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NONLINEAR ESTIMATES APPROACH TO THE NON-LIPSCHITZ GAP BETWEEN BOUNDEDNESS AND CONTINUITY IN C^∞-PROPERTIES OF INFINITE DIMENSIONAL SEMIGROUPS

This paper is aimed to discuss an intrinsic effect on the presence of the certain *gap between* the *boundedness* and *continuity* topologies for derivatives of infinite dimensional semigroups that describe evolution of unbounded lattice spin systems.
This gap is influenced by the non-Lipschitz order of corresponding generator's coefficients and to control its precise value we develop the approach of quasi-contractive nonlinear estimates, e.g. [2, 3, 4], and achieve the *continuity* with respect to the initial data of associated variational equations. In fact the results point that the weighted hierarchies on the boundedness topologies [1, 2] are essential.

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1. Introduction.

There is a canonical approach, ascending to the papers of Cauchy, Liouville, Picard and others, to the investigation of continuous dependence and the arbitrary order differentiability with respect to the initial data for solutions to the parabolic differential equations. Due to the implicit function techniques it is closely related to, this approach permits a generalization to the case of differential equations on the infinite dimensional Banach space, working excellently for the equations with Lipschitz with bounded Frechet derivatives coefficients, e.g. [6, 10, 11, 12, 15, 19, 23, 25, 26]. It also leads to the C^∞ smooth properties of the associated semigroups, i.e. gives the preservation of spaces of smooth functions with *the same* topologization on the boundedness and continuity of derivatives: for some topologies $\mathcal{B}_i$ imposed onto derivatives of functions over Banach space $\mathcal{B}_0$

$$\forall\, i = 1,n \;\forall\, x, y \in \mathcal{B}_0 \quad \|\partial^{(i)} f(x)\|_{\mathcal{B}_i} \le K, \qquad \|\partial^{(i)} f(x) - \partial^{(i)} f(y)\|_{\mathcal{B}_i} \le K\|x-y\|_{\mathcal{B}_0},$$

the consideration of semigroup $(P_t f)(x) = f(y_t(x))$, generated by Lipschitz differential flow $y_t(x) = x - \int_0^t F(y_s(x))ds$, guarantees no more than exponential growth of constants

$$\exists\, M \;\forall\, t \ge 0 \;\; \forall\, i = 1,n \;\forall\, x, y \in \mathcal{B}_0 \quad \|\partial^{(i)}(P_t f)(x)\|_{\mathcal{B}_i} \le K e^{Mt},$$

$$\|\partial^{(i)}(P_t f)(x) - \partial^{(i)}(P_t f)(y)\|_{\mathcal{B}_i} \le K e^{Mt}\|x-y\|_{\mathcal{B}_0}.$$

In other words, in the terms of norm

$$\|f\|_{\mathcal{C}^n} = \max_{i=0,n}(\sup_{x\in\mathcal{B}_0} \|\partial^{(i)} f(x)\|_{\mathcal{B}_i},\; \sup_{x,y\in\mathcal{B}_0} \frac{\|\partial^{(i)} f(x) - \partial^{(i)} f(y)\|_{\mathcal{B}_i}}{\|x-y\|_{\mathcal{B}_0}})$$

there is a quasi-contractive property

$$\exists\, M \;\forall\, f \in \mathcal{C}^n \quad \|P_t f\|_{\mathcal{C}^n} \le e^{Mt}\|f\|_{\mathcal{C}^n}.$$

In this paper we show that the non-Lipschitz coefficients in equation cause the *depending on the nonlinearity parameters gap* between the topologies on the *boundedness* and *continuity* of semigroups derivatives.

To estimate below this gap we use an observation that the higher order ($n \geq 2$) calculi of variations assume certain *nonlinear symmetries*, namely that the high order variations of nonlinear functionals

$$d^n F(y) = \sum_{s=1}^{n} \sum_{j_1+...+j_s=n} F^{(s)}(y) d^{j_1}y...d^{j_s}y \tag{1.1}$$

contain simultaneously in the r.h.s. the n^{th} variation $d^n y$ and the 1^{st} variation in the n^{th} power $[dy]^n$ for $s = n$ and $s = 1$ correspondingly, d denotes a derivation functor. Formula (1.1) for $n = 2$ was already applied by Cauchy, because at $F' = 0$ the sign F'' determines a type of extremal point.

This idea was exploited in e.g. [2, 3, 4], where it was studied an important in applications class of semigroups, describing the evolutions of the unbounded lattice spin systems [8, 9, 13, 24]. Developing further these results consider the not strongly continuous Feller semigroup $(P_t f)(x) = \mathbf{E}\, f(\xi^0_{t,x})$, generated by the stochastic non-Lipschitz differential equation in infinite dimensional space

$$\xi^0_{t,x} = x + \int_0^t dW_t - \int_0^t (F(\xi^0_{t,x}) + B\xi^0_{t,x})dt$$

with monotone F and linear B. Observation (1.1), applied to the solutions of associated variational equations

$$\frac{\partial^{(n)}\xi^0_{t,x}}{(\partial x)^n} = \xi^{(n)}_{t,x} = x^{(n)} - \int_0^t (F'(\xi^0_{t,x}) + B)\xi^{(n)}_{t,x} dt - \sum_{j_1+...+j_s=n,\ s\geq 2} \int_0^t F^{(s)}(\xi^0_{t,x})\xi^{(j_1)}_{t,x}...\xi^{(j_s)}_{t,x} dt \tag{1.2}$$

permitted to derive the nonlinear estimates on variations $\xi^{(n)}_{t,x}$

$$\rho_n(\xi_x; t) = \sum_{j=1}^{n} \mathbf{E}\, p_j(\|\xi^0_{t,x}\|)\|\xi^{(j)}_{t,x}\|^{m/j}_{X_j} \leq e^{M_n t}\rho_n(\xi_x; 0) \tag{1.3}$$

and, using the representation of semigroups' derivatives in the terms of variations

$$\partial^{(n)} P_t f(x) = \sum_{j_1+...+j_s=n,\ s\geq 1} \mathbf{E}\, f^{(s)}(\xi^0_{t,x})\xi^{(j_1)}_{t,x}...\xi^{(j_s)}_{t,x}, \tag{1.4}$$

preserve by P_t the Banach spaces of differentiable functions [2], topologized by norms

$$\|f\|_{C^n} = \max_{j=0,...,n} \sup_x \frac{\|\partial^{(j)} f(x)\|_{\mathcal{B}_j}}{q_j(\|x\|)}.$$

In this paper we investigate the action of P_t in topologies

$$\|f\|_{\mathcal{E}^n} = \max_{j=0,...,n} [\sup_x \frac{\|\partial^{(j)} f(x)\|_{\mathcal{B}_j}}{q_j(\|x\|)}, \sup_{x,y} \frac{\|\partial^{(j)} f(x) - \partial^{(j)} f(y)\|_{\widetilde{\mathcal{B}}_j}}{\|x-y\| p_j(\|x\| + \|y\|)}] \tag{1.5}$$

and demonstrate that to achieve the smooth properties of non-Lipschitz semigroup in these scales, i.e. the quasi-contractive behaviour

$$\exists\, M = M_{\mathcal{E}^n}\ \forall f \in \mathcal{E}^n \quad \|P_t f\|_{\mathcal{E}^n} \leq e^{Mt}\|f\|_{\mathcal{E}^n}$$

it is necessary to introduce the gaps between the topologies of boundedness and continuity

$$p_j(z) = \mathcal{P}ol_{\mathbf{k}}(z) \cdot q_j(z) \quad \& \quad \widetilde{\mathcal{B}}_j = C_{\mathbf{k}}\mathcal{B}_j$$

depending on the non-Lipschitz order $\mathbf{k}$ of F. Remark that operator gap $C_{\mathbf{k}}$ displays an essentially *infinite dimensional* effect.

Let us shortly discuss the *key idea.* We need *continuity* of variations, i.e. estimates on $\xi^{(j)}_{t,x} - \xi^{(j)}_{t,y}$ for solutions of (1.2) to control the continuity of semigroups derivatives through the representation (1.4). The principal part of equation on $\xi^{(j)}_{t,x} - \xi^{(j)}_{t,y}$

$$\frac{d}{dt}(\xi^{(j)}_{t,x} - \xi^{(j)}_{t,y}) = -\frac{[F'(\xi^0_{t,x}) + F'(\xi^0_{t,y})]}{2}(\xi^{(j)}_{t,x} - \xi^{(j)}_{t,y}) - \frac{[F'(\xi^0_{t,x}) - F'(\xi^0_{t,y})]}{2}(\xi^{(j)}_{t,x} + \xi^{(j)}_{t,y}) + ...$$

points on the similarity of behaviour

$$\frac{\|\xi^{(j)}_{t,x} - \xi^{(j)}_{t,y}\|_{Y_j}}{\|\xi^0_{t,x} - \xi^0_{t,y}\|} \sim \|\xi^{(j)}_{t,x}\|_{X_j} + \|\xi^{(j)}_{t,y}\|_{X_j} \tag{1.6}$$

due to $F'(\xi^0_{t,x}) - F'(\xi^0_{t,y}) \sim \xi^0_{t,x} - \xi^0_{t,y}$ with accuracy of some $polynomial(\xi^0_{t,x}, \xi^0_{t,y})$ factor.

We write first a simple generalization of (1.3)

$$\rho^b_n(\xi_x, \xi_y; t) = \sum_{j=1}^{n} \mathbf{E}\, \|\xi^0_{t,x} - \xi^0_{t,y}\|^\delta p_j(\|\xi^0_{t,x}\| + \|\xi^0_{t,y}\|)\{\|\xi^{(j)}_{t,x}\|^{m/j}_{X_j} + \|\xi^{(j)}_{t,y}\|^{m/j}_{X_j}\} \le e^{M_n t}\rho^b_n(\xi_x, \xi_y; 0)$$

and using similarity (1.6) introduce

$$\rho^c_n(\xi_x, \xi_y; t) = \sum_{j=1}^{n} \mathbf{E}\, \|\xi^0_{t,x} - \xi^0_{t,y}\|^{\delta - m/j} q_j(\|\xi^0_{t,x}\| + \|\xi^0_{t,y}\|)\|\xi^{(j)}_{t,x} - \xi^{(j)}_{t,y}\|^{m/j}_{Y_j} \tag{1.7}$$

It appears that the expression $\widetilde{\rho}_n(\xi_x, \xi_y; t) = \rho^b_n(\xi_x, \xi_y; t) + \rho^c_n(\xi_x, \xi_y; t)$ fulfills a *nonlinear quasi-contractive estimate*

$$\forall\, \delta \ge m \quad \widetilde{\rho}_n(\xi_x, \xi_y; t) \le e^{\widetilde{M}_n t}\widetilde{\rho}_n(\xi_x, \xi_y; 0) \tag{1.8}$$

which gives the continuous behaviour of variations $\xi^{(n)}_{t,x}$ with respect to initial data x, necessary to control later the gap between boundedness and continuity topologies (1.5).

Theorem 1 achieves the continuity estimates on variations $\xi^{(n)}_{t,x}$ with respect to the initial data by proving the quasi-contractive nonlinear estimates (1.8) and precisely determines the influence of nonlinear parameters on the hierarchies between scales $\{X_j, Y_j\}$ and weights $\{p_j, q_j\}$ in (1.7).

Theorem 3 is completely devoted to the study of action of semigroup P_t in the scales of spaces of smooth functions. Here the main attention is devoted to estimate the *boundedness – continuity* topologization gap.

Theorem 5 plays an especial role. We derive estimates

$$\mathbf{E}\, q_n(\|\xi^0_{t,x}\| + \|\xi^0_{t,y}\|)\|\xi^{(n)}_{t,x} - \xi^{(n)}_{t,y}\|^{m/n}_{Y_n} \le e^{\widetilde{M}_n t}\|x - y\|^{m/n} p_1(\|x\| + \|y\|)2\|\widetilde{x}^{(1)}\|^m_{X_1} \tag{1.9}$$

which follow by formal appeal to (1.7), (1.8) at $\delta = m/n < m$, when we remain n^{th} term of ρ_n^c in the l.h.s. of (1.8) and choose initial data

$$\widetilde{x}^{(1)} = \widetilde{y}^{(1)} = 1, \quad \widetilde{x}^{(i)} = \widetilde{y}^{(i)} = 0, \quad i \geq 2 \tag{1.10}$$

in (1.2), therefore $\rho_n^c|_{t=0} = 0$ and summands on $j \geq 2$ in $\rho_n^b|_{t=0}$ disappear in the r.h.s. of (1.8).

Remark now that estimate (1.8) holds only *in the domain* $\delta \geq m$, otherwise one would face singular terms like $1/\|\xi_{t,x}^0 - \xi_{t,y}^0\|^{m-\delta}$ in expression (1.7). Having guessed in Theorem 1 the precise form of $\{p_j, q_j\}$ and $\{X_j, Y_j\}$ we apply in Theorem 5 the evolutionary equations techniques to the non-autonomous inhomogeneous equation (1.2) with a special initial data (1.10) to reach $\delta = m/n$ and obtain estimate (1.9), important in the proof of Theorem 3.

2. Description of the problem and nonlinear estimate on the continuity of variations.

This Section is devoted to the derivation of quasi-contractive nonlinear estimate which reflects the continuous dependence of variations with respect to the initial data.

Introduce notation $I\!\!P$ for vectors $c = \{c_k\}_{k\in\mathbb{Z}^d}$ such that $\delta_c = \sup_{|k-j|=1} |c_k/c_j| < \infty$. Let cylinder Wiener process $W = \{W_k(t)\}_{k\in\mathbb{Z}^d}$ with values in $\ell_2(a)$, $\sum_{k\in\mathbb{Z}^d} a_k = 1,\ a \in I\!\!P$ be canonically realized on measurable space $(\Omega = C_0([0,\infty), \ell_2(a)), \mathcal{F}, \mathcal{F}_t, \mathbf{P})$ with canonical filtration $\mathcal{F}_t = \sigma\{W(s)|0 \leq s \leq t\}$ and cylinder Wiener measure $\mathbf{P}$.

Consider the stochastic differential equation

$$\xi_{k,\emptyset}^x(t) = x_k + \int_0^t dW_k(s) - \int_0^t [F(\xi_{k,\emptyset}^x(s)) + (B\xi_\emptyset^x)_k] ds \tag{2.1}$$

with nonlinear diagonal map $\{F(x)\}_k = F(x_k),\ k \in \mathbb{Z}^d$ defined by a smooth monotone function $F \in C^\infty(I\!\!R^1),\ F(0) = 0$ such that

$$\exists\, \mathbf{k} \geq -1: \quad \forall\, n \in I\!\!N\ \exists\, C_n^F\ \forall\, i = 1, ..., n \quad |F^{(i)}(x) - F^{(i)}(y)| \leq C_n^F |x-y|(1+|x|+|y|)^{\mathbf{k}} \tag{2.2}$$

and linear finite diagonal map B:

$$\exists\, r_0 > 0 \quad (Bz)_k = \sum_{j:\ |j-k|\leq r_0} B(k-j) z_j, \qquad k \in \mathbb{Z}^d \text{ for some numbers } B(i),\ |i| \leq r_0.$$

The solvability of equations like (2.1) has already been studied in [7, 8, 20, 21, 22]. For example, for initial data $x \in \ell_{2(\mathbf{k}+1)^2}(a)$ there is a *unique strong solution* $\xi_\emptyset^x(t) = \{\xi_{k,\emptyset}^x(t)\}_{k\in\mathbb{Z}^d}$ to equation (2.1), i.e. $\ell_2(a)$-continuous $\mathcal{F}_t$-adapted process $\xi_\emptyset^x \in \mathcal{D}_{\ell_2(a)}(F)$, which a.e. fulfills equation (2.1) and for $x \in \ell_2(a)$ there is a *generalized solution*, obtained as uniform on $[0,T]$ $\mathbf{P}$ a.e. limit of strong solutions. By a slight modification of approach [7] it is easy to show (see [2, Th.3.11]) that for $x, y \in \ell_2(a)$ and any $q \geq 1$ there are constants M and K_q such that

$$\mathbf{E} \sup_{\sigma\in[0,T]} \|\xi_\emptyset^x(\sigma) - \xi_\emptyset^y(\sigma)\|_{\ell_2(a)}^q \leq e^{qMT} \|x-y\|_{\ell_2(a)}^q, \tag{2.3}$$

$$\mathbf{E} \sup_{\sigma\in[0,T]} (1 + \|\xi_\emptyset^x(\sigma)\|_{\ell_2(a)}^2 + \|\xi_\emptyset^y(\sigma)\|_{\ell_2(a)}^2)^q \leq K_q e^{qMT} (1 + \|x\|_{\ell_2(a)}^2 + \|y\|_{\ell_2(a)}^2)^q. \tag{2.4}$$

Consider associated to (2.1) variational equations

$$\xi^x_{k,\tau}(t) = x_{k,\tau} - \int_0^t [F'(\xi^x_{k,\emptyset})\xi^x_{k,\tau} + (B\xi^x_\tau)_k + \varphi^x_{k,\tau}]ds \tag{2.5}$$

with inhomogeneous part φ^x_τ defined by

$$\varphi^x_{k,\tau} = \sum_{\gamma_1\cup...\cup\gamma_s=\tau,\ s\geq 2} F^{(s)}(\xi^x_{k,\emptyset})\xi^x_{k,\gamma_1}...\xi^x_{k,\gamma_s}, \tag{2.6}$$

where the summation runs on all subdivisions of set τ onto the nonintersecting subsets $\gamma_1,...,\gamma_s \subset \tau,\ |\gamma_1|+...+|\gamma_s| = |\tau|,\ s\geq 2,\ |\gamma_1|\geq 1$ and $\xi^x_{\gamma_1},...,\xi^x_{\gamma_s}$ are the solutions of lower rank variational equations.

It is important to mention that the processes ξ^x_τ are variations of process $\xi^x_\emptyset$ with respect to the initial data x (2.1), i.e. have sense of partial derivatives $\{\xi_\tau\}_k = \xi_{k,\tau} = \dfrac{\partial^{|\tau|}\xi^x_{k,\emptyset}}{\partial x_{j_n}...\partial x_{j_1}}$ with respect to variables $\{x_{j_n},...,x_{j_1}\},\ \tau=\{j_1,...,j_n\}$, only for zero-one initial data in (2.5) [2, Th.3.6&3.7]

$$\widetilde{x}_{k,\tau} = \begin{cases} \delta_{kj},\ \tau=\{j\},\ |\tau|=1 \\ 0,\ |\tau|\geq 2 \end{cases}. \tag{2.7}$$

Suppose that vectors $\{c_\gamma\}_{\gamma\subset\tau}$ satisfy hierarchy: for any subdivision of set $\gamma=\alpha_1\cup...\cup\alpha_s,\ \gamma\subset\tau$ on nonempty nonintersecting subsets $\alpha_1,...,\alpha_s$ there is a constant $R_{\gamma,\alpha}$ such that

$$\forall\, k\in\mathbb{Z}^d \quad [c_{k,\gamma}]^{|\gamma|} a_k^{-\frac{\mathbf{k}+1}{2}m_1} \leq R_{\gamma,\alpha}[c_{k,\alpha_1}]^{|\alpha_1|}...[c_{k,\alpha_s}]^{|\alpha_s|}. \tag{2.8}$$

In [2, Th.3.1] it was shown that for the initial data $x_\gamma\in\ell_{m_\gamma}(dc_\gamma),\ \gamma\subset\tau$ with $d_k\geq a_k^{-\frac{\mathbf{k}+1}{2}m_1},\ m_\gamma = m_1/|\gamma|>1$ and vectors $\{c_\gamma\}_{\gamma\subset\tau}$ that fulfill (2.8) there is a family of *strong solutions* $\{\xi_\gamma\}_{\gamma\subset\tau}$ to system (2.5), i.e. $\mathcal{F}_t$-adapted processes $\{\xi^x_\gamma(t)\}_{\gamma\subset\tau}$ such that: 1) for $\mathbf{P}$ a.e. $\omega\in\Omega$ the map $[0,T]\ni t\to\xi^x_\gamma(t)\in\ell_{m_\gamma}(c_\gamma)$ is Lipschitz continuous; 2) for a.e. $t\in[0,T]$ $\xi^x_\gamma(t)\in\mathcal{D}_{\ell_{m_\gamma}(c_\gamma)}(F'(\xi^x_0(t))+B)$; 3) there is a strong $\ell_{m_\gamma}(c_\gamma)$ derivative $\dfrac{d\xi^x_\gamma(t)}{dt}$ a.e. on $t\in[0,T]$ and equation (2.5) holds in $\ell_{m_\gamma}(c_\gamma)$ sense. Moreover $\forall\, x\in\ell_2(a)\ \forall\, q\geq 1\ \forall\,\gamma\subset\tau\ \forall\, T>0$

$$\mathbf{E}\sup_{t\in[0,T]}\|\xi^x_\gamma(t)\|^q_{\ell_{m_\gamma}(c_\gamma)} < \infty$$

and ξ^x_γ permits representation

$$\xi^x_\gamma(t) = U^x(t,0)x_\gamma + \int_0^t U^x(t,s)\varphi^x_\gamma(s)ds \tag{2.9}$$

in the terms of strongly continuous in $X_\gamma=\ell_{m_\gamma}(c_\gamma)$ evolutional family $\{U^x(t,s),\ 0\leq s\leq t\}$, generated by $\{C^x(t)=F'(\xi^x_\emptyset(t,\omega))+B\}_{t\in[0,T]}$, such that

$$\|U^x(t,s)\|_{\mathcal{L}(X_\gamma)}\leq e^{\lambda(t-s)}, \quad \|U^x(t,s)\|_{\mathcal{L}(Y_\gamma)}\leq e^{\widetilde{\lambda}(t-s)} \tag{2.10}$$

for $Y_\gamma = \ell_{m_\gamma}(dc_\gamma)$, $d_k \geq a_k^{-\frac{\mathbf{k}_{+1}}{2}m_1}$, and constants $\lambda = \|B\|_{\mathcal{L}(X_\gamma)}$, $\widetilde{\lambda} = \|B\|_{\mathcal{L}(Y_\gamma)}$. For further consideration we will need the property of $\{U^x(t,s)\}$: for any $z \in Y_\gamma$ and $0 \leq s \leq t \leq T$

$$\|U^x(t,s)z - U^y(t,s)z\|_{X_\gamma} \leq |t-s| e^{2T(\lambda+\widetilde{\lambda})} \|z\|_{Y_\gamma} \sup_{\tau \in [s,t]} \|C^x(\tau) - C^y(\tau)\|_{\mathcal{L}(Y_\gamma, X_\gamma)}, \qquad (2.11)$$

which follows from the general theory of evolutional semigroups [14, 16, 17, 18].

To proceed further let $p_\gamma, q_\gamma \in C^\infty(I\!R_+)$, $\gamma \subset \tau$ be positive monotone functions of polynomial behaviour, i.e. such that

$$\exists\, \varepsilon > 0\ \forall\, z \in I\!R_+ \quad p_\gamma(z) \geq \varepsilon,\ p'_\gamma(z) \geq \varepsilon,$$

$$\exists\, C > 0 \quad (1+z)|p''_\gamma(z)| \leq C p'_\gamma(z), \quad (1+z) p'_\gamma(z) \leq C p_\gamma(z). \qquad (2.12)$$

Introduce nonlinear expression

$$\rho_\tau(\xi^x, \xi^y; t) = \rho^b_\tau(\xi^x, \xi^y; t) + \rho^c_\tau(\xi^x, \xi^y; t)$$

for respective boundedness ρ^b_τ and continuity ρ^c_τ parts

$$\rho^b_\tau(\xi^x, \xi^y; t) = \sum_{\gamma \subset \tau} \mathbf{E}\, \|\xi^x_\emptyset - \xi^y_\emptyset\|_{\ell_2(a)}{}^{\delta} p_\gamma(n^{x,y}_t) \{ \|\xi^x_\gamma\|^{m_\gamma}_{\ell_{m_\gamma}(c_\gamma)} + \|\xi^y_\gamma\|^{m_\gamma}_{\ell_{m_\gamma}(c_\gamma)} \}, \qquad (2.13)$$

$$\rho^c_\tau(\xi^x, \xi^y; t) = \sum_{\gamma \subset \tau} \mathbf{E}\, \|\xi^x_\emptyset - \xi^y_\emptyset\|_{\ell_2(a)}{}^{\delta - m_\gamma} q_\gamma(n^{x,y}_t) \|\xi^x_\gamma - \xi^y_\gamma\|^{m_\gamma}_{\ell_{m_\gamma}(a^{\frac{\mathbf{k}_{+1}}{2}m_\gamma} c_\gamma)}. \qquad (2.14)$$

Henceforth let $n^{x,y}_t$ denote the sum of norms

$$n^{x,y}_t = \|\xi^x_\emptyset(t)\|^2_{\ell_2(a)} + \|\xi^y_\emptyset(t)\|^2_{\ell_2(a)}$$

with corresponding sense of $n^{x,y}_0 = \|x\|^2_{\ell_2(a)} + \|y\|^2_{\ell_2(a)}$.

In the following Theorem we obtain the estimates of the continuous dependence of solutions ξ^x_γ to variational equations (2.5) with respect to the initial data x in (2.1) in the terms of quasi-contractive behaviour of expression $\rho_\tau(\xi^x, \xi^y; t)$. We'd like to turn the attention that the multiple $\|\xi^x_\emptyset - \xi^y_\emptyset\|^{\delta - m_\gamma}$ in (2.14) reflects the similarity of behaviour (1.6).

Theorem 1. *Let F satisfy (2.2), $\delta \geq m_1 \geq |\tau|$ and $\xi^x_\emptyset, \xi^y_\emptyset$, $\xi^x_\gamma, \xi^y_\gamma$, $\gamma \subset \tau$ be generalized solutions to (2.1) and (2.5) with initial data $x, y \in \ell_2(a)$ and $x_\gamma, y_\gamma \in \ell_{m_\gamma}(dc_\gamma)$, $d_k \geq a_k^{-\frac{\mathbf{k}_{+1}}{2}m_1}$ correspondingly.*

Suppose that weights $\{c_\gamma,\ \gamma \subset \tau\}$ fulfill hierarchy (2.8), functions $\{p_\gamma,\ \gamma \subset \tau\}$ of polynomial behaviour fulfill: $\forall\, \alpha_1 \cup \ldots \cup \alpha_s = \gamma \subset \tau$, $s \geq 2\ \exists\, K_p$

$$[p_\gamma(z)]^{|\gamma|} (1+z)^{\frac{\mathbf{k}_{+1}}{2}m_1} \leq K_p [p_{\alpha_1}(z)]^{|\alpha_1|} \ldots [p_{\alpha_s}(z)]^{|\alpha_s|}, \quad z \in I\!R_+ \qquad (2.15)$$

and functions $\{q_\gamma, \gamma \subset \tau\}$ are such that $q_\gamma(z)(1+z)^{\mathbf{k} m_\gamma/2} = p_\gamma(z)$.

Then $\exists\, M_\tau$ such that

$$\rho_\tau(\xi^x, \xi^y; t) \leq e^{M_\tau t} \rho_\tau(\xi^x, \xi^y; 0). \qquad (2.16)$$

Proof. The complete proof of this Theorem may be found in [5]. We give below its abridged version, because we will need some its steps in further calculations.

Theorem is proved by induction on the number of points in set τ. First suppose that initial data $x, y \in \ell_{2(\mathbf{k}+1)^2}(a)$, i.e. $\xi^x_\emptyset, \xi^y_\emptyset$ and $\xi^x_\gamma, \xi^y_\gamma$ form strong solutions to (2.1) and (2.5).

For $i = 1, ..., |\tau|$ introduce notations

$$h^i_\tau(\xi^x, \xi^y; t) = \begin{cases} 0, & i = 0 \\ \sum\limits_{\gamma\subset\tau,\ |\gamma|\le i} [g^b_\gamma(\xi^x, \xi^y; t) + g^c_\gamma(\xi^x, \xi^y; t)], & i \ge 1 \end{cases}$$

with

$$g^b_\gamma(t) = \ \mathbf{E}\ \|\xi^x_\emptyset - \xi^y_\emptyset\|_{\ell_2(a)}{}^{\delta} p_\gamma(n^{x,y}_t)\{\|\xi^x_\gamma\|^{m_\gamma}_{\ell_{m_\gamma}(c_\gamma)} + \|\xi^y_\gamma\|^{m_\gamma}_{\ell_{m_\gamma}(c_\gamma)}\},$$

$$g^c_\gamma(t) = \ \mathbf{E}\ \|\xi^x_\emptyset - \xi^y_\emptyset\|_{\ell_2(a)}{}^{\delta - m_\gamma} q_\gamma(n^{x,y}_t)\|\xi^x_\gamma - \xi^y_\gamma\|^{m_\gamma}_{\ell_{m_\gamma}(a^{\frac{\mathbf{k}+1}{2}m_\gamma} c_\gamma)}.$$

Similar to [2, 3] it is sufficient to derive estimate

$$g^b_\gamma(t) \le g^b_\gamma(0) + A_1 \int_0^t g^b_\gamma(s)ds + A_2 \int_0^t h^{i-1}_\tau(s)ds \tag{2.17}$$

for boundedness part g^b_γ of nonlinear expression and then, using the special symmetry (1.6) between g^b_γ and g^c_γ, obtain inequality

$$g^c_\gamma(t) \le g^c_\gamma(0) + B_1 \int_0^t g^c_\gamma ds + B_2 \int_0^t g^b_\gamma ds + B_3 \int_0^t h^{i-1}_\tau ds \tag{2.18}$$

for continuity part g^c_γ. Together with (2.17) this implies

$$g_\gamma(t) \le e^{C_1 t} g_\gamma(0) + C_2 \int_0^t e^{C_1(t-s)} h^{i-1}_\tau(\xi^x, \xi^y; s)ds \tag{2.19}$$

and nonlinear estimate (2.16).

Estimate (2.17). Ito formula for $g^b_\gamma = \ \mathbf{E}\ I^\delta_1 I_2 p_\gamma(n^{x,y}_t)$ with finite variation processes

$$I_1 = \|\xi^x_\emptyset - \xi^y_\emptyset\|_{\ell_2(a)} \qquad I_2 = \|\xi^x_\gamma\|^{m_\gamma}_{\ell_{m_\gamma}(c_\gamma)} + \|\xi^y_\gamma\|^{m_\gamma}_{\ell_{m_\gamma}(c_\gamma)} \tag{2.20}$$

implies

$$g^b_\gamma(t) = g^b_\gamma(0) + \ \mathbf{E}\ \int_0^t I^\delta_1 I_2 dp_\gamma(n^{x,y}_t) + p_\gamma(n^{x,y}_t) I_2 dI^\delta_1 + I^\delta_1 p_\gamma(n^{x,y}_t) dI_2 \tag{2.21}$$

To estimate first integral in (2.21) we write the stochastic differential of $p_\gamma(n^{x,y}_t)$

$$dp_\gamma(n^{x,y}_t) = -L^{x,y} p_\gamma(n^{x,y}_t)dt + 2p'_\gamma(n^{x,y}_t) < \xi^x_\emptyset + \xi^y_\emptyset, dW_t >_{\ell_2(a)} \tag{2.22}$$

where the second order differential operator $L^{x,y}$ acts by the rule

$$L^{x,y} p(n^{x,y}_0) = -2p'(n^{x,y}_0) \sum_{k\in\mathbb{Z}^d} a_k - 2p''(n^{x,y}_0) \sum_{k\in\mathbb{Z}^d} a^2_k (x_k + y_k)^2 + \tag{2.23}$$

$$+2p'(n^{x,y}_0)\{< x, F(x) + Bx >_{\ell_2(a)} + < y, F(y) + By >_{\ell_2(a)}\}.$$

Using (2.22) and estimate

$$L^{x,y}p(n_0^{x,y}) \geq -M_p p(n_0^{x,y}) \tag{2.24}$$

which is analogous to [1, Hint 9] we have

$$\mathbf{E}\int_0^t I_1^\delta I_2 dp_\gamma = -\int_0^t \mathbf{E}\, I_1^\delta I_2\, L^{x,y}p_\gamma(n_t^{x,y})dt \leq M_{p_\gamma}\int_0^t g_\gamma^b dt \tag{2.25}$$

Monotonicity of map F and Ito formula for the stochastic differential of I_1^δ give for the second term in (2.21)

$$\mathbf{E}\int_0^t p_\gamma(n_t^{x,y})I_2 dI_1^\delta \leq \delta\|B\|_{\mathcal{L}(\ell_2(a))}\int_0^t g_\gamma^b(t)dt \tag{2.26}$$

Ito formula for stochastic differential of I_2

$$dI_2 = m_\gamma\{ < [\xi_\gamma^x]^\#, d\xi_\gamma^x >_{\ell_{m_\gamma}(c_\gamma)} + < [\xi_\gamma^y]^\#, d\xi_\gamma^y >_{\ell_{m_\gamma}(c_\gamma)}\}dt$$

with $< \xi^\#, y >_{\ell_p(c)} = \sum\limits_{k\in \mathbb{Z}^d} c_k\xi_k|\xi_k|^{p-2}y_k$ and inequality

$$| < \xi^\#, \varphi >_{\ell_m} | \leq \tfrac{1}{m}\|\varphi\|_{\ell_m}^m + \tfrac{m-1}{m}\|\xi\|_{\ell_m}^m \tag{2.27}$$

together with $F' \geq 0$ imply

$$\mathbf{E}\int_0^t I_1^\delta p_\gamma(n_t^{x,y})dI_2 \leq (m_\gamma\|B\|_{\mathcal{L}(\ell_{m_\gamma}(c_\gamma))} + (m_\gamma - 1)K_\gamma)\int_0^t g_\gamma^b(t)dt + \int_0^t \mathbf{E}\, I_1^\delta p_\gamma(n_t^{x,y})\times$$

$$\times \sum_{\alpha_1\cup...\cup\alpha_s=\gamma,\ s\geq 2}\{\|F^{(s)}(\xi_\emptyset^x)\xi_{\alpha_1}^x...\xi_{\alpha_s}^x\|_{\ell_{m_\gamma}(c_\gamma)}^{m_\gamma} + \|F^{(s)}(\xi_\emptyset^y)\xi_{\alpha_1}^y...\xi_{\alpha_s}^y\|_{\ell_{m_\gamma}(c_\gamma)}^{m_\gamma}\}dt \tag{2.28}$$

where K_γ is a number of all possible subdivisions of set γ on subsets $\alpha_1, ..., \alpha_s$, $s \geq 2$.

Both terms in the second line of (2.28) are estimated in the same manner. Condition (2.2) implies

$$|F^{(s)}(\xi_{k,\emptyset}^x)|^{m_\gamma} \leq (C_n^F)^{m_\gamma} a_k^{-\frac{\mathbf{k}_{+1}}{2}m_\gamma}(1 + n_t^{x,y})^{\frac{\mathbf{k}_{+1}}{2}m_\gamma}$$

Using hierarchy (2.8) and representation $|\xi_\alpha|^{m_\gamma} = (|\xi_\alpha|^{m_\alpha})^{|\alpha|/|\gamma|}$ we estimate first term in (2.28) by

$$(2.28)_1 \leq$$

$$\leq (C_n^F)^{m_\gamma}\sum_{\alpha_1\cup...\cup\alpha_s=\gamma,\ s\geq 2} R_{\gamma,\alpha}^{1/|\gamma|}\int_0^t \mathbf{E}\, I_1^\delta p_\gamma(n_t^{x,y})(1 + n_t^{x,y})^{\frac{\mathbf{k}_{+1}}{2}m_\gamma}\sum_{k\in\mathbb{Z}^d}\prod_{\ell=1}^{s}(c_{k,\alpha_\ell}|\xi_{k,\alpha_\ell}^x|^{m_{\alpha_\ell}})^{|\alpha_\ell|/|\gamma|}dt \tag{2.29}$$

Hierarchy (2.15) and inequality $|x_1...x_n| \leq |x_1|^{q_1}/q_1 + ... + |x_n|^{q_n}/q_n$ with $q_\ell = |\gamma|/|\alpha_\ell|$ imply

$$(2.29) \leq K_p^{1/|\gamma|}(C_n^F)^{m_\gamma}\sum_{\alpha_1\cup...\cup\alpha_s=\gamma,\ s\geq 2} R_{\gamma,\alpha}^{1/|\gamma|}\int_0^t h_\tau^{i-1}(s)ds \tag{2.30}$$

Finally, (2.26), (2.30) and analogous estimates for second term in (2.28) lead to inequality (2.17).

Estimate (2.18). Similar to (2.17) $g^c_\gamma(t) = \mathbf{E}\, I_1^{\delta-m_\gamma} I_3 q_\gamma(n^{x,y}_t)$ with I_1 introduced in (2.20) and $I_3 = \|\xi^x_\gamma - \xi^y_\gamma\|^{m_\gamma}_{\ell_{m_\gamma}(a^{\frac{\mathbf{k}+1}{2}m_\gamma} c_\gamma)}$. Because $I_1^{\delta-m_\gamma}$ and I_3 are finite variation processes, applying Ito formula to $g^c_\gamma(t)$, we have

$$g^c_\gamma(t) = g^c_\gamma(0) + \mathbf{E}\int_0^t \{I_1^{\delta-m_\gamma} I_3 dq_\gamma(n^{x,y}_t) + q_\gamma(n^{x,y}_t) I_3 dI_1^{\delta-m_\gamma} + I_1^{\delta-m_\gamma} q_\gamma(n^{x,y}_t) dI_3\} \tag{2.31}$$

Representation (2.22) for $q_\gamma(n^{x,y}_t)$, inequality (2.24) and monotonicity of map F imply for the first and second integrals in (2.31):

$$\mathbf{E}\int_0^t I_1^{\delta-m_\gamma} I_3 dq_\gamma(n^{x,y}_t) \le M_{q_\gamma}\int_0^t g^c_\gamma(t)dt \tag{2.32}$$

$$\mathbf{E}\int_0^t q_\gamma(n^{x,y}_t) I_3 dI_1^{\delta-m_\gamma} \le (\delta - m_\gamma)\|B\|_{\mathcal{L}(\ell_2(a))}\int_0^t q^c_\gamma(s)ds \tag{2.33}$$

Estimation of the third term in (2.31) uses *proportionality* (1.6). Using Ito formula for I_3 with $\widetilde{c}_{k,\gamma} = a_k^{\frac{\mathbf{k}+1}{2}m_\gamma} c_{k,\gamma}$, representation (2.6) of φ_γ, monotonicity of map F ($F' \ge 0$), inequality (2.27) and, where necessary, adding and subtracting intermediate terms, we have

$$\mathbf{E}\int_0^t I_1^{\delta-m_\gamma} q_\gamma(n^{x,y}_t) dI_3 \le (m_\gamma\|B\|_{\mathcal{L}(\ell_{m_\gamma}(\widetilde{c}_\gamma))} + (m_\gamma - 1)(|\gamma|+1)K_\gamma)\int_0^t g^c_\gamma dt + \tag{2.34}$$

$$+\int_0^t \mathbf{E}\, I_1^{\delta-m_\gamma} q_\gamma(n^{x,y}_t)\|(F'(\xi^x_\emptyset) - F'(\xi^y_\emptyset))\xi^y_\gamma\|^{m_\gamma}_{\ell_{m_\gamma}(\widetilde{c}_\gamma)} dt + \tag{2.35}$$

$$+\sum_{\alpha_1\cup\ldots\cup\alpha_s=\gamma,\ s\ge 2}\int_0^t \mathbf{E}\, I_1^{\delta-m_\gamma} q_\gamma(n^{x,y}_t)\|\,[F^{(s)}(\xi^x_\emptyset) - F^{(s)}(\xi^y_\emptyset)]\xi^y_{\alpha_1}\ldots\xi^y_{\alpha_s}\|^{m_\gamma}_{\ell_{m_\gamma}(\widetilde{c}_\gamma)} dt + \tag{2.36}$$

$$+\sum_{\alpha_1\cup\ldots\cup\alpha_s=\gamma,\ s\ge 2}\sum_{j=1}^s\int_0^t \mathbf{E}\, I_1^{\delta-m_\gamma} q_\gamma(n^{x,y}_t)\,\|\,F^{(s)}(\xi^x_\emptyset)\xi^y_{\alpha_1}\ldots\xi^y_{\alpha_{j-1}}(\xi^x_{\alpha_j} - \xi^y_{\alpha_j})\xi^x_{\alpha_{j+1}}\ldots\xi^x_{\alpha_s}\|^{m_\gamma}_{\ell_{m_\gamma}(\widetilde{c}_\gamma)} dt \tag{2.37}$$

We use connection $q_\gamma(n^{x,y}_t)(1+n^{x,y}_t)^{\mathbf{k}m_\gamma/2} = p_\gamma(n^{x,y}_t)$ and assumption (2.2) on map F

$$\|\,[F'(\xi^x_\emptyset) - F'(\xi^y_\emptyset)]\xi^y_\gamma\|^{m_\gamma}_{\ell_{m_\gamma}(a^{\frac{\mathbf{k}+1}{2}m_\gamma} c_\gamma)} \le (C^F_n)^{m_\gamma}\|\xi^x_\emptyset - \xi^y_\emptyset\|_{\ell_2(a)}{}^{m_\gamma}(1+n^{x,y}_t)^{\mathbf{k}m_\gamma/2}\|\xi^y_\gamma\|^{m_\gamma}_{\ell_{m_\gamma}(c_\gamma)} \tag{2.38}$$

to obtain

$$(2.35) \le (C^F_n)^{m_\gamma}\int_0^t g^b_\gamma(t)dt \tag{2.39}$$

To estimate (2.36) we use $1 \le a_k^{-\frac{\mathbf{k}+1}{2}m_\gamma}$ and hierarchy (2.8) to get

$$\|\,[F^{(s)}(\xi^x_\emptyset) - F^{(s)}(\xi^y_\emptyset)]\xi^y_{\alpha_1}\ldots\xi^y_{\alpha_s}\|^{m_\gamma}_{\ell_{m_\gamma}(\widetilde{c}_\gamma)} \le$$

$$\le (C^F_n)^{m_\gamma}\|\xi^x_\emptyset - \xi^y_\emptyset\|_{\ell_2(a)}{}^{m_\gamma}(1+n^{x,y}_t)^{\mathbf{k}m_\gamma/2}\sum_{k\in\mathbb{Z}^d} c_{k,\gamma}|\xi^y_{k,\alpha_1}\ldots\xi^y_{k,\alpha_s}|^{m_\gamma} \le$$

$$\leq (C_n^F)^{m_\gamma} \|\xi_\emptyset^x - \xi_\emptyset^y\|_{\ell_2(a)}{}^{m_\gamma} (1+n_t^{x,y})^{\mathbf{k} m_\gamma/2} R_{\gamma,\alpha}^{1/|\gamma|} \prod_{j=1}^s [\|\xi_{\alpha_j}^y\|_{\ell_{m_{\alpha_j}}(c_{\alpha_j})}^{m_{\alpha_j}}]^{|\alpha_j|/|\gamma|} \tag{2.40}$$

and therefore by inequality $|x_1...x_n| \leq |x_1|^{q_1}/q_1 + ... + |x_n|^{q_n}/q_n$ with $q_j = |\gamma|/|\alpha_j|$, connection $q_\gamma(z)(1+z)^{\mathbf{k} m_\gamma/2} = p_\gamma(z)$ and using hierarchy (2.15) of polynomials $\{p_\gamma\}$ we have

$$(2.36) \leq (C_n^F)^{m_\gamma} K_p^{1/|\gamma|} \sum_{\alpha_1\cup...\cup\alpha_s=\gamma,\ s\geq 2} R_{\gamma,\alpha}^{1/|\gamma|} \int_0^t \mathbf{E}\, I_1^\delta \prod_{j=1}^s [p_{\alpha_j}(n_t^{x,y}) \|\xi_{\alpha_j}^y\|_{\ell_{m_{\alpha_j}}(c_{\alpha_j})}^{m_{\alpha_j}}]^{|\alpha_j|/|\gamma|} \leq$$

$$\leq (C_n^F)^{m_\gamma} K_p^{1/|\gamma|} \sum_{\alpha_1\cup...\cup\alpha_s=\gamma,\ s\geq 2} R_{\gamma,\alpha}^{1/|\gamma|} \sum_{j=1}^s \frac{|\alpha_j|}{|\gamma|} \int_0^t g_{\alpha_j}^b(t)dt \tag{2.41}$$

Finally, we estimate (2.37) for $\widetilde{c}_{k,\gamma} = a_k^{\frac{\mathbf{k}+1}{2} m_\gamma} c_{k,\gamma}$ by

$$\|F^{(s)}(\xi_\emptyset^x)\xi_{\alpha_1}^y ... \xi_{\alpha_{j-1}}^y (\xi_{\alpha_j}^x - \xi_{\alpha_j}^y) \xi_{\alpha_{j+1}}^x ... \xi_{\alpha_s}^x\|_{\ell_{m_\gamma}(\widetilde{c}_\gamma)}^{m_\gamma} \leq$$

$$\leq (C_n^F)^{m_\gamma} \sum_{k\in\mathbb{Z}^d} a_k^{\frac{\mathbf{k}+1}{2} m_\gamma} c_{k,\gamma} (1 + |\xi_{k,\emptyset}^x|^2 + |\xi_{k,\emptyset}^y|^2)^{\frac{\mathbf{k}+1}{2} m_\gamma} |\xi_{k,\alpha_1}^y|^{m_\gamma} ... |\xi_{k,\alpha_j}^x - \xi_{k,\alpha_j}^y|^{m_\gamma} ... |\xi_{k,\alpha_i}^x|^{m_\gamma} \leq$$

$$\leq (C_n^F)^{m_\gamma} (1+n_t^{x,y})^{\frac{\mathbf{k}+1}{2} m_\gamma} R_{\gamma,\alpha}^{1/|\gamma|} \|\xi_{\alpha_j}^x - \xi_{\alpha_j}^y\|_{\ell_{m_{\alpha_j}}(\widetilde{c}_{\alpha_j})}^{m_{\alpha_j}\cdot|\alpha_j|/|\gamma|} \times$$

$$\times \prod_{\ell=1,\ \ell\neq j}^s (1 + \|\xi_{\alpha_\ell}^x\|_{\ell_{m_{\alpha_\ell}}(c_{\alpha_\ell})}^{m_{\alpha_\ell}} + \|\xi_{\alpha_\ell}^y\|_{\ell_{m_{\alpha_\ell}}(c_{\alpha_\ell})}^{m_{\alpha_\ell}})^{|\alpha_\ell|/|\gamma|} \tag{2.42}$$

where on the last step we used

$$c_{k,\gamma} = c_{k,\gamma} a_k^{-\frac{\mathbf{k}+1}{2} m_\gamma} a_k^{\frac{\mathbf{k}+1}{2} m_\gamma} \leq R_{\gamma,\alpha}^{1/|\gamma|} [c_{k,\alpha_1}]^{|\alpha_1|/|\gamma|} ... [c_{k,\alpha_s}]^{|\alpha_s|/|\gamma|} a_k^{\frac{\mathbf{k}+1}{2} m_\gamma} =$$

$$= R_{\gamma,\alpha}^{1/|\gamma|} [c_{k,\alpha_j} a_k^{\frac{\mathbf{k}+1}{2} m_{\alpha_j}}]^{|\alpha_j|/|\gamma|} \prod_{\ell=1,\ \ell\neq j}^s [c_{k,\alpha_\ell}]^{|\alpha_\ell|/|\gamma|}$$

To estimate (2.37) we note that the hierarchy (2.15) of weights p_γ and the connection $q_\gamma(z)(1+z)^{\mathbf{k} m_\gamma/2} = p_\gamma(z)$ imply

$$q_\gamma(z)(1+z)^{\frac{\mathbf{k}+1}{2} m_\gamma} \leq (1+z)^{-\mathbf{k} m_\gamma/2} K_p^{1/|\gamma|} p_{\alpha_1}^{|\alpha_1|/|\gamma|} ... p_{\alpha_s}^{|\alpha_s|/|\gamma|} = K_p^{1/p} q_{\alpha_j}^{|\alpha_j|/|\gamma|} \prod_{\ell=1,\ \ell\neq j}^s p_{\alpha_\ell}^{|\alpha_\ell|/|\gamma|}$$

Finally, substituting (2.42) in (2.37) and using $\delta - m_\gamma = (\delta - m_{\alpha_j}) \dfrac{|\alpha_j|}{|\gamma|} + \sum_{\ell=1,\ \ell\neq j}^s \delta \dfrac{|\alpha_\ell|}{|\gamma|}$ we have

$$(2.37) \leq (C_n^F)^{m_\gamma} \sum_{\alpha_1\cup...\cup\alpha_s=\gamma,\ s\geq 2} R_{\gamma,\alpha}^{1/|\gamma|} K_p^{1/|\gamma|} \times$$

$$\times \sum_{j=1}^s \int_0^t \mathbf{E}\, (\|\xi_\emptyset^x - \xi_\emptyset^y\|_{\ell_2(a)}{}^{\delta - m_{\alpha_j}} q_{\alpha_j}(n_t^{x,y}) \|\xi_{\alpha_j}^x - \xi_{\alpha_j}^y\|_{\ell_{m_{\alpha_j}}(\widetilde{c}_{\alpha_j})}^{m_{\alpha_j}})^{|\alpha_j|/|\gamma|} \times$$

$$\times \prod_{\ell=1,\ \ell\neq j}^{s} (\ \|\xi^x_\emptyset - \xi^y_\emptyset\|_{\ell_2(a)}{}^{\delta} p_{\alpha_\ell}(n^{x,y}_t)(1+\|\xi^x_{\alpha_\ell}\|^{m_{\alpha_\ell}}_{\ell_{m_{\alpha_\ell}}(c_{\alpha_\ell})} + \|\xi^y_{\alpha_\ell}\|^{m_{\alpha_\ell}}_{\ell_{m_{\alpha_\ell}}(c_{\alpha_\ell})})\ )^{|\alpha_\ell|/|\gamma|} dt \leq$$

$$\leq (C^F_n)^{m_\gamma} \sum_{\alpha_1\cup\ldots\cup\alpha_s=\gamma,\ s\geq 2} R^{1/|\gamma|}_{\gamma,\alpha} K^{1/|\gamma|}_p \sum_{j=1}^{s} \int_0^t \{\frac{|\alpha_j|}{|\gamma|} g^c_{\alpha_j} + \sum_{\ell=1,\ \ell\neq j}^{s} \frac{|\alpha_\ell|}{|\gamma|} g^b_{\alpha_\ell}\} dt \leq$$

$$\leq (C^F_n)^{m_\gamma} \sum_{\alpha_1\cup\ldots\cup\alpha_s=\gamma,\ s\geq 2} R^{1/|\gamma|}_{\gamma,\alpha} K^{1/|\gamma|}_p \int_0^t h^{i-1}_\tau dt \tag{2.43}$$

Collecting together (2.32), (2.33), (2.34), (2.39), (2.41) and (2.43) we have (2.18) proved.

The possibility to close nonlinear estimate (2.16) from $x, y \in \ell_{2(\mathbf{k}+1)^2}(a)$ to $x, y \in \ell_2(a)$ follows from [2, Th.3.4&3.11]. ■

Next Corollary prepares the uniform on $\tau \subset \mathbb{Z}^d$, $|\tau| \leq n$ estimates which we need for the proof of smooth properties of semigroup P_t in Section 4.

COROLLARY 2. *Let F satisfy (2.2) and $\xi^x_\emptyset, \xi^y_\emptyset, \xi^x_\gamma, \xi^x_\gamma$ be generalized solutions to (2.1), (2.5). with initial data $x, y \in \ell_2(a)$ and zero-one initial data x_γ, y_γ (2.7).*

Then for function $Q(\cdot)$ of polynomial behaviour and vector $\psi = \{\psi_k\}_{k\in\mathbb{Z}^d} \in I\!P$ there are uniform on $\tau \subset \mathbb{Z}^d$ constants $K_1 = 2|\tau|\psi_0$ and $M_{|\tau|}$ such that $\forall\, \delta \geq 0\ \forall\, k \in \mathbb{Z}^d$

$$\mathbf{E}\ \|\xi^x_\emptyset - \xi^y_\emptyset\|_{\ell_2(a)}{}^{\delta} Q(n^{x,y}_t)\{|\xi^x_{k,\tau}|^{m_\tau} + |\xi^y_{k,\tau}|^{m_\tau}\} \leq$$

$$\leq \frac{K_1 e^{M_{|\tau|}t} \|x-y\|^{\delta}_{\ell_2(a)} Q(n^{x,y}_0)(1+n^{x,y}_0)^{\frac{\mathbf{k}+1}{2} m_1 \frac{|\tau|-1}{|\tau|}}}{a_k^{\frac{\mathbf{k}+1}{2} m_1 \frac{|\tau|-1}{|\tau|}} \prod_{j\in\tau} \psi^{m_1/|\tau|}_{k-j}} \tag{2.44}$$

Proof. Estimate (2.44) follows from inequality (2.17) which implies

$$\rho^b_\tau(\xi^x, \xi^y; t) \leq e^{M_\tau t} \rho^b_\tau(\xi^x, \xi^y; t) \tag{2.45}$$

with ρ^b_τ introduced in (2.13). In l.h.s of (2.45) we omit all lower terms, corresponding to sets $\gamma \subset \tau,\ |\gamma| < |\tau|$, choose $p_\gamma(z) = Q(z)(1+z)^{\frac{\mathbf{k}+1}{2}(m_\gamma - m_\tau)},\ \gamma \subset \tau$ and achieve

$$\mathbf{E}\ \|\xi^x_\emptyset - \xi^y_\emptyset\|_{\ell_2(a)}{}^{\delta} Q(n^{x,y}_t)\{\|\xi^x_\tau\|^{m_\tau}_{\ell_{m_\tau}(c_\tau)} + \|\xi^y_\tau\|^{m_\tau}_{\ell_{m_\tau}(c_\tau)}\} \leq$$

$$\leq e^{M_\tau t} \|x-y\|^{\delta}_{\ell_2(a)} Q(n^{x,y}_0)(1+n^{x,y}_0)^{\frac{\mathbf{k}+1}{2} m_1 \frac{|\tau|-1}{|\tau|}} 2 \sum_{j\in\tau} \|\widetilde{x}_{\{j\}}\|^{m_1}_{\ell_{m_1}(c_{\{j\}})}$$

Finally taking

$$c_{k,\gamma} = a_k^{\frac{\mathbf{k}+1}{2} m_1 \frac{|\gamma|-1}{|\gamma|}} \prod_{j\in\gamma} \psi^{m_1/|\gamma|}_{k-j} \tag{2.46}$$

which fulfill (2.8) with $R_{\gamma,\alpha} = 1$ we obtain estimate (2.44). Remark that by

$$\|Bx\|_{\ell_p(c)} = (\sum_{k\in\mathbb{Z}^d} c_k | \sum_{j:\,|k-j|\leq r_0} b(k-j) x_j|^p)^{1/p} =$$

$$= (\sum_{k\in\mathbb{Z}^d} c_k | \sum_{|i|\le r_0} b(i) x_{k-i}|^p)^{1/p} \le \sum_{|i|\le r_0} |b(i)| (\sum_{k\in\mathbb{Z}^d} c_k |x_{k-i}|^p)^{1/p}$$
$$= \sum_{|i|\le r_0} |b(i)| (\sum_{k\in\mathbb{Z}^d} \frac{c_k}{c_{k-i}} c_{k-i} |x_{k-i}|^p)^{1/p} \le \sum_{|i|\le r_0} |b(i)|\, \delta_c^{|i|/p}\, \|x\|_{\ell_p(c)}$$

and property $\delta_{ab} \le \delta_a \delta_b$ for $\delta_\psi = \sup_{|k-j|=1} |\psi_k/\psi_j| < \infty$, $\psi \in \mathbb{P}$ the choice (2.46) of vectors $\{c_\gamma\}_{\gamma\subset\tau}$ gives that $\|B\|_{\mathcal{L}(\ell_{m_\tau}(c_\tau))}$ is independent on $\tau : \ |\tau| \le n$.

Therefore constants A_1 and A_2 in (2.17) are uniform on $\tau : \ |\tau| \le n$ for any $n \in \mathbb{N}$ and this implies the same property of $M_\tau = M_{|\tau|}$ in (2.44). To end the proof one should also note that (2.17) works for any $\delta \ge 0$. ■

3. C^∞ properties of semigroup and non-Lipschitz gap between the boundedness and continuity topologies.

In this Section we obtain the preservation of spaces of continuously differentiable functions under the action of Feller semigroup $(P_t f)(x) = \mathbf{E}\, f(\xi_\emptyset^x(t))$. Using the quasi-contractive nonlinear estimates on the continuity of variations we estimate the gap between boundedness and continuity topologies on derivatives of functions, which ensures the quasi-contractive property of semigroup in non-Lipschitz case.

Let $Lip_r(\ell_2(a))$ denote the Banach space of continuous functions over $\ell_2(a)$ equipped with a norm

$$\|f\|_{Lip_r} = \sup_{x\in\ell_2(a)} \frac{|f(x)|}{(1+\|x\|_{\ell_2(a)})^{r+1}} + \sup_{x,y\in\ell_2(a)} \frac{|f(x)-f(y)|}{\|x-y\|_{\ell_2(a)}(1+\|x\|_{\ell_2(a)}+\|y\|_{\ell_2(a)})^r} < \infty$$

To introduce the norms on derivatives for function of $f \in Lip_r(\ell_2(a))$ consider for $n \in \mathbb{N}$ the array $\Theta = \Theta^1 \cup ... \cup \Theta^n$ of pairs $\{(q_m, \mathcal{G}^m) \in \Theta^m\}$ where q_m is a smooth polynomial which satisfies (2.12) and $\mathcal{G}^m = G^1 \otimes ... \otimes G^m$ is m-tensor constructed by vectors $G^i \in \mathbb{P}$, $i = 1, ..., m$.

The *Banach space* $\mathcal{E}_{\Theta,r}(\ell_2(a))$, $\Theta = \Theta_b \cup \Theta_c$ consists of functions $f \in Lip_r(\ell_2(a))$ which have partial derivatives up to n^{th} order $\{\partial^{(1)} f, ..., \partial^{(n)} f\}$, $\{\partial^{(m)} f\}_{k_1,...,k_m} = \partial_{\{k_1,...,k_m\}} f(x)$ and norm is finite

$$\|f\|_{\mathcal{E}_{\Theta,r}} = \|f\|_{Lip_r} + \max_{m=1,...,n} (\|\partial^{(m)} f\|_{\Theta_b^m}, \|\partial^{(m)} f\|_{\Theta_c^m}) < \infty$$

where

$$\|\partial^{(m)} f\|_{\Theta_b^m} = \max_{(q_m,\mathcal{G}^m)\in\Theta_b^m} \sup_{x\in\ell_2(a)} \frac{|\partial^{(m)} f(x)|_{\mathcal{G}^m}}{q_m(\|x\|^2_{\ell_2(a)})} \tag{3.1}$$

$$\|\partial^{(m)} f\|_{\Theta_c^m} = \max_{(q_m,\mathcal{H}^m)\in\Theta_c^m} \sup_{x,y\in\ell_2(a)} \frac{|\partial^{(m)} f(x) - \partial^{(m)} f(y)|_{\mathcal{H}^m}}{\|x-y\|_{\ell_2(a)} q_m(n_0^{x,y})(1+n_0^{x,y})^{\mathbf{k}/2}} \tag{3.2}$$

with

$$|\partial^{(m)} f|^2_{\mathcal{G}^m} = \sum_{\tau=\{j_1,...,j_m\}\subset\mathbb{Z}^d} G^1_{j_1} ... G^m_{j_m} |\partial_\tau f(x)|^2$$

for $\mathcal{G}^m = G^1 \otimes ... \otimes G^m$, $G^i \in \mathbb{P}$, and similar expression for $\mathcal{H}^m = H^1 \otimes ... \otimes H^m$.

The partial derivatives $\{\partial^{(1)} f, ..., \partial^{(n)} f\}$ of function $f \in \mathcal{E}_{\Theta,r}$ are understood in the sense of identities: $\forall\, x \in \ell_2(a)\ \forall\, h \in \mathbf{X}_\infty([a,b]) = \bigcap_{p\ge1,\ c\in\mathbb{P}} AC_\infty([a,b], \ell_p(c))$

$$f(x+h(\cdot))\Big|_a^b = \int_a^b ds \sum_{k\in\mathbb{Z}^d} \partial_k f(x+h(s)) h'_k(s) \tag{3.3}$$

and $\forall\,|\tau| \le n-1$

$$\partial_\tau f(x+h(\cdot))\Big|_a^b = \int_a^b ds \sum_{k\in\mathbb{Z}^d} \partial_{\tau\cup\{k\}} f(x+h(s))h'_k(s) \tag{3.4}$$

Here $AC_\infty([a,b],X) = \{h \in C([a,b],X):\ \exists\, h' \in L_\infty([a,b],X)\}$ for Banach space X.

The topologies of space $\mathcal{E}_{\Theta,r}$, or more precisely weights $\mathcal{G}^m, \mathcal{H}^m$ should fulfill additional assumptions to guarantee the quasi-contractive property of semigroup. Like in [1, 2] we say that the array $\Theta = \Theta^1 \cup ... \cup \Theta^n$ is *quasi-contractive* with parameter $\mathbf{k}$ iff $\forall\, m = 2,...,n$ $\forall\,(q,\mathcal{G}) \in \Theta^m$ and $\forall\, i,j \in \{1,...,m\}$, $i \ne j$ there is a pair $(\widetilde{q},\widetilde{\mathcal{G}}) \in \Theta^{m-1}$ such that

$$\begin{array}{ll} \exists\, K\ \ \forall\, z \in \mathbb{R}_+ & (1+z)^{\frac{\mathbf{k}+1}{2}}\widetilde{q}(z) \le Kq(z) \\ \exists\, L & (\widehat{\mathcal{G}}^{\{i,j\}})^\ell \le L\widetilde{G}^\ell, \quad \ell = 1,..,m-1 \end{array} \tag{3.5}$$

where the $(m-1)$ tensor $\widehat{\mathcal{G}}^{\{i,j\}}$ is constructed from m-tensor $\mathcal{G}$ by rule

$$\widehat{\mathcal{G}}^{\{i,j\}} = G^1 \otimes \underset{\widehat{i}}{....} \otimes \underset{\uparrow j}{A^{-(\mathbf{k}+1)}G^i G^j} \otimes .. \otimes G^m$$

Notation $G^1 \otimes \underset{\widehat{i}}{....} \otimes G^s$ means that in tensor product the i^{th} - vector is omitted and $G^1 \otimes .. \otimes \underset{\uparrow j}{B} \otimes .. \otimes G^s$ means that on j^{th} - place in tensor product it is inserted vector B. Inequality (3.5) is understood as a coordinate inequality between two vectors.

Henceforth we demand that *array Θ_b in (3.1) is generated by array Θ_c (3.2) by law*

$$\forall\, m = 1,...,n \quad \Theta_b^m = \{\ (q_m, \mathcal{G}_j^m)_{j=1}^m \ \ such\ \ that\ \ \mathcal{G}_j^m = H^1 \otimes ... \otimes A^{-(\mathbf{k}+1)}H^j \otimes ... \otimes H^m \tag{3.6}$$
$$for\ \ (q_m, \mathcal{H}^m = H^1 \otimes ... \otimes H^m) \in \Theta_c^m\}$$

Immediately remark that for quasi-contractive with parameter $\mathbf{k}$ array Θ_c the array $\Theta_b = \Theta_b^1 \cup ... \cup \Theta_b^n$ generated by (3.6) is also a quasi-contractive one, which could be directly checked.

The next theorem gives the smooth properties of semigroup P_t, associated with the stochastic differential equation (2.1), and estimates the gap between the boundedness and continuity topologizations in corresponding functional spaces.

Theorem 3. *Let F fulfill (2.2), $\Theta = \Theta_b \cup \Theta_c$ for Θ_c be quasi-contractive array with parameter $\mathbf{k}$ and Θ_b be generated by Θ_c by rule (3.6).*

Then $\forall\, t \ge 0$ $P_t : \mathcal{E}_{\Theta,r} \to \mathcal{E}_{\Theta,r}$ and $\exists\, K_{\Theta,r}, M_{\Theta,r}$ such that

$$\forall\, f \in \mathcal{E}_{\Theta,r} \quad \|P_t f\|_{\mathcal{E}_{\Theta,r}} \le K_{\Theta,r} e^{M_{\Theta,r}t} \|f\|_{\mathcal{E}_{\Theta,r}} \tag{3.7}$$

Proof. First remark that the definition of norm in $\mathcal{E}_{\Theta,r}$ implies that the function $f \in \mathcal{E}_{\Theta,r}$ has partial derivatives $\partial_\tau f \in C(\ell_2(a), \mathbb{R}^1)$, $|\tau| \le n$. In [2, Thm.3.9] we have shown that the spaces $C_{\Theta,r}$ equipped with norm

$$\|f\|_{C_{\Theta,r}} = \|f\|_{Lip_r} + \max_{m=1,...,n} \|\partial^{(m)} f\|_{\Theta_b}$$

and consisting of functions f with partial derivatives $\partial_\tau f \in C(\ell_2(a), I\!R^1)$, $|\tau| \le n$ in sense (3.3), (3.4) are preserved under the action of semigroup:

$$\forall\, t \ge 0 \quad P_t : C_{\Theta,r} \to C_{\Theta,r} \quad \text{and} \quad \exists\, K, M\; \forall\, f \in C_{\Theta,r} \;\; \|P_t f\|_{C_{\Theta,r}} \le K e^{Mt} \|f\|_{C_{\Theta,r}}. \tag{3.8}$$

Moreover for $f \in C_{\Theta,r}$ the partial derivatives $\partial_\tau P_t f \in C(\ell_2(a), I\!R^1)$ fulfill (3.3), (3.4) and have representation

$$\partial_\tau (P_t f)(x) = \sum_{s=1}^{m} \sum_{\gamma_1 \cup \ldots \cup \gamma_s = \tau} \mathbf{E} \;< \partial^{(s)} f(\xi_\emptyset^x), \xi_{\gamma_1}^x \otimes \ldots \otimes \xi_{\gamma_s}^x > \tag{3.9}$$

for $\tau = \{k_1, ..., k_m\}$ and $\partial_\tau = \partial^{|\tau|}/\partial x_{k_1} ... \partial x_{k_m}$, where we used notation

$$< \partial^{(s)} f(\xi_\emptyset^x), \xi_{\gamma_1}^x \otimes \ldots \otimes \xi_{\gamma_s}^x > = \sum_{j_1,..,j_s \in Z\!\!\!Z^d} (\partial_{\{j_1,...,j_s\}} f)(\xi_\emptyset^x)\, \xi_{j_1,\gamma_1}^x ... \xi_{j_s,\gamma_s}^x$$

Because $\mathcal{E}_{\Theta,r} \subset C_{\Theta,r}$ by the above considerations, we only have to obtain estimate (3.7). Due to (3.8) it is sufficient to show that

$$\max_{m=1,...,n} \|\partial^{(m)} P_t f\|_{\Theta_c^m} \le K_\Theta e^{M_\Theta t} \max_{m=1,...,n} (\|\partial^{(m)} f\|_{\Theta_b^m}, \|\partial^{(m)} f\|_{\Theta_c^m}) \tag{3.10}$$

To prove (3.10) we represent derivatives of semigroup $\partial^{(m)} P_t f(x) = \{\partial_{k_1} ... \partial_{k_m} P_t f(x)\}_{k_1,...,k_m}$ (3.9) as:

$$\partial^{(m)} P_t f(x) = \sum_{s=1}^{m} \sum_{\beta_1 \cup \ldots \cup \beta_s = \{1,...,m\}} \mathbf{E} \;< \partial^{(s)} f(\xi_\emptyset^x), \vec{\xi}_{\beta_1}^x ... \vec{\xi}_{\beta_s}^x >$$

where for any fixed set $\{j_1, ..., j_m\} \subset Z\!\!\!Z^d$ the subdivision $\gamma_1 \cup ... \cup \gamma_s = \{j_1, ..., j_m\}$ is replaced by subdivision $\beta_1 \cup ... \cup \beta_s = \{1, ..., m\}$ with corresponding identification $\gamma_i = \{j_\ell,\ \ell \in \beta_i\}$ and $\{\vec{\xi}_{\beta_\ell}^x\}_{j_1,...,j_m} = \xi_{\gamma_\ell}^x$, $\ell = 1, ..., s$. In other words expression $\mathbf{E} \;< \partial^{(s)} f(\xi_\emptyset^x), \vec{\xi}_{\beta_1}^x ... \vec{\xi}_{\beta_s}^x >$, $\beta_1 \cup ... \cup \beta_s = \{1, ..., m\}$ has coordinates

$$\{\, \mathbf{E} \;< \partial^{(s)} f(\xi_\emptyset^x), \vec{\xi}_{\beta_1}^x ... \vec{\xi}_{\beta_s}^x >\}_{j_1,...,j_m \in Z\!\!\!Z^d} = \mathbf{E} \;< \partial^{(s)} f(\xi_\emptyset^x), \xi_{\gamma_1}^x \otimes \ldots \otimes \xi_{\gamma_s}^x >$$

Now we take a fixed pair $(q_m, \mathcal{H}^m = H^1 \otimes ... \otimes H^m) \in \Theta_c^m$ in the expression (3.2) of $\|\partial^{(m)} P_t f\|_{\Theta_c^m}$, add and subtract intermediate terms to obtain

$$\frac{|\partial^{(m)} P_t f(x) - \partial^{(m)} P_t f(y)|_{\mathcal{H}^m}}{\|x-y\|_{\ell_2(a)} q_m(n_0^{x,y})(1+n_0^{x,y})^{\mathbf{k}/2}} \le$$

$$\le \sum_{s=1}^{m} \sum_{\beta_1 \cup \ldots \cup \beta_s = \{1,...,m\}} \Big\{ \frac{\Big|\, \mathbf{E} \sum_{k_1,...,k_s \in Z\!\!\!Z^d} [\partial_{k_1} ... \partial_{k_s} f(\xi_\emptyset^x) - \partial_{k_1} ... \partial_{k_s} f(\xi_\emptyset^y)] \vec{\xi}_{k_1,\beta_1}^x ... \vec{\xi}_{k_s,\beta_s}^x \Big|_{\mathcal{H}^m}}{\|x-y\|_{\ell_2(a)} q_m(n_0^{x,y})(1+n_0^{x,y})^{\mathbf{k}/2}} + \tag{3.11}$$

$$+ \sum_{j=1}^{s} \frac{\Big|\, \mathbf{E} \sum_{k_1,...,k_s \in Z\!\!\!Z^d} \partial_{k_1} ... \partial_{k_s} f(\xi_\emptyset^y) \vec{\xi}_{k_1,\beta_1}^x ... \vec{\xi}_{k_{j-1},\beta_{j-1}}^x (\vec{\xi}_{k_j,\beta_j}^x - \vec{\xi}_{k_j,\beta_j}^y) \vec{\xi}_{k_{j+1},\beta_{j+1}}^y ... \vec{\xi}_{k_s,\beta_s}^y \Big|_{\mathcal{H}^m}}{\|x-y\|_{\ell_2(a)} q_m(n_0^{x,y})(1+n_0^{x,y})^{\mathbf{k}/2}} \Big\} \tag{3.12}$$

To finish the proof we have to estimate each term in (3.11) and (3.12).

To estimate (3.11) we apply Corollary 2. For fixed $s \in \{1, ..., m\}$, and subdivision $\beta_1 \cup ... \cup \beta_s = \{1, ..., m\}$

$$(3.11) \leq \Big| \sum_{k_1,...,k_s \in \mathbb{Z}^d} \vec{B}^{\beta_1,...,\beta_s}_{k_1,...,k_s} \Big(\mathbf{E} \frac{|\partial_{k_1}...\partial_{k_s} f(\xi^x_\emptyset) - \partial_{k_1}...\partial_{k_s} f(\xi^y_\emptyset)|^2}{\|\xi^x_\emptyset - \xi^y_\emptyset\|_{\ell_2(a)}{}^2 q_s^2(n_t^{x,y})(1+n_t^{x,y})^{\mathbf{k}}} \Big)^{1/2} \Big|_{\mathcal{H}^m}$$

where expression $\vec{B}^{\beta_1,...,\beta_s}_{k_1,...,k_s}$ has coordinates: $\{\vec{B}^{\beta_1,...,\beta_s}_{k_1,...,k_s}\}_{j_1,...,j_m \in \mathbb{Z}^d} = B^{\gamma_1,...,\gamma_s}_{k_1,...,k_s}$ for $\gamma_i = \{j_\ell,\ \ell \in \beta_i\}$ and

$$B^{\gamma_1,...,\gamma_s}_{k_1,...,k_s} = \frac{\big(\mathbf{E}\, \|\xi^x_\emptyset - \xi^y_\emptyset\|_{\ell_2(a)}{}^2 q_s^2(n_t^{x,y})(1+n_t^{x,y})^{\mathbf{k}} |\xi^x_{k_1,\gamma_1}...\xi^x_{k_s,\gamma_s}|^2\big)^{1/2}}{\|x-y\|_{\ell_2(a)} q_m(n_0^{x,y})(1+n_0^{x,y})^{\mathbf{k}/2}} \leq$$

$$\leq \frac{\prod_{\ell=1}^s \big(\mathbf{E}\, \|\xi^x_\emptyset - \xi^y_\emptyset\|_{\ell_2(a)}{}^2 q_s^2(n_t^{x,y})(1+n_t^{x,y})^{\mathbf{k}} |\xi^x_{k_\ell,\gamma_\ell}|^{2m/|\gamma_\ell|}\big)^{|\gamma_\ell|/2m}}{\|x-y\|_{\ell_2(a)} q_m(n_0^{x,y})(1+n_0^{x,y})^{\mathbf{k}/2}}$$

Above we applied Hölder inequality with

$$x_i = \|\xi^x_\emptyset - \xi^y_\emptyset\|_{\ell_2(a)}{}^{2|\gamma_i|/m} q_s^{2|\gamma_i|/m}(n_t^{x,y})(1+n_t^{x,y})^{\mathbf{k}|\gamma_i|/m} |\xi^x_{k_i,\gamma_i}|^2, \quad \sum |\gamma_i|/m = 1$$

Taking in (2.44) $\delta = 2$, $Q(z) = q_s^2(z)(1+z)^{\mathbf{k}}$, $m_1 = 2m$ and m_τ replaced by $2m/|\gamma_\ell|$ we obtain

$$B^{\gamma_1,...,\gamma_s}_{k_1,...,k_s} \leq \frac{\prod_{\ell=1}^s \{K_1 e^{M_m t} \|x-y\|^2_{\ell_2(a)} q_s^2(n_0^{x,y})(1+n_0^{x,y})^{\mathbf{k}+\frac{\mathbf{k}+1}{2} m_1 \frac{|\gamma_\ell|-1}{|\gamma_\ell|}}\}^{|\gamma_\ell|/2m}}{\|x-y\|_{\ell_2(a)} q_m(n_0^{x,y})(1+n_0^{x,y})^{\mathbf{k}/2} \prod_{\ell=1}^s (a_{k_\ell}^{\frac{\mathbf{k}+1}{2} m_1 \frac{|\gamma_\ell|-1}{|\gamma_\ell|}} \prod_{j\in\gamma_\ell} \psi_{k_\ell - j}^{m_1/|\gamma_\ell|})^{|\gamma_\ell|/2m}} =$$

$$= K_1^{1/2} e^{M_m t/2} \frac{q_s(n_0^{x,y})(1+n_0^{x,y})^{\frac{\mathbf{k}+1}{2}(m-s)}}{q_m(n_0^{x,y})} \prod_{\ell=1}^s (a_{k_\ell}^{-\frac{\mathbf{k}+1}{2}(|\gamma_\ell|-1)} \prod_{j\in\gamma_\ell} \psi^{-1}_{k_\ell-j}) \qquad (3.13)$$

Quasi-contractivity of array Θ_c implies that for any pair $(q_m, \mathcal{H}^m) \in \Theta_c^m$ and $\forall\, s \leq m$ and subdivision $\beta_1 \cup ... \cup \beta_s = \{1, ..., m\}$ there is a pair $(q_s, \widetilde{\mathcal{H}}^s = \widetilde{H}^1 \otimes ... \otimes \widetilde{H}^s) \in \Theta_c^s$ such that

$$q_s(z)(1+z)^{\frac{\mathbf{k}+1}{2}(m-s)} \leq K^{m-s} q_m(s) \qquad (3.14)$$

$$\forall\, k_1, ..., k_s \in \mathbb{Z}^d \quad \prod_{i=1}^s a_{k_i}^{-(\mathbf{k}+1)(|\beta_i|-1)} H^{(\beta_i)}_{k_i} \leq L^{m-s} \prod_{i=1}^s \widetilde{H}^i_{k_i} \qquad (3.15)$$

with $H^{(\beta)}_k = \prod_{i\in\beta} H^i_k,\ k \in \mathbb{Z}^d$.

Applying (3.14) to (3.13) we continue

$$B^{\gamma_1,...,\gamma_s}_{k_1,...,k_s} \leq K^{m-s} K_1^{1/2} e^{M_m t/2} \prod_{\ell=1}^s (a_{k_\ell}^{-\frac{\mathbf{k}+1}{2}(|\gamma_\ell|-1)} \prod_{j\in\gamma_\ell} \psi^{-1}_{k_\ell-j}) \qquad (3.16)$$

and substituting (3.16) into (3.11)

$$(3.11) \le K^{m-s} K_1^{1/2} e^{M_m t/2} [\sum_{j_1,...,j_m \in \mathbb{Z}^d} H^1_{j_1} ... H^m_{j_m} \{ \sum_{k_1,...,k_s \in \mathbb{Z}^d} (\prod_{\ell=1}^s a_{k_\ell}^{-\frac{\mathbf{k}+1}{2}(|\gamma_\ell|-1)} \prod_{j\in\gamma_\ell} \psi^{-1}_{k_\ell - j})$$

$$(\mathbf{E} \frac{|\partial_{k_1}...\partial_{k_s} f(\xi^x_\emptyset) - \partial_{k_1}...\partial_{k_s} f(\xi^y_\emptyset)|^2}{\|\xi^x_\emptyset - \xi^y_\emptyset\|_{\ell_2(a)}{}^2 q_s^2(n_t^{x,y})(1+n_t^{x,y})^{\mathbf{k}}})^{1/2} \}^2]^{1/2} \tag{3.17}$$

Combinatorial Lemma 7 with $b_a = \psi_a^{-1}$

$$x_{k_1,...,k_s} = \prod_{\ell=1}^s a_{k_\ell}^{-\frac{\mathbf{k}+1}{2}(|\gamma_\ell|-1)} (\mathbf{E} \frac{|\partial_{k_1}...\partial_{k_s} f(\xi^x_\emptyset) - \partial_{k_1}...\partial_{k_s} f(\xi^y_\emptyset)|^2}{\|\xi^x_\emptyset - \xi^y_\emptyset\|_{\ell_2(a)}{}^2 q_s^2(n_t^{x,y})(1+n_t^{x,y})^{\mathbf{k}}})^{1/2}$$

and vector $\psi \in I\!\!P$ chosen so that

$$K_\psi = \max_{m=1,...,n} \max_{(q_m,\mathcal{H}^m)\in\Theta^m} \max_{s=1,...,m} \sum_{a\in\mathbb{Z}^d} \psi_a^{-1} [\delta_{H^s}]^{|a|} < \infty$$

imply

$$(3.17) \le K_1^{1/2} K^{m-s} e^{M_m t/2} (1+K_\psi)^m \{ \sum_{k_1,...,k_s \in \mathbb{Z}^d} H^{(\gamma_1)}_{k_1} ... H^{(\gamma_s)}_{k_s} [\prod_{\ell=1}^s a_{k_\ell}^{-\frac{\mathbf{k}+1}{2}(|\gamma_\ell|-1)}$$

$$(\mathbf{E} \frac{|\partial_{k_1}...\partial_{k_s} f(\xi^x_\emptyset) - \partial_{k_1}...\partial_{k_s} f(\xi^y_\emptyset)|^2}{\|\xi^x_\emptyset - \xi^y_\emptyset\|_{\ell_2(a)}{}^2 q_s^2(n_t^{x,y})(1+n_t^{x,y})^{\mathbf{k}}})^{1/2}]^2 \}^{1/2} \tag{3.18}$$

with $H^{(\gamma)}_k = \prod_{j\in\gamma} H^j_k$. Using the quasi-contractivity of array $\Theta_c = \{(q_m, \mathcal{H}^m),\ m = 1, ..., n\}$ and its consequence (3.15) to (3.18) we finally have

$$(3.18) \le K_1^{1/2} K^{m-s} e^{M_m t/2} (1+K_\psi)^m L^{m-s} (\mathbf{E} \frac{|\partial^{(s)} f(\xi^x_\emptyset) - \partial^{(s)} f(\xi^y_\emptyset)|^2_{\widetilde{\mathcal{H}}^s}}{\|\xi^x_\emptyset - \xi^y_\emptyset\|_{\ell_2(a)}{}^2 q_s^2(n_t^{x,y})(1+n_t^{x,y})^{\mathbf{k}}})^{1/2} \le$$

$$\le K_1^{1/2} K^{m-s} L^{m-s} e^{M_m t/2} (1+K_\psi)^m \max_{m=1,...,n} \|\partial^{(m)} f\|_{\Theta^m_c} \tag{3.19}$$

To estimate (3.12) we apply Theorem 5 and its Corollary 6, i.e. nonlinear estimate continued into domain $\delta = m_\tau$. For fixed $s \in \{1, ..., m\}$, subdivision $\beta_1 \cup ... \cup \beta_s = \{1, ..., m\}$ and $j \in \{1, ..., s\}$ expression (3.12) is estimated by

$$(3.12) \le | \sum_{k_1,...,k_s \in \mathbb{Z}^d} \vec{D}^{\beta_1,...,\beta_s}_{k_1,...,k_s} (\mathbf{E} \frac{|\partial_{k_1}...\partial_{k_s} f(\xi^x_\emptyset)|^2}{q_s^2(\|\xi^x_\emptyset\|^2_{\ell_2(a)})})^{1/2} |_{\mathcal{H}^m} \tag{3.20}$$

where $\vec{D}^{\beta_1,...,\beta_s}_{k_1,...,k_s}$ has coordinates: $\{\vec{D}^{\beta_1,...,\beta_s}_{k_1,...,k_s}\}_{j_1,...,j_m\in\mathbb{Z}^d} = D^{\gamma_1,...,\gamma_s}_{k_1,...,k_s}$ for $\gamma_i = \{j_\ell,\ \ell \in \beta_i\}$

$$D^{\gamma_1,...,\gamma_s}_{k_1,...,k_s} = \frac{(\mathbf{E}\, q_s^2(\|\xi^x_\emptyset\|^2_{\ell_2(a)}) |\xi^x_{k_1,\gamma_1} ... \xi^x_{k_{j-1},\gamma_{j-1}} (\xi^x_{k_j,\gamma_j} - \xi^y_{k_j,\gamma_j}) \xi^y_{k_{j+1},\gamma_{j+1}} ... \xi^y_{k_s,\gamma_s}|^2)^{1/2}}{\|x-y\|_{\ell_2(a)} q_m(n_0^{x,y})(1+n_0^{x,y})^{\mathbf{k}/2}} \le$$

$$\leq \frac{(\ \mathbf{E}\ q_s^2(n_t^{x,y})|\xi^x_{k_j,\gamma_j}-\xi^y_{k_j,\gamma_j}|^{2m/|\gamma_j|})^{|\gamma_j|/2m}}{\|x-y\|_{\ell_2(a)}q_m(n_0^{x,y})(1+n_0^{x,y})^{\mathbf{k}/2}} \tag{3.21}$$

$$\prod_{\ell=1,\ \ell\neq j}[\ \mathbf{E}\ q_s^2(n_t^{x,y})(|\xi^x_{k_\ell,\gamma_\ell}|^{2m/|\gamma_\ell|}+|\xi^y_{k_\ell,\gamma_\ell}|^{2m/|\gamma_\ell|})]^{|\gamma_\ell|/2m} \tag{3.22}$$

Above we used $q_s(\|\xi^x_\emptyset\|^2_{\ell_2(a)})\leq q_s(n_t^{x,y})$ and Hölder inequality with $x_i=q_s^{|\gamma_i|/m}(n_t^{x,y})|\xi^x_{k_i,\gamma_i}|^2$, $i\neq j$, and $x_j=q_s^{|\gamma_j|/m}(n_t^{x,y})|\xi^x_{k_j,\gamma_j}-\xi^y_{k_j,\gamma_j}|^2$, $\alpha_i=2m/|\gamma_i|$.

To expressions (3.21) and (3.22) we apply Corollary 6 (4.27) and Corollary 2 (2.44) with $Q(z)=q_s^2(z)$, $m_1=2m$ and m_τ replaced by $2m/|\gamma_j|$ or $2m/|\gamma_\ell|$ correspondingly. Collecting together the inequalities obtained and using hierarchy (3.14) for weights $\{q_s\}$ we derive

$$D^{\gamma_1,\dots,\gamma_s}_{k_1,\dots,k_s}\leq(\max\{K_1,K_2\})^{1/2}e^{M_mt/2}\frac{q_s(n_0^{x,y})(1+n_0^{x,y})^{\frac{\mathbf{k}+1}{2}(m-s)}}{q_m(n_0^{x,y})}\cdot$$

$$\cdot a_{k_j}^{-(\mathbf{k}+1)/2}\prod_{\ell=1}^{s}(a_{k_\ell}^{-\frac{\mathbf{k}+1}{2}(|\gamma_\ell|-1)}\prod_{r\in\gamma_\ell}\psi^{-1}_{k_\ell-r})\leq$$

$$\leq(\max\{K_1,K_2\})^{1/2}K^{m-s}e^{M_mt/2}a_{k_j}^{-(\mathbf{k}+1)/2}\prod_{\ell=1}^{s}(a_{k_\ell}^{-\frac{\mathbf{k}+1}{2}(|\gamma_\ell|-1)}\prod_{r\in\gamma_\ell}\psi^{-1}_{k_\ell-r}) \tag{3.23}$$

Substitution of (3.23) into (3.20) gives for fixed $j\in\{1,...,s\}$

$$(3.12)\leq K'e^{M't}[\sum_{j_1,\dots,j_m}H^1_{j_1}...H^m_{j_m}\{\sum_{k_1,\dots,k_s}a_{k_j}^{-\frac{\mathbf{k}+1}{2}}\times$$

$$\times\prod_{\ell=1}^{s}(a_{k_\ell}^{-\frac{\mathbf{k}+1}{2}(|\gamma_\ell|-1)}\prod_{b\in\gamma_\ell}\psi^{-1}_{k_\ell-b})(\ \mathbf{E}\ \frac{|\partial_{k_1}...\partial_{k_s}f(\xi^x_\emptyset)|^2}{q_s^2(\|\xi^x_\emptyset\|^2_{\ell_2(a)})})^{1/2}\}^2]^{1/2}\leq$$

$$\leq K'e^{M't}(1+K_\psi)^m(\sum_{k_1,\dots,k_s}H^{(\gamma_1)}_{k_1}...H^{(\gamma_s)}_{k_s}a_{k_j}^{-(\mathbf{k}+1)}\prod_{\ell=1}^{s}a_{k_\ell}^{-(\mathbf{k}+1)(|\gamma_\ell|-1)}\ \mathbf{E}\ \frac{|\partial_{k_1}...\partial_{k_s}f(\xi^x_\emptyset)|^2}{q_s^2(\|\xi^x_\emptyset\|^2_{\ell_2(a)})})^{1/2} \tag{3.24}$$

On the last step we applied Lemma 7 with $b_a=\psi_a^{-1}$

$$x_{k_1,\dots,k_s}=a_{k_j}^{-\frac{\mathbf{k}+1}{2}}(\prod_{\ell=1}^{s}a_{k_\ell}^{-\frac{\mathbf{k}+1}{2}(|\gamma_\ell|-1)})(\ \mathbf{E}\ \frac{|\partial_{k_1}...\partial_{k_s}f(\xi^x_\emptyset)|^2}{q_s^2(\|\xi^x_\emptyset\|^2_{\ell_2(a)})})^{1/2}$$

and vector $\psi\in I\!\!P$ was chosen so that

$$\widetilde{K}_\psi=\max_{m=1,\dots,n}\ \max_{(q_m,\mathcal{H}^m)\in\Theta^m_c}\ \max_{s=1,\dots,m}\sum_{a\in\mathbb{Z}^d}\psi_a^{-1}[\delta_{H^s}]^{|a|}<\infty$$

Due to the quasi-contractivity (3.15) of array Θ_c for tensor $\mathcal{H}^m=H^1\otimes...\otimes H^m$ there is tensor $\widetilde{\mathcal{H}}^s\in\Theta^s_c$ such that

$$H^{(\gamma_1)}_{k_1}...H^{(\gamma_s)}_{k_s}\prod_{\ell=1}^{s}a_{k_\ell}^{-(\mathbf{k}+1)(|\gamma_\ell|-1)}\leq L^{m-s}\widetilde{H}^1_{k_1}...\widetilde{H}^s_{k_s}$$

Furthermore, by construction of array Θ_b (3.6) for any $\widetilde{\mathcal{H}}^s \in \Theta_c^s$ tensor $\{\mathcal{G}_j^s\}_{k_1,...,k_s} = \widetilde{H}_{k_1}...\widetilde{H}_{k_s} a_{k_j}^{-(\mathbf{k}+1)}$ belongs to Θ_b^s and we have

$$(3.24) \leq (\max\{K_1, K_2\})^{1/2} K^{m-s} L^{m-s} e^{M_m t/2} (1+\widetilde{K}_\psi)^m \sup_{\xi_\emptyset^x} \frac{|\partial^{(s)} f(\xi_\emptyset^x)|_{\mathcal{G}^s}}{q_s(\|\xi_\emptyset^x\|^2_{\ell_2(a)})} \leq$$

$$\leq (\max\{K_1, K_2\})^{1/2} K^{m-s} L^{m-s} e^{M_m t/2} (1+\widetilde{K}_\psi)^m \max_{m=1,...,n} \|\partial^{(m)} f\|_{\Theta_b^m} \tag{3.25}$$

From (3.19) and (3.25) we have estimate (3.10) proved. ■

4. Continuation of nonlinear estimate in the domain $\delta = m_\tau$.

We give a proof of estimate on the continuity of variations in the domain $\delta = m_\tau \leq m_1$ (2.16) by application of evolutionary semigroups techniques. We show that at the optimal choice of parameters $\{p_\gamma, c_\gamma, q_\gamma\}$, achieved in the quasi-contractive nonlinear estimate (2.16), one can extend its consequence in form (1.9) in the domain of smaller values for constant $\delta = m_\tau < m_1$, i.e. when the direct monotone methods are not applicable.

We pay a special attention below to control the uniformity of constants with respect to $\tau: \ |\tau| \leq n$, necessary for the estimation of (3.21).

First Lemma provides uniform estimates on the boundedness of ξ_τ^x.

Lemma 4. *Let vectors $\{c_\gamma\}_{\gamma\subset\tau}$ satisfy hierarchy (2.8) and $\{\xi_\gamma^x\}_{\gamma\subset\tau}$ be strong solutions to system (2.5) with zero-one initial data $\{\widetilde{x}_\gamma, \ \gamma \subset \tau\}$ (2.7) and initial data $x \in \ell_2(a)$ for equation (2.1). Then $\forall\, q \geq 1 \ \exists\, M_\tau$ such that*

$$\mathbf{E} \sup_{\sigma\in[0,t]} \|\xi_\tau^x(\sigma)\|^q_{\ell_{m_\tau}(c_\tau)} \leq e^{M_\tau t} (1+\|x\|^2_{\ell_2(a)})^{\frac{\mathbf{k}+1}{2} q(|\tau|-1)} \sum_{j\in\tau} \|\widetilde{x}_{\{j\}}\|^{q|\tau|}_{\ell_{m_1}(c_{\{j\}})} \tag{4.1}$$

Proof. We prove this Lemma by induction on $|\tau|$. For $\tau = \{j\}, \ |\tau| = 1$ and $\varphi_\tau \equiv 0$ representation (2.9) implies

$$\mathbf{E} \sup_{\sigma\in[0,t]} \|\xi_\tau^x(\sigma)\|^q_{\ell_{m_1}(c_{\{j\}})} \leq e^{\lambda q t} \|\widetilde{x}_{\{j\}}\|^q_{\ell_{m_1}(c_{\{j\}})}$$

with $\lambda = \|B\|_{\mathcal{L}(\ell_{m_1}(c_{\{j\}}))}$. This gives an inductive base.

Let (4.1) be proved for all $\gamma \subset \tau, \ |\gamma| < |\tau|$. By (2.9) we have

$$\mathbf{E} \sup_{[0,t]} \|\xi_\tau^x\|^q_{\ell_{m_\tau}(c_\tau)} \leq \ \mathbf{E}\, (e^{\lambda t} \|\widetilde{x}_\tau\|_{\ell_{m_\tau}(c_\tau)} + t e^{\lambda t} \sup_{[0,t]} \|\varphi_\tau^x\|_{\ell_{m_\tau}(c_\tau)})^q \leq$$

$$\leq t^q e^{\lambda q t}\, \mathbf{E} \sup_{[0,t]} \|\varphi_\tau^x\|^q_{\ell_{m_\tau}(c_\tau)} \tag{4.2}$$

Above we used that for zero-one initial data $\widetilde{x}_\tau$ if $|\tau| \geq 2$ then $\widetilde{x}_{k,\tau} = 0$ and $\|\widetilde{x}_\tau\|_{\ell_{m_\tau}(c_\tau)} = 0$. Expression (2.6) for φ_τ, property (2.2) of map F and hierarchy (2.8) give, like in (2.28)-(2.29) an estimate

$$\mathbf{E} \sup_{[0,t]} \|\varphi_\tau^x\|^q_{\ell_{m_\tau}(c_\tau)} = \ \mathbf{E} \sup_{[0,t]} \| \sum_{\gamma_1\cup...\cup\gamma_s=\tau, \ s\geq 2} F^{(s)}(\xi_\emptyset^x) \xi_{\gamma_1}^x ... \xi_{\gamma_s}^x \|^q_{\ell_{m_\tau}(c_\tau)} \leq$$

$$\leq (C_n^F)^q\,\mathbf{E}\sup_{[0,t]}(1+\|\xi_\emptyset^x\|^2_{\ell_2(a)})^{\frac{\mathbf{k}+1}{2}q}(\sum_{\gamma_1,\ldots,\gamma_s} R_{\tau,\gamma}\prod_{\ell=1}^s \|\xi^x_{\gamma_\ell}\|^{m_\tau}_{\ell_{m_{\gamma_\ell}}(c_{\gamma_\ell})})^{q/m_\tau}\leq$$

$$\leq (C_n^F)^q(\,\mathbf{E}\sup_{[0,t]}(1+\|\xi_\emptyset^x\|^2_{\ell_2(a)})^{\frac{\mathbf{k}+1}{2}q(s+1)})^{1/(s+1)}\times$$

$$\times(\sum_{\gamma_1\cup\ldots\cup\gamma_s=\tau,\ s\geq 2} R_{\tau,\gamma})^{q/m_\tau}(\sup_{\gamma_1,\ldots,\gamma_s}\prod_{\ell=1}^s\,\mathbf{E}\sup_{[0,t]}\|\xi^x_{\gamma_\ell}\|^{(s+1)q}_{\ell_{m_{\gamma_\ell}}(c_{\gamma_\ell})})^{1/(s+1)} \tag{4.3}$$

By application to (4.3) of inductive assumption and estimate (2.4), using that for $s\geq 2$ $\alpha=\sum_{\ell=1}^s(|\gamma_\ell|-1)+1\leq|\tau|-1$ and therefore $(1+\|x\|^2)^\alpha\leq(1+\|x\|^2)^{|\tau|-1}$ we obtain

$$\mathbf{E}\sup_{[0,t]}\|\varphi^x_\tau\|^q_{\ell_{m_\tau}(c_\tau)}\leq M_1e^{M_2t}(1+\|x\|^2_{\ell_2(a)})^{\frac{\mathbf{k}+1}{2}q(|\tau|-1)}\sup\prod_{\ell=1}^s(\sum_{j\in\gamma_\ell}\|\widetilde{x}_{\{j\}}\|^{(s+1)q|\gamma_\ell|}_{\ell_{m_1}(c_{\{j\}})})^{1/(s+1)} \tag{4.4}$$

with constants

$$M_1=(C_n^F)^qK^{1/(s+1)}_{\frac{\mathbf{k}+1}{2}q(s+1)}(\sum_{\gamma_1,\ldots,\gamma_s}R_{\tau,\gamma})^{q/m_\tau} \tag{4.5}$$

$$M_2=(s+1)(\mathbf{k}+1)Mq+\sup_{\gamma_1,\ldots,\gamma_s}\tfrac{1}{s+1}\sum_{\ell=1}^s M_{\gamma_\ell}$$

The equivalence of all norms in $I\!R^{|\tau|}$ gives inequality

$$\prod_{\ell=1}^s(\sum_{j\in\tau}\|\widetilde{x}_{\{j\}}\|^{(s+1)q|\gamma_\ell|}_{\ell_{m_1}(c_{\{j\}})})^{1/(s+1)}\leq C_{|\tau|}\sum_{j\in\tau}\|\widetilde{x}_{\{j\}}\|^{q\sum_{\ell=1}^s|\gamma_\ell|}_{\ell_{m_1}(c_{\{j\}})} \tag{4.6}$$

so we estimate (4.4) by

$$(4.4)\leq 2C_{|\tau|}M_1e^{M_2t}(1+\|x\|^2_{\ell_2(a)})^{q\frac{\mathbf{k}+1}{2}(|\tau|-1)}\sum_{j\in\tau}\|\widetilde{x}_{\{j\}}\|^{q|\tau|}_{\ell_{m_1}(c_{\{j\}})} \tag{4.7}$$

Substituting (4.3)-(4.7) into (4.2) and applying $Ct^q\leq\exp(q\sqrt[q]{C}t)$ we have statement (4.1) of Lemma 4 with

$$M_\tau=q\sqrt[q]{M_1C_{|\tau|}}+\lambda q+M_2 \qquad \blacksquare$$

Theorem 5. (Continuation of nonlinear estimate to $\delta=m_\tau$). *Let conditions of Theorem 1 hold. Then for zero-one initial data (2.7) the variations fulfill*

$$\exists\,M_\tau\qquad \mathbf{E}\,q(n_t^{x,y})\sup_{t\in[0,T]}\|\xi^x_\tau-\xi^y_\tau\|^{\delta m_\tau}_{\ell_{m_\tau}(a^{\frac{\mathbf{k}+1}{2}m_\tau}c_\tau)}\leq$$

$$\leq e^{M_\tau T}\|x-y\|^{\delta m_\tau}_{\ell_2(a)}q(n_0^{x,y})(1+n_0^{x,y})^{\delta(\frac{\mathbf{k}+1}{2}m_1\frac{|\tau|-1}{|\tau|}+\mathbf{k}m_\tau/2)}\sum_{j\in\tau}\|\widetilde{x}_{\{j\}}\|^{\delta m_1}_{\ell_{m_1}(c_{\{j\}})} \tag{4.8}$$

Proof. To obtain the inductive base at $\tau=\{j\},\ m_\tau=m_1$ we use representation (2.9) and estimate (2.11) with $\varphi_\tau\equiv 0,\ |\tau|=1$

$$\mathbf{E}\,q(n_t^{x,y})\sup_{t\in[0,T]}\|\xi^x_{\{j\}}-\xi^y_{\{j\}}\|^{\delta m_1}_{\ell_{m_1}(a^{\frac{\mathbf{k}+1}{2}m_1}c_{\{j\}})}=$$

$$= \mathbf{E}\, q(n_t^{x,y}) \sup_{t\in[0,T]} \|\mathcal{U}^x(t,0)\widetilde{x}_{\{j\}} - \mathcal{U}^y(t,0)\widetilde{x}_{\{j\}}\|^{\delta m_1}_{\ell_{m_1}(a^{\frac{\mathbf{k}+1}{2}m_1}c_{\{j\}})} \leq$$

$$\leq (T\exp 2T(\lambda+\widetilde{\lambda}))^{\delta m_1}\ \mathbf{E}\, q(n_t^{x,y}) \sup_{\sigma,t\in[0,T]} \|F'(\xi^x_\emptyset) - F'(\xi^y_\emptyset)\|^{\delta m_1}_{\mathcal{L}(Y_{\{j\}},X_{\{j\}})} \|\widetilde{x}_{\{j\}}\|^{\delta m_1}_{Y_{\{j\}}} \tag{4.9}$$

where $X_{\{j\}} = \ell_{m_1}(a^{\frac{\mathbf{k}+1}{2}m_1}c_{\{j\}})$ and $Y_{\{j\}} = \ell_{m_1}(da^{\frac{\mathbf{k}+1}{2}m_1}c_{\{j\}}) = \ell_{m_1}(c_{\{j\}})$ for a weight $d_k = a_k^{-\frac{\mathbf{k}+1}{2}m_1}$ and $\lambda = \|B\|_{\mathcal{L}(X_{\{j\}})}$, $\widetilde{\lambda} = \|B\|_{\mathcal{L}(Y_{\{j\}})}$.

Using properties of nonlinear F and proceeding like in (2.38) we have for $u \in Y_{\{j\}}$

$$\|[F'(\xi^x_\emptyset) - F'(\xi^y_\emptyset)]u\|^{\delta m_1}_{X_{\{j\}}} \leq (C_1^F)^{\delta m_1} \|\xi^x_\emptyset - \xi^y_\emptyset\|^{\delta m_1}_{\ell_2(a)} (1+n_t^{x,y})^{\delta \mathbf{k} m_1/2} \|u\|^{\delta m_1}_{\ell_{m_1}(c_{\{j\}})}$$

and estimate (4.9) by

$$(4.9) \leq (Te^{2T(\lambda+\widetilde{\lambda})}C_1^F)^{\delta m_1} \|\widetilde{x}_{\{j\}}\|^{\delta m_1}_{\ell_{m_1}(c_{\{j\}})}\ \mathbf{E}\, q(n_t^{x,y}) \sup_{\sigma\in[0,T]} \|\xi^x_\emptyset - \xi^y_\emptyset\|^{\delta m_1}_{\ell_2(a)} (1+n_t^{x,y})^{\delta \mathbf{k} m_1/2} \leq$$

$$\leq (Te^{2T(\lambda+\widetilde{\lambda})}C_1^F)^{\delta m_1} \|\widetilde{x}_{\{j\}}\|^{\delta m_1}_{\ell_{m_1}(c_{\{j\}})} (\ \mathbf{E}\, q^3(n_t^{x,y}))^{1/3} \times$$

$$\times (\ \mathbf{E} \sup_{\sigma\in[0,T]} \|\xi^x_\emptyset - \xi^y_\emptyset\|^{3\delta m_1}_{\ell_2(a)})^{1/3} (\ \mathbf{E} \sup_{\sigma\in[0,T]} (1+n_t^{x,y})^{3\delta m_1 \mathbf{k}/2})^{1/3} \tag{4.10}$$

Representation (2.22) of stochastic differential of $q^3(n_t^{x,y})$ and inequality (2.24) lead to

$$\mathbf{E}\, q^3(n_t^{x,y}) = q^3(n_0^{x,y}) -\ \mathbf{E} \int_0^t L^{x,y} q^3(n_t^{x,y}) dt \leq q^3(n_0^{x,y}) + M_{q^3} \int_0^t\ \mathbf{E}\, q^3(n_t^{x,y}) dt \tag{4.11}$$

with second order differential operator $L^{x,y}$ given by (2.23). By Gronwall-Bellmann lemma inequality (4.11) implies

$$\mathbf{E}\, q^3(n_t^{x,y}) \leq e^{2M_{q^3}t} q^3(n_0^{x,y}) \tag{4.12}$$

Finally (4.12) and (2.3), (2.4) applied to (4.10) give

$$(4.10) \leq e^{M_{\{j\}}T} \|\widetilde{x}_{\{j\}}\|^{\delta m_1}_{\ell_{m_1}(c_{\{j\}})} q(n_0^{x,y}) \|x-y\|^{\delta m_1}_{\ell_2(a)} (1+n_0^{x,y})^{\delta \mathbf{k} m_1/2} \tag{4.13}$$

with

$$M_{\{j\}} = 2\delta m_1(\lambda+\widetilde{\lambda}) + \frac{2}{3}M_{q^3} + \delta m_1 M(1+\mathbf{k}) + C_1^F \delta m_1 \sqrt[3\delta m_1]{2K_{3\delta m_1 \mathbf{k}/2}} \tag{4.14}$$

where we used that $CT^q \leq \exp(q\sqrt[q]{C}T)$.

Inequality (4.13) gives the statement of Theorem 5 at $\tau = \{j\}$.

Inductive step. Suppose that for all sets $\gamma \subset \tau, |\gamma| < |\tau|$ estimate (4.8) is already proved. Now we check it for τ. In terms of notation $X_\tau = \ell_{m_\tau}(\widetilde{c}_\tau)$ for $\widetilde{c}_\tau = a^{\frac{\mathbf{k}+1}{2}m_\tau}c_\tau$, $Y_\tau = \ell_{m_\tau}(d\widetilde{c}_\tau) = \ell_{m_\tau}(c_\tau)$, $d_k = a_k^{-\frac{\mathbf{k}+1}{2}m_\tau}$, for zero-one initial data x, y (2.7) using representation (2.9) and properties (2.10) and (2.11) of evolutional families $U^x(t,s)$, $U^y(t,s)$ we have

$$\mathbf{E}\, q(n_t^{x,y}) \sup_{t\in[0,T]} \|\xi^x_\tau - \xi^y_\tau\|^{\delta m_\tau}_{X_\tau} =$$

$$= \mathbf{E}\, q(n_t^{x,y}) \sup_{t\in[0,T]} \| \int_0^t \{\mathcal{U}^x(t,\sigma)\varphi_\tau^x(\sigma) - \mathcal{U}^y(t,\sigma)\varphi_\tau^y(\sigma)\} d\sigma\|_{X_\tau}^{\delta m_\tau} \leq$$

$$\leq T^{\delta m_\tau} 2^{\delta m_\tau - 1}\, \mathbf{E}\, q(n_t^{x,y}) \sup_{t,\sigma\in[0,T]} \|\{\mathcal{U}^x(t,\sigma) - \mathcal{U}^y(t,\sigma)\}\varphi_\tau^x(\sigma)\|_{X_\tau}^{\delta m_\tau} +$$

$$+T^{\delta m_\tau} 2^{\delta m_\tau - 1}\, \mathbf{E}\, q(n_t^{x,y}) \sup_{t,\sigma\in[0,T]} \|\mathcal{U}^y(t,\sigma)\{\varphi_\tau^x(\sigma) - \varphi_\tau^y(\sigma)\}\|_{X_\tau}^{\delta m_\tau} \leq$$

$$\leq T^{2\delta m_\tau} 2^{\delta m_\tau - 1} e^{2\delta m_\tau(\lambda+\widetilde{\lambda})T}\, \mathbf{E}\, q(n_t^{x,y}) \sup_{t\in[0,T]} \|F'(\xi_\emptyset^x) - F'(\xi_\emptyset^y)\|_{\mathcal{L}(Y_\tau,X_\tau)}^{\delta m_\tau} \|\varphi_\tau\|_{Y_\tau}^{\delta m_\tau} + \tag{4.15}$$

$$+T^{\delta m_\tau} 2^{\delta m_\tau - 1} e^{\lambda \delta m_\tau T}\, \mathbf{E}\, q(n_t^{x,y}) \sup_{t\in[0,T]} \|\varphi_\tau^x - \varphi_\tau^y\|_{X_\tau}^{\delta m_\tau} \tag{4.16}$$

To estimate (4.15) we apply (2.38), (2.3), (2.4) and achieve

$$(\,\mathbf{E} \sup_{t\in[0,T]} \|F'(\xi_\emptyset^x) - F'(\xi_\emptyset^y)\|_{\mathcal{L}(Y_\tau,X_\tau)}^{3\delta m_\tau})^{1/3} \leq$$

$$\leq (C_n^F)^{\delta m_\tau} e^{\delta m_\tau MT} \|x-y\|_{\ell_2(a)}^{\delta m_\tau} K_{3\delta \mathbf{k} m_\tau}^{1/6} e^{\delta m_\tau \mathbf{k} MT} (1 + n_0^{x,y})^{\delta \mathbf{k} m_\tau/2} \tag{4.17}$$

Estimate (4.4) with $q = 3\delta m_\tau$ and inequality (4.6) give

$$(\,\mathbf{E} \sup_{t\in[0,T]} \|\varphi_\tau^x\|_{\ell_{m_\tau}(c_\tau)}^{3\delta m_\tau})^{1/3} \leq M_1^{1/3} e^{M_2 T/3} (1 + \|x\|_{\ell_2(a)}^2)^{\delta m_\tau \frac{\mathbf{k}+1}{2}(|\tau|-1)}$$

$$\sup_{\gamma_1,\ldots,\gamma_s} \prod_{\ell=1}^s (\sum_{j\in\gamma_\ell} \|\widetilde{x}_{\{j\}}\|_{\ell_{m_1}(c_{\{j\}})}^{3\delta(s+1)m_\tau|\gamma_\ell|})^{1/3(s+1)} \leq$$

$$\leq M_1^{1/3} C_{|\tau|} e^{M_2 T/3} (1 + \|x\|_{\ell_2(a)}^2)^{\delta m_\tau \frac{\mathbf{k}+1}{2}(|\tau|-1)} \sum_{j\in\tau} \|\widetilde{x}_{\{j\}}\|_{\ell_{m_1}(c_{\{j\}})}^{\delta m_1} \tag{4.18}$$

with constants $M_1,\ M_2$ given by (4.5). Above we used that $\delta m_\tau \sum_{\ell=1}^s |\gamma_\ell| = \delta m_\tau |\tau| = \delta m_1$.

Applying (4.11) we finally have from (4.17) and (4.18)

$$(4.15) \leq M_1' T^{2\delta m_\tau} e^{M_2' T} \|x-y\|_{\ell_2(a)}^{\delta m_\tau} (1 + n_0^{x,y})^{\delta m_\tau \mathbf{k}/2 + \delta m_\tau \frac{\mathbf{k}+1}{2}(|\tau|-1)} q(n_0^{x,y}) \sum_{j\in\tau} \|\widetilde{x}_{\{j\}}\|_{\ell_{m_1}(c_{\{j\}})}^{\delta m_1} \tag{4.19}$$

where

$$M_1' = (C_n^F)^{\delta m_\tau} K_{3\delta \mathbf{k} m_\tau}^{1/6} 2^{\delta m_\tau - 1} M_1^{1/3} C_{|\tau|}$$

$$M_2' = 2\delta m_\tau(\lambda + \widetilde{\lambda}) + \delta m_\tau M(\mathbf{k}+1) + \frac{2}{3} M_{q^3} + M_2/3$$

From inequality $CT^q \leq \exp(q\sqrt[q]{C}T)$ we transform (4.19) in required form (4.8).

Estimation of (4.16). Using representation (2.6) for φ_τ and adding and subtracting necessary terms we obtain for some combinatorial constant $C_{|\tau|,\delta m_1}$

$$\|\varphi_\tau^x - \varphi_\tau^y\|_{X_\tau}^{\delta m_\tau} \leq C_{|\tau|,\delta m_1} \sum_{\gamma_1\cup\ldots\cup\gamma_s=\tau,\ s\geq 2} \{\| [F^{(s)}(\xi_\emptyset^x) - F^{(s)}(\xi_\emptyset^y)] \xi_{\gamma_1}^y \ldots \xi_{\gamma_s}^y \|_{\ell_{m_\tau}(\widetilde{c}_\tau)}^{\delta m_\tau} + \tag{4.20}$$

$$+\sum_{i=1}^{s}\|F^{(s)}(\xi_\emptyset^x)\xi_{\gamma_1}^y...\xi_{\gamma_{i-1}}^y(\xi_{\gamma_i}^x-\xi_{\gamma_i}^y)\xi_{\gamma_{i+1}}^x...\xi_{\gamma_s}^x\|_{\ell_{m_\tau}(\widetilde{c}_\tau)}^{\delta m_\tau}\} \tag{4.21}$$

In analog to (2.40) we have

$$\|[F^{(s)}(\xi_\emptyset^x)-F^{(s)}(\xi_\emptyset^y)]\xi_{\gamma_1}^y...\xi_{\gamma_s}^y\|_{\ell_{m_\tau}(\widetilde{c}_\tau)}^{\delta m_\tau}\le$$

$$\le (C_n^F)^{\delta m_\tau}\|\xi_\emptyset^x-\xi_\emptyset^y\|_{\ell_2(a)}^{\delta m_\tau}(1+n_t^{x,y})^{\delta m_\tau\mathbf{k}/2}R_{\tau,\gamma}^{\delta/|\tau|}\prod_{i=1}^{s}\|\xi_{\gamma_i}^y\|_{\ell_{m_{\gamma_i}}(\widetilde{c}_{\gamma_i})}^{\delta m_\tau} \tag{4.22}$$

Substituting (4.22) into (4.16) and applying (2.3), (2.4), (4.12) and Lemma 4 (4.1) we have

$$T^{\delta m_\tau}2^{\delta m_\tau-1}e^{\lambda\delta m_\tau T}\,\mathbf{E}\,q(n_t^{x,y})\sup_{\sigma\in[0,T]}(4.20)\le$$

$$\le M_1''T^{\delta m_\tau}e^{M_2''T}q(n_0^{x,y})(1+n_0^{x,y})^{\delta m_\tau\mathbf{k}/2+\delta m_\tau\frac{\mathbf{k}+1}{2}(|\tau|-1)}\sum_{j\in\tau}\|\widetilde{x}_{\{j\}}\|_{\ell_{m_1}(\widetilde{c}_{\{j\}})}^{\delta m_1} \tag{4.23}$$

with constants

$$M_1''=2^{\delta m_\tau-1}(C_n^F)^{\delta m_\tau}C_{|\tau|,\delta m_1}\sum_{\gamma_1,\dots,\gamma_s}R_{\tau,\gamma}^{\delta/|\tau|}K_{4\delta\mathbf{k}m_\tau/2}^{1/4}C_{|\tau|}$$

$$M_2''=\lambda\delta m_\tau+M_{q^4}/4+\delta m_\tau M(\mathbf{k}+1)+\sup_{\gamma_1,\dots,\gamma_s}\frac{1}{4s}\sum_{j=1}^{s}M_{\gamma_j}$$

Using $CT^q\le\exp(q\sqrt[q]{C}T)$ and that by $a_k\le 1:\|\cdot\|_{\ell_{m_1}(\widetilde{c}_{\{j\}})}\le\|\cdot\|_{\ell_{m_1}(c_{\{j\}})}$ we have (4.23) in required form (4.8).

It remains to estimate (4.21). Proceeding like in (2.42) we obtain

$$(4.21)\le$$

$$\le\sum_{i=1}^{s}(C_n^F)^{\delta m_\tau}(1+n_t^{x,y})^{\delta m_\tau\frac{\mathbf{k}+1}{2}}R_{\tau,\gamma}^{\delta/|\tau|}\|\xi_{\gamma_1}^y\|_{\ell_{m_{\gamma_1}}(\widetilde{c}_{\gamma_1})}^{\delta m_\tau}...\|\xi_{\gamma_i}^x-\xi_{\gamma_i}^y\|_{\ell_{m_{\gamma_i}}(\widetilde{c}_{\gamma_i})}^{\delta m_\tau}...\|\xi_{\gamma_s}^x\|_{\ell_{m_{\gamma_s}}(\widetilde{c}_{\gamma_s})}^{\delta m_\tau} \tag{4.24}$$

By inductive assumption we have

$$(\,\mathbf{E}\,q^{s+1}(n_t^{x,y})\sup_{\sigma\in[0,T]}\|\xi_{\gamma_i}^x-\xi_{\gamma_i}^y\|_{\ell_{m_{\gamma_i}}(\widetilde{c}_{\gamma_i})}^{\delta(s+1)m_\tau})^{1/(s+1)}\le e^{M_{\gamma_i}T/(s+1)}\|x-y\|_{\ell_2(a)}^{\delta m_\tau}q(n_0^{x,y})\times$$

$$\times(1+n_0^{x,y})^{\delta m_\tau\frac{\mathbf{k}+1}{2}(|\gamma_i|-1)+\delta\mathbf{k}m_\tau/2}(\sum_{j\in\gamma_i}\|\widetilde{x}_{\{j\}}\|_{\ell_{m_1}(\widetilde{c}_{\{j\}})}^{\delta m_1\frac{|\gamma_i|}{|\tau|}(s+1)})^{1/(s+1)} \tag{4.25}$$

due to

$$(\tfrac{\mathbf{k}+1}{2}(m_1-m_{\gamma_i})+\tfrac{\mathbf{k}m_{\gamma_i}}{2})\delta\tfrac{|\gamma_i|}{|\tau|}=\delta m_\tau\tfrac{\mathbf{k}+1}{2}(|\gamma_i|-1)+\tfrac{\delta\mathbf{k}m_\tau}{2}$$

Substituting (4.24) in (4.16) and using (4.25), (2.4) and Lemma 4 (4.1) we have

$$(4.16)\le M_1'''T^{\delta m_\tau}e^{M_2'''T}(1+n_0^{x,y})^{\delta m_\tau\frac{\mathbf{k}+1}{2}(|\tau|-1)+\delta\mathbf{k}m_\tau/2}q(n_0^{x,y})\times$$

$$\times\|x-y\|_{\ell_2(a)}^{\delta m_\tau}\prod_{\ell=1}^{s}(\sum_{j\in\gamma_\ell}\|\widetilde{x}_{\{j\}}\|_{\ell_{m_1}(\widetilde{c}_{\{j\}})}^{\delta m_\tau(s+1)|\gamma_\ell|})^{1/(s+1)} \tag{4.26}$$

with constants

$$M_1''' = 2^{\delta m_\tau} s C_{|\tau|,\delta m_1} (C_n^F)^{\delta m_\tau} K^{1/(s+1)}_{\delta m_\tau \frac{\mathbf{k}+1}{2}(s+1)} \sum_{\gamma_1,\dots,\gamma_s} R^{\delta/|\tau|}_{\tau,\gamma}$$

$$M_2''' = \lambda m_\tau \delta + \delta m_\tau (\mathbf{k}+1) M + \frac{1}{s+1} \sum_{\ell=1}^{s} M_{\gamma_\ell}$$

Applying to (4.26) estimate (4.6), inequality $CT^q \le \exp(q\sqrt[q]{C}T)$ and $\|\cdot\|_{\ell_{m_1}(\widetilde{c}_{\{j\}})} \le \|\cdot\|_{\ell_{m_1}(c_{\{j\}})}$ we finally have (4.26) in the required form (4.8). ■

The next result states the uniform on $|\tau| \le n$ estimates on variations, used to derive the smooth properties of semigroup P_t in step (3.21).

COROLLARY 6. *Let F fulfill (2.2) and $\xi^x_\emptyset, \xi^y_\emptyset, \xi^x_\gamma, \xi^y_\gamma$, $\gamma \subset \tau$ be generalized solutions to (2.1), (2.5) with initial data $x, y \in \ell_2(a)$ and zero-one initial data x_γ, y_γ. Then the following estimate holds*

$$\mathbf{E}\, Q(n_t^{x,y}) |\xi^x_{k,\tau} - \xi^y_{k,\tau}|^{m_\tau} \le \frac{K_2 e^{M_{|\tau|}t} \|x-y\|^{m_\tau}_{\ell_2(a)} Q(n_0^{x,y}) (1+n_0^{x,y})^{\frac{\mathbf{k}+1}{2} m_1 \frac{|\tau|-1}{|\tau|} + \mathbf{k} m_\tau/2}}{a_k^{\frac{\mathbf{k}+1}{2} m_1/|\tau|} a_k^{\frac{\mathbf{k}+1}{2} m_1 \frac{|\tau|-1}{|\tau|}} \prod_{a\in\tau} \psi_{k-a}^{m_1/|\tau|}} \tag{4.27}$$

with uniform on $|\tau| \le n$, $n \in I\!N$ constants $K_2 = |\tau|\psi_0$ and $M_{|\tau|}$.

Proof. Estimate (4.27) is a direct coordinate consequence of (4.8) if we choose $q(\cdot) = Q(\cdot)$, vectors c_τ in (2.46). Note that vectors $\{c_\gamma\}$ (2.46) fulfill hierarchy (2.8) with uniform on sets τ, γ constant $R_{\tau,\gamma} \equiv 1$. This implies that constants λ, $\widetilde{\lambda}$ and M_τ in Theorem 5 are uniform with respect to $\tau : |\tau| \le n$. Uniformity of $\|B\|$ is similar to Corollary 2. ■

LEMMA 7. ([2, LEMMA 3.12]). *Let $\delta_d \stackrel{def}{=} \sup_{|k-j|=1} |d_k/d_j|$ and $b \in I\!P$ be such that $|b(k)| \le 1$, $k \in \mathbb{Z}^d$. Suppose that for $d^{(i)} \in I\!P$, $i = 1, ..., n$ constant*

$$C_b(d) = \prod_{\ell=1}^{n} \{1 + \sum_{a\in\mathbb{Z}^d} b_a |\delta_{d^{(\ell)}}|^{|a|}\} < \infty \tag{4.28}$$

Then $\forall\, \beta_1 \cup ... \cup \beta_s = \{1, ..., n\}$, $s \ge 1$ we have inequality with $d_k^{(\beta)} = \prod_{i\in\beta} d_k^{(i)}$

$$\Big(\sum_{j_1\dots j_n\in\mathbb{Z}^d} d^{(1)}_{j_1} \dots d^{(n)}_{j_n} \Big| \sum_{k_1\dots k_s\in\mathbb{Z}^d} x_{k_1\dots k_s} \prod_{i=1}^{s} \prod_{\ell\in\beta_i} b_{k_i - j_\ell}\Big|^2\Big)^{1/2} \le \tag{4.29}$$

$$\le C_b(d) \cdot \Big(\sum_{k_1\dots k_s\in\mathbb{Z}^d} d^{(\beta_1)}_{k_1} \dots d^{(\beta_s)}_{k_s} |x_{k_1\dots k_s}|^2\Big)^{1/2}$$

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