doi: https://doi.org/10.15407/dopovidi2017.03.014
UDC 517.5

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Finite mean oscillation on Finsler manifolds

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We study functions of the finite mean oscillation in Finsler spaces with respect to the boundary behavior of ring Q-homeomorphisms.

Keywords: *Finsler manifolds, FMO class functions, ring Q-homeomorphisms.*

In this article, we continue our study of mappings on Finsler manifolds (M^n, Φ) started in [1]. For historical remarks, we refer to [2]. Recall some needed definitions. By a *domain* in the topological space *T*, we mean an open linearly connected set. A domain *D* is called locally connected at a point $x_0 \in \partial D$, if, for any neighborhood U of x_0, there is a neighborhood $V \subseteq U$ of x_0 such that $V \bigcap D$ is connected (cf. [3]). Similarly, we say that a domain *D* is *locally linearly connected at a point* $x_0 \in \partial D$ if, for any neighborhood U of x_0, there exists a neighborhood $V \subseteq U$ of x_0 such that $V \bigcap D$ is linearly connected. Recall that the *n-dimensional topological manifold* M^n means a Hausdorff topological space with countable base such that every point has a neighborhood homeomorphic to R^n. The manifold of the class C^r with $r \geqslant 1$ is called *smooth*.

Let D denote a domain in the Finsler space (M^n, Φ), $n \geqslant 2$, and let $TM^n = \cup T_x M^n$ be a tangent bundle of (M^n, Φ), $\forall x \in M^n$. By a *Finsler manifold* (M^n, Φ), $n \geqslant 2$, we mean a smooth manifold of the class C^∞ with defined Finsler structure $\Phi(x, \xi)$, where $\Phi(x, \xi) : TM^n \to R^+$ is a function satisfying the following conditions:

1) $\Phi \in C^\infty (TM^n \setminus \{0\})$;

2) $\forall a > 0$ hold $\Phi(x, a\xi) = a\, \Phi(x, \xi)$ and $\Phi(x, \xi) > 0$ for $\xi \neq 0$;

3) the $n \times n$ Hessian matrix $g_{ij}(x, \xi) = \dfrac{1}{2} \dfrac{\partial^2 \Phi^2(x, \xi)}{\partial \xi_i \partial \xi_j}$ is positive definite at every point of $TM^n \setminus \{0\}$ (cf. [4]).

By the *geodesic distance* $d_\Phi(x, y)$, we mean the infimum of lengths of piecewise-smooth curves joining x and y in (M^n, Φ), $n \geqslant 2$. It is well known that, for such metric, only two axioms of metric spaces hold, namely the identity and triangle inequality axioms. Therefore, the Finsler manifold provides a quasimetric space, for which the symmetry axiom fails.

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Remark 1. Later, we consider a Finsler structure of the type

$$\tilde{\Phi}(x,\xi)=\frac{1}{2}(\Phi(x,\xi)+\Phi(x,-\xi)),$$

thereby obtaining a Finsler manifold $(M^n,\tilde{\Phi})$ with symmetrized (reversible) function $\tilde{\Phi}$. Clearly, if $\tilde{\Phi}$ is reversible, then the induced distance function $d_{\tilde{\Phi}}$ is reversible, i.e., $d_{\tilde{\Phi}}(x,y)=d_{\tilde{\Phi}}(y,x)$, for all pairs of points $x,y\in M^n$, see [5]. It is also known that the reversible Finsler metric coincides with the Riemannian one, see, e.g., [6]. Therefore, in our further discussion, we can rely on the results of [2].

Later, $\gamma:[a,b]\to M^n$ is a piecewise-smooth curve, and $x(t)$ is its parametrization. An *element of length* in $(M^n,\tilde{\Phi})$, $n\geqslant 2$, is defined as a differential of the path for an infinitesimal measured part of a curve $\gamma\in D$ by

$$ds^2_{\tilde{\Phi}}=\sum_{i,j=1}^{n} g_{ij}(x,\xi)d\eta_i\,d\eta_j;$$

see, e.g., [7]. So, the distance $ds_{\tilde{\Phi}}$ in the Finsler space, as in the case of a Riemannian space, is determined by a metric tensor. Using the quadratic form $ds_{\tilde{\Phi}}$, we determine the length of $\gamma\subset D$ by

$$s_{\tilde{\Phi}}(\gamma)=\int_{\gamma} ds_{\tilde{\Phi}}=\int_{t_1}^{t_2}\tilde{\Phi}(x,dx)dt,$$

see, e.g., [8, 9]. The invariance of this integral requires above-given restrictions 2—3 on the Lagrangian $\tilde{\Phi}(x,dx)$.

Following [10], in view of Remark 1, an element of *volume* on the Finsler manifold is defined by $d\sigma_{\tilde{\Phi}}(x)=\sqrt{\det g_{ij}(x,\xi)}\,dx^1\ldots dx^n$. It is known that the volume in the Finsler space coincides with its Hausdorff measure induced by the metric $d_{\Phi}(x,y)$, if $\Phi(x,\xi)$ is an invertible function, see, e.g., [5].

Let Γ be a family of curves in a domain *D*. By the family of curves Γ, we mean a fixed set of curves γ, and, for an arbitrary mapping $f:M^n\to M^n_*$, $f(\Gamma):=\{f\circ\gamma\,|\,\gamma\in\Gamma\}$. The *modulus* of the family Γ is defined by

$$M(\Gamma):=\inf_{\rho\in\operatorname{adm}\Gamma}\int_D \rho^n(x)d\sigma_{\tilde{\Phi}}(x),$$

where the infimum is taken over all nonnegative Borel functions ρ such that the condition

$$\int_{\gamma}\rho\tilde{\Phi}(x,dx)=\int_{\gamma}\rho\,ds_{\tilde{\Phi}}\geqslant 1$$

holds for any curve $\gamma\in\Gamma$. The functions ρ satisfying this condition are called *admissible* for Γ, cf. [4].

Later, for sets *A*, *B*, and *C* from $(M^n,\tilde{\Phi})$, $n\geqslant 2$, by $\Delta(A,B;C)$, we denote a set of all curves $\gamma:[a,b]\to M^n$, which join *A* and *B* in *C*, i.e. $\gamma(a)\in A$, $\gamma(b)\in B$, and $\gamma(t)\in C$ for all $t\in(a,b)$.

By Remark 1, one can apply the following well-known facts: Proposition 1 and Remark 1 in [2]. Thus, we assume that the geodesic spheres $S(x_0,r)$, geodesic balls $B(x_0,r)$, and geodesic rings $A=A(x_0,r_1,r_2)$ lie in a normal neighborhood of the point x_0.

Let *D* and *D'* be domains on the Finsler manifolds $(M^n,\tilde{\Phi})$ and $(M^n_*,\tilde{\Phi}_*)$, $n\geqslant 2$, respectively, and let $Q:M^n\to(0,\infty)$ be a measurable function. We say that a homeomorphism $f:D\to D'$ is the *ring Q-homeomorphism at a point* $x_0 \in \overline{D}$, if

$$M(\Delta(f(C_0), f(C_1); D')) \leqslant \int\limits_{A\cap D} Q(x)\cdot\eta^{\alpha}(d(x,x_0))\,d\mu(x) \tag{1}$$

holds for any geodesic ring $A = A(x_0,\varepsilon,\varepsilon_0)$, $0<\varepsilon<\varepsilon_0$, any two continua (compact connected sets) $C_0 \subset \overline{B(x_0,r_1)} \cap D$ and $C_1 \subset D\setminus B(x_0,r_2)$, and each Borel function $\eta:(r_1,r_2)\to[0,\infty]$ such that

$$\int\limits_{r_1}^{r_2}\eta(r)dr \geqslant 1.$$

We say that *f* is a *ring Q-homeomorphism* in *D*, if (1) holds for all points $x_0 \in \overline{D}$.

We say that the boundary of the domain *D* is *weakly flat at a point* $x_0 \in \partial D$, if, for any number $P>0$ and any neighborhood U of x_0, there exists a neighborhood $V \subset U$ such that $M(\Delta(E,F;D)) \geqslant P$ for any continua E and F in D intersecting ∂U and ∂V. We also say that the boundary D is *strongly accessible at a point* $x_0 \in \partial D$, if, for any neighborhood U of x_0, there are a compactum U of $E\subset D$, a neighborhood $V\subset U$ of x_0, and a number $\delta>0$ such that $M(\Delta(E,F;D)) \geqslant \delta$ for any continuum F in D intersecting ∂U and ∂V. The boundary of D is called *strongly accessible* and *weakly flat*, if it has the corresponding property at every its point, cf. [11].

Similarly to [11], we say that a function $\phi: M^n \to R$ has the *finite mean oscillation at a point* $x_0 \in M^n$, abbr. $\phi \in \mathrm{FMO}(x_0)$, if

$$\overline{\lim_{\varepsilon\to 0}}\frac{1}{\sigma_{\tilde{\Phi}}(B(x_0,\varepsilon))}\int\limits_{B(x_0,\varepsilon)}|\phi(x)-\tilde{\phi}_{\varepsilon}|\,d\sigma_{\tilde{\Phi}}(x)<\infty,$$

where

$$\tilde{\phi}_{\varepsilon} = \frac{1}{\sigma_{\tilde{\Phi}}(B(x_0,\varepsilon))}\int\limits_{B(x_0,\varepsilon)}\phi(x)\,d\sigma_{\tilde{\Phi}}(x)$$

is the mean value of the function $\phi(x)$ over the $B(x_0,\varepsilon)$ with respect to the measure $\sigma_{\tilde{\Phi}}$.

Theorem 1. *Let D be locally connected at a point* $x_0 \in \partial D$, *let* $\partial D'$ *be strongly accessible, and let the closure* $\overline{D}'$ *be compact. If* $Q \in \mathrm{FMO}(x_0)$, *then any ring Q-homeomorphism* $f: D\to D'$ *can be continued to the point* x_0 *by continuity on* $(M^n_*, \tilde{\Phi}_*)$.

Corollary 1. *Let* D *be locally connected at the point* $x_0 \in \partial D$, let $\partial D'$ *be strongly accessible, and let* $\overline{D}'$ *be compact. If*

$$\overline{\lim_{\varepsilon\to 0}}\;\frac{1}{\sigma_{\tilde{\Phi}}(B(x_0,\varepsilon))}\int\limits_{B(x_0,\varepsilon)} Q\,d\sigma_{\tilde{\Phi}}(x)<\infty,$$

any ring Q-homeomorphism $f:D\to D'$ *can be continued to the point* x_0 *by continuity on* $(M^n_*, \tilde{\Phi}_*)$.

Theorem 2. *Let* D *be locally connected on the boundary, let* $\partial D'$ *be strongly accessible, and let* $\overline{D}'$ *be compact. If Q belongs to* FMO, *then any ring Q-homeomorphism* $f:D\to D'$ *admits a continuous continuation* $\overline{f}:\overline{D}\to\overline{D}'$.

Theorem 3. *Let* D *be locally connected on the boundary, let* $\partial D'$ *be weakly flat, and let* $\overline{D}$ *and* $\overline{D}'$ *be compact. If Q belongs to* FMO, *then any ring Q-homeomorphism* $f:D\to D'$ *admits the continuation to the homeomorphism* $\overline{f}:\overline{D}\to\overline{D}'$.

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Received 19.11.2016

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СКІНЧЕННЕ СЕРЕДНЄ КОЛИВАННЯ У ФІНСЛЕРОВИХ МНОГОВИДАХ

Вивчаються функції скінченного середнього коливання у фінслерових просторах відносно граничної поведінки кільцевих Q-гомеоморфізмів.

Ключові слова: *фінслерові многовиди, функції класу FMO, кільцеві Q-гомеоморфізми.*

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КОНЕЧНОЕ СРЕДНЕЕ КОЛЕБАНИЕ НА ФИНСЛЕРОВЫХ МНОГООБРАЗИЯХ

Изучаются функции конечного среднего колебания в финслеровых пространствах относительно граничного поведения кольцевых Q-гомеоморфизмов.

Ключевые слова: *финслеровы многообразия, функции класса FMO, кольцевые Q-гомеоморфизмы.*