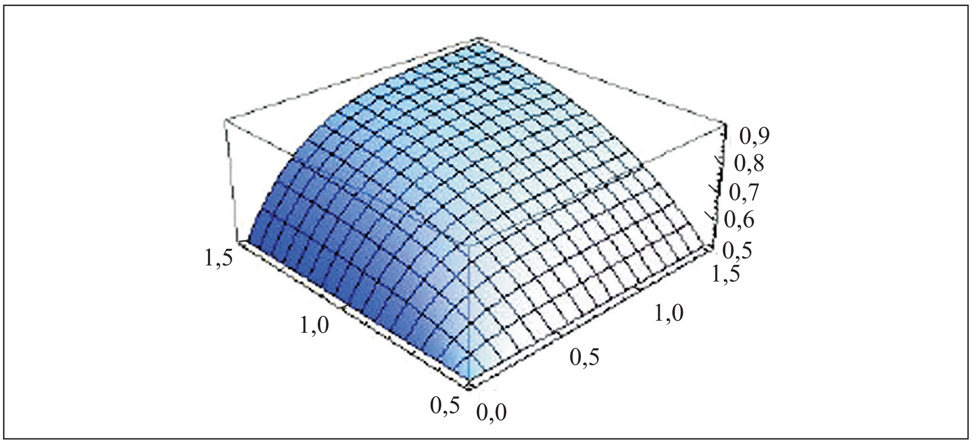

Fig. 2

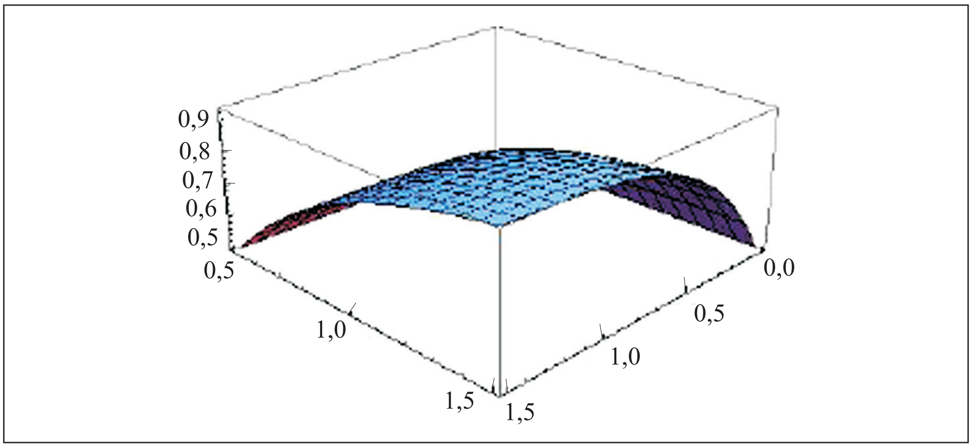

Fig. 3

where J_μ is the Bessel function of order μ, and the relation between the Bessel and modified Bessel functions:

$$I_\mu(z) = e^{\frac{i\mu\pi}{2}} J_\mu(ze^{\frac{i\pi}{2}}).$$

Take μ = 0 and set $\sigma^2 = B > 0,\ \alpha = 2\sqrt{A}i$. Then, (27) will reduce to

$$\int_{t=0}^{\infty} t\, e^{-Bt^2} J_0(2\sqrt{A}\, t\, e^{\frac{i\pi}{2}})\, dt = \frac{1}{2B} e^{\frac{A}{B}}$$

and on the other hand, in terms of the modified Bessel function of order zero, being

$$\int_{t=0}^{\infty} te^{-Bt^2} I_0(2\sqrt{A}\,t)\,dt. \tag{28}$$

With the substitution $t=\sqrt{u}$ we have (28) further reduce to

$$\int_{t=0}^{\infty} \frac{1}{2} e^{-Bu} J_0(2\sqrt{Au})\,du = \frac{1}{2B} e^{\frac{A}{B}}.$$

The latter proves the lemma.

Using Lemma 2 we render the integration of

$$\int_{y=0}^{\infty} e^{-\delta y} I_0\left(2\sqrt{\frac{\delta\lambda y(x-Y_0)}{2}}\right) dy = \frac{1}{\delta} e^{\frac{\lambda(x-Y_0)}{2}}$$

and finally arrive at the marginal pdf of $B_{\nu-1}$:

$$f_{B_{\nu-1}}(x) = \frac{1}{2}\mathbf{1}_{(Y_0,\infty)}(x)\,\delta_D\,(x-Y_0) + \frac{\lambda}{4} e^{-\frac{\lambda}{2}(x-Y_0)} \mathbf{1}_{(Y_0,\infty)}. \tag{29}$$

Further integrating (29) we get the marginal PDF of $B_{\nu-1}$:

$$F_{B_{\nu-1}}(x) = \int_{u=0}^{x} f_{B_{\nu-1}}(u)\,du = \mathbf{1}_{(Y_0,\infty)}(x)\left[1-\frac{1}{2} e^{-\frac{\lambda}{2}(x-Y_0)}\right] \tag{30}$$

which looks very simple compared to the joint PDF of (25).

Theorem 3. Under the condition of Theorem 2, the marginal pdf and PDF of the highest value $B_{\nu-1}$ of $\mathcal{B}$ prior to the first drop satisfy formulas (29) and (30).

4. The Marginal Distribution of $\tau_{\nu-1}$. To find the marginal pdf of $\tau_{\nu-1}$ integrate the joint pdf $f(x, y)$ with respect to x:

$$f_{\tau_{\nu-1}}(y) = \frac{1}{2}\int_{x=0}^{\infty} \mathbf{1}_{(Y_0,\infty)}(x)\,\delta_D\,(x-Y_0)\,\delta_D\,(y) +$$

$$+\mathbf{1}_{(Y_0,\infty)}(x)\frac{\delta\lambda}{4} e^{-\delta y} \int_{x=0}^{\infty} e^{-\lambda(x-Y_0)} I_0\left(2\sqrt{\frac{\delta\lambda y\,(x-Y_0)}{2}}\right) dx =$$

$$= \left[\frac{1}{2} \delta_D (y) + \frac{\delta}{4} e^{-\frac{\delta y}{2}} \right] \mathbf{1}_{(0,\infty)}(y). \tag{31}$$

The result easily follows form Lemma 2:

$$\int_{x=0}^{\infty} e^{-\lambda(x-Y_0)} I_0 \left(2 \sqrt{\frac{\delta \lambda y (x - Y_0)}{2}} \right) dx = \frac{1}{\lambda} \; e^{\frac{\delta y}{2}} \mathbf{1}_{(0,\infty)}(y).$$

Proceeding analogously (as in section 3) we have the marginal PDF of $\tau_{\nu-1}$

$$F_{\tau_{\nu-1}}(y) = \int_{u=0}^{y} f_{\tau_{\nu-1}}(u) \, du =$$

$$= \frac{1}{2} \mathbf{1}_{(0,\infty)}(y) + \frac{1}{2} \left(1 - e^{-\frac{\delta y}{2}} \right) \mathbf{1}_{(0,\infty)}(y) = \mathbf{1}_{(0,\infty)}(y) \left[1 - \frac{1}{2} e^{-\frac{\delta y}{2}} \right]. \tag{32}$$

Theorem 4. Under the condition of Theorem 2, the marginal pdf and PDF of the time $\tau_{\nu-1}$, when the highest value $B_{\nu-1}$ of $\mathcal{B}$ prior to the first drop is attained, satisfy formulas (31) and (32).

Досліджено біваріантний рекурентний процес, за допомогою якого може бути описане поводження немонотонного фінансового інструмента, що спостерігається у випадкові моменти часу. З використанням теоретико-грального підходу явно визначено об’єднане розподілення найбільшої величини інструмента, що передує його першому падінню.

1. *Dshalalow J. H., Robinson R.* On One-sided Stochastic Games and their Applications to Finance, Submitted.
2. *Bingham N. H.* Random Walk and Fluctuation Theory//Handbook of Statistics/Eds. D.N. Shanbhag, Rao C. R. Volume 19. — Elsevier Science, 2001. — P. 171—213.
3. *Kyprianou A. E., Pistorius M. R.* Perpetual Options and Canadization Through Fluctuation Theory// Ann. Appl. Prob. — 2003. — **13**, N 3. — P. 1077—1098.
4. *Redner S.* A Guide to First-Passage Processes. — Cambridge : Cambridge University Press, 2001.
5. *Takács L.* On Fluctuations of Sums of Random Variables//Studies in Probability and Ergodic Theory. Advances in Mathematics; Supplementary Studies. Volume 2/ Ed. by G.-C. Rota. — 1978. — P. 45—93.
6. *Dshalalow J. H.* Fluctuations of Recurrent Processes and their Application to the Stock Marke// Stochastic Analysis and Applications. — 2004. — **22**, N 1. — P. 67—79.
7. *Dshalalow J. H.* On Exit Times of a Multivariate Random Walk with Some Applications to Finance// Nonlinear Analysis, Series A. — 2005. — **63**. — P. 569—577.

8. *Dshalalow J. H., Liew A.* On Fluctuations of a Multivariate Random Walk with Some Applications to Stock Options Trading and Hedging// Math. and Comp. Modeling. — 2006. — **44**, N 10. — P. 931—944.
9. *Fishburn P. C.* Non-cooperative Stochastic Dominance Games// International Journal of Game Theory. — 1978. — **7**, N 1. — P. 51—61.
10. *Konstantinov R. V., Polovinkin E. S.* Mathematical Simulation of a Dynamic Game in the Enterprise Competition Problem// Cybernetics and Systems Analysis. — 2004. — **40**, N 5. — P. 720—725.
11. *Shashikin V. N.* Antagonistic Game with Interval Payoff Functions// Cybernetics and Systems Analysis. — 2004. — **40**, N 4. — P. 556—564.
12. *Bateman H., Erdélyi.* Higher Transcendental Functions. Volume 2. — McGraw-Hill, 1953.
13. *Bateman H., Erdélyi.* Tables of Integral Transforms. Volume I. — McGraw-Hill, 1954.

Submitted 05.02.10