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Hardy Type Spaces Estimates for Multilinear Marcinkiewicz Operators

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Abstract. The purpose of this paper is to prove the boundedness for some multilinear operators generated by Marcinkiewicz integral operators and Lipschitz functions on Hardy and Herz-Hardy spaces.

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1. Introduction and Results

In this paper, we will study a class of multilinear operators related to Marcinkiewicz integral operators, which are defined by the following.

Fix $\delta > 0$, $\lambda > 1$ and $0 < \gamma \leq 1$. Suppose that S^{n-1} is the unit sphere of $R^n (n \geq 2)$ equipped with normalized Lebesgue measure $d\sigma = d\sigma(x')$. Let Ω be homogeneous of degree zero and satisfy the following two conditions:

(i) $\Omega(x)$ is continuous on S^{n-1} and satisfies the Lip_γ condition on S^{n-1}, i.e.

$$|\Omega(x') - \Omega(y')| \leq M|x' - y'|^\gamma, \quad x', y' \in S^{n-1};$$

(ii) $\int_{S^{n-1}} \Omega(x')\, dx' = 0$.

Let m be a positive integer and A be a function on R^n. Set

$$R_{m+1}(A; x, y) = A(x) - \sum_{|\alpha| \leq m} \frac{1}{\alpha!} D^\alpha A(y)(x - y)^\alpha$$

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and

$$Q_{m+1}(A;x,y) = R_m(A;x,y) - \sum_{|\alpha|=m} \frac{1}{\alpha!} D^\alpha A(x)(x-y)^\alpha.$$

We denote $\Gamma(x) = \{(y,t) \in R_+^{n+1} : |x-y| < t\}$ and the characteristic function of $\Gamma(x)$ by $\chi_{\Gamma(x)}$. The multilinear Marcinkiewicz integral operators are defined by

$$\mu_\lambda^A(f)(x) = \left[\iint_{R_+^{n+1}} \left(\frac{t}{t+|x-y|} \right)^{n\lambda} |F_t^A(f)(x,y)|^2 \frac{dy\,dt}{t^{n+3}} \right]^{1/2},$$

where

$$F_t^A(f)(x,y) = \int_{|y-z|\le t} \frac{\Omega(y-z)}{|y-z|^{n-\delta-1}} \frac{R_{m+1}(A;x,z)}{|x-z|^m} f(z)\,dz.$$

The variants of μ_λ^A is defined by

$$\tilde{\mu}_\lambda^A(f)(x) = \left[\iint_{R_+^{n+1}} \left(\frac{t}{t+|x-y|} \right)^{n\lambda} |\tilde{F}_t^A(f)(x,y)|^2 \frac{dy\,dt}{t^{n+3}} \right]^{1/2},$$

where

$$\tilde{F}_t^A(f)(x,y) = \int_{|y-z|\le t} \frac{\Omega(y-z)}{|y-z|^{n-\delta-1}} \frac{Q_{m+1}(A;x,z)}{|x-z|^m} f(z)\,dz.$$

Set

$$F_t(f)(y) = \int_{|y-z|\le t} \frac{\Omega(y-z)}{|y-z|^{n-\delta-1}} f(z)\,dz.$$

We also define that

$$\mu_\lambda(f)(x) = \left(\iint_{R_+^{n+1}} \left(\frac{t}{t+|x-y|} \right)^{n\lambda} |F_t(f)(y)|^2 \frac{dy\,dt}{t^{n+3}} \right)^{1/2},$$

which are the Marcinkiewicz integral operator (see [14]).

Note that when $m = 0$ and $\delta = 0$, μ_λ^A is just the commutator of Marcinkiewicz integral operators (see [9, 14]), while when $m > 0$, they are non-trivial generalizations of the commutators. It is well known that multilinear operators are of great interest in harmonic analysis and have been widely studied by many authors when A has derivatives of order m in $BMO(R^n)$ (see [2–5]). In [1, 15], authors obtain the boundedness of multilinear singular integral operators generated by singular integrals and Lipschitz functions on L^p $(p > 1)$ and some Hardy spaces. The main purpose of this paper is to discuss the boundedness properties of the multilinear Marcinkiewicz integral operators on Hardy and Herz-Hardy spaces. Let us first introduce some definitions (see [6,7,11–13]). Throughout this paper, $M(f)$ will denote the Hardy-Littlewood maximal function of f, Q will denote a cube of R^n with side parallel to the axes. Denote the Hardy spaces by $H^p(R^n)$. It is well known that $H^p(R^n)(0 < p \leq 1)$ has the atomic decomposition characterization (see [6]). For $\beta > 0$, the Lipschitz space $\mathrm{Lip}_\beta(R^n)$ is the space of functions f such that (see [13])

$$\|f\|_{\mathrm{Lip}_\beta} = \sup_{\substack{x,h\in R^n \\ h\neq 0}} |f(x+h) - f(x)|/|h|^\beta < \infty.$$

Let $B_k = \{x \in R^n : |x| \leq 2^k\}, C_k = B_k \setminus B_{k-1}, k \in Z$.

Definition 1. *Let $0 < p$, $q < \infty$, $\alpha \in R$.*

(1) *The homogeneous Herz space is defined by*

$$\dot{K}_q^{\alpha,p}(R^n) = \{f \in L^q_{loc}(R^n \setminus \{0\}) : \|f\|_{\dot{K}_q^{\alpha,p}(R^n)} < \infty\},$$

where

$$\|f\|_{\dot{K}_q^{\alpha,p}} = \left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p} \|f\chi_k\|^p_{L^q}\right]^{1/p};$$

(2) *The nonhomogeneous Herz space is defined by*

$$K_q^{\alpha,p}(R^n) = \{f \in L^q_{loc}(R^n) : \|f\|_{K_q^{\alpha,p}(R^n)} < \infty\},$$

where

$$\|f\|_{K_q^{\alpha,p}} = \left[\sum_{k=1}^{\infty} 2^{k\alpha p} \|f\chi_k\|^p_{L^q} + \|f\chi_{B_0}\|^p_{L^q}\right]^{1/p}.$$

Definition 2. *Let $\alpha \in R$, $0 < p, q < \infty$.*

(1) *The homogeneous Herz type Hardy space is defined by*

$$H\dot{K}_q^{\alpha,p}(R^n) = \{f \in S'(R^n) : G(f) \in \dot{K}_q^{\alpha,p}(R^n)\},$$

and

$$\|f\|_{H\dot{K}_q^{\alpha,p}} = \|G(f)\|_{\dot{K}_q^{\alpha,p}};$$

(2) *The nonhomogeneous Herz type Hardy space is defined by*

$$HK_q^{\alpha,p}(R^n) = \{f \in S'(R^n) : G(f) \in K_q^{\alpha,p}(R^n)\},$$

and

$$\|f\|_{HK_q^{\alpha,p}} = \|G(f)\|_{K_q^{\alpha,p}};$$

where $G(f)$ is the grand maximal function of f.

The Herz type Hardy spaces have the atomic decomposition characterization.

Definition 3. *Let $\alpha \in R$, $1 < q < \infty$. A function $a(x)$ on R^n is called a central (α, q)-atom (or a central (α, q)-atom of restrict type), if*

1) $\operatorname{Supp} a \subset B(0, r)$ *for some $r > 0$ (or for some $r \geq 1$),*
2) $\|a\|_{L^q} \leq |B(0, r)|^{-\alpha/n}$,
3) $\int_{R^n} a(x)x^\eta\, dx = 0$ *for $|\eta| \leq [\alpha - n(1 - 1/q)]$.*

Lemma 1 (see [12]). *Let $0 < p < \infty$, $1 < q < \infty$ and $\alpha \geq n(1 - 1/q)$. A temperate distribution f belongs to $H\dot{K}_q^{\alpha,p}(R^n)$ (or $HK_q^{\alpha,p}(R^n)$) if and only if there exist central (α, q)-atoms (or central (α, q)-atoms of restrict type) a_j supported on $B_j = B(0, 2^j)$ and constants λ_j, $\sum_j |\lambda_j|^p < \infty$ such that $f = \sum_{j=-\infty}^{\infty} \lambda_j a_j$ (or $f = \sum_{j=0}^{\infty} \lambda_j a_j$) in the $S'(R^n)$ sense, and*

$$\|f\|_{H\dot{K}_q^{\alpha,p}} (\text{ or } \|f\|_{HK_q^{\alpha,p}}) \sim \left(\sum_j |\lambda_j|^p\right)^{1/p}.$$

Now we can state our results as following.

Theorem 1. *Let $0 < \beta \leq 1$, $0 \leq \delta < n - \beta$, $\max(n/(n + \beta), n/(n + \gamma), n/(n + 1/2)) < p \leq 1$ and $1/p - 1/q = (\delta + \beta)/n$. If $D^\alpha A \in \mathrm{Lip}_\beta(R^n)$ for $|\alpha| = m$. Then μ_λ^A is bounded from $H^p(R^n)$ to $L^q(R^n)$.*

Theorem 2. *Let $0 < \beta < \min(1/2, \gamma)$, $0 \leq \delta < n - \beta$. If $D^\alpha A \in \mathrm{Lip}_\beta(R^n)$ for $|\alpha| = m$. Then $\tilde{\mu}_\lambda^A$ is bounded from $H^{n/(n+\beta)}(R^n)$ to $L^{n/(n-\delta)}(R^n)$.*

Theorem 3. *Let $0 < \beta < \min(1/2, \gamma)$, $0 < \delta < n - \beta$. If $D^\alpha A \in \mathrm{Lip}_\beta(R^n)$ for $|\alpha| = m$. Then μ_λ^A is bounded from $H^{n/(n+\beta)}(R^n)$ to weak $L^{n/(n-\delta)}(R^n)$.*

Theorem 4. *Let $0 < \beta \leq 1$, $0 < \delta < n - \beta$, $0 < p < \infty$, $1 < q_1, q_2 < \infty$, $1/q_1 - 1/q_2 = (\delta + \beta)/n$ and $\min(n(1 - 1/q_1) + \beta, n(1 - 1/q_1) + \gamma, n(1 - 1/q_1) + 1/2)$. If $D^\alpha A \in \mathrm{Lip}_\beta(R^n)$ for $|\alpha| = m$. Then μ_λ^A is bounded from $H\dot{K}_{q_1}^{\alpha,p}(R^n)$ to $\dot{K}_{q_2}^{\alpha,p}(R^n)$.*

Remark 1. Theorem 4 also hold for the nonhomogeneous Herz type Hardy space.

2. Some Lemmas

We begin with some preliminary lemmas.

Lemma 2 (see [4]). *Let A be a function on R^n and $D^\alpha A \in L^q(R^n)$ for $|\alpha| = m$ and some $q > n$. Then*

$$|R_m(A; x, y)| \le C|x-y|^m \sum_{|\alpha|=m} \left(\frac{1}{|\tilde{Q}(x,y)|} \int_{\tilde{Q}(x,y)} |D^\alpha A(z)|^q \, dz \right)^{1/q},$$

where $\tilde{Q}(x, y)$ is the cube centered at x and having side length $5\sqrt{n}|x-y|$.

Lemma 3. *Let $0 < \beta \le 1$, $1 < p < n/(\delta+\beta)$, $1/q = 1/p - (\delta+\beta)/n$ and $D^\alpha A \in Lip_\beta(R^n)$ for $|\alpha| = m$. Then μ_λ^A is bounded from $L^p(R^n)$ to $L^q(R^n)$.*

Proof. By Minkowski inequality and note that $|x-z| \le t(1+2^{k+1}) \le 2^{k+2}t$, $|y-z| \ge |x-z| - 2^{k+3}t$ when $|x-y| \le 2^{k+1}t$ and $|y-z| \le t$, we have

$$\mu_\lambda^A(f)(x) \le \int_{R^n} \Bigg[\iint_{R_+^{n+1}} \left(\frac{t}{t+|x-y|} \right)^{n\lambda}$$

$$\times \left(\frac{|\Omega(y-z)||R_{m+1}(A; x, z)||f(z)|}{|y-z|^{n-\delta-1}|x-z|^m} \right)^2 \chi_{\Gamma(z)}(y,t) \frac{dy\,dt}{t^{n+3}} \Bigg]^{1/2} dz$$

$$\le C \int_{R^n} \frac{|R_{m+1}(A; x, z)||f(z)|}{|x-z|^m} \Bigg[\int_0^\infty \int_{|x-y|\le t} \left(\frac{t}{t+|x-y|} \right)^{n\lambda}$$

$$\times \frac{\chi_{\Gamma(z)}(y,t)}{(|x-z|-t)^{2n-2\delta-2}} \frac{dy\,dt}{t^{n+3}} \Bigg]^{1/2} dz + C \int_{R^n} \frac{|R_{m+1}(A; x, z)||f(z)|}{|x-z|^m}$$

$$\times \Bigg[\int_0^\infty \sum_{k=0}^\infty \int_{\substack{2^k t<|x-y| \\ \le 2^{k+1}t}} \left(\frac{t}{t+|x-y|} \right)^{n\lambda} \frac{\chi_{\Gamma(z)}(y,t) t^{-n-3}\,dy\,dt}{(|x-z|-2^{k+2}t)^{2n-2\delta-2}} \Bigg]^{1/2} dz$$

$$\le C \int_{R_n} \frac{|R_{m+1}(A; x, z)||f(z)|}{|x-z|^{m+1/2}} \Bigg[\int_{|x-z|/2}^\infty \frac{dt}{(|x-z|-3t)^{2n-2\delta}} \Bigg]^{1/2} dz$$

$$+ C \int_{R_n} \frac{|R_{m+1}(A; x, z)||f(z)|}{|x-z|^{m+1/2}} \Bigg[\sum_{k=0}^\infty \int_{2^{-2-k}|x-z|}^\infty 2^{-kn\lambda}(2^k t)^n t^{-n}$$

$$\times \frac{2^k dt}{(|x-z|-2^{k+3}t)^{2n-2\delta}}\Bigg]^{1/2} dz \leq C \int\limits_{R_n} \frac{|R_{m+1}(A;x,z)||f(z)|}{|x-z|^{m+n-\delta}}\, dz$$

$$+ C \int\limits_{R_n} \frac{|R_{m+1}(A;x,z)||f(z)|}{|x-z|^{m+n-\delta}}\, dz \Bigg[\sum_{k=0}^{\infty} 2^{kn(1-\lambda)}\Bigg]^{1/2}$$

$$= C \int\limits_{R_n} \frac{|R_{m+1}(A;x,z)|}{|x-z|^{m+n-\delta}} |f(z)|\, dz.$$

Thus, the lemma follows from [1]. □

3. Proofs of Theorems

3.1. Proof of Theorem 1

It suffices to show that there exists a constant $C > 0$ such that for every H^p-atom a,

$$\|\mu_\lambda^A(a)\|_{L^q} \leq C.$$

Let a be a H^p-atom, that is that a supported on a cube $Q = Q(x_0, r)$, $\|a\|_{L^\infty} \leq |Q|^{-1/p}$ and $\int_{R^n} a(x)x^\eta dx = 0$ for $|\eta| \leq [n(1/p-1)]$. We write

$$\int\limits_{R^n} [\mu_\lambda^A(a)(x)]^q\, dx = \Bigg(\int\limits_{2Q} + \int\limits_{(2Q)^c}\Bigg)[\mu_\lambda^A(a)(x)]^q\, dx = I + II.$$

For I, taking $1 < p_1 < n/(\delta+\beta)$ and q_1 such that $1/p_1 - 1/q_1 = (\delta+\beta)/n$, by Holder's inequality and the (L^{p_1}, L^{q_1})-boundedness of μ_λ^A (see Lemma 3), we see that

$$I \leq C\|\mu_\lambda^A(a)\|_{L^{q_1}}^q |2Q|^{1-q/q_1} \leq C\|a\|_{L^{p_1}}^q |Q|^{1-q/q_1} \leq C.$$

To obtain the estimate of II, we need to estimate $\mu_\lambda^A(a)(x)$ for $x \in (2Q)^c$. Let $\tilde{Q} = 5\sqrt{n}Q$ and $\tilde{A}(x) = A(x) - \sum_{|\alpha|=m} \frac{1}{\alpha!}(D^\alpha A)_{\tilde{Q}} x^\alpha$. Then $R_m(A;x,y) = R_m(\tilde{A};x,y)$ and $D^\alpha \tilde{A}(y) = D^\alpha A(y) - (D^\alpha A)_Q$. We have, by the vanishing moment of a,

$$|F_t^A(a)(x,y)| \leq \int \left|\frac{\Omega(y-z)}{|y-z|^{n-1-\delta}|x-z|^m} - \frac{\Omega(y-x_0)}{|y-x_0|^{n+m-1-\delta}}\right|$$

$$\times \chi_{\Gamma(y)}(z,t)\big|R_m(\tilde{A};x,z)\big||a(z)|\, dz$$

$$+ \int \frac{\chi_{\Gamma(y)}(x_0,t)|\Omega(y-x_0)|}{|y-x_0|^{n+m-1-\delta}}\big|R_m(\tilde{A};x,z) - R_m(\tilde{A};x,x_0)\big||a(z)|\, dz$$

$$+\left|\int(\chi_{\Gamma(y)}(z,t)-\chi_{\Gamma(y)}(x_0,t))\frac{\Omega(y-x_0)R_m(\tilde{A};x,x_0)}{|y-x_0|^{n+m-1-\delta}}a(z)\,dz\right|$$

$$+\sum_{|\alpha|=m}\frac{1}{\alpha!}\int\limits_{\Gamma(y)}\frac{\Omega(y-z)(x-z)^\alpha D^\alpha\tilde{A}(z)}{|y-z|^{n-1-\delta}|x-z|^m}\,a(z)\,dz=II_1+II_2+II_3+II_4.$$

For II_1, by Lemma 2 and the following inequality, for $b\in \mathrm{Lip}_\beta(R^n)$,

$$|b(x)-b_Q|\le\frac{1}{|Q|}\int\limits_Q\|b\|_{\mathrm{Lip}_\beta}|x-y|^\beta dy\le\|b\|_{\mathrm{Lip}_\beta}(|x-x_0|+r)^\beta,$$

we get

$$|R_m(\tilde{A};x,z)|\le\sum_{|\alpha|=m}\|D^\alpha A\|_{\mathrm{Lip}_\beta}(|x-z|+r)^{m+\beta},$$

on the other hand, by the following inequality (see [14]):

$$\left|\frac{\Omega(y-z)}{|y-z|^n}-\frac{\Omega(y-x_0)}{|y-x_0|^n}\right|\le\left(\frac{r}{|y-x_0|^{n+1}}+\frac{r^\gamma}{|x-x_0|^{n+\gamma}}\right)$$

and note that $|x-z|\sim|x-x_0|$ for $z\in Q$ and $x\in R^n\setminus 2Q$, we obtain, similar to the proof of Lemma 3,

$$\left[\iint\limits_{R_+^{n+1}}\left(\frac{t}{t+|x-y|}\right)^{n\lambda}|II_1|^2\,dy\,dt/t^{n+3}\right]^{1/2}$$

$$\le C\int\left(\frac{r}{|x-x_0|^{n+m+1-\delta}}+\frac{r^\gamma}{|x-x_0|^{n+m+\gamma-\delta}}\right)$$
$$\times\sum_{|\alpha|=m}\|D^\alpha A\|_{\mathrm{Lip}_\beta}|x-x_0|^{m+\beta}|a(y)|\,dy$$

$$\le C\sum_{|\alpha|=m}\|D^\alpha A\|_{\mathrm{Lip}\beta}\left(\frac{|Q|^{\beta/n+1-1/p}}{|x-x_0|^{n-\delta}}+\frac{|Q|^{\gamma/n+1-1/p}}{|x-x_0|^{n+\gamma-\delta-\beta}}\right);$$

For II_2, by the following equality (see [4]):

$$R_m(\tilde{A};x,y)-R_m(\tilde{A};x,x_0)=\sum_{|\eta|<m}\frac{1}{\eta!}R_{m-|\eta|}(D^\eta\tilde{A};x_0,y)(x-x_0)^\eta$$

we obtain

$$\left[\iint\limits_{R_+^{n+1}}\left(\frac{t}{t+|x-y|}\right)^{n\lambda}|II_2|^2\,dy\,dt/t^{n+3}\right]^{1/2}$$

$$\leq C\sum_{|\alpha|=m}\|D^\alpha A\|_{\mathrm{Lip}_\beta}\int\bigg(\sum_{|\eta|<m}\frac{|y-x_0|^{m+\beta-|\eta|}}{|x-x_0|^{n+m-|\eta|-\delta}}\bigg)|a(y)|\,dy$$

$$\leq C\sum_{|\alpha|=m}\|D^\alpha A\|_{\mathrm{Lip}_\beta}\frac{|Q|^{\beta/n+1-1/p}}{|x-x_0|^{n-\delta}};$$

For II_3, we obtain, similar to the estimates of Lemma 3 and II_1,

$$\bigg[\iint_{R^{n+1}_+}\Big(\frac{t}{t+|x-y|}\Big)^{n\lambda}|II_3|^2\,dy\,dt/t^{n+3}\bigg]^{1/2}$$

$$\leq C\int_{R^n}\bigg(\iint_{R^{n+1}_+}\Big(\frac{t}{t+|x-y|}\Big)^{n\lambda}\bigg[\frac{\big|R_m(\tilde{A};x,x_0)\big||a(z)|}{|y-x_0|^{n+m-1-\delta}}$$

$$\times\big(\chi_{\Gamma(y)}(z,t)-\chi_{\Gamma(y)}(x_0,t)\big)\bigg]^2dy\,dt/t^{n+3}\bigg)^{1/2}dz$$

$$\leq C\int\big|R_m(\tilde{A};x,x_0)\big||a(z)|\bigg|\iint_{R^{n+1}_+}\Big(\frac{t}{t+|x-y|}\Big)^{n\lambda}\frac{t^{-n-3}\chi_{\Gamma(y)}(z,t)\,dy\,dt}{|y-x_0|^{2n+2m-2-2\delta}}$$

$$-\iint_{R^{n+1}_+}\Big(\frac{t}{t+|x-y|}\Big)^{n\lambda}\frac{t^{-n-3}\chi_{\Gamma(y)}(x_0,t)\,dy\,dt}{|y-x_0|^{2n+2m-2-2\delta}}\bigg|^{1/2}dz$$

$$\leq C\int\big|R_m(\tilde{A};x,x_0)\big||a(z)|\bigg|\frac{1}{|x-x_0|^{2n+2m-2\delta}}-\frac{1}{|x-z|^{2n+2m-2\delta}}\bigg|^{1/2}dz$$

$$\leq C\sum_{|\alpha|=m}\|D^\alpha A\|_{\mathrm{Lip}_\beta}\frac{|Q|^{1+1/(2n)-1/p}}{|x-x_0|^{n+1/2-\delta-\beta}};$$

Similarly,

$$\bigg[\iint_{R^{n+1}_+}\Big(\frac{t}{t+|x-y|}\Big)^{n\lambda}|II_4|^2\,dy\,dt/t^{n+3}\bigg]^{1/2}\leq C\sum_{|\alpha|=m}\|D^\alpha A\|_{\mathrm{Lip}\beta}$$

$$\times\bigg(\frac{|Q|^{\beta/n+1-1/p}}{|x-x_0|^{n-\delta}}+\frac{|Q|^{1+\gamma/n-1/p}}{|x-x_0|^{n+\gamma-\delta-\beta}}+\frac{|Q|^{1+1/(2n)-1/p}}{|x-x_0|^{n+1/2-\delta-\beta}}\bigg).$$

Thus

$$II\leq\sum_{k=1}^{\infty}\int_{2^{k+1}Q\setminus 2^kQ}[\mu^A_\lambda(a)(x)]^q\,dx\leq C\bigg(\sum_{|\alpha|=m}\|D^\alpha A\|_{\mathrm{Lip}_\beta}\bigg)^q$$

$$\times \sum_{k=1}^{\infty} \left[2^{kqn(1/p-(n+\beta)/n)} + 2^{kqn(1/p-(n+\gamma)/n)} + 2^{kqn(1/p-(n+1/2)/n)}\right]$$
$$\leq C\Big(\sum_{|\alpha|=m} \|D^\alpha A\|_{\mathrm{Lip}_\beta}\Big)^q,$$

which together with the estimate for I yields the desired result. This finishes the proof of Theorem 1.

3.2. Proof of Theorem 2

It suffices to show that there exists a constant $C > 0$ such that for every $H^{n/(n+\beta)}$-atom a supported on $Q = Q(x_0, r)$, we have

$$\|\tilde{\mu}_\lambda^A(a)\|_{L^{n/(n-\delta)}} \leq C.$$

We write

$$\int_{R^n} [\tilde{\mu}_\lambda^A(a)(x)]^{n/(n-\delta)} dx = \left[\int_{2Q} + \int_{(2Q)^c}\right] [\tilde{\mu}_\lambda^A(a)(x)]^{n/(n-\delta)} dx := J + JJ.$$

For J, by the following equality

$$Q_{m+1}(A; x, y) = R_{m+1}(A; x, y) - \sum_{|\alpha|=m} \frac{1}{\alpha!}(x-y)^\alpha (D^\alpha A(x) - D^\alpha A(y)),$$

we have, similar to the proof of Lemma 3,

$$\tilde{\mu}_\lambda^A(a)(x) \leq \mu_\lambda^A(a)(x) + C \sum_{|\alpha|=m} \int_{R^n} \frac{|D^\alpha A(x) - D^\alpha A(y)|}{|x-y|^{n-\delta}} |a(y)|\, dy,$$

thus, $\tilde{\mu}_\lambda^A$ is (L^p, L^q)-bounded by Lemma 3 and [8], where $1 < p < n/(\delta + \beta)$ and $1/q = 1/p - (\delta+\beta)/n$. We see that

$$J \leq C\|\tilde{\mu}_\lambda^A(a)\|_{L^q}^{n/(n-\delta)} |2Q|^{1-n/((n-\delta)q)} \leq C\|a\|_{L^p}^{n/(n-\delta)} |Q|^{1-n/((n-\delta)q)} \leq C.$$

To obtain the estimate of JJ, set $\tilde{A}(x) = A(x) - \sum_{|\alpha|=m} \frac{1}{\alpha!}(D^\alpha A)_{2Q} x^\alpha$. Then $Q_m(A; x, z) = Q_m(\tilde{A}; x, z)$. We write, by the vanishing moment of a and $Q_{m+1}(A; x, z) = R_m(A; x, z) - \sum_{|\alpha|=m} \frac{1}{\alpha!}(x-z)^\alpha D^\alpha A(x)$, for $x \in (2Q)^c$,

$$\tilde{F}_t^A(a)(x, y) = \int_{\Gamma(y)} \frac{\Omega(y-z) R_m(\tilde{A}; x, z)}{|y-z|^{n-1-\delta} |x-z|^m}\, a(z)\, dz$$

$$- \sum_{|\alpha|=m} \frac{1}{\alpha!} \int\limits_{\Gamma(y)} \frac{\Omega(y-z) D^\alpha \tilde{A}(x)(x-z)^\alpha}{|y-z|^{n-1-\delta}|x-z|^m} a(z)\,dz$$

$$= \int \left[\frac{\chi_{\Gamma(y)}(z,t)\Omega(y-z)R_m(\tilde{A};x,z)}{|y-z|^{n-1-\delta}|x-z|^m} - \frac{\chi_{\Gamma(y)}(x_0,t)\Omega(y-x_0)R_m(\tilde{A};x,x_0)}{|x-x_0|^{n+m-1-\delta}} \right] a(z)\,dz$$

$$- \sum_{|\alpha|=m} \frac{1}{\alpha!} \int \left[\frac{\chi_{\Gamma(y)}(z,t)\Omega(y-z)(x-z)^\alpha}{|y-z|^{n-1-\delta}|x-z|^m} - \frac{\chi_{\Gamma(y)}(x_0,t)\Omega(y-x_0)(x-x_0)^\alpha}{|x-x_0|^m} \right] D^\alpha \tilde{A}(x)\, a(z)\,dz,$$

thus, similar to the proof of Theorem 1, we obtain, for $x \in (2Q)^c$

$$|\tilde{\mu}_\lambda^A(a)(x)| \le C|Q|^{-\beta/n} \sum_{|\alpha|=m} \Bigg[\|D^\alpha A\|_{\mathrm{Lip}_\beta} \left(\frac{|Q|^{1/n}}{|x-x_0|^{n+1-\delta-\beta}} + \frac{|Q|^{1/(2n)}}{|x-x_0|^{n+1/2-\delta-\beta}} + \frac{|Q|^{\gamma/n}}{|x-x_0|^{n+\gamma-\delta-\beta}} \right) + |D^\alpha \tilde{A}(x)| \left(\frac{|Q|^{1/n}}{|x-x_0|^{n+1-\delta}} + \frac{|Q|^{1/(2n)}}{|x-x_0|^{n+1/2-\delta-\beta}} + \frac{|Q|^{\gamma/n}}{|x-x_0|^{n+\gamma-\delta}} \right) \Bigg],$$

so that,

$$JJ \le C \left(\sum_{|\alpha|=m} \|D^\alpha A\|_{\mathrm{Lip}_\beta} \right)^{n/(n-\delta)} \sum_{k=1}^\infty \left[2^{kn(\beta-1)/(n-\delta)} + 2^{kn(\beta-1/2)/(n-\delta)} + 2^{kn(\beta-\gamma)/(n-\delta)} \right] \le C,$$

which together with the estimate for J yields the desired result. This finishes the proof of Theorem 2.

3.3. Proof of Theorem 3

By the following equality

$$R_{m+1}(A;x,y) = Q_{m+1}(A;x,y) + \sum_{|\alpha|=m} \frac{1}{\alpha!}(x-y)^\alpha (D^\alpha A(x) - D^\alpha A(y))$$

and similar to the proof of Lemma 3, we get

$$\mu_\lambda^A(f)(x) \le \tilde{\mu}_\lambda^A(f)(x) + C \sum_{|\alpha|=m} \int\limits_{R^n} \frac{|D^\alpha A(x) - D^\alpha A(y)|}{|x-y|^{n-\delta}} |f(y)|\,dy,$$

from Theorem 1 and 2 with [8], we obtain

$$|\{x \in R^n : \mu_\lambda^A(f)(x) > \eta\}| \le |\{x \in R^n : \tilde{\mu}_\lambda^A(f)(x) > \eta/2\}|$$
$$+ \left|\left\{x \in R^n : \sum_{|\alpha|=m} \int_{R^n} \frac{|D^\alpha A(x) - D^\alpha A(y)|}{|x-y|^{n-\delta}} |f(y)|\, dy > C\eta\right\}\right|$$
$$\le C(\eta^{-1}\|f\|_{H^{n/(n+\beta)}})^{n/(n-\delta)}.$$

This completes the proof of Theorem 3.

3.4. Proof of Theorem 4

Let $f \in H\dot{K}_{q_1}^{\alpha,p}(R^n)$ and $f(x) = \sum_{j=-\infty}^{\infty} \lambda_j a_j(x)$ be the atomic decomposition for f as in Lemma 1. We write

$$\|\mu_\lambda^A(f)\|_{\dot{K}_q^{\alpha,p}}^p \le \sum_{k=-\infty}^{\infty} 2^{k\alpha p}\Big(\sum_{j=-\infty}^{k-3} |\lambda_j| \|\mu_\lambda^A(a_j)\chi_k\|_{L^{q_2}}\Big)^p$$
$$+ \sum_{k=-\infty}^{\infty} 2^{k\alpha p}\Big(\sum_{j=k-2}^{\infty} |\lambda_j| \|\mu_\lambda^A(a_j)\chi_k\|_{L^{q_2}}\Big)^p = L_1 + L_2.$$

For L_2, by the (L^{q_1}, L^{q_2}) boundedness of μ_λ^A(see Lemma 3), we have

$$\begin{aligned}
L_2 &\le C \sum_{k=-\infty}^{\infty} 2^{k\alpha p}\Big(\sum_{j=k-2}^{\infty} |\lambda_j| \|a_j\|_{L^{q_1}}\Big)^p \\
&\le C \sum_{j=-\infty}^{\infty} |\lambda_j|^p \Big(\sum_{k=-\infty}^{j+2} 2^{(k-j)\alpha p}\Big), \quad 0 < p \le 1 \\
\text{(or)} \quad &\le C \sum_{j=-\infty}^{\infty} |\lambda_j|^p \Big(\sum_{k=-\infty}^{j+2} 2^{(k-j)\alpha p/2}\Big)\Big(\sum_{k=-\infty}^{j+2} 2^{(k-j)\alpha p'/2}\Big)^{p/p'}, \ p>1 \\
&\le C \sum_{j=-\infty}^{\infty} |\lambda_j|^p \\
&\le C\|f\|_{H\dot{K}_{q_1}^{\alpha,p}}^p.
\end{aligned}$$

For L_1, similar to the proof of Theorem 1, we have, for $x \in C_k$, $j \le k-3$,

$$\mu_\lambda^A(a_j)(x) \le C\Big(\frac{|B_j|^{\beta/n}}{|x|^{n-\delta}} + \frac{|B_j|^{1/(2n)}}{|x|^{n+1/2-\delta-\beta}} + \frac{|B_j|^{\gamma/n}}{|x|^{n+\gamma-\delta-\beta}}\Big) \int |a_j(y)|\, dy$$
$$\le C\big(2^{j(\beta+n(1-1/q_1)-\alpha)}|x|^{\delta-n} + 2^{j(1/2+n(1-1/q_1)-\alpha)}|x|^{\delta+\beta-n-1/2}$$
$$+ 2^{j(\gamma+n(1-1/q_1)-\alpha)}|x|^{\delta+\beta-n-\gamma}\big),$$

thus

$$\|\mu_\lambda^A(a_j)\chi_k\|_{L^{q_2}} \le C2^{-k\alpha}\big(2^{(j-k)(\beta+n(1-1/q_1)-\alpha)} + 2^{(j-k)(1/2+n(1-1/q_1)-\alpha)} + 2^{(j-k)(\gamma+n(1-1/q_1)-\alpha)}\big),$$

and

$$\begin{aligned}
L_1 &\le C\sum_{k=-\infty}^{\infty}\bigg(\sum_{j=-\infty}^{k-3}|\lambda_j|\big(2^{(j-k)(\beta+n(1-1/q_1)-\alpha)} \\
&\qquad + 2^{(j-k)(1/2+n(1-1/q_1)-\alpha)} + 2^{(j-k)(\gamma+n(1-1/q_1)-\alpha)}\big)\bigg)^p \\
&\le C\sum_{j=-\infty}^{\infty}|\lambda_j|^p\sum_{k=j+3}^{\infty}\big(2^{(j-k)(\beta+n(1-1/q_1)-\alpha)} \\
&\qquad + 2^{(j-k)(1/2+n(1-1/q_1)-\alpha)} + 2^{(j-k)(\gamma+n(1-1/q_1)-\alpha)}\big)^p,\ 0<p\le 1 \\
\text{(or)}\ &\le C\sum_{j=-\infty}^{\infty}|\lambda_j|^p\sum_{k=j+3}^{\infty}\big(2^{(j-k)p(\beta+n(1-1/q_1)-\alpha)/2} \\
&\qquad + 2^{(j-k)p(1/2+n(1-1/q_1)-\alpha)/2} + 2^{(j-k)p(\gamma+n(1-1/q_1)-\alpha)/2}\big),\ p>1 \\
&\le C\sum_{j=-\infty}^{\infty}|\lambda_j|^p \\
&\le C\|f\|^p_{H\dot{K}^{\alpha,p}_{q_1}}.
\end{aligned}$$

This finishes the proof of Theorem 4.

4. Examples

Now we give two examples related to the Marcinkiewicz operators.

Fixed $0 \le \delta < n$, $\lambda > 1$ and $0 < \gamma \le 1$. Let Ω be homogeneous of degree zero on R^n with $\int_{S^{n-1}}\Omega(x')\,d\sigma(x') = 0$. Assume that $\Omega \in \mathrm{Lip}_\gamma(S^{n-1})$. The multilinear Marcinkiewicz Ω-operators and S-operators are defined by (see [12, 14])

$$\mu_\Omega^A(f)(x) = \left(\int_0^\infty |F_t^A(f)(x)|^2\,\frac{dt}{t^3}\right)^{1/2}$$

and

$$\mu_S^A(f)(x) = \left[\iint_{\Gamma(x)} |F_t^A(f)(x,y)|^2\,\frac{dy\,dt}{t^{n+3}}\right]^{1/2},$$

where

$$F_t^A(f)(x) = \int\limits_{|x-y|\le t} \frac{\prod_{j=1}^{l} R_{m_j+1}(A_j; x, y)}{|x-y|^m} \frac{\Omega(x-y)}{|x-y|^{n-1-\delta}} f(y)\,dy$$

and

$$F_t^A(f)(x,y) = \int\limits_{|y-z|\le t} \frac{\prod_{j=1}^{l} R_{m_j+1}(A_j; y, z)}{|y-z|^m} \frac{\Omega(y-z)}{|y-z|^{n-\delta-1}} f(z)\,dz;$$

The variants of μ_Ω^A and μ_S^A are defined by

$$\tilde{\mu}_\Omega^A(f)(x) = \left(\int\limits_0^\infty |\tilde{F}_t^A(f)(x)|^2 \frac{dt}{t^3} \right)^{1/2}$$

and

$$\tilde{\mu}_S^A(f)(x) = \left[\iint\limits_{\Gamma(x)} |\tilde{F}_t^A(f)(x,y)|^2 \frac{dy\,dt}{t^{n+3}} \right]^{1/2},$$

where

$$\tilde{F}_t^A(f)(x) = \int\limits_{|x-y|\le t} \frac{\prod_{j=1}^{l} Q_{m_j+1}(A_j; x, y)}{|x-y|^m} \frac{\Omega(x-y)}{|x-y|^{n-1-\delta}} f(y)\,dy$$

and

$$\tilde{F}_t^A(f)(x,y) = \int\limits_{|y-z|\le t} \frac{\prod_{j=1}^{l} Q_{m_j+1}(A_j; y, z)}{|y-z|^m} \frac{\Omega(y-z)}{|y-z|^{n-\delta-1}} f(z)\,dz.$$

Set

$$F_t(f)(x) = \int\limits_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1-\delta}} f(y)\,dy;$$

We also define that

$$\mu_\Omega(f)(x) = \left(\int\limits_0^\infty |F_t(f)(x)|^2 \frac{dt}{t^3} \right)^{1/2}$$

and

$$\mu_S(f)(x) = \left(\iint\limits_{\Gamma(x)} |F_t(f)(y)|^2 \frac{dy\,dt}{t^{n+3}} \right)^{1/2},$$

which are the Marcinkiewicz operators (see [14]). Let H be the space

$$H = \left\{ h : \|h\| = \left(\int_0^\infty |h(t)|^2 \, dt/t^3 \right)^{1/2} < \infty \right\}$$

or

$$H = \left\{ h : \|h\| = \left(\iint_{R_+^{n+1}} |h(t)|^2 \, dy \, dt/t^{n+3} \right)^{1/2} < \infty \right\}.$$

Then, it is clear that

$$\mu_\Omega^A(f)(x) = \left\| F_t^A(f)(x) \right\|, \quad \mu_\Omega(f)(x) = \left\| F_t(f)(x) \right\|$$

and

$$\mu_S^A(f)(x) = \left\| \chi_{\Gamma(x)} F_t^A(f)(x,y) \right\|, \quad \mu_S(f)(x) = \left\| \chi_{\Gamma(x)} F_t(f)(y) \right\|.$$

It is easily to see that μ_Ω and μ_S satisfy the conditions of Theorem 1, 2, 3 and 4, thus Theorem 1, 2, 3 and 4 hold for μ_Ω^A and $\tilde{\mu}_\Omega^A$, μ_S^A and $\tilde{\mu}_S^A$.

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