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The Cauchy problem for degenerate parabolic equations with source and damping

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Abstract. We prove optimal estimates for the decay of mass of solutions to the Cauchy problem for a wide class of quasilinear parabolic equations with damping terms. In the degenerate case, we also prove estimates for the finite speed of propagation. When the equation contains also a blow up term, we discuss existence and nonexistence of global solutions.

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1. Introduction

We study the following Cauchy problem in $Q_T = \mathbb{R}^N \times (0,T)$, $T > 0$,

$$u_t - \operatorname{div}\bigl(u^{m-1}|Du|^{\lambda-1}Du\bigr) = -\epsilon|Du^\nu|^q + \delta u^p\,, \quad \text{in } Q_T, \tag{1.1}$$

$$u(x,0) = u_0(x) \geq 0\,, \qquad x \in \mathbb{R}^N\,. \tag{1.2}$$

We assume throughout the paper that ε, $\delta \geq 0$, $m+\lambda-2 \geq 0$, $\lambda > 0$, $1 < q < \lambda+1$, $\nu q > m+\lambda-1$, $p > 1$, and that $u_0 \in L^1(\mathbb{R}^N)$, with $u_0 \geq 0$, $u_0 \not\equiv 0$.

The special case of (1.1)

$$u_t = \Delta u - \varepsilon|Du|^q + \delta u^p\,, \tag{1.3}$$

has been studied by many authors (see [1, 2, 9–13, 16, 18, 19, 21, 24–29]). The equation (1.3) was introduced in [16] in order to investigate the effect of the damping term $|Du|^q$ on the existence (or nonexistence) of a global solution to the Cauchy–Dirichlet problem with blow up sources. Equation (1.3) has also been proposed as a model in population dynamics, where the damping term accounts for the presence of predators, attracted by the flow of preys (see [26]).

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From the mathematical point of view, the interest of (1.1), when $\delta = 0$, is to understand how the damping gradient term changes the long time behaviour of solutions. We show that a critical exponent q^* exists such that if $q \leq q^*$ the influence of the damping term is determinant (for the semilinear case see [16]). For example the total mass of the solution decays as $t \to \infty$. When $q > q^*$ this is not the case (see Theorem 1.4), and the behaviour of solutions, from our point of view, is the same as in the homogeneous case when ε, $\delta = 0$.

When $\delta > 0$, blow up of solutions in a finite time may take place. We identify the critical threshold p^* such that for $p > p^*$, $q < q^*$ solutions corresponding to small initial data are globally bounded, while for $p = p^*$, $q < q^*$ all non trivial solutions blow up in a finite time. Note that p^* is strictly less than the corresponding critical exponent in the case $\varepsilon = 0$, due to the effect of the damping term (see [2, 9, 11, 16]).

To the best of our knowledge, the problem (1.1)–(1.2) has never been treated in the range of parameters we consider. Our goal here is to obtain first optimal bounds for the mass of a solution to (1.1)–(1.2) without blow up term. This decay estimate leads to the optimal bound of the L^∞ norm of the solution as well. Next, we prove the results about global solvability and nonexistence of global solutions.

Our approach is based on the natural development of the ideas in [6–8], and relies on energetical arguments. The methods are flexible enough to apply even to higher order equations, how we point out briefly in Section 7. Moreover, as it follows from the proofs, they apply to equations of more general form, like

$$u_t - \operatorname{div} \mathbf{a}(x, t, u, Du) = f(x, t, u, Du)\,,$$

under structure assumptions that preserve the key features of (1.1).

Definition 1.1. *A nonnegative function $u \in L^\infty_{loc}(Q_T)$ is a (weak) solution to (1.1) if*

$$u \in C(0, T; L^2_{loc}(\mathbb{R}^N)); \qquad |Du^\sigma|^{\lambda+1},\ \varepsilon|Du^\nu|^q,\ \delta u^p \in L^1_{loc}(Q_T),$$

where $\sigma = (m + \lambda - 1)/\lambda$, and if for any $\eta \in C^1_0(Q_T)$

$$\iint_{Q_T} \Big\{ -u\eta_t + u^{m-1}|Du|^{\lambda-1}Du\cdot D\eta + \varepsilon|Du^\nu|^q\eta \Big\}\,\mathrm{d}x\,\mathrm{d}t = \delta \iint_{Q_T} u^p\eta\,\mathrm{d}x\,\mathrm{d}t.$$

Moreover u is a solution to (1.1)–(1.2) if $u(\cdot, t) \to u_0$ as $t \to 0$ in $L^1(\mathbb{R}^N)$.

Remark 1.1. A solution to (1.1)–(1.2) is locally Hölder continuous in Q_T. Although this result is not contained explicitly in [23, 30], the techniques employed therein can be easily adapted to the case with a damping term. Then, using also the arguments in [4], one can prove existence of a solution to (1.1)–(1.2), at least if $\lambda = 1$. Such a a solution is global in time if $\delta = 0$. The question of existence and uniqueness of solutions, in the general case $\lambda \neq 1$, however, is not trivial, and will be discussed in a forthcoming paper.

We use in the following the notation

$$\mathcal{K} = N(m+\lambda-2)+\lambda+1\,, \quad q^* = \frac{\mathcal{K}+N}{N\nu+1}\,,$$
$$\mathcal{H} = (\lambda+1)(\nu q-1)-q(m+\lambda-2)\,, \quad \mathcal{A} = \frac{q^*-q}{\mathcal{H}}(N\nu+1)\,.$$

Here $\mathcal{K}$ is the Barenblatt's exponent for the homogeneous problem ε, $\delta=0$. The constant $\mathcal{H}$ is positive under our assumptions. The role of the parameters introduced above will be commented upon in Remark 1.2. Moreover we set

$$\|u\|_{1,\Omega} = \int_\Omega |u|\,\mathrm{d}x, \qquad \|u\|_1 = \|u\|_{1,\mathbb{R}^N}.$$

The symbols γ, γ_*, ..., denote generic positive constants depending on N, m, λ, q, ν, p.

We begin by stating a result that gives optimal estimates for the finite speed of propagation in the degenerate case.

Theorem 1.1. *Let u be a weak solution of* (1.1)–(1.2) *with $\delta=0$, $\epsilon=1$, $m+\lambda-2>0$,* $\operatorname{supp} u_0 \subset B_{\rho_0}$*, $\rho_0<\infty$. Then we have for all large enough t*

$$Z(t) := \inf\{r>0 \mid u(x,t)=0 \text{ a.e. } |x|>r\} \le \gamma t^{\frac{\nu q-(m+\lambda-1)}{\mathcal{H}}}. \tag{1.4}$$

Remark 1.2. The exponent q^* is critical under many respects, as shown by the results below. We assume here $q<q^*$.

The growth rate in (1.4) can be heuristically obtained by looking for self-similar radial solutions in the form $t^{-\alpha}f(\rho t^{-\beta})$. One finds that β equals the power in (1.4), that is

$$\frac{\nu q-m-\lambda+1}{\mathcal{H}} \sim \text{ speed of propagation.}$$

Moreover α equals the power given in (1.9), that is

$$-\frac{\lambda+1-q}{\mathcal{H}} \sim L^\infty \text{ decay rate.}$$

The constant $\mathcal{A}$ will be relevant to us, as (see Theorem 1.2)

$$-\mathcal{A} \sim L^1 \text{ decay rate.}$$

Remark 1.3. The estimate

$$Z(t) \le \gamma\big(\|u_0\|_1^{\frac{m+\lambda-2}{\mathcal{K}}} t^{\frac{1}{\mathcal{K}}} + \rho_0\big)\,, \qquad \text{for all } t>0, \tag{1.5}$$

is classical under the assumptions of Theorem 1.1 (see e. g., [6]), for solutions of the homogeneous equation where ε, $\delta=0$. Obviously it is valid for u under the assumptions of Theorem 1.1, since u is a subsolution to the homogeneous equation. For t large enough, when $q<q^*$, (1.4) gives better results than (1.5), while the latter is optimal for $q>q^*$. Also note that the bound in (1.4) does not depend on the initial data.

Exploiting the estimate (1.4) we are able to prove our first results on decay of mass.

Theorem 1.2. *Let u be a weak solution of* (1.1)–(1.2) *with* $\delta = 0$, $\epsilon = 1$, $m + \lambda - 2 > 0$, $\operatorname{supp} u_0 \subset B_{\rho_0}$, $\rho_0 < \infty$. *Then we have for all large enough t*

$$\text{if } q < q^* \text{ then } \quad \|u(t)\|_1 \le \gamma t^{-\mathcal{A}}\,, \tag{1.6}$$

$$\text{if } q = q^* \text{ then } \quad \|u(t)\|_1 \le \gamma [\ln t]^{-\frac{1}{\nu q - 1}}\,. \tag{1.7}$$

Remark 1.4. The sup estimate

$$\|u(t)\|_{\infty,\mathbb{R}^N} \le \gamma \sup_{\frac{t}{2}<\tau<t} \|u(\tau)\|_1^{\frac{\lambda+1}{\mathcal{K}}} t^{-\frac{N}{\mathcal{K}}}\,, \qquad t > 0\,, \tag{1.8}$$

is well known for subsolutions of the homogeneous equation ε, $\delta = 0$ (see e. g., [6]). Thus, it holds for solutions of our equation with $\delta = 0$.

Combining (1.8) with the decay results of, for example, (1.6), we obtain for large t the sup estimate

$$\|u(t)\|_{\infty,\mathbb{R}^N} \le \gamma t^{-\frac{\lambda+1-q}{\mathcal{H}}}\,, \tag{1.9}$$

valid under the assumptions of Theorem 1.2. Note that (1.9) implies a faster decay rate, for large t, than (1.8) if and only if $q < q^*$. For $q = q^*$, one can still obtain a decay rate better than $t^{-N/\mathcal{K}}$ by substituting (1.7) in (1.8).

Even dropping the assumption of compactly supported initial data, and of degeneracy of the equation, we prove the following result of decay of mass.

Theorem 1.3. *Let u be a weak solution of* (1.1)–(1.2) *with* $\delta = 0$, $\epsilon = 1$, $q < q^*$. *Then for any t large enough we have*

$$\|u(t)\|_1 \le \gamma \int_{|x|>R(t)} u_0 \,\mathrm{d}x + \gamma t^{-\mathcal{A}}\,, \tag{1.10}$$

where $R(t) = t^{\frac{\nu q - (m+\lambda-1)}{\mathcal{H}}}$, *provided either (a)* $\nu = \sigma = (m + \lambda - 1)/\lambda$, *or (b)* $\lambda = 1$, $N \ge 2$.

In case (b) of Theorem 1.3 we need the special structure of the porous media equation, since we integrate twice by parts in the proof.

The threshold $q < q^*$ in Theorem 1.3 is optimal, as we show in our next result.

Theorem 1.4. *Let u be a weak solution of* (1.1)–(1.2) *with* $q > q^*$, $\delta = 0$, $\epsilon = 1$. *Then for all $t > 0$ we have*

$$\|u(t)\|_1 \ge c > 0\,, \tag{1.11}$$

where c is a positive constant depending on u_0.

Remark 1.5. In the case $\nu, m, \lambda = 1$ the critical exponent $q^* = (N+2)/(N+1)$ is well known, as the critical p^* appearing in the blow up results below, which in the mentioned case equals $q/(2-q)$ (see [2, 9, 11, 16]). However, Theorem 1.3 seems to be new even in that case.

Next we state our results concerning the blow up problem $\delta = 1$. First we prove, for supercritical p, a priori estimates which are instrumental in the proof of existence of global solutions.

Theorem 1.5. *Let u be a solution to* (1.1)–(1.2) *which can be a.e. approximated by globally bounded subsolutions. Let $\delta = \epsilon = 1$, $\nu = \sigma$, $p > m + \lambda - 1$. We also assume that*

$$p > p^* := \frac{q(\nu(\lambda+1) - (m+\lambda-1))}{\lambda+1-q}\,, \qquad q < q^*\,. \tag{1.12}$$

Moreover assume that a constant $M > 0$ exists so that for all $\rho > 1$,

$$\int_{|x|>\rho} u_0(x)\,\mathrm{d}x \le M\rho^{-\frac{\mathcal{A}\mathcal{H}}{\sigma q-(m+\lambda-1)}}\,. \tag{1.13}$$

Assume also that for $\beta > N(p-m-\lambda+1)/(\lambda+1)$ we have $\|u_0\|_{\beta,\mathbb{R}^N} < \sigma_1$, where $\sigma_1(N, \lambda, m, q, p, M) > 0$ is small enough. Then the following a priori bound holds

$$\|u(t)\|_{\infty,\mathbb{R}^N} \le \gamma t^{-\frac{\lambda+1-q}{\mathcal{H}}}\,, \qquad t > 1\,. \tag{1.14}$$

Some extra regularity assumption, like $u_0 \in L^\beta(\mathbb{R}^N)$, is needed, even for existence of local in time solutions, when the equation contains non linear sources (see [3]).

Remark 1.6. If $\lambda = 1$, the a priori estimates of Theorem 1.5 are enough to prove existence of global solutions to (1.1)–(1.2), applying the arguments in [4] (see also Remark 1.1).

Remark 1.7. The p^* appearing in (1.12) is critical because for $p > p^*$

$$\int^\infty \|u(t)\|^{p-1}_{\infty,\mathbb{R}^N}\,\mathrm{d}t < \infty\,,$$

according to (1.14) (we need $q < q^*$ here, too), a fact which is exploited in the proof of Theorem 1.5.

Remark 1.8. When $q > q^*$, estimate (1.14) does not provide any further information than estimate (1.8). However, in this case, it follows immediately from the methods of [3, 6] that global a priori bounds may be obtained for $p > m + \lambda - 1 + (\lambda+1)/N$.

Finally, let us state our blow up result, which shows that the restriction on p of Theorem 1.5 is optimal.

Theorem 1.6. *Let $\delta = 1$ and $0 < \varepsilon < \bar\varepsilon$, where $\bar\epsilon = \bar\varepsilon(N, \nu, \lambda, m, q) > 0$. Assume also $p = p^*$, $q < q^*$. Then any non trivial solution to* (1.1)–(1.2) *blows up in a finite time.*

Remark 1.9. The requirement $\varepsilon < \bar{\varepsilon}$ in Theorem 1.6 does not seem to be only a technical restriction, in view of the existence of a (formal) stationary solution $u(x) = |x|^{-\theta}$. Here we must assume $p = p^*$, $\theta = (\lambda+1-q)/(\nu q-m-\lambda+1)$, and ε is a suitable positive constant depending on the other parameters.

The material is organized as follows: Section 2 is devoted to the proof of the finite speed of propagation property, Theorem 1.1. Section 3 contains the proofs of the Theorems 1.2 and 1.3 about decay of mass, while the connected result Theorem 1.4 is proven in Section 4. The proof of the global existence Theorem 1.5 is given in Section 5, and the proof of the blow up Theorem 1.6 is given in Section 6. Finally, Section 7 is devoted to some remarks about higher order equations.

2. Proof of Theorem 1.1

We proceed as in [6–8]. Consider the sequence $r_i = 2\rho(1-2^{-i-1})$, $i = 0, 1, \dots$, where $\rho > 2\rho_0$. Let $\tilde{r}_i = (r_i + r_{i+1})/2$, and let ζ_i be a cut off function in $C^1(\mathbb{R}^N)$ such that $\zeta_i \equiv 0$ when $|x| < r_i$, $\zeta_i \equiv 1$ when $|x| > \tilde{r}_i$, and $|D\zeta_i| \le \gamma 2^i \rho^{-1}$. Then routine calculations give

$$\sup_{0<\tau<t} \int_{\tilde{U}_i} u^{\theta+1} \,\mathrm{d}x + \int_0^t \int_{\tilde{U}_i} u^{m+\theta-2} |Du|^{\lambda+1} \,\mathrm{d}x \,\mathrm{d}\tau$$
$$+ \int_0^t \int_{\tilde{U}_i} u^{\theta} |Du^{\nu}|^q \,\mathrm{d}x \,\mathrm{d}\tau \le \gamma \frac{2^{i(\lambda+1)}}{\rho^{\lambda+1}} \int_0^t \int_{U_i \setminus \tilde{U}_i} u^{m+\lambda+\theta-1} \,\mathrm{d}x \,\mathrm{d}\tau \,, \tag{2.1}$$

where $U_i = \{|x| > r_i\}$, $\tilde{U}_i = \{|x| > \tilde{r}_i\}$.

Define $v = (u\tilde{\zeta}_i)^{\omega}$, $\omega = (m+\lambda+\theta-1)/(\lambda+1)$, $\beta = (\theta+1)/\omega > 1$ (due to the choice of θ), where $\tilde{\zeta}_i$ is a cut off function analogous to ζ_i, but such that $\tilde{\zeta}_i \equiv 0$ for $|x| < \tilde{r}_i$, $\tilde{\zeta}_i \equiv 1$ for $|x| > r_{i+1}$.

Next apply Nirenberg–Gagliardo inequality to find

$$\int_{\mathbb{R}^N} v^{\lambda+1} \,\mathrm{d}x \le \gamma \Big(\int_{\mathbb{R}^N} |Dv|^{\lambda+1} \,\mathrm{d}x \Big)^{\alpha} \Big(\int_{\mathbb{R}^N} v^{\beta} \,\mathrm{d}x \Big)^{\frac{(1-\alpha)(\lambda+1)}{\beta}} \,, \tag{2.2}$$

where $\alpha = N(m+\lambda-2)/\mathcal{K}_\theta$, $\mathcal{K}_\theta = N(m+\lambda-2) + (\theta+1)(\lambda+1)$. Integrating (2.2) in time and using also Hölder's inequality, we obtain

$$\int_0^t \int_{\mathbb{R}^N} v^{\lambda+1} \,\mathrm{d}x \,\mathrm{d}\tau \le \gamma t^{1-\alpha} y_i^{1+(1-\alpha)(\frac{\lambda+1}{\beta}-1)} \,, \tag{2.3}$$

where

$$y_i := \sup_{0<\tau<t} \int_{U_i} u^{\theta+1} \,\mathrm{d}x + \int_0^t \int_{U_i} u^{m+\theta-2} |Du|^{\lambda+1} \,\mathrm{d}x \,\mathrm{d}\tau$$
$$+ \int_0^t \int_{U_i} u^{\theta} |Du^{\nu}|^q \,\mathrm{d}x \,\mathrm{d}\tau + \frac{2^{i(\lambda+1)}}{\rho^{\lambda+1}} \int_0^t \int_{U_i \setminus U_{i+1}} u^{m+\lambda+\theta-1} \,\mathrm{d}x \,\mathrm{d}\tau \,.$$

Therefore, recalling the definition of v, and (2.1),

$$y_{i+1} \le \gamma \frac{2^{i(\lambda+1)}}{\rho^{\lambda+1}} t^{\frac{(1+\theta)(\lambda+1)}{\mathcal{K}_\theta}} y_i^{1+\frac{(\lambda+1)(m+\lambda-2)}{\mathcal{K}_\theta}} . \tag{2.4}$$

Next we use the damping term to find an additional recursive inequality. Let $w = u^{(\nu q+\theta)/q}$. We already know that the support of $u(\cdot, t)$ is bounded, i.e., that $Z(t) < \infty$ (because u is a subsolution to the homogeneous equation; see e.g., [6]). Then Poincaré's inequality gives

$$\int_{U_i} w^q \,\mathrm{d}x \le \gamma Z^q \int_{U_i} |Dw|^q \,\mathrm{d}x$$

(we denote $Z = Z(t)$ for ease of notation). Then, by Hölder's inequality

$$\begin{aligned}
&\int_0^t \int_{U_i \setminus U_{i+1}} w^{\frac{q(m+\lambda+\theta-1)}{\nu q+\theta}} \,\mathrm{d}x \,\mathrm{d}\tau \\
&\qquad \le \gamma \Big(\int_0^t \int_{U_i} w^q \,\mathrm{d}x \,\mathrm{d}\tau \Big)^{\frac{m+\lambda+\theta-1}{\nu q+\theta}} (t\rho^N)^{\frac{\nu q-(m+\lambda-1)}{\nu q+\theta}} \\
&\qquad\qquad \le \gamma (t\rho^N)^{\frac{\nu q-(m+\lambda-1)}{\nu q+\theta}} Z^{q\frac{m+\lambda+\theta-1}{\nu q+\theta}} y_i^{\frac{m+\lambda+\theta-1}{\nu q+\theta}} .
\end{aligned} \tag{2.5}$$

Thus, (2.1) and (2.5) give

$$y_{i+1} \le \gamma 2^{i(\lambda+1)} t^{\frac{\nu q-(m+\lambda-1)}{\nu q+\theta}} \rho^{N\frac{\nu q-(m+\lambda-1)}{\nu q+\theta}-\lambda-1} Z^{q\frac{m+\lambda+\theta-1}{\nu q+\theta}} y_i^{\frac{m+\lambda+\theta-1}{\nu q+\theta}} . \tag{2.6}$$

Let

$$a = \frac{\mathcal{K}_\theta}{\mathcal{K}_\theta + (\lambda+1)(m+\lambda-2)} , \qquad A = \big[t^{\frac{(1+\theta)(\lambda+1)}{\mathcal{K}_\theta}} \rho^{-\lambda-1} \big]^a ,$$

$$b = \frac{\nu q+\theta}{m+\lambda+\theta-1} > 1 , \qquad B = \big[(t\rho^N)^{\frac{\nu q-(m+\lambda-1)}{\nu q+\theta}} \rho^{-\lambda-1} Z^{q\frac{m+\lambda+\theta-1}{\nu q+\theta}} \big]^b .$$

Then from (2.4) and (2.6) it follows that

$$\frac{y_{i+1}^a}{A} + \frac{y_{i+1}^b}{B} \le \gamma C^i y_i ,$$

for a suitable $C > 1$. On applying Young's inequality we obtain

$$\frac{y_{i+1}^{\varepsilon_1}}{A^{\varepsilon_1} B^{1-\varepsilon_1}} \le \gamma C^i y_i , \tag{2.7}$$

where $\varepsilon_1 = b/(b+1-a) < 1$. Therefore from the iterative Lemma 5.6, Chapter II of [22] we conclude that $y_i \to 0$ if

$$(y_0 B)^{\frac{1-a}{b}} A \le \gamma_0 . \tag{2.8}$$

Of course this would imply that $u(x,t)=0$ for $|x|\geq 2\rho$.

In order to find the sharp bound of y_0 we need to proceed as follows. Let

$$y^{(i)}(\tilde\rho) := \sup_{0<\tau<t}\int_{|x|>\tilde\rho_i} u(x,\tau)^{1+\theta}\,\mathrm{d}x + \int_0^t\int_{|x|>\tilde\rho_i} u^{m+\theta-2}|Du|^{\lambda+1}\,\mathrm{d}x\,\mathrm{d}\tau$$
$$+\int_0^t\int_{|x|>\tilde\rho_i} u^\theta|Du^\nu|^q\,\mathrm{d}x\,\mathrm{d}\tau + \frac{2^{i(\lambda+1)}}{\tilde\rho^{\lambda+1}}\int_0^t\int_{\tilde\rho_i>|x|>\tilde\rho_{i+1}} u^{m+\lambda+\theta-1}\,\mathrm{d}x\,\mathrm{d}\tau\,,$$

where $\tilde\rho_i=\tilde\rho(1+2^{-i})/2$. Proceeding exactly as in the proof of (2.6) we get

$$y^{(i)}(\tilde\rho)\leq \gamma b_1^i t^{\frac{\nu q-(m+\lambda-1)}{\nu q+\theta}}\tilde\rho^{N\frac{\nu q-(m+\lambda-1)}{\nu q+\theta}-\lambda-1} Z^{q\frac{m+\lambda+\theta-1}{\nu q+\theta}}[y^{(i+1)}(\tilde\rho)]^{\frac{m+\lambda+\theta-1}{\nu q+\theta}}\,.$$

A simple iterative process leads to the bound

$$y^{(0)}(\tilde\rho)\leq\gamma t\tilde\rho^{\frac{N(\nu q-(m+\lambda-1))-(\lambda+1)(\nu q+\theta)}{\nu q-(m+\lambda-1)}} Z^{q\frac{m+\lambda+\theta-1}{\nu q-(m+\lambda-1)}}\,.$$

We check by direct inspection that

$$y_0\leq\gamma[y^{(0)}(\rho)+y^{(0)}(4\rho/3)+y^{(0)}(3\rho/2)]$$
$$\leq\gamma t\rho^{\frac{N(\nu q-(m+\lambda-1))-(\lambda+1)(\nu q+\theta)}{\nu q-(m+\lambda-1)}} Z^{q\frac{m+\lambda+\theta-1}{\nu q-(m+\lambda-1)}}.$$

Substituting this estimate in (2.8) one checks (after lenghty but trivial calculations) that $u(x,t)\equiv 0$ if $|x|\geq 2\rho$, with ρ as in

$$2\rho\Big(\frac{2\rho}{Z}\Big)^{\gamma_1}=\gamma t^{\frac{\nu q-(m+\lambda-1)}{\mathcal{H}}},$$

for a constant $\gamma_1>0$ which we do not reproduce here. Note that this implies $2\rho\geq Z$ by definition of Z. Estimate (1.4) follows immediately.

3. Decay of mass

3.1. Proof of Theorem 1.2

1) First we prove (1.6). We multiply both sides of equation (1.1) by u^θ and integrate over B_ρ, with

$$\rho=R(t):=\gamma t^{\frac{\nu q-(m+\lambda-1)}{\mathcal{H}}}\geq Z(t)\,,$$

for large enough t. We find for $0<\tau<t$,

$$\frac{1}{\theta+1}\frac{\mathrm{d}}{\mathrm{d}\tau}\int_{B_\rho} u(x,\tau)^{1+\theta}\,\mathrm{d}x\leq -\int_{B_\rho} u^\theta|Du^\nu|^q\,\mathrm{d}x\leq -\gamma\int_{B_\rho}|Du^{\frac{\nu q+\theta}{q}}|^q\,\mathrm{d}x\,. \tag{3.1}$$

Applying Hölder's and Poincaré's inequalities, we get from (3.1)

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\int_{B_\rho} u^{\theta+1}(x,\tau)\,\mathrm{d}x \le -\gamma\rho^{-\frac{q(1+\theta)+N(\nu q-1)}{1+\theta}}\Big(\int_{B_\rho} u^{\theta+1}\,\mathrm{d}x\Big)^{\frac{\nu q+\theta}{1+\theta}},$$

and therefore, integrating over $(t/2,t)$,

$$\int_{B_{R(t)}} u^{\theta+1}(x,t)\,\mathrm{d}x \le \gamma t^{-\frac{1+\theta}{\nu q-1}} R(t)^{\frac{N(\nu q-1)+q(1+\theta)}{\nu q-1}}. \tag{3.2}$$

Finally we infer (1.6) from an application of Hölder's inequality, together with (1.4) and (3.2).

2) Next we prove (1.7). Integrate the equation (1.1) over $\mathbb{R}^N$ to find

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_1 = -\int_{\mathbb{R}^N}|Du(x,t)^\nu|^q\,\mathrm{d}x \le -\gamma Z(t)^{-[N(\nu q-1)+q]}\|u(t)\|_1^{\nu q},$$

where we have used also Hölder's and Poincaré's inequalities.

Recalling the estimate (1.4), and that $q=q^*$, we find for large enough t

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_1 \le -\gamma t^{-1}\|u(t)\|_1^{\nu q},$$

whence the desired result follows by integration.

3.2. Proof of Theorem 1.3

Fix $t>0$. Let us split the total mass as follows:

$$\|u(\tau)\|_1 = \int_{B_{2\rho}} u(x,\tau)\,\mathrm{d}x + \int_{|x|>2\rho} u(x,\tau)\,\mathrm{d}x =: E_1(\rho,\tau)+E_2(\rho,\tau). \tag{3.3}$$

We obtain from Hölder's, Nirenberg-Gagliardo and Young's inequalities

$$\begin{aligned} E_1(\rho,\tau) &\le \gamma\Big(\int_{\mathbb{R}^N} u^{\nu q}(x,\tau)\,\mathrm{d}x\Big)^{\frac{1}{\nu q}}\rho^{\frac{N(\nu q-1)}{\nu q}} \\ &\le \gamma\Big(\int_{\mathbb{R}^N}|Du^\nu|^q\,\mathrm{d}x\Big)^{\frac{\alpha}{\nu q}}\|u(\tau)\|_1^{1-\alpha}\rho^{\frac{N(\nu q-1)}{\nu q}} \\ &\le \frac{1}{2}\|u(\tau)\|_1 + \gamma\Big(\int_{\mathbb{R}^N}|Du^\nu|^q\,\mathrm{d}x\Big)^{\frac{1}{\nu q}}\rho^{\frac{N(\nu q-1)+q}{\nu q}}, \end{aligned} \tag{3.4}$$

where

$$\alpha = N(\nu q-1)/[N(\nu q-1)+q].$$

On the other hand, integrating (1.1) over $\mathbb{R}^N$ we get

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\int_{\mathbb{R}^N} u\,\mathrm{d}x = -\int_{\mathbb{R}^N}|Du^\nu|^q\,\mathrm{d}x. \tag{3.5}$$

Thus (3.4), (3.5) yield

$$\|u(\tau)\|_1 \le \gamma\Big(-\frac{\mathrm{d}}{\mathrm{d}\tau}\|u(\tau)\|_1\Big)^{\frac{1}{\nu q}}\rho^{\frac{N(\nu q-1)+q}{\nu q}} + 2E_2(\rho,\tau), \qquad (3.6)$$

In order to bound E_2 in (3.6) we distinguish between the two cases listed in the statement of the Theorem.

Case (a): $\nu=\sigma$. First we note that if $\nu=\sigma=\frac{m+\lambda-1}{\lambda}$ then the equation (1.1) may be rewritten as

$$u_t = \sigma^\lambda \operatorname{div}\big(|Du^\sigma|^{\lambda-1}Du^\sigma\big) - |Du^\sigma|^q\,. \qquad (3.7)$$

Moreover, the assumption $\nu q > m+\lambda-1$ leads to $\lambda<q$. Let ζ be a cut off function vanishing inside B_ρ and such that $\zeta(x)\equiv 1$ for $|x|>2\rho$, $|D\zeta|\le\gamma/\rho$. Then, on multiplying both sides of (3.7) by ζ^s, where $s=q/(q-\lambda)>\lambda+1$, and integrating by parts over $\mathbb{R}^N$, we get

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\int_{\mathbb{R}^N}\zeta^s u\,\mathrm{d}x + \int_{\mathbb{R}^N}|Du^\sigma|^q\zeta^s\,\mathrm{d}x \le \frac{\gamma}{\rho}\int_{\rho<|x|<2\rho}|Du^\sigma|^\lambda\zeta^{s-1}\,\mathrm{d}x\,. \qquad (3.8)$$

The right hand side in (3.8) can be bound above by

$$\frac{1}{2}\int_{\mathbb{R}^N}|Du^\sigma|^q\zeta^s\,\mathrm{d}x + \gamma\rho^{N-\frac{q}{q-\lambda}}\,.$$

Thus (3.8) yields for $0<\tau<t$

$$\begin{aligned} E_2(\rho,\tau) = \int_{|x|>2\rho} u(x,\tau)\,\mathrm{d}x &\le \int_{|x|>\rho} u_0(x)\,\mathrm{d}x + \gamma t\rho^{N-\frac{q}{q-\lambda}} \\ &= \int_{|x|>\rho} u_0(x)\,\mathrm{d}x + \gamma t^{-\mathcal{A}} =: \tilde{E}(t)\,, \qquad (3.9)\end{aligned}$$

where we have set $\rho=R(t)=t^{\frac{\sigma q-(m+\lambda-1)}{\mathcal{H}}}$. Then denoting $F(\tau)=\|u(\tau)\|_1-2\tilde{E}(t)$, we get from (3.6)

$$F(\tau)\le\Big(-\frac{\mathrm{d}F}{\mathrm{d}\tau}\Big)^{\frac{1}{\nu q}}R(t)^{\frac{N(\nu q-1)+q}{\nu q}}\,, \qquad 0<\tau<t\,. \qquad (3.10)$$

If $F(t)\le 0$, estimate (1.10) follows immediately. Also keeping in mind that $F'\le 0$, we may therefore assume that $F>0$ in $(0,t)$. Integrating (3.10) we find

$$F(t)\le\gamma t^{-\frac{1}{\sigma q-1}}R(t)^{\frac{N(\sigma q-1)+q}{\sigma q-1}} = \gamma t^{-\mathcal{A}}\,,$$

whence (1.10).

Case (b): $\lambda=1$, $N\ge 2$. Let ζ_n be a cut off function such that $\zeta_n=0$ for $|x|<\rho_{n+1}$, $\zeta_n=1$ for $|x|>\rho_n$, where $\rho_n=\rho(1+2^{-n})$. We may assume that $|D\zeta_n| \le 2^n\gamma/\rho$, $|\Delta\zeta_n| \le 2^{2n}\gamma/\rho^2$. On using ζ_n^s, where $s > 2$, as a testing function in (1.1), we have

$$\begin{aligned} \int_{\mathbb{R}^N} \zeta_n^s u(x,\tau)\,\mathrm{d}x + \iint_{Q_\tau} \zeta_n^s |Du^\nu|^q \,\mathrm{d}x\,\mathrm{d}\eta \\ = \int_{\mathbb{R}^N} \zeta_n^s u_0 \,\mathrm{d}x + \iint_{Q_\tau} u^m \Delta\zeta_n^s \,\mathrm{d}x\,\mathrm{d}\eta \\ \le \int_{|x|>\rho} u_0\,\mathrm{d}x + \gamma\frac{2^{2n}}{\rho^2}\int_0^\tau \int_{\rho_n>|x|>\rho_{n+1}} u^m \,\mathrm{d}x\,\mathrm{d}\eta =: K_0 + 2^{2n}K_1. \end{aligned} \tag{3.11}$$

Using Hölder's inequality we get

$$K_1 \le \gamma t^{\frac{\nu q-m}{\nu q}} \rho^{\frac{N(\nu q-m)+mq}{\nu q}-2} \Big(\int_0^\tau \int_{\rho_n>|x|>\rho_{n+1}} \frac{u^{\nu q}}{|x|^q} \,\mathrm{d}x\,\mathrm{d}\eta \Big)^{\frac{m}{\nu q}} . \tag{3.12}$$

Next we apply Hardy's inequality, obtaining

$$\int_{|x|>\rho_{n+1}} \frac{u^{\nu q}}{|x|^q}\,\mathrm{d}x \le \gamma \int_{|x|>\rho_{n+1}} |Du^\nu|^q \,\mathrm{d}x\,. \tag{3.13}$$

Here we use the fact that $q < \lambda + 1 = 2 \le N$.

Denote for all fixed $\tau \in (0,t)$

$$H_n(\tau) = \int_{|x|>\rho_n} u(x,\tau)\,\mathrm{d}x + \int_0^\tau \int_{|x|>\rho_n} |Du^\nu|^q \,\mathrm{d}x\,, \qquad n \ge 0\,.$$

Thus, it follows from (3.11)–(3.13) that for $n \ge 0$

$$\begin{aligned} H_n(\tau) \le K_0 + \gamma 2^{2n} t^{\frac{\nu q-m}{\nu q}} \rho^{\frac{N(\nu q-m)+mq}{\nu q}-2} H_{n+1}(\tau)^{\frac{m}{\nu q}} \\ \le K_0 + \omega H_{n+1}(\tau) + C_\omega 2^{\frac{2n\nu q}{\nu q-m}} t \rho^{N+\frac{mq}{\nu q-m}-\frac{2\nu q}{\nu q-m}} . \end{aligned} \tag{3.14}$$

A simple iteration procedure, when $0 < \omega < 1$ is chosen small enough, yields

$$\int_{|x|>2\rho} u(x,\tau)\,\mathrm{d}x \le H_0(\tau) \le K_0 + \gamma t \rho^{N+\frac{mq}{\nu q-m}-\frac{2\nu q}{\nu q-m}}\,, \qquad 0 < \tau < t\,,$$

where we select $\rho = R(t)$ to obtain

$$\int_{|x|>2R(t)} u(x,\tau)\,\mathrm{d}x \le \int_{|x|>R(t)} u_0(x)\,\mathrm{d}x + \gamma t^{-\mathcal{A}} =: \tilde{E}(t)\,. \tag{3.15}$$

Denoting again $F(\tau) = \|u(\tau)\|_1 - 2\tilde{E}(t)$, the proof can be concluded as in Case (a) above.

4. No decay of mass: Proof of Theorem 1.4

As a preliminary step to the proof, we recall the well known bound

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \le \gamma \|u_0\|_1^{\frac{\lambda+1}{\mathcal{K}}} t^{-\frac{N}{\mathcal{K}}}\,, \qquad t>0\,, \tag{4.1}$$

following from (1.8).

Integrating the equation (1.1) over $\mathbb{R}^N \times (t_1,t_2)$ for any $0<t_1<t_2$, we have

$$\|u(t_1)\|_1 = \|u(t_2)\|_1 + \int_{t_1}^{t_2}\int_{\mathbb{R}^N} |Du^\nu|^q \,\mathrm{d}x\,\mathrm{d}\tau\,. \tag{4.2}$$

Using Hölder's inequality, we find for $0<\theta<1$

$$\begin{aligned}\int_{t_1}^{t_2}\int_{\mathbb{R}^N} |Du^\nu|^q \,\mathrm{d}x\,\mathrm{d}\tau &\le \left[\int_{t_1}^{t_2}\int_{\mathbb{R}^N} |Du|^{\lambda+1} u^{\theta+m-2}\,\mathrm{d}x\,\mathrm{d}\tau\right]^{\frac{q}{\lambda+1}} \\ &\times \left[\int_{t_1}^{t_2}\int_{\mathbb{R}^N} u^{\frac{q[(\lambda+1)\nu-(m+\lambda-1)-\theta]}{\lambda+1-q}}\,\mathrm{d}x\,\mathrm{d}\tau\right]^{\frac{\lambda+1-q}{\lambda+1}} \equiv J_1^{\frac{q}{\lambda+1}} J_2^{\frac{\lambda+1-q}{\lambda+1}}\,. \end{aligned} \tag{4.3}$$

We estimate J_1 by multiplying the equation by u^θ and integrating over $(t_1,t_2)\times\mathbb{R}^N$; this yields

$$\begin{aligned}\int_{\mathbb{R}^N} u^{\theta+1}(x,t_2)\,\mathrm{d}x &- \int_{\mathbb{R}^N} u^{\theta+1}(x,t_1)\,\mathrm{d}x \\ &+ \theta(\theta+1)\int_{t_1}^{t_2}\int_{\mathbb{R}^N} |Du|^{\lambda+1}u^{\theta+m-2}\,\mathrm{d}x\,\mathrm{d}\tau \\ &+ (\theta+1)\int_{t_1}^{t_2}\int_{\mathbb{R}^N} u^\theta |Du^\nu|^q\,\mathrm{d}x\,\mathrm{d}\tau = 0\,. \end{aligned} \tag{4.4}$$

Next we apply the estimate (4.1) to get

$$J_1 \le \frac{1}{\theta(\theta+1)}\int_{\mathbb{R}^N} u^{\theta+1}(x,t_1)\,\mathrm{d}x \le \gamma \|u_0\|_1^{\frac{(\lambda+1)\theta}{\mathcal{K}}} \|u(t_1)\|_1 t_1^{-\frac{N\theta}{\mathcal{K}}}\,. \tag{4.5}$$

The power of u in the integral J_2 amounts to $(\mathcal{H}-\theta q)/(\lambda+1-q)+1$, and thus it is greater than 1, under our assumptions, at least for small θ. Therefore we may apply again (4.1) and estimate

$$J_2 \le \gamma \|u_0\|_1^{\frac{(\mathcal{H}-\theta q)(\lambda+1)}{\mathcal{K}(\lambda+1-q)}} \|u(t_1)\|_1 t_1^{1-\frac{N(\mathcal{H}-\theta q)}{\mathcal{K}(\lambda+1-q)}}\,. \tag{4.6}$$

Indeed, we invoke here the obvious statement

$$t \mapsto \|u(t)\|_1 \quad \text{is non increasing,} \tag{4.7}$$

and the fact that the exponent of t_1 in (4.6) is negative. The latter, in turn, is a consequence of $q>q^*$, provided $\theta>0$ is small enough.

From (4.2)–(4.6) it follows

$$\|u(t_1)\|_1 \leq \|u(t_2)\|_1 + \gamma_1 \|u_0\|_1^{\frac{\mathcal{H}}{\mathcal{K}}} \|u(t_1)\|_1 \, t_1^{\frac{\mathcal{A}\mathcal{H}}{\mathcal{K}}} \,. \tag{4.8}$$

Since $\mathcal{A} < 0$, owing again to the assumption $q > q^*$, we finally obtain

$$\|u(t_2)\|_1 \geq \frac{1}{2} \|u(\bar{t})\|_1 \,, \qquad \text{for all } t_2 > \bar{t} := \left[(2\gamma_1)^{-1} \|u_0\|_1^{-\frac{\mathcal{H}}{\mathcal{K}}} \right]^{\frac{\mathcal{K}}{\mathcal{A}\mathcal{H}}} . \tag{4.9}$$

This, also taking into account (4.7), implies the statement, provided we show that $\|u(\bar{t})\|_1 > 0$. This is the content of next lemma.

Lemma 4.1. *A solution to* (1.1) *with* $\delta = 0$, $\varepsilon = 1$ *cannot satisfy* $u(x, t_0) \equiv 0$ *over the whole space* $\mathbb{R}^N$ *for any finite* $t_0 > 0$.

Note that the special assumption $q > q^*$ is not needed in this lemma.

Proof. Assume, by contradiction, that a finite time $t_0 > 0$ exists such that

$$\|u(t_0)\|_1 = 0, \qquad \|u(t)\|_1 > 0, \quad t < t_0.$$

Define for $0 < \theta < 1$

$$E_\theta(t) = \int_{\mathbb{R}^N} u(x,t)^{1+\theta} \, \mathrm{d}x.$$

By means of calculations similar to the ones performed in the proof of Theorem 1.4, we obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \|u(t)\|_1 &= - \int_{\mathbb{R}^N} |Du^\nu|^q \, \mathrm{d}x \\ &\geq - \left(\int_{\mathbb{R}^N} |Du|^{\lambda+1} u^{\theta+m-2} \, \mathrm{d}x \right)^{\frac{q}{\lambda+1}} \left(\int_{\mathbb{R}^N} u^{\frac{q[(\lambda+1)\nu - (m+\lambda-1) - \theta]}{\lambda+1-q}} \, \mathrm{d}x \right)^{1-\frac{q}{\lambda+1}} \\ &\geq - \Big(- \frac{\mathrm{d}}{\mathrm{d}t} E_\theta(t) \Big)^{\frac{q}{\lambda+1}} \Big(C \|u(t)\|_1 \Big)^{1-\frac{q}{\lambda+1}} \geq \omega \frac{\mathrm{d}}{\mathrm{d}t} E_\theta(t) - C_\omega \|u(t)\|_1 \,. \end{aligned} \tag{4.10}$$

Here $\omega \in (0,1)$ will be chosen later, and the constants C, C_ω depend on a uniform L^∞ bound for $u(t)$, $t_0/2 < t < t_0$; such a bound can be found for example invoking (4.1). Integrate (4.10) over (t, t_0), for t close to t_0, to obtain

$$\begin{aligned} \|u(t)\|_1 &\leq \omega E_\theta(t) + C_\omega \int_t^{t_0} \|u(\tau)\|_1 \, \mathrm{d}\tau \\ &\leq \omega C \|u(t)\|_1 + C_\omega (t_0 - t) \|u(t)\|_1 \,. \end{aligned} \tag{4.11}$$

Here C is as above, and we have taken advantage of (4.7) (this last detail is in fact inessential). Obviously, dividing (4.11) by $\|u(t)\|_1$, and then letting $t \to t_0$ we get an inconsistency, provided ω is so small as $\omega C < 1$ in (4.11). □

5. Proof of Theorem 1.5

We remark preliminarily that under our current assumptions,

$$\frac{N}{\mathcal{K}_\beta}(p-1) < 1 < \frac{\lambda+1-q}{\mathcal{H}}(p-1)\,, \tag{5.1}$$

where $\mathcal{K}_\beta = N(m+\lambda-2)+\beta(\lambda+1)$.

Define T as the supremum of the times $t>1$ such that

$$\tau\|u(\tau)\|^{p-1}_{\infty,\mathbb{R}^N} \le \frac{1}{2}, \qquad 0<\tau<t, \tag{5.2}$$

$$\|u(\tau)\|_{\infty,\mathbb{R}^N} \le c_0\tau^{-\frac{\lambda+1-q}{\mathcal{H}}}, \qquad 1<\tau<t. \tag{5.3}$$

Here c_0 is a constant depending only on N, m, λ, q, p and M, which is known a priori, but will be specified later. It is known (see [3, 6]) that, given any $t_1>1$, (5.2) is satisfied for all $0<t<t_1$, provided $\|u_0\|_{\beta,\mathbb{R}^N}$ is chosen small enough. Next we recall the estimate (see [3, 6])

$$\|u(\tau)\|_{\infty,\mathbb{R}^N} \le \gamma\|u_0\|^{\frac{\lambda+1}{\mathcal{K}_\beta}}_{\beta,\mathbb{R}^N}\tau^{-\frac{N}{\mathcal{K}_\beta}}, \qquad 0<\tau<t, \tag{5.4}$$

which is valid under assumption (5.2). Indeed the quoted results of [3, 6] were proven for subsolutions to the equation without damping term, and thus hold true even in the present case. Note that, for any fixed $t_1>1$, we have

$$\gamma\|u_0\|^{\frac{\lambda+1}{\mathcal{K}_\beta}}_{\beta,\mathbb{R}^N}\tau^{-\frac{N}{\mathcal{K}_\beta}} \le c_0\tau^{-\frac{\lambda+1-q}{\mathcal{H}}}, \qquad 1<\tau<t_1,$$

provided $\|u_0\|_{\beta,\mathbb{R}^N}$ is small enough in dependence of t_1. Then, for $1<t<t_1$, (5.3) is a consequence of (5.2).

Let us summarize what we have obtained: the supremum T defined above satisfies $T\ge t_1>1$, where the finite number t_1 can be chosen as large as we need, provided the bound for $\|u_0\|_{\beta,\mathbb{R}^N}$ is redefined accordingly. We always assume $t_1>2$. Our goal is to show that $T=\infty$.

Let now $\zeta\in C^1(\mathbb{R}^N)$ be a cut off function such that $\zeta\equiv 0$ inside B_ρ, $\rho>1$, and $\zeta\equiv 1$ in $\mathbb{R}^N\setminus B_{2\rho}$, with $|D\zeta|\le 2/\rho$. Taking ζ^s, $s>q/(q-\lambda)$, as a cut off function in (1.1) we get for $t<T$

$$\sup_{0<\tau<t}\int_{\mathbb{R}^N}\zeta^s u(x,\tau)\,\mathrm{d}x + \int_0^t\int_{\mathbb{R}^N}\zeta^s|Du^\sigma|^q\,\mathrm{d}x\,\mathrm{d}\tau \le \int_{\mathbb{R}^N}\zeta^s u_0(x)\,\mathrm{d}x \\ + \frac{2s}{\rho}\int_0^t\int_{\mathbb{R}^N}\zeta^{s-1}|Du^\sigma|^\lambda\,\mathrm{d}x\,\mathrm{d}\tau + \int_0^t\int_{\mathbb{R}^N}\zeta^s u^p\,\mathrm{d}x\,\mathrm{d}\tau\,. \tag{5.5}$$

Next we proceed to bound the three terms on the right hand side of (5.5). The first term is bound by means of assumption (1.13). An application of Young's inequality yields for the second term the estimate

$$\frac{1}{2}\int_0^t\int_{\mathbb{R}^N}\zeta^s|Du^\sigma|^q\,\mathrm{d}x\,\mathrm{d}\tau + \gamma t\rho^{N-\frac{q}{q-\lambda}}.$$

The third term is more critical: first, we bound it by

$$\int_0^t \|u(\tau)\|_{\infty,\mathbb{R}^N}^{p-1} \,\mathrm{d}\tau \sup_{0<\tau<t} \int_{\mathbb{R}^N} \zeta^s u(x,\tau) \,\mathrm{d}x.$$

Using both (5.4) (for $\tau < t_1$), and (5.3) (for $\tau > t_1$) we obtain

$$\begin{aligned}
&\int_0^t \|u(\tau)\|_{\infty,\mathbb{R}^N}^{p-1} \,\mathrm{d}\tau \\
&\qquad \leq \gamma \int_0^{t_1} \|u_0\|_{\beta,\mathbb{R}^N}^{\frac{\lambda+1}{\mathcal{K}_\beta}} \tau^{-\frac{N}{\mathcal{K}_\beta}(p-1)} \,\mathrm{d}\tau + c_0 \int_{t_1}^t \tau^{-\frac{\lambda+1-q}{\mathcal{H}}(p-1)} \,\mathrm{d}\tau \\
&\leq \gamma \|u_0\|_{\beta,\mathbb{R}^N}^{\frac{\lambda+1}{\mathcal{K}_\beta}} t_1^{1-\frac{N}{\mathcal{K}_\beta}(p-1)} + \frac{c_0 \mathcal{H}}{(\lambda+1-q)(p-1)-\mathcal{H}} t_1^{1-\frac{\lambda+1-q}{\mathcal{H}}(p-1)} \leq \frac{1}{2},
\end{aligned}$$

provided we first select $t_1 = t_1(c_0)$ (recall (5.1)), and next a small enough $\|u_0\|_{\beta,\mathbb{R}^N}$. Of course for $t < t_1$ the estimate above is still valid.

As a consequence of the estimates we have found, the right hand side of (5.5) can be partially absorbed into the left hand side, leading us to

$$\sup_{0<\tau<t} \int_{|x|>2\rho} u(x,\tau) \,\mathrm{d}x \leq 2M\rho^{-\frac{\mathcal{A}\mathcal{H}}{\sigma q - m - \lambda + 1}} + \gamma t \rho^{N - \frac{q}{q-\lambda}} =: C(\rho, t). \quad (5.6)$$

It is important to note that the constants in (5.6) do not depend on t_1 or c_0; this applies to all the constants appearing below and denoted by γ. Employing this bound, and reasoning as in (3.3)–(3.4), we arrive at

$$\|u(\tau)\|_1 \leq 2C(\rho,t) + \gamma \Big(\int_{\mathbb{R}^N} |Du^\sigma|^q \,\mathrm{d}x \Big)^{\frac{1}{\sigma q}} \rho^{\frac{N(\sigma q - 1) + q}{\sigma q}}. \quad (5.7)$$

Next, it follows from (5.2) and from the definition of T that for all $\tau < T$,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\tau} \|u(\tau)\|_1 &= -\int_{\mathbb{R}^N} |Du^\sigma|^q \,\mathrm{d}x + \int_{\mathbb{R}^N} u^p \,\mathrm{d}x \\
&\leq -\int_{\mathbb{R}^N} |Du^\sigma|^q \,\mathrm{d}x + \frac{1}{2\tau} \|u(\tau)\|_1. \quad (5.8)
\end{aligned}$$

Setting $f(\tau) = \tau^{-\frac{1}{2}} \|u(\tau)\|_1$, this yields (on multiplying (5.8) by $\tau^{-1/2}$)

$$f'(\tau) \leq -\tau^{-\frac{1}{2}} \int_{\mathbb{R}^N} |Du^\sigma|^q \,\mathrm{d}x.$$

Thus, assuming also $t/2 < \tau < t < T$, (5.7) amounts to

$$f(\tau) \leq 2\sqrt{2} t^{-\frac{1}{2}} C(\rho,t) + \gamma \big(-t^{\frac{1}{2}} f'(\tau) \big)^{\frac{1}{\sigma q}} \rho^{\frac{N(\sigma q - 1)+q}{\sigma q}} t^{-\frac{1}{2}}. \quad (5.9)$$

Define also $F(\tau) = f(\tau) - 2\sqrt{2}t^{-\frac{1}{2}}C(\rho, t)$. Assume provisionally $F(t) > 0$, and therefore, due to the monotonic character of F, $F(\tau) > 0$ for $\tau < t$. Then (5.9) gives

$$F'(\tau) \leq -\gamma t^{\frac{\sigma q-1}{2}} F(\tau)^{\sigma q} \rho^{-N(\sigma q-1)-q}. \tag{5.10}$$

Then, integrating (5.10) over $(t/2, t)$, we get

$$F(t) \leq \gamma t^{-\frac{\sigma q+1}{2(\sigma q-1)}} \rho^{N+\frac{q}{\sigma q-1}},$$

i.e.,

$$\|u(t)\|_1 \leq 2\sqrt{2}C(\rho, t) + \gamma t^{\frac{1}{2}-\frac{\sigma q+1}{2(\sigma q-1)}} \rho^{N+\frac{q}{\sigma q-1}}. \tag{5.11}$$

If $F(t) \leq 0$ we still have (5.11), obviously. Finally we select in (5.11) $\rho = \rho(t) = t^{(\sigma q-m-\lambda+1)/\mathcal{H}}$, obtaining

$$\|u(t)\|_1 \leq \gamma t^{-\mathcal{A}}, \qquad 1 < t < T. \tag{5.12}$$

Substituting this estimate in the L^1–L^∞ estimate of Remark 1.8, which is valid for our u under assumption (5.2) (see [3, 6]) we have

$$\|u(t_1)\|_{\infty,\mathbb{R}^N} \leq \gamma \sup_{\frac{t}{2}<\tau<t} \|u(\tau)\|_1^{\frac{\lambda+1}{\mathcal{K}}} t^{-\frac{N}{\mathcal{K}}} \leq \gamma_* t^{-\frac{\lambda+1-q}{\mathcal{H}}}, \quad 2 < t < T. \tag{5.13}$$

The importance of the estimate (5.13) is twofold. First, owing to well known compactness results (see Remark 1.1), it can be employed to prove existence of a solution up to time T, by means of standard approximation techniques with solutions to smoothed problems.

Second, it permits us to prove that $T = \infty$. Indeed, choose $c_0 = 2\gamma_*$ in (5.3): here γ_* is the constant appearing in (5.13). This can be done safely, because, as we already remarked, γ_* does not depend either on c_0 or on t_1. By this choice, we have actually shown that (5.3) holds with c_0 formally replaced by $c_0/2$ up to the time T. Therefore, if $T < \infty$, (5.2) must fail for some finite time. But, for $t_1 < \tau < T$, taking into account (5.13) again, we have

$$\tau \|u(\tau)\|_{\infty,\mathbb{R}^N}^{p-1} \leq \gamma_* \tau^{1-\frac{\lambda+1-q}{\mathcal{H}}(p-1)} < \gamma_* t_1^{1-\frac{\lambda+1-q}{\mathcal{H}}(p-1)} \leq \frac{1}{4},$$

where the last inequality is guaranteed by a suitable choice of t_1 (recall (5.1)). Again, this can be done without any danger of circular reasoning, as γ_* does not depend on t_1.

The proof is concluded.

6. Proof of Theorem 1.6

Following [6, 20], we use as a testing function in the equation $\zeta^s u^{-\theta}$, where $0 < \theta < 1$, and $s > \lambda + 1$ will be chosen later. Here $\zeta(x)$ is a standard cut off function in $B_{2\rho}$, such that $\zeta \equiv 1$ in B_ρ, and $|D\zeta| \leq \gamma\rho^{-1}$. We have

$$\frac{1}{1-\theta}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^N}\zeta^s u^{1-\theta}\,\mathrm{d}x = -\int_{\mathbb{R}^N} u^{m-1}|Du|^{\lambda-1}Du\cdot D(\zeta^s u^{-\theta})\,\mathrm{d}x \\ -\varepsilon\int_{\mathbb{R}^N}|Du^\nu|^q u^{-\theta}\zeta^s\,\mathrm{d}x + \int_{\mathbb{R}^N} u^{p-\theta}\zeta^s\,\mathrm{d}x =: I_1+I_2+I_3. \quad (6.1)$$

Next we write I_1 as

$$I_1 = \theta\int_{\mathbb{R}^N} u^{m-\theta-2}|Du|^{\lambda+1}\zeta^s\,\mathrm{d}x \\ - s\int_{\mathbb{R}^N} u^{m-1}|Du|^{\lambda-1}Du\cdot D\zeta\zeta^{s-1}u^{-\theta}\,\mathrm{d}x =: \theta I_4 - sI_5. \quad (6.2)$$

Applying Young's inequality we get for $\varepsilon_1>0$

$$I_5 \le \varepsilon_1 I_4 + \gamma\frac{C_{\varepsilon_1}}{\rho^{\lambda+1}}\int_{B_{2\rho}}\zeta^{s-(\lambda+1)}u^{m+\lambda-\theta-1}\,\mathrm{d}x, \quad (6.3)$$

and by the same token, for $\varepsilon_2>0$,

$$\frac{1}{\rho^{\lambda+1}}\int_{B_{2\rho}}\zeta^{s-(\lambda+1)}u^{m+\lambda-\theta-1}\,\mathrm{d}x \le \varepsilon_2\int_{B_{2\rho}}\zeta^s u^{p-\theta}\,\mathrm{d}x + C_{\varepsilon_2}\rho^{N-\frac{(\lambda+1)(p-\theta)}{p-(m+\lambda-1)}}, \quad (6.4)$$

where $s=(\lambda+1)(p-\theta)/(p-m-\lambda+1)>\lambda+1$ if θ is small enough. Moreover, for $\varepsilon_3>0$,

$$-I_2 = \varepsilon\int_{B_{2\rho}}|Du^\nu|^q u^{-\theta}\zeta^s\,\mathrm{d}x \le \varepsilon\varepsilon_3\int_{B_{2\rho}}\zeta^s u^{m-2-\theta}|Du|^{\lambda+1}\,\mathrm{d}x \\ + \varepsilon C_{\varepsilon_3}\int_{B_{2\rho}}\zeta^s u^{p^*-\theta}\,\mathrm{d}x. \quad (6.5)$$

Thus, (6.2)–(6.5) show that we may absorb into I_4 all negative terms on the right hand side of (6.1), excepting only the last term in (6.4), provided ε, ε_1, ε_2, ε_3 are small enough. Of course we are also using $p=p^*$ here. We have proven that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{B_{2\rho}}\zeta^s u^{1-\theta}\,\mathrm{d}x \ge \tilde\gamma\int_{B_{2\rho}}\zeta^s u^{p^*-\theta}\,\mathrm{d}x - \gamma\rho^{N-\frac{(\lambda+1)(p^*-\theta)}{p^*-(m+\lambda-1)}}. \quad (6.6)$$

Then apply Hölder's inequality to get

$$I(t) := \int_{B_{2\rho}}\zeta^s(x)u^{1-\theta}(x,t)\,\mathrm{d}x \le \gamma\Big(\int_{B_{2\rho}}\zeta^s u^{p^*-\theta}\,\mathrm{d}x\Big)^{\frac{1-\theta}{p^*-\theta}}\rho^{N\frac{p^*-1}{p^*-\theta}}.$$

This and (6.6) imply

$$\frac{\mathrm{d}}{\mathrm{d}t}I(t) \ge \gamma_0 I(t)^{\frac{p^*-\theta}{1-\theta}}\rho^{-N\frac{p^*-1}{1-\theta}} - \gamma_1\rho^{N-\frac{(\lambda+1)(p^*-\theta)}{p^*-(m+\lambda-1)}}. \quad (6.7)$$

Fix any $\bar{t} > 0$. If for all $\rho > 0$ the right hand side of (6.7) is small, i.e., if

$$I(\bar{t}) \leq \left(\frac{2\gamma_1}{\gamma_0}\right)^{\frac{1-\theta}{p^*-\theta}} \rho^{N - \frac{(\lambda+1)(1-\theta)}{p^*-(m+\lambda-1)}}, \tag{6.8}$$

then we get $u(x, \bar{t}) \equiv 0$ on letting $\rho \to \infty$. Indeed, if $q < q^*$, the power of ρ in (6.8) is negative, for small enough θ. As we are dealing with non trivial solutions, we may therefore assume, for some large ρ,

$$\frac{\mathrm{d}}{\mathrm{d}t} I(t) \geq \frac{\gamma_0}{2} I(t)^{\frac{p^*-\theta}{1-\theta}} \rho^{-N\frac{p^*-1}{1-\theta}}. \tag{6.9}$$

Note that if (6.9) holds at $t = \bar{t}$, it holds for all $t > \bar{t}$, because $I(t)$ is an increasing function, as (6.9) itself implies. Then, I is a supersolution to a non linear ordinary differential equation whose positive solutions blow up in a finite time. The proof is concluded.

7. Higher order equations

The approach introduced above can be applied to find optimal decay estimates for higher order parabolic problems like

$$\left(|v|^{\beta-1}v\right)_t + (-1)^l \sum_{|\alpha|=l} D^\alpha\left(|D^l v|^{\lambda-1} D^\alpha v\right) = -|Dv^\mu|^q v, \quad \text{in } Q_T, \tag{7.1}$$

$$v(x, 0) = v_0(x), \qquad \text{in } \mathbb{R}^N, \tag{7.2}$$

where $\mathrm{supp} v_0 \subset B_{\rho_0}$, $v_0 \in L^{\beta+1}(\mathbb{R}^N)$, and $l > 1$. In (7.1) the sum is extended to all the derivatives of order l and we denote

$$|D^l v| = \left(\sum_{|\alpha|=l} |D^\alpha v|^{\lambda+1}\right)^{\frac{1}{\lambda+1}}.$$

We assume that

$$1 < 1 + \beta < 1 + \lambda < \mu q + 1, \qquad 1 < q < \lambda + 1,$$

and

$$q < q^* = \frac{N(\lambda - 1) + \beta l(\lambda + 1)}{\mu N + \beta},$$

and consider energy solutions, which can be defined as in [8, 14], with slight modifications.

Then one can prove the following bounds for large enough t:

$$Z(t) \leq \gamma t^{(\mu q+1-\lambda)/\Lambda}, \tag{7.3}$$

where $\Lambda = l(\lambda + 1)(\mu q + 1 - \beta) + q(\beta - \lambda)$, and Z has been defined in Theorem 1.1, and

$$\int_{\mathbb{R}^N} |v(x, t)|^{\beta+1} \,\mathrm{d}x \leq \gamma t^{-(1+\beta)(l(\lambda+1)-q)/\Lambda}. \tag{7.4}$$

As a consequence of (7.3), (7.4) one obtains the mass decay rate

$$\int_{\mathbb{R}^N} |v(x,t)|^\beta \, \mathrm{d}x \le \gamma t^{-(q^*-q)(\mu N+\beta)/\Lambda}. \tag{7.5}$$

The proof follows closely the arguments in Sections 2, 3, borrowing some techniques from [8].

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