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doi: <https://doi.org/10.15407/dopovidi2017.01.017>

UDC 517.5

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## Approximation of $2\pi$ -periodic functions by Taylor – Abel – Poisson operators in the integral metric

*Presented by Academician of the NAS of Ukraine V.L. Makarov*

*We obtain direct and inverse approximation theorems of  $2\pi$ -periodic functions by Taylor – Abel – Poisson operators in the integral metric.*

**Keywords:** *direct approximation theorem, inverse approximation theorem,  $K$ -functional, linear approximation method.*

Let  $L_p = L_p(\mathbb{T})$ ,  $1 \leq p \leq \infty$ , be the space of all functions  $f$ , given on the torus  $\mathbb{T}$ , with the usual norm

$$\|f\|_p := \|f\|_{L_p(\mathbb{T})} := \begin{cases} \left( \frac{1}{2\pi} \int_0^{2\pi} |f(x)|^p dx \right)^{1/p}, & 1 \leq p < \infty, \\ \operatorname{ess\,sup}_{x \in [0, 2\pi]} |f(x)|, & p = \infty. \end{cases}$$

Further, let  $f \in L_1$ . The Fourier coefficients of  $f$  are given by

$$\hat{f}_k := \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-ikx} dx, \quad k \in \mathbb{Z}.$$

Here, we study approximative properties of the Taylor – Abel – Poisson operators  $A_{\varrho, r}$ , which are defined in the following way [1, 2]:

For  $\varrho \in [0, 1)$ ,  $r \in \mathbb{N}$  and  $f \in L_1$ , we set

$$A_{\varrho, r}(f)(x) := \sum_{k \in \mathbb{Z}} \lambda_{|k|, r}(\varrho) \hat{f}_k e^{ikx},$$

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ISSN 1025-6415. *Доп. НАН України. 2017. № 1*

where, for  $k = 0, 1, \dots, r-1$ , the numbers  $\lambda_{k,r}(\varrho) \equiv 1$  and

$$\lambda_{k,r}(\varrho) := \sum_{j=0}^{r-1} \binom{k}{j} (1-\varrho)^j \varrho^{k-j}, \quad k = r, r+1, \dots, \quad \varrho \in [0,1].$$

We denote, by  $f(\varrho, x)$ ,  $0 \leq \varrho < 1$ , the Poisson integral (the Poisson operator) of  $f$ , i.e.,

$$f(\varrho, x) := \frac{1}{2\pi} \int_0^{2\pi} f(t) P(\varrho, x-t) dt,$$

where  $P(\varrho, t) = \frac{1-\varrho^2}{|1-\varrho e^{it}|^2}$  is the Poisson kernel.

Leis [3] considered the transformation

$$L_{\varrho,r}(f)(x) = \sum_{k=0}^{r-1} \frac{d^k f(x)}{dn^k} \frac{(1-\varrho)^k}{k!}, \quad r \in \mathbb{N},$$

where

$$\frac{df(x)}{dn} = - \left. \frac{\partial f(\varrho, x)}{\partial \varrho} \right|_{\varrho=1}$$

is the normal derivative of the function  $f$ .

Butzer and Sunouchi [4] considered the transformation

$$B_{\varrho,r}(f)(x) := \sum_{k=0}^{r-1} (-1)^{\frac{k+1}{2}} f^{(k)}(x) \frac{(-\ln \varrho)^k}{k!},$$

where  $f^{\{k\}} = f^{(k)}$ , if  $k \in 2\mathbb{Z}_+$  and  $f^{\{k\}} = \tilde{f}^{(k)}$ , if  $(k-1) \in 2\mathbb{Z}_+$ .

The relation between the operators  $A_{\varrho,r}$  and the operators  $L_{\varrho,r}$  and  $B_{\varrho,r}$  is shown in the following relation:

$$A_{\varrho,r}(f)(x) = \sum_{k=0}^{r-1} \frac{\partial^k f(\varrho, x)}{\partial \varrho^k} \frac{(1-\varrho)^k}{k!}, \tag{1}$$

which holds for any function  $f \in L_1$  and for all numbers  $r \in \mathbb{N}$ ,  $\varrho \in [0,1)$ , and  $x \in \mathbb{T}$ .

If, for a function  $f \in L_1$  and for a positive integer  $n$ , there exists the function  $g \in L_1$  such that

$$\hat{g}_k = 0, \text{ if } |k| < n \text{ and } \hat{g}_k = \frac{|k|!}{(|k|-n)!} \hat{f}_k, \text{ if } |k| \geq n, \quad k \in \mathbb{Z},$$

then we say that, for the function  $f$ , there exists the radial derivative  $g$  of order  $n$ , for which we use the notation  $f^{[n]}$ . Here, we use the term “radial derivative» in view of the following fact.

If the function  $f^{[r]} \in L_1$ , then its Poisson integral can be presented as

$$f^{[r]}(\varrho, x) = (f(\varrho, \cdot))^{[r]}(x) = \varrho^r \frac{\partial^r f(\varrho, x)}{\partial \varrho^r}, \quad \varrho \in [0,1), \quad x \in \mathbb{T}.$$

Hence, by virtue of the theorem of limit values of a Poisson integral (see, for example, [5, p. 27]), for almost all  $x \in \mathbb{T}$ , we have  $f^{[r]}(x) = \lim_{\varrho \rightarrow 1-} f^{[r]}(\varrho, x)$ .

In the space  $L_p$ , the  $K$ -functional of a function  $f$  (see, for example, [6, Ch. 6]) generated by the radial derivative of order  $n$  is the following quantity:

$$K_n(\delta, f)_p := \inf \left\{ \|f - h\|_p + \delta^n \|h^{[n]}\|_p : h^{[n]} \in L_p \right\}, \quad \delta > 0.$$

**Theorem 1.** Assume that  $f \in L_p$ ,  $1 \leq p \leq \infty$ ,  $n, r \in \mathbb{N}$ ,  $n \leq r$ , and  $0 < \alpha < n$ . If

$$K_n(f^{[r-n]})_p = O(\delta^\alpha), \quad \delta \rightarrow 0+, \tag{2}$$

then

$$\|f - A_{\varrho, r}(f)\|_p = O((1-\varrho)^{r-n+\alpha}), \quad \varrho \rightarrow 1-. \tag{3}$$

**Theorem 2.** Assume that  $f \in L_p$ ,  $1 \leq p \leq \infty$ ,  $n, r \in \mathbb{N}$ ,  $n \leq r$ , and  $0 < \alpha < n$ . If relation (3) holds, then  $f^{[r-n]} \in L_p$ , and relation (2) also holds.

Note that the relation  $\|f - A_{\varrho, r}(f)\|_p = o((1-\varrho)^r)$ ,  $\varrho \rightarrow 1-$ , holds only in the trivial case where

$$f = \sum_{|k| \leq r-1} \hat{f}_k e^{ikx}.$$

In such case, the theorems are easily true. This fact is related to the so-called saturation property of the approximation method generated by the operator  $A_{\varrho, r}$ . In particular, in [1], it was shown that the operator  $A_{\varrho, r}$  generates the linear approximation method of holomorphic functions, which is saturated in the space  $H_p$  with the saturation order  $(1-\varrho)^r$  and the saturation class  $H_p^{r-1} \text{Lip}1$ .

It is of interest to consider the case  $n = 1$ . In this case, by virtue of Theorem 2.4 (Ch. 6 §2 [6]), the set of all functions  $f \in L_p$  satisfying the condition  $K_1(\delta, f) = O(\delta^\alpha)$ ,  $\delta \rightarrow 0+$ ,  $\alpha > 0$ , is equivalent to the Lipschitz class

$$\text{Lip}(\alpha, p) = \left\{ f \in L_p : \omega(f, \delta) := \sup_{|h| \leq \delta} \|f(\cdot + h) - f(\cdot)\|_p = O(\delta^\alpha), \quad \delta \rightarrow 0+ \right\}.$$

**Corollary 1.** Assume that  $f \in L_p$ ,  $1 \leq p \leq \infty$ ,  $r \in \mathbb{N}$  and  $0 < \alpha < 1$ . The following statements are equivalent:

- 1)  $\|f - A_{\varrho, r}(f)\|_p = O((1-\varrho)^{r-1+\alpha})$ ,  $\varrho \rightarrow 1-$ ;
- 2)  $f^{[r-1]} \in \text{Lip}(\alpha, p)$ .

*This work was supported in part by the FP7-People-2011-IRSES project number 295164 (EU-MLS: EU-Ukrainian Mathematicians for Life Sciences).*

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Received 06.09.2016

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НАБЛИЖЕННЯ  $2\pi$ -ПЕРІОДИЧНИХ ФУНКЦІЙ ОПЕРАТОРАМИ  
ТЕЙЛОРА – АБЕЛЯ – ПУАССОНА В ІНТЕГРАЛЬНІЙ МЕТРИЦІ

Отримано прямі та обернені теореми наближення  $2\pi$ -періодичних функцій операторами Тейлора – Абе-  
ля – Пуассона в інтегральній метриці.

**Ключові слова:** пряма теорема наближення, обернена теорема наближення,  $K$ -функціонал, лінійний метод наближення.

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