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# $n$-distributivity and $n$-modularity in lattices

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**Abstract.** In this paper we consider some forbidden sublattices for $n$-distributive, but non-modular lattices. We define the new notion of $n$-modularity (weaker than $n$-distributivity). We also consider some forbidden sublattice for an $n$-modular lattice. We prove that $n$-modularity implies $(n+1)$-modularity. The counter-examples for the inverse implication are shown.

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## 1. Introduction

We recall the $n$-distributivity notion, which was introduced by G. M. Bergman (in [1]) and A. P. Huhn (in [4]) as a generalization of the ordinary distributivity (for $n = 1$), for modular lattices:

A lattice $(L, \vee, \wedge)$ is *$n$-distributive* if for every $x, y_0, \ldots, y_n \in L$ the condition is satisfied:

**($\mathbf{D}_n$)** $$x \wedge \bigvee_{i=0}^{n} y_i = \bigvee_{j=0}^{n} (x \wedge \bigvee_{i=0; j \neq i}^{n} y_i).$$

A lattice $L$ is *dually $n$-distributive* if for every $x, y_0, \ldots, y_n \in L$ the following equality is satisfied:

$$x \vee \bigwedge_{i=0}^{n} y_i = \bigwedge_{j=0}^{n} (x \vee \bigwedge_{i=0; i \neq j}^{n} y_i).$$

A lattice $L$ is *modular*, if for every $x, y, z \in L$, $x \leq y$ implies $x \wedge (y \vee z) = x \vee (x \wedge z)$.

The condition **($\mathbf{D}_n$)** is equivalent to the dual $n$-distributivity condition iff a lattice $L$ is modular (see [4]).

It is easy to show that every $n$-distributive lattice (dually $n$-distributive) is also $(n+1)$-distributive (dually $(n+1)$-distributive, respectively). For standard terminology, see [3].

We introduce two notions weaker than notion of $n$-distributivity and dual $n$-distributivity, respectively:

1) A lattice $(L, \vee, \wedge)$ is *$n$-modular* if for every $x, y_0, \ldots, y_n \in L$ the following implication is true:

$[\bigvee_{i=0}^{n-1} y_i \leq x] \Rightarrow [x \wedge \bigvee_{i=0}^{n} y_i = (\bigvee_{i=0}^{n-1} y_i) \vee \bigvee_{j=0}^{n-1}(x \wedge \bigvee_{i=0;i\neq j}^{n} y_i)]$.

2) A lattice $(L, \vee, \wedge)$ is *dually $n$-modular* if for every $x, y_0, \ldots, y_n \in L$ the implication:

$[\bigwedge_{i=0}^{n-1} y_i \geq x] \Rightarrow [x \vee \bigwedge_{i=0}^{n} y_i = (\bigwedge_{i=0}^{n-1} y_i) \wedge \bigwedge_{j=0}^{n-1}(x \vee \bigwedge_{i=0;i\neq j}^{n} y_i)]$

is valid.

The 1-modular lattices and dually 1-modular lattices are exactly modular.

If $P$ is a poset and for $a,\ b,\ c \in P$ the conditions $a < b,\ a \leq c \leq b$ imply $c = a$ or $c = b$, then we say, that $b$ *covers* $a$ in the set $P$ (or $a$ is *covered* by $b$).

## 2. Some properties for $n$-distributive and $n$-modular lattices; Characterization of an $n$-modular lattice by the forbidden sublattice

In 1972 A. P. Huhn (see [4]) proved that a modular lattice $L$ is not $n$-distributive iff it contains a sublattice $B$ isomorphic to the $2^{n+1}$-element Boolean lattice and an element $x$ such that $x \wedge a = \bigwedge B$, $x \vee a = \bigvee B$, for every atom $a$ of $B$. For $n = 1$, it is the well-known criterion of distributivity.

The following proposition without the modularity assumption is some partial generalization for the above Huhn's result.

**Proposition 1.** *A lattice $(L, \vee, \wedge)$ is not $n$-distributive whenever it contains a sublattice $B$ isomorphic to the $2^{n+1}$-element Boolean lattice and an element $x$ such that $x > b$, for some $b \in B$ and $\bigvee B$ is the only element in $B$, which covers $x$ in $L$.*

*Proof.* Let $\{y_0, \ldots, y_n\}$ be the set of atoms in the algebra $B$. Then

$$x \wedge \bigvee_{i=0}^{n} y_i = x \wedge \bigvee B = x.$$

According to the assumption there is an element $b_0 \in B$ such that $x$ covers $b_0$ in the poset $B \cup \{x\}$. Hence, $x \wedge \bigvee_{i=0;j\neq i}^{n} y_i \leq b_0 < x$, for $0 \leq j \leq n$ and $\bigvee_{j=0}^{n}(x \wedge \bigvee_{i=0;j\neq i}^{n} y_i) \leq b_0 < x$, which contradicts the $n$-distributivity. □

**Corollary 1.** *A lattice $(L, \vee, \wedge)$ is not dually $n$-distributive whenever it contains a sublattice $B$ isomorphic to the $2^{n+1}$-element Boolean lattice and an element $x$ covering $\bigwedge B$ in $L$ such that $x < b_0$, for some $b_0 \in B$.*

The inverse implication in the above theorem seems true, but it is still an open problem.

**Proposition 2.** *A lattice* $(L, \vee, \wedge)$ *is* $n$*-modular iff for comparable elements* $x$ *and* $\bigvee_{i=0}^{n-1} y_i$ *the following equality is satisfied:*

$$(\mathbf{M}_n) \qquad x \wedge [(x \wedge \bigvee_{i=0}^{n-1} y_i) \vee y_n] = \bigvee_{j=0}^{n}(x \wedge \bigvee_{i=0; i\neq j}^{n} y_i).$$

*Proof.* Assuming $\bigvee_{i=0}^{n-1} y_i \leq x$ in $(\mathbf{M}_n)$ we get $x \wedge \bigvee_{i=0}^{n} y_i = (\bigvee_{i=0}^{n-1} y_i) \vee \bigvee_{j=0}^{n-1}(x \wedge \bigvee_{i=0;i\neq j}^{n} y_i)$, what gives $n$-modularity. Let $\bigvee_{i=0}^{n-1} y_i > x$, then we get $n$-modularity applying the absorbtion laws. Now, let $\bigvee_{i=0}^{n-1} y_i \leq x$ and assume that $(\mathbf{M}_n)$ fails, for some $x,\ y_0, \ldots, y_n \in L$. Then

$$x \wedge \bigvee_{i=0}^{n} y_i = x \wedge [(x \wedge \bigvee_{i=0}^{n-1} y_i) \vee y_n] \neq \bigvee_{j=0}^{n}(x \wedge \bigvee_{i=0;\ i\neq j}^{n} y_i) =$$
$$= (\bigvee_{i=0}^{n-1} y_i) \vee \bigvee_{j=0}^{n-1}(x \wedge \bigvee_{i=0;\ i\neq j}^{n} y_i),$$

which contradicts the $n$-modularity. □

**Corollary 2.** *A lattice* $(L, \vee, \wedge)$ *is dually* $n$*-modular iff for comparable elements* $x$ *and* $\bigwedge_{i=0}^{n-1} y_i$ *the following equality is valid:*

$$x \vee [(x \vee \bigwedge_{i=0}^{n-1} y_i) \wedge y_n] = \bigwedge_{j=0}^{n}(x \vee \bigwedge_{i=0;i\neq j}^{n} y_i).$$

**Proposition 3.** *Let* $n \geq 1$*. Then:*

**(i)** *Every* $n$*-distributive (dually* $n$*-distributive) lattice is* $n$*-modular (dually* $n$*-modular, respectively).*

**(ii)** *Every* $n$*-modular (dually* $n$*-modular) lattice is* $(n+1)$*-modular (dually* $(n+1)$*-modular, respectively).*

*Proof.* First implication is obvious. Now we prove that the usual modularity implies $n$-modularity for $n > 1$. Let $\bigvee_{i=0}^{n-1} y_i \leq x$. Then using modularity, we get

$$x \wedge \bigvee_{i=0}^{n} y_i = x \wedge (\bigvee_{i=0}^{n-1} y_i \vee \bigvee_{i=0;j\neq i}^{n} y_i) = \bigvee_{i=0}^{n-1} y_i \vee \left(x \wedge \bigvee_{i=0;j\neq i}^{n} y_i\right),$$

for every $0 \leq j \leq n-1$. Hence, $x \wedge \bigvee_{i=0}^{n} y_i = (\bigvee_{i=0}^{n-1} y_i) \vee \bigvee_{j=0}^{n-1}(x \wedge \bigvee_{i=0;i\neq j}^{n} y_i)$, what gives $n$-modularity. Now, let $x, y_0, \ldots, y_n, y_{n+1} \in L$, $\bigvee_{i=0}^{n} y_i \leq x$ and let $0 \leq l, k \leq n$ be fixed indices. Then assuming $n$-modularity and treating $y_l \vee y_k$ as a single element we get the equality:

$$x \wedge \bigvee_{i=0}^{n+1} y_i = \bigvee_{i=0}^{n} y_i \vee [\bigvee_{j=0;j\neq l,k}^{n}(x \wedge \bigvee_{i=0;i\neq j}^{n+1} y_i)] \vee (x \wedge \bigvee_{i=0;i\neq l,k}^{n+1} y_i).$$

The supremum over all $0 \leq l, k \leq n$ of the right-hand side of this equality is exactly equal to $\bigvee_{i=0}^{n} y_i \vee \bigvee_{j=0}^{n}(x \wedge \bigvee_{i=0;i\neq j}^{n+1} y_i)$. Hence we get $(n+1)$-modularity. Analogously, inverting operations we prove the dual theorem. □

**Remark.** *The inverse implications in the Proposition 3 are not always true!*

The lattices $L_1$, $L_2$, $L_3$ (see Figure 1) are not modular;
$L_2$, $L_3$ are not 2-distributive, but they are 2-modular;
$L_1$ is not 2-distributive and not 2-modular.

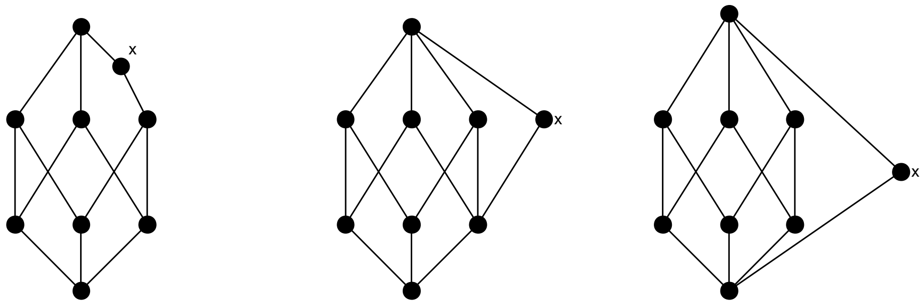


Figure 1.

**Proposition 4.** *A lattice* $(L, \vee, \wedge)$ *is not* $n$*-modular whenever it contains a sublattice* $B$ *isomorphic to the* $2^{n+1}$*-element Boolean lattice and an element* $x$ *such that* $c_0 < x < \bigvee B$*, for some coatom* $c_0$ *of* $B$*. A lattice* $L$ *is not* 2*-modular if and only if it contains an isomorphic copy of* $L_1$ *as a poset (see Figure 1).*

*Proof.* Let $A = \{y_0, y_1, \ldots, y_n\} \subseteq B$ be the set of atoms of $B$. Since $c_0 = \bigvee_{i=0}^{n-1} y_i < x$, hence $x \wedge \bigvee_{i=0}^{n} y_i = x$. An element $\bigvee_{i=0;j\neq i}^{n} y_i$ is a coatom of $B$, for $0 \leq j \leq n$. Hence $x \wedge \bigvee_{i=0;j\neq i}^{n} y_i \leq c_0 < x$, for every $0 \leq j \leq n$ and $(\bigvee_{i=0}^{n-1} y_i) \vee (\bigvee_{j=0}^{n-1}(x \wedge \bigvee_{i=0;j\neq i}^{n} y_i)) = \bigvee_{i=0}^{n-1} y_i = c_0$, which contradicts the $n$-modularity.

Now, we prove the inverse implication, in the case $n = 2$. Assume that $L$ is not 2-modular. Then for some $x, y_1, y_2, y_3 \in L$, $y_1 \vee y_2 \leq x$ we get the inequality (*) $x \wedge (y_1 \vee y_2 \vee y_3) > (y_1 \vee y_2) \vee [x \wedge (y_1 \vee y_3)] \vee [x \wedge (y_2 \vee y_3)]$.

Notice that comparability of every pair of elements $y_1, y_2, y_3$ contradicts this inequality. Now, let $y_1 \vee y_2 = y_1 \vee y_3$. Then $(y_1 \vee y_2) \geq (y_2 \vee y_3)$, $x \wedge (y_1 \vee y_2 \vee y_3) = y_1 \vee y_2$ and $(y_1 \vee y_2) \vee [x \wedge (y_1 \vee y_3)] \vee [x \wedge (y_2 \vee y_3)] = (y_1 \vee y_2) \vee [x \wedge (y_2 \vee y_3)] = (y_1 \vee y_2)$.

If $(y_1 \vee y_3) = (y_2 \vee y_3)$, then $(y_1 \vee y_2) \leq (y_2 \vee y_3)$, $x \wedge (y_1 \vee y_2 \vee y_3) = x \wedge (y_1 \vee y_3)$ and $(y_1 \vee y_2) \vee [x \wedge (y_1 \vee y_3)] \vee [x \wedge (y_2 \vee y_3)] = (y_1 \vee y_2) \vee [x \wedge (y_1 \vee y_3)] = x \wedge (y_1 \vee y_3)$. Hence, elements $y_1 \vee y_2$, $y_1 \vee y_3$, $y_2 \vee y_3$ must be different. Similarly, if any two of the following elements $y_1 \vee y_2$, $y_1 \vee y_3$, $y_2 \vee y_3$ are comparable, then it contradicts the inequality (*). Three incomparable elements $y_1 \vee y_2$, $y_1 \vee y_3$, $y_2 \vee y_3$ generate a lattice isomorphic to the $2^3$-element Boolean lattice (see. [3], p. 48).

Hence, $L$ must contain $L_1$. □

**Corollary 3.** *A lattice* $(L, \vee, \wedge)$ *is not dually* $n$*-modular whenever it contains a sublattice* $B$ *isomorphic to the* $2^{n+1}$*-element Boolean lattice and an element* $x$ *such that* $\bigwedge B < x < a_0$*, for some atom* $a_0$ *of* $B$ *(the inverse implication is true for* $n = 1$ *and* $n = 2$*).*

The inverse implication of Proposition 4 seems true also for $n > 2$, but it is still an open problem (for $n = 1$ it is the well-known criterion of modularity).

A. P. Huhn proved, that for a modular lattice $L$ the equality:
$\bigwedge_{j=0}^{n+1} \bigvee_{i=0;i\neq j}^{n+1} y_i = \bigvee_{k=0}^{n+1} \bigwedge_{j=0;j\neq k}^{n+1} \bigvee_{i=0;i\neq j,k}^{n+1} y_i$, for $y_0, ..., y_{n+1} \in L$ is equivalent to $(\mathbf{D}_n)$ (see [4], [5]). The next proposition gives the equality condition implying $(\mathbf{D}_n)$ without modularity assumption:

**Proposition 5.** *A lattice $L$ is $n$-distributive whenever for every $y_0, ..., y_{n+1} \in L$ the following equality is satisfied:*
$\bigwedge_{j=0}^{n+1} \bigvee_{i=0;i\neq j}^{n+1} y_i = (\bigwedge_{i=0}^{n} y_i) \vee \bigvee_{j=0}^{n}(y_{n+1} \wedge \bigvee_{i=0;j\neq i}^{n} y_i)$.

*Proof.* Denote the left-hand side of the above equality by $a$, and the right-hand one by $b$. Assuming $\bigvee_{i=0}^{n-1} y_i \leq y_{n+1}$ in $a = b$ and using the absorbtion laws we get $y_{n+1} \wedge \bigvee_{i=0}^{n} y_i = (\bigvee_{i=0}^{n-1} y_i) \vee (\bigvee_{j=0}^{n-1}(y_{n+1} \wedge \bigvee_{i=0;j\neq i}^{n} y_i))$, what gives $n$-modularity. Notice, that $y_{n+1} \wedge a = y_{n+1} \wedge (\bigwedge_{j=0}^{n+1} \bigvee_{i=0;i\neq j}^{n+1} y_i) =$
$= y_{n+1} \wedge \bigvee_{i=0}^{n} y_i \wedge \bigwedge_{j=0}^{n}(y_{n+1} \vee \bigvee_{i=0;j\neq i}^{n} y_i) = y_{n+1} \wedge \bigvee_{i=0}^{n} y_i$.
Since $L$ is $n$-modular and $\bigvee_{j=0}^{n}(y_{n+1} \wedge \bigvee_{i=0;j\neq i}^{n} y_i) \leq y_{n+1}$, hence
$y_{n+1} \wedge b = y_{n+1} \wedge [(\bigwedge_{i=0}^{n} y_i) \vee \bigvee_{j=0}^{n}(y_{n+1} \wedge \bigvee_{i=0;j\neq i}^{n} y_i)] =$
$= [\bigvee_{j=0}^{n}(y_{n+1} \wedge \bigvee_{i=0;j\neq i}^{n} y_i)] \vee \bigvee_{j=0}^{n}\{y_{n+1} \wedge [(\bigwedge_{i=0}^{n} y_i) \vee (y_{n+1} \wedge \bigvee_{i=0;i\neq j}^{n} y_i)]\}$.
Because of the inequality
$\{y_{n+1} \wedge [(\bigwedge_{i=0}^{n} y_i) \vee (y_{n+1} \wedge \bigvee_{i=0;i\neq j}^{n} y_i)]\} \leq (y_{n+1} \wedge \bigvee_{i=0;i\neq j}^{n} y_i)$, $0 \leq j \leq n$, which is valid for an arbitrary lattice, we deduce
$y_{n+1} \wedge b = \bigvee_{j=0}^{n}(y_{n+1} \wedge \bigvee_{i=0;i\neq j}^{n} y_i)$.
The equality $y_{n+1} \wedge a = y_{n+1} \wedge b$ gives $n$-distributivity. □

There are some useful applications for $(\mathbf{D}_n)$ condition in lattices of closed sets with respect to a given closure operator. For example, the $n$-distributivity property can be asocciated to the Carathéodory number, which is some parameter describing a closure operator on a given set (see [2], [6]–[8]).

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