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Some endpoint inequalities for multilinear integral operators

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Abstract. In this paper, the endpoint estimates for some multilinear operators related to certain fractional singular integral operators are obtained. The operators include Calderón–Zygmund singular integral operator and fractional integral operator.

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1. Introduction

Let T be the Calderón–Zygmund singular integral operator, the classical result by Coifman, Rochberg and Weiss (see [6]) states that the commutator $[b, T](f) = T(bf) - bT(f)$ (where $b \in BMO(R^n)$) is bounded on $L^p(R^n)$ for $1 < p < \infty$; Chanillo (see [1]) has proved a similar result when T was replaced by the fractional integral operator; in [9], the endpoint boundedness of the commutators was obtained. The main purpose of this paper is to establish the endpoint boundedness of some multilinear operators related to certain non-convolution type fractional singular integral operators. As an application, the endpoint boundedness of the multilinear operators related to the Calderón–Zygmund singular integral operator and fractional integral operator is obtained.

2. Notations and results

Throughout this paper, Q will denote a cube of R^n with sides parallel to the axes. For a cube Q and a locally integrable function f, let $f_Q = |Q|^{-1} \int_Q f(x)\, dx$ and $f^{\#}(x) = \sup_{x \in Q} |Q|^{-1} \int_Q |f(y) - f_Q|\, dy$. For

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a weight function w, f is said to belong to $BMO(w)$ if $f^{\#} \in L^{\infty}(w)$. Set $\|f\|_{BMO(w)} = \|f^{\#}\|_{L^{\infty}(w)}$. Note that $BMO(w) = BMO(R^n)$ if $w = 1$. A function a is called an H^1 atom if there exists a cube Q such that a is supported in Q, $\|a\|_{L^{\infty}(w)} \le w(Q)^{-1}$ and $\int a(x)dx = 0$. It is well known that the Hardy space $H^1(w)$ has the atomic decomposition characterization (see [8, 12]).

In this paper, we consider a class of multilinear integral operators defined in the following way.

First, given a fixed locally integrable function $K(x, y)$ on $R^n \times R^n$, set

$$T_K(f)(x) = \int_{R^n} K(x, y)f(y)\, dy$$

for every bounded and compactly supported function f. We write $K \in \Sigma_\delta$ for $\delta \ge 0$ if

$$|K(x, y)| \le C|x - y|^{-n+\delta}$$

and

$$|K(y, x) - K(z, x)| + |K(x, y) - K(x, z)| \le C|y - z|^{\varepsilon}|x - z|^{-n-\varepsilon+\delta}$$

and $2|y - z| \le |x - z|$ for a fixed $\varepsilon > 0$. T_K is called a fractional singular integral operator if $K \in \Sigma_\delta$ for some $\delta \ge 0$.

Now, let m be a positive integer and A be a function on R^n. Set

$$R_{m+1}(A; x, y) = A(x) - \sum_{|\alpha| \le m} \frac{1}{\alpha!} D^{\alpha}A(y)(x - y)^{\alpha},$$

and

$$Q_{m+1}(A; x, y) = R_m(A; x, y) - \sum_{|\alpha| = m} \frac{1}{\alpha!} D^{\alpha}A(x)(x - y)^{\alpha}.$$

The multilinear operator associated with the fractional singular integral operator T_K is defined by

$$T_K^A(f)(x) = \int \frac{R_{m+1}(A; x, y)}{|x - y|^m} K(x, y)f(y)\, dy.$$

We also consider the variant of T_K^A, which is defined by

$$\tilde{T}_K^A(f)(x) = \int_{R^n} \frac{Q_{m+1}(A; x, y)}{|x - y|^m} K(x, y)f(y)\, dy.$$

Note that $\tilde{T}_K^A$ is closely related to T_K^A, for

$$R_{m+1}(A;x,y) - Q_{m+1}(A;x,y) = \sum_{|\alpha|=m} \frac{1}{\alpha!}(x-y)^\alpha (D^\alpha A(x) - D^\alpha A(y)).$$

Note that when $m = 0$, T_K^A is just the commutators of T_K and A(see [1, 6, 9]). It is well known that multilinear operator, as an extension of commutator, is of great interest in harmonic analysis and has been widely studied by many authors(see, e.g. [2–5]). In [7] and [10], the weighted L^p $(p > 1)$ and $H^p(0 < p \le 1)$ boundedness of the multilinear operator related to the Calderón–Zygmund singular integral operator was obtained; in [2], the weak $(H^1,\ L^1)$ boundedness of the multilinear operator related to some singular integral operator was obtained.

Now we state our results as following.

Theorem 2.1. *Let $0 \le \delta < n$ and $D^\alpha A \in BMO(R^n)$ for all α with $|\alpha| = m$. Suppose T_K is bounded from $L^p(R^n)$ to $L^q(R^n)$ for any $p, q \in (1, +\infty)$ and $1/q = 1/p - \delta/n$. If $K \in \Sigma_\delta$, then*

(a) T_K^A is bounded from $L^{n/\delta}(R^n)$ to $BMO(R^n)$;

(b) $\tilde{T}_K^A$ is bounded from $H^1(R^n)$ to $L^{n/(n-\delta)}(R^n)$;

(c) T_K^A is bounded from $H^1(R^n)$ to weak $L^{n/(n-\delta)}(R^n)$.

Theorem 2.2. *Let $D^\alpha A \in BMO(R^n)$ for all α with $|\alpha| = m$ and $w \in A_1$. Suppose T_K is bounded on $L^p(w)$ for all $1 < p \le \infty$. If $K \in \Sigma_0$, then*

(i) T_K^A is bounded from $L^\infty(w)$ to $BMO(w)$;

(ii) $\tilde{T}_K^A$ is bounded from $H^1(w)$ to $L^1(w)$;

(iii) T_K^A is bounded from $H^1(w)$ to weak $L^1(w)$.

Remark 2.1. The boundedness is uniform with respect to $K \in \Sigma_\delta$ and $K \in \Sigma_0$, respectively. In general, T_K^A is not $(H^1, L^{n/(n-\delta)})$ or $(H^1(w), L^1(w))$ bounded.

3. Proofs of the theorems

To prove these theorems, we need the following lemmas.

Lemma 3.1 (see [5, p. 448]). *Let A be a function on R^n and $D^\alpha A \in L^q(R^n)$ for $|\alpha| = m$ and some $q > n$. Then*

$$|R_m(A;x,y)| \le C|x-y|^m \sum_{|\alpha|=m} \left(\frac{1}{|\tilde{Q}(x,y)|} \int\limits_{\tilde{Q}(x,y)} |D^\alpha A(z)|^q \, dz \right)^{1/q},$$

where $\tilde{Q}(x,y)$ is the cube centered at x and having side length $5\sqrt{n}|x-y|$.

Lemma 3.2 (see [1, p. 8]). *Let $b \in BMO(R^n)$ and C_b be the commutator defined by*

$$C_b(f)(x) = \int\limits_{R^n} \frac{b(x)-b(y)}{|x-y|^{n-\delta}} f(y)\, dy.$$

(1) *If $0 \le \delta < n$, $1 < p < \infty$ and $1/q = 1/p - \delta/n$, then C_b is bounded from $L^p(R^n)$ to $L^q(R^n)$ and from $H^1(R^n)$ to weak $L^{n/(n-\delta)}(R^n)$.*

(2) *If $\delta = 0$, $1 < p < \infty$ and $w \in A_1$, then C_b is bounded on $L^p(w)$ and from $H^1(w)$ to weak $L^1(w)$.*

Lemma 3.3 (see [5, p. 454(28)] and [12, p. 222]). *Let Q be a cube and $\tilde{A}(x) = A(x) - \sum_{|\alpha|=m} \frac{1}{\alpha!}(D^\alpha A)_{\tilde{Q}} x^\alpha$. Then $R_{m+1}(A;x,y) = R_{m+1}(\tilde{A};x,y)$.*

Lemma 3.4 (see [3, p. 695, Lemma 2.2]). *Let Q_1 and Q_2 be the cubes with $Q_1 \subset Q_2$. Then*

$$|b_{Q_1} - b_{Q_2}| \le C\left(1 + |\log(|Q_1|/|Q_2|)|\right) \|b\|_{BMO}.$$

Proof of Theorem 2.1. (a) It suffices to prove that there exists a constant C_Q such that

$$\frac{1}{|Q|} \int\limits_Q |T_K^A(f)(x) - C_Q| \, dx \le C\|f\|_{L^{n/\delta}}$$

holds for any cube Q. Fix a cube $Q = Q(x_0, d)$. Let $\tilde{Q} = 5\sqrt{n}Q$ and $\tilde{A}(x) = A(x) - \sum_{|\alpha|=m} \frac{1}{\alpha!}(D^\alpha A)_{\tilde{Q}} x^\alpha$, then $R_{m+1}(A;x,y) = R_{m+1}(\tilde{A};x,y)$ by induction and $D^\alpha \tilde{A} = D^\alpha A - (D^\alpha A)_{\tilde{Q}}$ for all α with $|\alpha| = m$. We write, for $f_1 = f\chi_{\tilde{Q}}$ and $f_2 = f\chi_{R^n \setminus \tilde{Q}}$,

$$T_K^A(f)(x) = \int\limits_{R^n} \frac{R_{m+1}(\tilde{A};x,y)}{|x-y|^m} K(x,y)f(y)\,dy$$
$$= \int\limits_{R^n} \frac{R_m(\tilde{A};x,y)}{|x-y|^m} K(x,y)f_1(y)\,dy$$
$$- \sum_{|\alpha|=m} \frac{1}{\alpha!} \int\limits_{R^n} \frac{K(x,y)(x-y)^\alpha}{|x-y|^m} D^\alpha \tilde{A}(y) f_1(y)\,dy$$
$$+ \int\limits_{R^n} \frac{R_{m+1}(\tilde{A};x,y)}{|x-y|^m} K(x,y) f_2(y)\,dy,$$

then

$$\left|T_K^A(f)(x) - T_K^{\tilde{A}}(f_2)(x_0)\right| \le \left|T_K\left(\frac{R_m(\tilde{A};x,\cdot)}{|x-\cdot|^m} f_1\right)(x)\right|$$
$$+ \sum_{|\alpha|=m} \frac{1}{\alpha!} \left|T_K\left(\frac{(x-\cdot)^\alpha}{|x-\cdot|^m} D^\alpha \tilde{A} f_1\right)(x)\right| + \left|T_K^{\tilde{A}}(f_2)(x) - T_K^{\tilde{A}}(f_2)(x_0)\right|$$
$$:= I(x) + II(x) + III(x),$$

and, thus,

$$\frac{1}{|Q|} \int\limits_Q \left|T_K^A(f)(x) - T_K^{\tilde{A}}(f_2)(x_0)\right| dx$$
$$\le \frac{1}{|Q|} \int\limits_Q I(x)\,dx + \frac{1}{|Q|} \int\limits_Q II(x)\,dx + \frac{1}{|Q|} \int\limits_Q III(x)\,dx$$
$$:= I + II + III.$$

Now, let us estimate I, II and III, respectively. First, we have known (see [12, p. 144]), for $b \in BMO(R^n)$,

$$\|b\|_{BMO} \approx \sup_Q \left(\frac{1}{|Q|} \int\limits_Q |b(y) - b_Q|^p\,dy\right)^{1/p},$$

then, for $x \in Q$ and $y \in \tilde{Q}$, using Lemma 3.1 and Lemma 3.4, we get

$$R_m(\tilde{A};x,y) \le C|x-y|^m \sum_{|\alpha|=m} \left[\frac{1}{|\tilde{Q}(x,y)|} \int\limits_{\tilde{Q}(x,y)} \left(|D^\alpha A(z) - (D^\alpha A)_{\tilde{Q}(x,y)}|\right.\right.$$
$$\left.\left. + |(D^\alpha A)_{\tilde{Q}(x,y)} - (D^\alpha A)_{\tilde{Q}}|\right)^q dz\right]^{1/q}$$

$$\leq C|x-y|^m \sum_{|\alpha|=m} \left(\|D^\alpha A\|_{BMO} + 1 + \left|\log |Q(x,y)|/|\tilde{Q}|\right|\right)$$
$$\leq C|x-y|^m \sum_{|\alpha|=m} \|D^\alpha A\|_{BMO},$$

thus, by the $(L^{n/\delta}, L^\infty)$-boundedness of T_K, we have

$$I \leq \frac{C}{|Q|} \int_Q \left| T_\delta\Big(\sum_{|\alpha|=m} \|D^\alpha A\|_{BMO} f_1 \Big)(x) \right| dx$$
$$\leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{BMO} \|T_\delta(f_1)\|_{L^\infty}$$
$$\leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{BMO} \|f\|_{L^{n/\delta}};$$

Secondly, by the (L^p, L^q)-boundedness of T_K for $1/q = 1/p - \delta/n$, $p > 1$ and Hölder's inequality, we gain

$$II \leq \frac{C}{|Q|} \int_Q |T_\delta\Big(\sum_{|\alpha|=m} (D^\alpha A - (D^\alpha A)_{\tilde{Q}}) f_1 \Big)(x)|\, dx$$
$$\leq C \sum_{|\alpha|=m} \left(\frac{1}{|Q|} \int_Q |T_\delta((D^\alpha A - (D^\alpha A)_{\tilde{Q}}) f_1)(x)|^q \, dx \right)^{1/q}$$
$$\leq C|Q|^{-1/q} \sum_{|\alpha|=m} \|(D^\alpha A - (D^\alpha A)_{\tilde{Q}}) f_1\|_{L^p}$$
$$\leq C \sum_{|\alpha|=m} \left(\frac{1}{|Q|} \int_{\tilde{Q}} |D^\alpha A(y) - (D^\alpha A)_{\tilde{Q}}|^q \, dy \right)^{1/q} \|f\|_{L^{n/\delta}}$$
$$\leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{BMO} \|f\|_{L^{n/\delta}}.$$

To estimate III, we write

$$\begin{aligned} &T_K^{\tilde{A}}(f_2)(x) - T_K^{\tilde{A}}(f_2)(x_0) \\ &\quad = \int_{R^n} \left[\frac{K(x,y)}{|x-y|^m} - \frac{K(x_0,y)}{|x_0-y|^m} \right] R_m(\tilde{A}; x, y) f_2(y)\, dy \\ &\quad + \int_{R^n} \frac{K(x_0,y) f_2(y)}{|x_0-y|^m} \left[R_m(\tilde{A}; x, y) - R_m(\tilde{A}; x_0, y) \right] dy \end{aligned}$$

$$- \sum_{|\alpha|=m} \frac{1}{\alpha!} \int_{R^n} \left(\frac{K(x,y)(x-y)^\alpha}{|x-y|^m} - \frac{K(x_0,y)(x_0-y)^\alpha}{|x_0-y|^m} \right) D^\alpha \tilde{A}(y) f_2(y)\, dy$$

$$:= III_1 + III_2 + III_3;$$

By Lemma 3.1 and Lemma 3.4, we know that, for $x \in Q$ and $y \in 2^{k+1}\tilde{Q} \setminus 2^k \tilde{Q}$,

$$|R_m(\tilde{A}; x, y)| \le C|x-y|^m \sum_{|\alpha|=m} (\|D^\alpha A\|_{BMO} + |(D^\alpha A)_{\tilde{Q}(x,y)} - (D^\alpha A)_{\tilde{Q}}|)$$
$$\le Ck|x-y|^m \sum_{|\alpha|=m} \|D^\alpha A\|_{BMO}.$$

Note that $|x-y| \sim |x_0 - y|$ for $x \in Q$ and $y \in R^n \setminus \tilde{Q}$, we obtain, by the condition on K,

$$|III_1| \le C \int_{R^n} \left(\frac{|x-x_0|}{|x_0-y|^{m+n+1-\delta}} + \frac{|x-x_0|^\varepsilon}{|x_0-y|^{m+n+\varepsilon-\delta}} \right) |R_m(\tilde{A}; x, y)| |f_2(y)|\, dy$$
$$\le C \sum_{|\alpha|=m} \|D^\alpha A\|_{BMO} \sum_{k=0}^{\infty} \int_{2^{k+1}\tilde{Q} \setminus 2^k \tilde{Q}} k \left(\frac{|x-x_0|}{|x_0-y|^{n+1-\delta}} + \frac{|x-x_0|^\varepsilon}{|x_0-y|^{n+\varepsilon-\delta}} \right) |f(y)|\, dy$$
$$\le C \sum_{|\alpha|=m} \|D^\alpha A\|_{BMO} \|f\|_{L^{n/\delta}} \sum_{k=1}^{\infty} k(2^{-k} + 2^{-\varepsilon k})$$
$$\le C \sum_{|\alpha|=m} \|D^\alpha A\|_{BMO} \|f\|_{L^{n/\delta}};$$

For III_2, by the formula (see (39) in [5]):

$$R_m(\tilde{A}; x, y) - R_m(\tilde{A}; x_0, y) = \sum_{|\beta|<m} \frac{1}{\beta!} R_{m-|\beta|}(D^\beta \tilde{A}; x, x_0)(x-y)^\beta$$

and Lemma 3.1, we have

$$|R_m(\tilde{A}; x, y) - R_m(\tilde{A}; x_0, y)|$$
$$\le C \sum_{|\beta|<m} \sum_{|\alpha|=m} |x-x_0|^{m-|\beta|} |x-y|^{|\beta|} \|D^\alpha A\|_{BMO},$$

similar to the estimates of III_1, we get

$$|III_2| \le C \sum_{|\alpha|=m} \|D^\alpha A\|_{BMO} \sum_{k=0}^{\infty} \int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}} \frac{|x-x_0|}{|x_0-y|^{n+1-\delta}} |f(y)|\,dy$$

$$\le C\|D^\alpha A\|_{BMO}\|f\|_{L^{n/\delta}};$$

For III_3, by taking $r>1$ such that $1/r+\delta/n=1$, similar to the estimates of III_1, we get

$$|III_3| \le C \sum_{|\alpha|=m} \sum_{k=0}^{\infty} \int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}} \left(\frac{|x-x_0|}{|x_0-y|^{n+1-\delta}} + \frac{|x-x_0|^\varepsilon}{|x_0-y|^{n+\varepsilon-\delta}}\right)|D^\alpha \tilde{A}(y)||f(y)|\,dy$$

$$\le C\sum_{|\alpha|=m}\sum_{k=1}^{\infty}(2^{-k}+2^{-\varepsilon k})\left(|2^k\tilde{Q}|^{-1}\int_{2^k\tilde{Q}} |D^\alpha A(y)-(D^\alpha A)_{\tilde{Q}}|^r\,dy\right)^{1/r}\|f\|_{L^{n/\delta}}$$

$$\le C\sum_{|\alpha|=m}\|D^\alpha A\|_{BMO}\|f\|_{L^{n/\delta}}.$$

Thus

$$III \le C\sum_{|\alpha|=m}\|D^\alpha A\|_{BMO}\|f\|_{L^{n/\delta}}.$$

(b) It is only to show that there exists a constant $C>0$ such that for every H^1-atom a(that is that a satisfies: $\operatorname{supp} a \subset Q = Q(x_0,d)$, $\|a\|_{L^\infty} \le |Q|^{-1}$ and $\int a(y)\,dy = 0$ (see [8])), the following holds:

$$\|\tilde{T}_K^A(a)\|_{L^{n/(n-\delta)}} \le C.$$

We write

$$\int_{R^n} \left[\tilde{T}_K^A(a)(x)\right]^{n/n-\delta} dx$$

$$= \left[\int_{|x-x_0|\le 2r} + \int_{|x-x_0|>2r}\right]\left[\tilde{T}_K^A(a)(x)\right]^{n/(n-\delta)} dx := J + JJ.$$

For J, by the following equality

$$Q_{m+1}(A;x,y) = R_{m+1}(A;x,y) + \sum_{|\alpha|=m}\frac{1}{\alpha!}(x-y)^\alpha(D^\alpha A(x)-D^\alpha A(y)),$$

we have,

$$|\tilde{T}_K^A(a)(x)| \le |T_K^A(a)(x)| + C \sum_{|\alpha|=m} \int_{R^n} \frac{|D^\alpha A(x) - D^\alpha A(y)|}{|x-y|^{n-\delta}} |a(y)|\, dy,$$

thus, $\tilde{T}_K^A$ is (L^p, L^q)-bounded by Lemma 3.2 and (a), where $1/q = 1/p - \delta/n$. We see that

$$J \le C \|\tilde{T}_K^A(a)\|_{L^q}^{n/((n-\delta)q)} |2Q|^{1-n/((n-\delta)q)} \le C \|a\|_{L^p}^{n/(n-\delta)} |Q|^{1-n/((n-\delta)q)} \le C.$$

To obtain the estimate of JJ, we denote $\tilde{A}(x) = A(x) - \sum_{|\alpha|=m} \frac{1}{\alpha!} \times (D^\alpha A)_{2Q} x^\alpha$. Then $Q_m(A; x, y) = Q_m(\tilde{A}; x, y)$. We write, by the vanishing moment of a,

$$\begin{aligned}
\tilde{T}_K^A(a)(x) &= \int_{R^n} \frac{K(x,y) R_m(A; x, y)}{|x-y|^m} a(y) dy \\
&\quad - \sum_{|\alpha|=m} \frac{1}{\alpha!} \int_{R^n} \frac{K(x,y) D^\alpha \tilde{A}(x)(x-y)^\alpha}{|x-y|^m} a(y)\, dy \\
&= \int_{R^n} \left[\frac{K(x,y)}{|x-y|^m} - \frac{K(x,x_0)}{|x-x_0|^m} \right] R_m(\tilde{A}; x, y) a(y)\, dy \\
&\quad + \int_{R^n} \frac{K(x,x_0)}{|x-x_0|^m} [R_m(\tilde{A}; x, y) - R_m(\tilde{A}; x, x_0)] a(y)\, dy \\
&\quad - \sum_{|\alpha|=m} \frac{1}{\alpha!} \int_{R^n} \left[\frac{K(x,y)(x-y)^\alpha}{|x-y|^m} - \frac{K(x,x_0)(x-x_0)^\alpha}{|x-x_0|^m} \right] D^\alpha \tilde{A}(x) a(y)\, dy, \\
&:= JJ_1 + JJ_2 + JJ_3.
\end{aligned}$$

Now, similar to the proof of III, we obtain, for $x \in (2Q)^c$

$$\begin{aligned}
|JJ_1| &\le C \int_{R^n} \left[\frac{|y-x_0|}{|x-y|^{n+m+1-\delta}} + \frac{|y-x_0|^\varepsilon}{|x-y|^{n+m+\varepsilon-\delta}} \right] |R_m(\tilde{A}; x, y)| |a(y)|\, dy \\
&\le C \sum_{|\alpha|=m} \|D^\alpha A\|_{BMO} (|Q|^{1/n} |x-x_0|^{-n-1+\delta} + |Q|^{\varepsilon/n} |x-x_0|^{-n-\varepsilon+\delta}),
\end{aligned}$$

$$|JJ_2| \le C \int_{R^n} \frac{|R_m(\tilde{A}; x, y) - R_m(\tilde{A}; x, x_0)| |a(y)|}{|x-y|^{m+n-\delta}}\, dy$$

$$\leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{BMO} \int_{R^n} \frac{|x_0 - y||a(y)|}{|x - x_0|^{n+1-\delta}}\, dy$$
$$\leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{BMO} |Q|^{1/n} |x - x_0|^{-n-1+\delta}$$

and

$$|JJ_3| \leq C \int_{R^n} \frac{|x_0 - y|}{|x - y|^{n+1-\delta}} \sum_{|\alpha|=m} |D^\alpha \tilde{A}(x)||a(y)|\, dy$$
$$\leq C \sum_{|\alpha|=m} |D^\alpha \tilde{A}(x)|(|Q|^{1/n}|x - x_0|^{-n-1+\delta} + |Q|^{\varepsilon/n}|x - x_0|^{-n-\varepsilon+\delta}).$$

Thus

$$JJ \leq \int_{(2Q)^c} (|JJ_1 + JJ_2 + JJ_3|)^{n/(n-\delta)}\, dx$$
$$\leq C\Big(\sum_{|\alpha|=m} \|D^\alpha A\|_{BMO}\Big)^{n/(n-\delta)} \sum_{k=1}^{\infty} k[2^{-kn/(n-\delta)} + 2^{-kn\varepsilon/(n-\delta)}] \leq C.$$

(c) By the following equality

$$R_{m+1}(A; x, y) = Q_{m+1}(A; x, y) + \sum_{|\alpha|=m} \frac{1}{\alpha!}(x - y)^\alpha (D^\alpha A(x) - D^\alpha A(y)),$$

we have

$$|T_K^A(f)(x)| \leq |\tilde{T}_K^A(f)(x)| + C \sum_{|\alpha|=m} \int_{R^n} \frac{|D^\alpha A(x) - D^\alpha A(y)|}{|x - y|^{n-\delta}} |f(y)|\, dy,$$

thus, by Lemma 3.2 and (b), we obtain

$$|\{x \in R^n : |T_K^A(f)(x)| > \lambda\}| \leq |\{x \in R^n : |\tilde{T}_K^A(f)(x)| > \lambda/2\}|$$
$$+ \left|\left\{x \in R^n : \sum_{|\alpha|=m} \int_{R^n} \frac{|D^\alpha A(x) - D^\alpha A(y)|}{|x - y|^{n-\delta}} |f(y)|\, dy > C\lambda\right\}\right|$$
$$\leq C(\|f\|_{H^1}/\lambda)^{n/(n-\delta)}.$$

This completes the proof of Theorem 2.1. □

Proof of Theorem 2.2. (i) It is only to prove that there exists a constant C_Q such that

$$\frac{1}{w(Q)}\int_Q |T_K^A(f)(x)-C_Q|w(x)\,dx \le C\|f\|_{L^\infty(w)}$$

holds for any cube Q. Fix a cube $Q=Q(x_0,d)$. Let $\tilde{Q}$ and $\tilde{A}(x)$ be the same as the proof of Theorem 2.1. We have, similar to the proof of Theorem 2.1, for $f_1=f\chi_{\tilde{Q}}$ and $f_2=f\chi_{R^n\setminus\tilde{Q}}$,

$$\begin{aligned}\left|T_K^A(f)(x)-T_K^{\tilde{A}}(f_2)(x_0)\right| &\le \left|T_K\left(\frac{R_m(\tilde{A};x,\cdot)}{|x-\cdot|^m}f_1\right)(x)\right| \\ &+\sum_{|\alpha|=m}\frac{1}{\alpha!}\left|T_K\left(\frac{(x-\cdot)^\alpha}{|x-\cdot|^m}D^\alpha\tilde{A}f_1\right)(x)\right|+\left|T_K^{\tilde{A}}(f_2)(x)-T_K^{\tilde{A}}(f_2)(x_0)\right| \\ &:= I(x)+II(x)+III(x),\end{aligned}$$

and, thus,

$$\begin{aligned}&\frac{1}{w(Q)}\int_Q\left|T_K^A(f)(x)-T_K^{\tilde{A}}(f_2)(x_0)\right|w(x)\,dx \\ &\le \frac{1}{w(Q)}\int_Q I(x)w(x)\,dx+\frac{1}{w(Q)}\int_Q II(x)w(x)\,dx \\ &\quad+\frac{1}{w(Q)}\int_Q III(x)w(x)\,dx := I+II+III.\end{aligned}$$

First, using Lemma 3.1 and the $L^\infty(w)$-boundedness of T_K, we have

$$\begin{aligned}I &\le \frac{C}{w(Q)}\int_Q\left|T_K\Bigl(\sum_{|\alpha|=m}\|D^\alpha A\|_{BMO}f_1\Bigr)(x)\right|w(x)\,dx \\ &\le C\sum_{|\alpha|=m}\|D^\alpha A\|_{BMO}\|Tf_1\|_{L^\infty(w)}\le C\sum_{|\alpha|=m}\|D^\alpha A\|_{BMO}\|f\|_{L^\infty(w)};\end{aligned}$$

Secondly, since $w\in A_1$, w satisfies the reverse of Hölder's inequality:

$$\left(\frac{1}{|Q|}\int_Q w(x)^q\,dx\right)^{1/q}\le\frac{C}{|Q|}\int_Q w(x)\,dx$$

for all cube Q and some $1 < q < \infty$ (see [12]), thus, taking $p > 1$ and $1/p + 1/p' = 1$, by the $L^p(w)$-boundedness of T_K and Hölder's inequality, we gain

$$\begin{aligned}
II &\le \frac{C}{w(Q)} \int_Q \left| T\Big(\sum_{|\alpha|=m} (D^\alpha A - (D^\alpha A)_{\tilde{Q}}) f_1 \Big)(x) \right| w(x)\,dx \\
&\le C \sum_{|\alpha|=m} \left(\frac{1}{w(Q)} \int_Q |T((D^\alpha A - (D^\alpha A)_{\tilde{Q}}) f_1)(x)|^p w(x) dx \right)^{1/p} \\
&\le C \sum_{|\alpha|=m} \left(\frac{1}{w(Q)} \int_Q |(D^\alpha A(x) - (D^\alpha A)_{\tilde{Q}}) f_1(x)|^p w(x)\,dx \right)^{1/p} \\
&\le C \sum_{|\alpha|=m} w(Q)^{-1/p} \left(\int_{\tilde{Q}} |D^\alpha A(x) - (D^\alpha A)_{\tilde{Q}}|^{pq'}\,dx \right)^{1/pq'} \\
&\qquad \times \left(\int_{\tilde{Q}} w(x)^q\,dx \right)^{1/pq} \|f\|_{L^\infty(w)} \\
&\le C \sum_{|\alpha|=m} \left(\frac{1}{|Q|} \int_{\tilde{Q}} |D^\alpha A(x) - (D^\alpha A)_{\tilde{Q}}|^{pq'}\,dx \right)^{1/pq'} \\
&\qquad \times \left(\frac{1}{|Q|} \int_{\tilde{Q}} w(x)^q\,dx \right)^{1/pq} \left(\frac{|Q|}{w(Q)} \right)^{1/p} \|f\|_{L^\infty(w)} \\
&\le C \sum_{|\alpha|=m} \|D^\alpha A\|_{BMO} \left(\frac{1}{|Q|} \int_{\tilde{Q}} w(x)\,dx \right)^{1/p} \left(\frac{|Q|}{w(Q)} \right)^{1/p} \|f\|_{L^\infty(w)} \\
&\le C \sum_{|\alpha|=m} \|D^\alpha A\|_{BMO} \|f\|_{L^\infty(w)};
\end{aligned}$$

For III, similar to the proof of Theorem 2.1, we obtain

$$\begin{aligned}
III &\le C \sum_{|\alpha|=m} \|D^\alpha A\|_{BMO} \frac{1}{w(Q)} \\
&\times \int_Q \sum_{k=0}^{\infty} \int_{2^{k+1}\tilde{Q} \setminus 2^k \tilde{Q}} k \left(\frac{|x - x_0|}{|x_0 - y|^{n+1}} + \frac{|x - x_0|^\varepsilon}{|x_0 - y|^{n+\varepsilon}} \right) |f(y)|\,dy\, w(x)\,dx
\end{aligned}$$

$$+C\sum_{|\alpha|=m}\frac{1}{w(Q)}\int_Q\sum_{k=0}^{\infty}\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}\left(\frac{|x-x_0|}{|x_0-y|^{n+1}}+\frac{|x-x_0|^{\varepsilon}}{|x_0-y|^{n+\varepsilon}}\right)$$
$$\times|D^\alpha\tilde{A}(y)||f(y)|\,dy\,w(x)\,dx$$
$$\le C\sum_{|\alpha|=m}\|D^\alpha A\|_{BMO}\|f\|_{L^\infty(w)}\sum_{k=1}^{\infty}k(2^{-k}+2^{-k\varepsilon})$$
$$\le C\sum_{|\alpha|=m}\|D^\alpha A\|_{BMO}\|f\|_{L^\infty(w)}.$$

(ii) It suffices to show that there exists a constant $C>0$ such that for every $H^1(w)$-atom a (that is that a satisfy: $\operatorname{supp} a\subset Q=Q(x_0,r)$, $\|a\|_{L^\infty(w)}\le w(Q)^{-1}$ and $\int a(y)dy=0$ (see [8])), we have

$$\|\tilde{T}_K^A(a)\|_{L^1(w)}\le C.$$

We write

$$\int_{R^n}\tilde{T}_K^A(a)(x)w(x)\,dx=\left[\int_{2Q}+\int_{(2Q)^c}\right]\tilde{T}_K^A(a)(x)w(x)\,dx:=J+JJ.$$

For J, similar to the proof of Theorem 2.1, we get

$$|\tilde{T}_K^A(a)(x)|\le|T^A(a)(x)|+C\sum_{|\alpha|=m}\int_{R^n}\frac{|D^\alpha A(x)-D^\alpha A(y)|}{|x-y|^n}|a(y)|\,dy,$$

thus, $\tilde{T}_K^A$ is $L^p(w)$-bounded by Lemma 3.2 and (i). We see that

$$J\le C\|\tilde{T}_K^A(a)\|_{L^\infty(w)}w(2Q)\le C\|a\|_{L^\infty(w)}w(Q)\le C;$$

For JJ, notice that if $w\in A_1$, then $\frac{w(Q_2)}{|Q_2|}\frac{|Q_1|}{w(Q_1)}\le C$ for all cubes Q_1,Q_2 with $Q_1\subset Q_2$. Thus, by Hölder's inequality and the reverse of Hölder's inequality for $w\in A_1$ and some $1<q<\infty$, taking $p>1$ and $1/p+1/p'=1$, similarly, we obtain

$$JJ\le C\sum_{|\alpha|=m}\|D^\alpha A\|_{BMO}\sum_{k=1}^{\infty}(2^{-k}+2^{-\varepsilon k})\left(\frac{|Q|}{w(Q)}\frac{w(2^{k+1}Q)}{|2^{k+1}Q|}\right)$$
$$+C\sum_{|\alpha|=m}\sum_{k=1}^{\infty}(2^{-k}+2^{-\varepsilon k})\frac{|Q|}{w(Q)}\left(\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}|D^\alpha\tilde{A}(x)|^p\,dx\right)^{1/p}$$

$$\times\left(\frac{1}{|2^{k+1}Q|}\int\limits_{2^{k+1}Q} w(x)^{p'}\,dx\right)^{1/p'}$$

$$\le C\sum_{|\alpha|=m}\|D^\alpha A\|_{BMO}\sum_{k=1}^{\infty}k(2^{-k}+2^{-\varepsilon k})\left(\frac{w(2^{k+1}Q)}{|2^{k+1}Q|}\frac{|Q|}{w(Q)}\right)\le C.$$

(iii) Similarly, we know

$$|T_K^A(f)(x)|\le|\tilde{T}^A(f)(x)|+C\sum_{|\alpha|=m}\int\limits_{R^n}\frac{|D^\alpha A(x)-D^\alpha A(y)|}{|x-y|^n}|f(y)|\,dy,$$

by Lemma 3.2 and (ii), we obtain

$$w(\{x\in R^n:|T_K^A(f)(x)|>\lambda\})\le w(\{x\in R^n:|\tilde{T}_K^A(f)(x)|>\lambda/2\})$$
$$+\,w\left(\left\{x\in R^n:\sum_{|\alpha|=m}\int\limits_{R^n}\frac{|D^\alpha A(x)-D^\alpha A(y)|}{|x-y|^n}|f(y)|\,dy>C\lambda\right\}\right)$$
$$\le C\|f\|_{H^1(w)}/\lambda.$$

This completes the proof of Theorem 2.2. □

4. Applications

In this section we shall apply the Theorem 2.1 and 2.2 to some particular operators such as the Calderón–Zygmund singular integral operator and fractional integral operator.

Aplication 1 (Calderón–Zygmund singular integral operator). Let T be the Calderón–Zygmund operator defined by (see [8, 12])

$$T(f)(x)=\int K(x,y)f(y)\,dy,$$

the multilinear operator related to T is defined by

$$T^A(f)(x)=\int\frac{R_{m+1}(A;x,y)}{|x-y|^m}K(x,y)f(y)\,dy.$$

Then it is easily to see that T_K satisfies the conditions in Theorem 2.2, thus that T^A is bounded from $L^\infty(w)$ to $BMO(w)$ and from $H^1(w)$ to weak $L^1(w)$ and that $\tilde{T}^A$ is bounded from $H^1(w)$ to $L^1(w)$ for $w\in A_1$ and $D^\alpha A\in BMO(R^n)$ with $|\alpha|=m$.

Aplication 2 (Fractional integral operator with rough kernel). For $0 \le \delta < n$, let T_δ be the fractional integral operator with rough kernel defined by (see [7, 9, 10])

$$T_\delta f(x) = \int_{R^n} \frac{\Omega(x-y)}{|x-y|^{n-\delta}} f(y)\,dy,$$

the multilinear operator related to T_δ is defined by

$$T_\delta^A f(x) = \int_{R^n} \frac{R_{m+1}(A;x,y)}{|x-y|^{m+n-\delta}} \Omega(x-y) f(y)\,dy,$$

where Ω is homogeneous of degree zero on R^n, $\int_{S^{n-1}} \Omega(x')d\sigma(x') = 0$ and $\Omega \in Lip_\gamma(S^{n-1})$ for $0 < \gamma \le 1$, that is there exists a constant $M > 0$ such that for any $x, y \in S^{n-1}$, $|\Omega(x) - \Omega(y)| \le M|x-y|^\gamma$. Then T_δ satisfies the conditions in Theorem 3.1. In fact, for $\operatorname{supp} f \subset (2Q)^c$ and $x \in Q = Q(x_0, d)$, by the condition of Ω, we have (see [12])

$$\left| \frac{\Omega(x-y)}{|x-y|^{n-\delta}} - \frac{\Omega(x_0-y)}{|x_0-y|^{n-\delta}} \right| \le C \left(\frac{|x-x_0|^\gamma}{|x_0-y|^{n+\gamma-\delta}} + \frac{|x-x_0|}{|x_0-y|^{n+1-\delta}} \right),$$

thus, similar to the proof of Theorem 2.1,

$$\begin{aligned} &|T_\delta^A(f)(x) - T_\delta^A(f)(x_0)| \\ &\qquad \le C \sum_{k=1}^{\infty} k(2^{-\gamma k} + 2^{-k}) \|D^\alpha A\|_{BMO} \|f\|_{L^{n/\delta}} \\ &\qquad\qquad \le C \|D^\alpha A\|_{BMO} \|f\|_{L^{n/\delta}}. \end{aligned}$$

Therefore that T_δ^A is bounded from $L^{n/\delta}(R^n)$ to $BMO(R^n)$ and from $H^1(R^n)$ to weak $L^{n/(n-\delta)}(R^n)$ and $\tilde{T}_\delta^A$ is bounded from $H^1(R^n)$ to $L^{n/(n-\delta)}(R^n)$ for all $D^\alpha A \in BMO(R^n)$ with $|\alpha| = m$.

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