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Operator pencils of the second order and linear fractional relations

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(*Presented by M. M. Malamud*)

Abstract. The notions of a pencil of the second order and a linear fractional relation (LFR) are defined in spaces of linear bounded operators acting between Banach spaces. It is shown that these notions are closely connected with various theoretical and applied problems and have diverse applications. A number of the open problems, both for pencils and LFR, are posed in this paper. Some of the above problems are solved and applied to the study of dichotomic behavior of dynamical systems.

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1. Basic definitions, history of the subject

1.1. Introduction

It is well known that publication of unsolved problems accompanied by a discussion is a specific sort of mathematical literature which has been of great importance (see, for example, the review of the book "Arnold's Problems", Bulletin of the AMS, vol. 43, No 1 (2006), p. 101). Following this line, in the first part of the paper we formulate 15 problems in the topic of operator pencils of the second order and operator linear fractional relations. In its second part we show, how the the solutions of some of these problems can be applied to the study of dynamical systems with continuous and discrete time. Herewith we establish for such systems a so called dichotomy compatible with the signature. These results are natural continuation and development of the corresponding results of the recent work [46] and also of the papers [49, 51–54].

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The purpose of this paper is to study two relatively new notions, those of second order operator pencils whose domains are operator spaces (see Section 1.2) and of linear fractional relations (LFR) (see Section 1.3). These objects are closely related (see below Section 2.3). We are going to study these notions in their interaction, because even at this, rather early stage of the study of these notions, it is already clear that the study of these notions in their interaction leads to remarkable insights into the structure of each of them.

Linear fractional relations are generalizations of linear fractional transformations (LFT), which are defined by the same formula as linear fractional functions:

$$W(Z) = (AZ + B)(CZ + D)^{-1},$$

but now A, B, C, D, Z, and W are operators acting on infinite-dimensional spaces. Linear fractional transformations were studied in depth by M. G. Krein and Yu. L. Shmul'yan [83] and were used to solve important problems in the theory of spaces with an indefinite metric (spaces with an indefinite metric were introduced and the relevant problems were posed by Sobolev and Pontryagin in their study of a very concrete applied problem, see Section 1.4 for more history of this direction of research).

Later LFT were used to solve problems on Abel–Schröder type functional equations, the Koenigs embedding problem (see [11–13]), dichotomy of non-autonomous differential equations in a Hilbert space (see [49, 51–53] for more information). Applicability of LFT to these problems is restricted by the possible non-invertibility of the values of the operator pencil $CZ + D$.

It is quite difficult (in the infinite-dimensional setting) to extend domains of LFT by means of the analytic continuation or other function-theoretic methods. In this connection we extend the domain of an LFT just rewriting the defining formula for it as a relation:

$$AZ + B = W(CZ + D), \tag{1.1}$$

and regarding the set of all W satisfying (1.1) as the image of Z. (Formal definitions and the notation we use can be found in Section 1.3.)

The paper is naturally divided into three parts.

In the first part we give basic definitions of the operator LFR and the operator pencil of the second order, and tell about history of the topic.

In the second part we pose a number of questions, concerning LFR and pencils, and we answer a part of these questions. In particular, we describe the domain and the image of LFR as the set of non- positivity of the corresponding pencil P. We use this fact to find conditions supplying to dom F_T and im F_T with non-emptiness, convexity and compactness in the weak operator topology (WOT).

The final part of the paper is devoted to applications of the above results to dynamical systems with continuous and discrete time. We establish dichotomous behaviour of trajectories of such systems under some natural conditions.

1.2. Operator pencils which domains are operator spaces

Usually (see, e.g. [72]) an operator pencil is defined as an operator-valued function of the form $A(\lambda) = A_0 + \lambda A_1 + \lambda^2 A_2 + \cdots + \lambda^n A_n$, where λ is a complex variable and $A_0, A_1, \ldots, A_n$ are given linear operators acting on a Hilbert space. In this paper operator pencils are understood as polynomial operator-valued functions defined on some subsets of spaces of operators. In particular, by an operator pencil of the second order we mean an operator-valued function of the form

$$P(X,Y) = P_{A,B,C,D}(X,Y) = A + XB + CY + XDY, \tag{1.2}$$

where $X \in L(\mathcal{X},\mathcal{Y})$, $Y \in L(\mathcal{U},\mathcal{V})$, $A \in L(\mathcal{U},\mathcal{Y})$, $B \in L(\mathcal{U},\mathcal{X})$, $C \in L(\mathcal{V},\mathcal{Y})$, $D \in L(\mathcal{V},\mathcal{X})$, $\mathcal{X},\mathcal{Y},\mathcal{U},\mathcal{V}$ are Banach spaces, and by $L(\mathcal{X},\mathcal{Y})$ we denote the space of all bounded linear operators from $\mathcal{X}$ to $\mathcal{Y}$. In this formula A, B, C, and D are fixed, and P is considered as a function on $L(\mathcal{X},\mathcal{Y}) \times L(\mathcal{U},\mathcal{V})$. The formula for $P(X,Y)$ can be written in a matrix form:

$$P(X,Y) = \begin{pmatrix} I & X \end{pmatrix} \begin{pmatrix} A & C \\ B & D \end{pmatrix} \begin{pmatrix} I \\ Y \end{pmatrix}.$$

1.3. Linear fractional relations

In our discussion of LFR it will be convenient to use the following notation.

Let E_i, E_i', $i = 1, 2$, be Banach spaces. The space of bounded linear operators acting from E_j to E_i' is denoted by $\mathcal{L}(E_j, E_i')$.

Let T be a 2×2 operator matrix of the form

$$T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}, \tag{1.3}$$

where $T_{ij} \in \mathcal{L}(E_j, E_i')$, $i, j = 1, 2$. For each such matrix we introduce three set-valued maps. All these maps will be called (with some abuse of terminology) *linear fractional relations* (LFR).

Definition 1.1. *The set-valued (multi-valued) map G_T from $\mathcal{L}(E_1, E_2)$ into the set of closed affine subspaces of $\mathcal{L}(E_1', E_2')$ is defined by*

$$G_T(X) = \{Y \in \mathcal{L}(E_1', E_2') : \quad T_{21} + T_{22}X = Y(T_{11} + T_{12}X)\}. \qquad (1.4)$$

Definition 1.2. *The LFR F_T is the set-valued mapping of the closed unit ball $\mathcal{K}_+$ of $\mathcal{L}(E_1, E_2)$ into the set of subsets of the unit ball $\mathcal{K}_+'$ of $\mathcal{L}(E_1', E_2')$ defined by*

$$F_T(K) = \{K' \in \mathcal{K}_+' : \quad T_{21} + T_{22}K = K'(T_{11} + T_{12}K)\}. \qquad (1.5)$$

Definition 1.3. *The LFR H_T is the set-valued mapping of $\mathcal{K}_-$ (the unit ball of $\mathcal{L}(E_2, E_1)$) onto sets of operators in $\mathcal{K}_-'$ (the unit ball of $\mathcal{L}(E_2', E_1')$) defined by*

$$H_T(Q) = \{Q' : \ T_{11}Q + T_{12} = Q'(T_{21}Q + T_{22})\} \qquad (1.6)$$

Definition 1.4. *A linear fractional relation G_T is said to be defined at X if $G_T(X) \neq \emptyset$. The set of all $X \in \mathcal{L}(E_1, E_2)$ at which G_T is defined will be denoted by* $\mathrm{dom}(G_T)$.

The same terminology and notation will be used for F_T and H_T.

One of the main reason of our interest to these concepts is their relations with the theory of operators in spaces with an indefinite metric (see Section 2.1 for discussion of these relations).

1.4. History of the topic

At the beginning the theory of linear operators in spaces with an indefinite metric concentrated around two classes of operators: symmetries and isometries. In a space with $\mathcal{J}$-metric ($\mathcal{H}$ is a Hilbert space, $\mathcal{J} = \mathcal{J}^* = \mathcal{J}^{-1}$, $[x, y] = (\mathcal{J}x, y)$, $x, y \in \mathcal{H}$, $(\cdot, \cdot)$ is the standard inner product on $\mathcal{H}, \mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, $\mathcal{H}_i = P_i\mathcal{H}$, $i = 1, 2$, where $P_1 = \frac{I+\mathcal{J}}{2}$, $P_2 = \frac{I-\mathcal{J}}{2}$) they are called *$\mathcal{J}$-symmetries* and *$\mathcal{J}$-isometries.*

The pioneering papers [80, 85] are devoted to description of maximal invariant semidefinite subspaces (non-positive and non-negative) of a $\mathcal{J}$ selfadjoint operator A and the structure of the spectra of the restrictions of A to such subspaces. Let

$$\kappa = \min\{\dim \mathcal{H}_1, \dim \mathcal{H}_2\}.$$

The paper [85] is devoted to the case $\kappa = 1$, the paper [80] is devoted to the case $\kappa < \infty$. Now the spaces with finite κ are called *Pontryagin spaces* and denoted $\prod_\kappa$.

Soon afterwards I. S. Iokhvidov [22] developed a theory of Cayley-Newmann linear fractional transformations and established connections between the classes of $\mathcal{J}$-symmetries and $\mathcal{J}$-isometries, later these results were generalized to classes of $\mathcal{J}$-dissipative and $\mathcal{J}$-nonexpanding operators, and even for plus-operators.

It is worth mentioning that the papers [80] and [85] are examples of a delicate work "by hand". Their authors prove the main results using deep, complicated, lengthy, mostly algebraic, computations (especially [80]).

In 1950 M. G. Krein [58] started to use "modern technologies" to study invariant subspaces and spectra of $\mathcal{J}$-unitary operators. He used fixed point theorems originated in work of Bohl, Brouwer, Schauder, and Tikhonov. Later he developed this method in [59]. The papers [61, 62], and [63] laid the foundation of the theory of linear fractional transformations. This theory also serves as a basis for the whole theory of linear operators in $\mathcal{J}$-spaces. Now such spaces are called *Krein spaces.*

Methods developed in [61, 62], and [63] motivated many other works continuing this line of research (see, e.g., [4, 5, 21, 23, 24, 32], and [33].)

As an instructive example of usefulness of methods of linear fractional transformations we discuss how effective were they in applications to the well-known Phillips problem. In [77–79] he studied systems of PDEs in Hilbert spaces and stated [79] the following conjecture:

Any invariant with respect to a commutative group G of J-unitary operators dual pair of subspaces can be extended to an invariant dual pair of maximal subspaces. (Let us recall that a pair of subspaces L_1, L_2 of a Krein space H is called a dual pair, if L_1 is nonnegative, L_2 is nonpositive and they are orthogonal each other with respect to the indefinite metric $[x, y] = (Jx, y)$).

R. Phillips proved this statement under an additional restriction that G is uniformly bounded. On the other hand, respectively to the general case of a group G he wrote: "We have been able to verify this conjecture for finite dimensional H, ...however a proof for infinite dimensional H has thus far eluded us".

He obtained the result using a complicated algebraic argument involving spaces of maximal ideals, etc.

Only after publication of the papers [61, 62], and [63], and using the methods developed in these papers, the results of Phillips and Naimark [75] were extended to the general case, see [4, 21, 32], and [33].

Later the theory of linear fractional transformations was extended both for the general case of operators — matrices of these transformations — acting in Banach indefinite spaces, see [42, 43, 82] and for the general case of linear fractional mappings, even unbounded, but for finite dimensional Krein spaces, see [11] and [12].

Simultaneously the theory found further applications. C. Cowen [11] proved that in diverse areas of Complex Analysis, for example, in the Koenigs' problem on embedding of a discrete iteration semigroup into a continuous one-parameter semigroup, in Abel–Schroeder equations linear fractional functions form a representative class of holomorphic functions (in many cases the validity of a statement for linear fractional functions implies its validity for any holomorphic function).

Later C. Cowen and B. MacCluer [13] started to extend the results of [11] to the case of many variables.

Results of [11, 12], and [13] inspired a series of papers in which linear fractional transformations were applied to the study of composition operators on Hilbert and Banach function spaces (such as Hardy and Bergman spaces), see [2], to study of generators of non-linear one-parameter semigroups, see [86] and [39], to general Abel–Schroeder equations [2, 40–43], to Koenigs' embedding problem, see [40] and [41].

The study of the image of a linear fractional transformation, in particular, their convexity and compactness in the weak operator topology made it possible to use linear fractional mappings to the study of the dychotomy of non-autonomous differential equations in a Hilbert space [36, 49, 51–53].

On the other hand, the expansion of the areas of applicability of the results motivated the study of objects that are more general than linear fractional mappings. We mean linear fractional relations between operators acting in Hilbert and Banach spaces. In terms of indefinite metrics the purpose is to study classes of operators that are more general than bi-strict plus operators, that induce holomorphic linear fractional mappings. We mean all strict plus operators, or even more generally, all operators of the form

$$T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$$

defining a linear fractional relation F_T of the form (1.5). Study of such linear fractional relations was undertaken in [25, 33, 35, 36, 38], (see also [37, 76]).

Another important problem which is connected with our considerations is the Koenigs embedding problem. Let D be a domain in a complex Banach space X. We denote by $\mathrm{Hol}(D, X)$ the set of holomorphic mappings on D with values in X and by $\mathrm{Hol}(D)$ the semigroup (with respect to composition) of holomorphic self-mappings of D. The following problem is classical: given a mapping $\varphi \in \mathrm{Hol}(D)$, find a continuous family $\{F(t)\} \subset \mathrm{Hol}(D)$, $t \geq 0$ with $F(1) = \varphi$, $F(0) = I$ (where I is the identity mapping on D), and satisfying the semigroup property

$F(s+t) = F(s) \circ F(t)$, $s, t \geq 0$.

For more than one hundred years this problem has been considered by many mathematicians in the framework of different theories (see, for example, [10–12, 18, 20, 30, 55, 69]).

The answer depends on what one requires precisely and is known to be “yes” in some cases, “no” in others. As far as we know the problem is still of interest, even for the one-dimensional case.

In 1884 G. Koenigs showed how the problem may be solved (locally) by using the solution of *Schröder's equation*

$$f \circ \varphi = \lambda f. \tag{1.7}$$

Namely, if $z_0 \in D$ is a fixed point of φ and $|\varphi'(z_0)| \neq 0, 1$, then (1.7) with $\lambda = \varphi'(z_0)$ has a unique solution $f \in \mathrm{Hol}(D, \mathbb{C})$ normalized by the condition $f'(z_0) = 1$. This function f is called the *Koenigs function* of φ.) Hence (locally) $\varphi^n = f^{-1} \circ (\lambda^n f)$. This expression then serves as a definition of φ^n when n is not a natural number.

Now consider the more general case of the so-called Abel–Schröder equation. Let Δ be the open unit disk in the complex plane $\mathbb{C}$. The equation

$$f \circ \varphi = \psi \circ f, \tag{1.8}$$

where φ and ψ, which belong to $\mathrm{Hol}(\Delta, \mathbb{C})$ are given, is called [11] the *Abel–Schröder equation*. In the particular case where $\varphi \in \mathrm{Hol}(\Delta)$ fixes 0, that is $\varphi(0) = 0$, and $\psi = \varphi'(0) = \lambda$, (1.8) becomes Schröder's equation (1.7) with the constraint $\varphi(0) = 0$, and in the case where $\psi \circ f = f + 1$, (1.8) becomes Abel equation (see [11]).

In 1981 C. C. Cowen [11] showed that in order to solve Schröder's equation it is sufficient to solve it in the particular case where $\varphi = \varphi_A$ is a linear fractional mapping. Using this fact he gave in [11] the general solution of the Schröder equation.

These observations lead us to the following idea: To look for sufficient and necessary geometric and analytic conditions for $\varphi \in \mathrm{Hol}(D)$ to be embeddable in a one-parameter semigroup in terms of its Koenigs function f. It turns out that this approach can work not only for the one-dimensional case, but also for multi-dimensional and infinite-dimensional cases whenever we know how to find explicitly the solution to Schröder's equation. The latter problem was solved for fractional-linear mappings (see [41, 43, 45]). On the other hand even in the one-dimensional case φ is embeddable if and only if the corresponding Schröder's equation has so-called *spiral like solution* [41]. The problem of existence of such a solution is very difficult and no efficient approaches to it are known, it restricts the applicability of this approach.

We suggest another approach. For fractional linear mappings the Koenigs problem admits the following equivalent reformulation.

Problem. *Let A be a plus-operator.*

- **(a)** *Under what conditions are all fractional powers A^t defined?*
- **(b)** *When is A^t a plus-operator for each $t \geq 0$?*

An answer to question **(a)** is usually given in terms of existence of $\log A$. One can get an answer to the question **(b)** studying the set $\{X : P(X) \leq 0\}$, where $P(X)$ is a self-adjoint operator pencil of the second order.

2. LFR and operator pencils

2.1. Relations between LFR and the theory of operators in spaces with an indefinite metric

There is a natural correspondence between LFR F_T and plus-operators (see [6]). Let us recall the definition of plus-operators. We introduce on $E = E_1 \oplus E_2$ and $E' = E_1' \oplus E_2'$ indefinite structures in the following way. Let P_1, P_2 be the canonical projections corresponding to the decomposition $E = E_1 \oplus E_2$ and P_1', P_2' be the canonical projections corresponding to the decomposition $E = E_1' \oplus E_2'$. We define the sets $\mathcal{P}_+$ and $\mathcal{P}_+'$ of non-negative vectors in E and E' by

$$\mathcal{P}_+ = \{x \in E : \|P_1 x\| \geq \|P_2 x\|\} \text{ and } \mathcal{P}_+' = \{x \in E' : \|P_1' x\| \geq \|P_2' x\|\},$$

respectively. The sets of non-positive vectors in E and E' are defined by

$$\mathcal{P}_- = \{x \in E : \|P_1 x\| \leq \|P_2 x\|\} \text{ and } \mathcal{P}_-' = \{x \in E' : \|P_1' x\| \leq \|P_2' x\|\}.$$

Definition 2.1. *A linear continuous operator $T : E \to E'$ is called a plus-operator if $T\mathcal{P}_+ \subset \mathcal{P}_+'$, and is called a minus-operator if $T\mathcal{P}_- \subset \mathcal{P}_-'$.*

Definition 2.2. *Denote by $\mathcal{K}_+$ and $\mathcal{K}_+'$ the closed unit ball of $\mathcal{L}(E_1, E_2)$ and $\mathcal{L}(E_1', E_2')$, respectively, and by $\mathcal{K}_-$ and $\mathcal{K}_-'$ the closed unit ball of $\mathcal{L}(E_2, E_1)$ and $\mathcal{L}(E_2', E_1')$, respectively. These sets are the sets of angular operators of maximal non-negative and maximal non-positive subspaces, respectively.*

The mentioned above correspondence between LFR and plus-operators can be described in the following way:

An operator $T : E \to E'$ is a plus-operator if and only if the LFR F_T (see (1.5)*) is defined at each point of $\mathcal{K}_+$.*

Using projections $P_1^*, P_2^*, (P_1')^*$, and $(P_2')^*$ we introduce an indefinite structure on E^* and $(E')^*$. For each LFR there are two dual objects:

1) F_{T^*}, where F_{T^*} is induced by the matrix T^* and maps contractions from $\mathcal{L}(E_1^*, E_2^*)$ onto sets of contractions in $\mathcal{L}((E_1')^*, (E_2')^*)$.

2) H_T (see Definition 1.3)

Properties of F_T correspond to plus-properties of T, whereas properties of H_T correspond to its minus-properties.

Observe that each of these relations can be empty, in the sense that their domains are empty.

We shall use the following terminology. If an LFR F_T is univalent (that is, the set $F_T(K)$ contains exactly one element K' for each $K \in \operatorname{dom} F_T$), we call it a *quasi linear fractional mapping* (QLFM). If $T_{11} + T_{12}K$ is invertible for each $K \in \operatorname{dom} F_T$, we call it a *linear fractional mapping* (LFM).

If an LFR F_T is defined everywhere in $\mathcal{K}_+$ (that is, T is a plus-operator), we call F_T a *plus-LFR*. If, in addition, $E = E'$, and hence $\mathcal{K}_+ = \mathcal{K}'_+$, and F_T is a plus-QLFM, we shall call it also a *quasi linear fractional transformation* (QLFT) of the ball $\mathcal{K}_+$. If, in addition, $T_{11} + T_{12}K$ is invertible for all $K \in \mathcal{K}_+$, we call F_T a *linear fractional transformation* (LFT) of the ball $\mathcal{K}_+$.

It is natural to consider two groups of problems:

1) global properties of pencils and LFR in the space $\mathcal{L}(E_1, E_2)$;

2) local properties of pencils and LFR acting between the unit balls $\mathcal{K}_+$ and $\mathcal{K}'_+$.

The first group is connected with the problems of the optimal control theory, in particular, with operator Riccati equation (see, e.g., [1, 56, 57, 64]), with the problems of extension of J-nonnegative definite operators up to J-selfadjoint operators (see, e.g. [70, 71]) and with the generalized Liouville theorem as well.

The second group of problems is closely connected with various problems of the operator theory in a space with indefinite metric, with Köenig's embedding problem, as well as the problem of extension of the invariant pairs of dual subspaces up to the maximal subspaces. This group of problems is also connected with applications to dynamical systems ([11–13, 40–43, 45–47, 49, 51–53]).

We consider in this paper pencils and relations between the unit balls. Since LFR is the divisibility relation between two linear pencils it is natural to discuss linear pencils. We are also interested in quadratic pencils and selfadjoint quadratic pencils.

2.2. Linear pencils and general properties of LFR

2.2.1. Basic results

Let F_T be a an LFR corresponding to the operator $T = (T_{ij})_{i,j=1}^2 : E \to E'$. The following is true (see [38, 44, 82]).

1. If $\operatorname{coker}(T_{11} + T_{12}K) = \{0\}$, $K \in \operatorname{dom}(F_T)$, then F_T is a QLFM.

2. If F_T is a plus-QLFM, then F_{T^*} is a plus-LFM.

3. If E, E' are reflexive, F_T is a plus-QLFM and $\ker(T_{11} + T_{12}K) = \{0\}$, then F_{T^*} is a plus-QLFM.

4. If E, E' are reflexive, F_T is a plus-LFR, then F_{T^*} is a plus-LFM.

So we see that properties of LFR are closely related to the behavior of linear pencils obtained from (1.2) when $D = 0 = B$.

2.2.2. General problems

Taking in account statements 1–4 on the first stage of study of linear operator pencils it is natural to ask the following questions for pencils $P(K) = A + CK$, where A and C are fixed, $K \in \mathcal{K}_+$.

Problem 2.1. A criterion for $\ker P(K) = \{0\}$.

Problem 2.2. A criterion for $\operatorname{coker} P(K) = \{0\}$.

Problem 2.3. A criterion for invertibility of $P(K)$.

Since the invertibility of A is a necessary condition for invertibility of $P(K)$ on $\mathcal{K}_+$, Problem 2.3 is equivalent to the following problem.

Problem 2.3′. A criterion for invertibility of $\tilde{P}(K) = I + C'K$.

Criteria for 2.1 and 2.2 in terms of inequalities $\|Cx\| \le \|Ax\|$ and $\|C^*y\| \le \|A^*y\|$ and in terms of embeddings of images of unit balls were found in [50] (see Theorems 5, 14, 16, 17, 19).

In particular, it would be interesting to find criteria for divisibility of operators in Banach spaces generalizing the Hilbert space case criteria found by Yu. Shmul'yan [83].

The well-known criterion of divisibility of operators in a Hilbert space can be stated in the following way: $A = KB$ if and only if $A^*A \le kB^*B$, where k is a positive real number satisfying $\|K\| \le k$. Corollary 9 in [50] is a generalization of this result to the Banach space case.

In further study of LFR, it is natural to study the following questions.

Problem 2.4. Criteria for $\dim \operatorname{coker} P(K) = \text{const}$.

Problem 2.5. Criteria for stability of $\operatorname{ind} P(K)$ on $\mathcal{K}_+$.

2.3. Quadratic Pencils and fixed points of LFR

2.3.1. Definitions

In this section we consider the quadratic pencil

$$P(X) = A + BX + XC + XDX \tag{2.1}$$

which is obtained from (1.2) when $Y = X$. Let F_T be an LFR and let K_+ be its fixed points, that is, $K_+ \in F_T(K_+)$. This condition can be written as

$$T_{21} + T_{22}K_+ = K_+(T_{11} + T_{12}K_+). \tag{2.2}$$

Opening the brackets in (2.2) and moving all terms to the left-hand side, we get the equation

$$P(K_+) = 0, \tag{2.3}$$

where $P(K_+)$ is a pencil of the form (2.1) with $A = T_{21}$, $B = -T_{11}$, $C = T_{22}$, $D = -T_{12}$. Hence (2.3) is an equation for fixed points of LFR.

Analogous equation takes place for a fixed point K_- of an LFR H_T. These fixed points are, respectively, angular operators of invariant maximal non-negative L_+ and maximal non-positive L_- subspaces of an operator T, that is,

$$L_+ = (P_1 + K_+)E_1, \quad L_- = (P_2 + K_-)E_2, \quad TL_+ \subset L_+, \quad TL_- \subset L_-.$$

The matrix quadratic equation

$$A + BX + XC + XDX = 0 \tag{2.4}$$

and for the Hilbert space case its symmetric version (known as algebraic Riccati equation)

$$A + BX + XB^* + XDX = 0 \tag{2.5}$$

where $D \geq 0$ and $A = A^*$ also arise in another kind of problems dealing with connections between the set of solutions of the equations (2.4) or (2.5) and factorizations of certain matrix functions of the Popov type associated with (2.4) or (2.5). (See, e.g., [27–29, 68]). The connection between the problem of determining solutions of the matrix quadratic equation and the problem of describing the factorizations of a matrix function is provided by generalized Bezoutians. The notion of generalized Bezoutians based on representations of the functions in realized form was introduced and developed in some papers (see, e.g., [17, 19, 27, 67]), as a generalization of the coefficient Bezoutian for quadruple of matrix polynomials introduced in [3].

One of the results in this regard ([27]) establishes the one-to-one correspondence between the set of solutions of (2.4) and the set of factorizations of the rational matrix function of the Popov type associated with (2.4)

$$G_{C,K}(\lambda) = I - \begin{bmatrix} K & \Phi \end{bmatrix} \left(\lambda I - \begin{bmatrix} -C - \Psi K & 0 \\ A + NK & B - N\Phi \end{bmatrix}\right)^{-1} \begin{bmatrix} \Psi \\ N \end{bmatrix}. \tag{2.6}$$

where K and N are arbitrary feedback matrices of appropriate sizes, and $D = \Psi\Phi$ is a rank decomposition of the matrix D. Such correspondence is provided by the generalized Bezoutian introduced and studied in [27].

It is worth to mention the polynomial approach, where the role of the rational matrix function (2.6) plays a matrix polynomial of the certain type (see, e.g., [28, 68]). One of the advantages of such approach is that it provides more insight into the structure of hermitian solutions of the homogeneous algebraic Riccati equation

$$BX + XB^* + XDX = 0. \tag{2.7}$$

2.3.2. Problems

The problem of finding maximal invariant semi-definite subspaces of a linear operator is one of the central problems in the theory of operators in spaces with indefinite metric. In this section we pose the following problems:

Problem 2.6. Find criteria for existence of zeros of a quadratic pencil.

Problem 2.7. Creation of methods of finding zeros of quadratic pencils.

Problem 2.8. The description of a geometric and topological structure of zero sets of quadratic pencils.

Problem 2.9. To study the structure of hermitian solutions of the nonhomogeneous algebraic Riccati equation (2.5).

There is an extensive literature devoted to Problems 2.6 and 2.8 stated in terms of fixed points of LFT and, sometimes, LFR. See [6] and references therein. Additional information on Problem 2.8 can be found in [8] and [48]. The mentioned papers employed methods of two types: methods of the theory of spaces with an indefinite metric and holomorphic analysis. In particular, it was shown in [48] that the set of fixed points of an LFT is a smooth manifold (let us recall that an LFT is holomorphic on $\mathcal{K}_+$).

The corresponding purposes are:

(i) To find new methods for proving the existence of fixed points of LFT, based on the study of zeros of operator pencils.

(ii) To develop new methods for description of zeros of an operator pencil.

(iii) To describe the geometric and topological structure of the set of fixed points of general LFR in terms of zeros of the pencil $P(K_+)$.

2.4. Selfadjoint quadratic pencils and geometrical and topological structure of the image and the domain of LFR, the Hilbert space case

2.4.1. Description of the image and the domain of LFR in terms of pencils

There is a simple, popular and useful theory of linear operator equations of the form $AX = B$ or $XC = B$ which describes the properties of the sets of solutions $X \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ and, in particular, the conditions for solvability (existence of such X). On the other hand, many problems from different areas (extensions of operators [71], indefinite metric spaces [63], linear fractional relations [49], operator functional equations [11], control theory and systems theory [14, 26, 64–66], and others) lead to quadratic equations and inequalities of the forms:

$$X^*AX = C, \tag{2.8}$$

$$X^*AX \leq C, \tag{2.9}$$

and, more generally,

$$X^*AX + B^*X + X^*B + C = 0, \tag{2.10}$$

$$X^*AX + B^*X + X^*B + C \leq 0. \tag{2.11}$$

The equation (2.10) is very close to the equation $XAX + B^*X + X^*B + C = 0$ which self-adjoint A and C which is usually called a *continuous algebraic Riccati equation.* It is clear that a self-adjoint solution of (2.10) is also a solution of the corresponding continuous algebraic Riccati equation. Since self-adjoint solutions are the most important in systems theory, our study can be of interest for infinite-dimensional systems theory. In control theory and systems theory there are many results stating that under certain assumptions the continuous algebraic Riccati equation has a non-negative solution (see, e.g. [66, Theorem 2.2.1], [14, Theorem 6.2.7]). Many other results on continuous algebraic Riccati equations can be found in [14,26], and [65]. Another object studied in systems theory is an inequality of the form $XAX + B^*X + X^*B + C \leq 0$ $(A \geq 0)$, called a *Riccati inequality*, see [15], [65, Section 9.1], [81]. The mentioned sources contain results on conditions of solvability of Riccati inequality and comparison of the sets of self-adjoint solutions of the continuous algebraic Riccati equation and the corresponding Riccati inequality. Scalar versions of the inequality (2.11) for an infinite dimensional Hilbert space were studied in [87]. This survey of the literature shows that information about properties of sets $M(A,B,C)$ can be of interest from different points of view. Our interest in this topic was motivated by the fact that such sets appear in the study of linear fractional relations (LFR), and we consider the present paper as a continuation of [38], in which we started systematic study of properties of LFR. Namely, in terms of $M(A,B,C)$ one can express the domains of LFR and the images of balls under LFR.

Relatively few results are known on topological and geometric properties of sets of solutions of the inequalities (2.9) and (2.11). We will denote by $E(A,C)$, $N(A,C)$ and $M(A,B,C)$ the sets of all solutions for (2.8), (2.9) and (2.11), respectively. It is known, that $M(A,B,C) = \mathcal{L}(\mathcal{H}_1,\mathcal{H}_2)$ if and only if A, C are non-positive and $B = (-A)^{1/2}T(-C)^{1/2}$ for some operator $T \in \mathcal{L}(\mathcal{H}_1,\mathcal{H}_2)$ with $\|T\| \leq 1$. If the operators A, C are invertible, this condition is equivalent to $C - B^*A^{-1}B \leq 0$. It is the so-called *generalized Sylvester criterion* (see [9], [60, pp. 732–733], and [84]). If A is invertible and positive, then the sets $E(A,C)$ and $N(A,C)$ are non-void only for $C \geq 0$ and in this case $E(A,C) = A^{-1/2}\mathcal{U}C^{1/2}$, $N(A,C) = A^{-1/2}\mathcal{R}C^{1/2}$ where $\mathcal{U}$ is the set of all isometric operators from $\overline{C\mathcal{H}_1}$ into $\mathcal{H}_2$, and $\mathcal{R}$ is the unit ball of $\mathcal{L}(\overline{C\mathcal{H}_1},\mathcal{H}_2)$. Hence $N(A,C)$ is convex and weak operator topology-compact. If A is non-negative but not invertible, then $N(A,C)$ is still convex (see, for example, [49]), and hence weak operator topology-closed.

The analysis of $M(A,B,C)$, and even of $N(A,C)$ for an arbitrary self-adjoint operator A, are much more complicated. Our purpose is to find conditions under which the sets $N(A,C)$ and $M(A,B,C)$ are non-empty, closed, or convex, and to characterize their interior points.

In this section we consider pencils of the form

$$P(X) = A + B^*X^* + XB + XDX^*, \tag{2.12}$$

where $E = \mathcal{H}$, $E' = \mathcal{H}'$ are Hilbert spaces, $A = A^*$, $D = D^*$. This pencil is obtained from (1.2) when $Y = X^*$ and $C = B^*$. Let F_T be an LFR, that is,

$$F_T(K_+) = \{K'_+ : T_{21} + T_{22}K_+ = K'_+(T_{11} + T_{12}K_+)\}, \tag{2.13}$$
$$K_+ \in \operatorname{dom} F_T,\ K'_+ \in \operatorname{im} F_T.$$

Let us describe $\operatorname{dom} F_T$ and $\operatorname{im} F_T$ in terms of the pencil (2.12). The equality from (2.13), combined with the Shmul'yan divisibility criterion [83], implies

$$(T_{21}+T_{22}K_+)^*(T_{21}+T_{22}K_+) \leq (T_{11}+T_{12}K_+)^*(T_{11}+T_{12}K_+). \quad (2.14)$$

Opening brackets in (2.14) and moving everything to the left-hand side we get

$$P(K_+) \leq 0, \quad (2.15)$$

where P is a pencil of the form (2.12) with $X = K_+^*, D = T_{22}^*T_{22} - T_{12}^*T_{12},\ B^* = T_{21}^*T_{22} - T_{11}T_{12},\ A = T_{21}^*T_{21} - T_{11}^*T_{11}$. Hence $\operatorname{dom} F_T = M_d := \{K_+ \in \mathcal{K}_+ :\ P(K_+) \leq 0\}$, where P is a pencil of the form (2.12) with coefficients described after (2.15) and $X = K_+^*$.

Let us rewrite the equality from (2.13) in the form

$$T_{21} - K_+'T_{11} = (K_+T_{12} - T_{22})K,$$

from where, as before, we get

$$(T_{21} - K_+'T_{11})(T_{21} - K_+'T_{11})^* \leq (K_+T_{12} - T_{22})(K_+T_{12} - T_{22})^*.$$

This implies

$$\operatorname{im} F_T = M_{Im} := \{K_+' \in \mathcal{K}_+ :\ P(K_+') \leq 0\}, \quad (2.16)$$

where P is a pencil of the form (2.12) with $X = K_+'$ and

$$A = T_{21}T_{21}^* - T_{22}T_{22}^*,$$

$$B = -T_{21}T_{11}^* + T_{22}T_{12}^*,$$

$$D = T_{11}T_{11}^* - T_{12}T_{12}^*.$$

Let us find the connections between the relations F_{T^*} and H_T which are dual to F_T.

Theorem 2.1. $-\operatorname{dom} H_T = (\operatorname{im} F_{T^*})^*,\ \ -\operatorname{im} H_T = (\operatorname{dom} F_{T^*})^*.$

In the case of invertible T we have

Theorem 2.2. *Let T be an invertible operator. If F_T is a plus-QLFM, then F_T is a plus-LFM, $H_{T^{-1}}$ is a minus-LFM, $F_{T^{-1}}$ and H_T are LFM and the following equalities hold; $F_{T^{-1}} = (F_T)^{-1}$, $H_{T^{-1}} = (H_T)^{-1}$.*

Corollary 2.1. *Let F_T be a plus-LFR and let T be invertible. If at least one of the relations F_T and $H_{T^{-1}}$ is univalent (that is, is a QLFM), then F_T and F_{T^*} are plus-LFM, $H_{T^{-1}}$ is a minus-LFM, $F_{T^{-1}}$ and H_T are LFM, and the following conditions are satisfied:*

$$\operatorname{im} H_{T^{-1}} = -(\operatorname{im} F_{T^*})^*, \operatorname{im} F_T = -(\operatorname{im} H_{(T^{-1})^*})^*.$$

Proof of Theorem 2.1. By means of criterion of divisibility of Shmul'yan we obtain from (1.6) that for any $Q \in H_T$ the following inequality takes place:

$$(T_{11}Q + T_{12})^*(T_{11}Q + T_{12}) \leq (T_{12}Q + T_{22})^*(T_{21}Q + T_{22}). \tag{2.17}$$

Setting all the terms of (2.17) in the left-hand side we have

$$\begin{aligned} Q_*(T_{11}^*T_{11} - T_{21}^*T_{21})Q + Q^*(T_{11}^*T_{12} - T_{21}^*T_{22}) \\ + (T_{12}^*T_{11} - T_{22}^*T_{21})Q + T_{12}^*T_{12} - T_{22}^*T_{22} \leq 0. \end{aligned} \tag{2.18}$$

Using (2.16) rewrite $\operatorname{im} F_{T^*}$ in the form

$$\begin{aligned} \operatorname{im} F_{T^*} = \{K'_+ \in K_+ : (K'_+)(T_{11}^*T_{11} - T_{21}^*T_{21})((K'_+)^* + K'_+)(T_{21}^*T_{22} \\ - T_{11}^*T_{12}) + (T_{22}^*T_{21} - T_{12}^*T_{11})(K'_+)^* + T_{12}^*T_{12} - T_{22}^*T_{22} \leq 0\}. \end{aligned} \tag{2.19}$$

We see that (2.18) and (2.19) coincide for $Q = -(K'_+)^*$, and the first equality of Theorem 2.1 is proved. The other equalities can be proved in just the same way. $\square$

In this section we pose the following problems.

Problem 2.10. Find criteria for semi-definitness of the pencil P on the whole space.

Problem 2.11. Find criteria for semi-definitness of the pencil P on the ball $\mathcal{K}_+$.

Problem 2.12. Find conditions under which the set $M := \{X : P(X) \leq 0\}$ is non-empty.

Problem 2.13. Find condition under which the intersection of M and $\mathcal{K}_+$ is non-empty.

Problem 2.14. Find conditions under which M is convex.

Problem 2.15. Find conditions under which M is closed in the weak operator topology.

2.4.2. Non-emptiness, convexity and compactness of the image and the domain of LFR

Now let us study conditions for compactness in the weak operator topology and for convexity of the sets $\operatorname{im} F_T$ and $\operatorname{dom} H_T$. From now on, we set $D = T_{11}T_{11}^* - T_{12}T_{12}^*$. Also we will use two notions of conjugation for a given linear bounded operator T: T^* with respect to usual scalar product $(\cdot,\cdot)$ and T^c with respect to indefinite metric $[\cdot,\cdot]$.

Denote by γ_∞ the set of all compact operators.

Theorem 2.3. Suppose that $T \in L(\mathfrak{H})$ and $D = R + C$, where $R \geq 0$ and $C \in \gamma_\infty$. Then $\operatorname{im} F_T$ is compact in the weak operator topology on the space $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$.

Proof. Let $X \in \mathcal{K}_+$. We use

$$T_{21} + T_{22}K_+ = K'_+(T_{11} + T_{12}K_+) \tag{2.20}$$

to rewrite the condition "$X \in \operatorname{im} F_T$" as follows: "there exists an operator $K_+ \in \mathcal{K}_+$ such that $(XT_{12} - T_{22})K_+ = T_{21} - XT_{11}$." The latter means (e.g., see [83]) that $(XT_{12}-T_{22})(XT_{12}-T_{22})^* \geq (T_{21}-XT_{11})(T_{21}-XT_{11})^*$. Thus, it suffices to prove the closedness in the weak operator topology of the set $\{X\}$ of all operators $X \in \mathcal{K}_+$ satisfying the inequality

$$X(R+C)X^* + B^*X^* + XB + A \leq 0, \tag{2.21}$$

that is $P(X) \leq 0$, where $P(X)$ is a pencil (2.12) with $D = R+C$, where R, C, B, $A \in L(\mathfrak{H})$, $R \geq 0$, $C \in \gamma_\infty$, and $A = A^*$.

We rewrite (2.21) as

$$(XRX^*x, x) \leq ((-XCX^* + B^*X^* + XB + A)x, x), \tag{2.22}$$

where $x \in \mathfrak{H}$.

Since the operator C is compact, the function on the right-hand side of (2.22) is continuous in the weak operator topology. We transform the left-hand side as follows: $(XRX^*x, x) = \|R'X^*x\|^2$, where R' is the nonnegative root of R. Let $X_n \xrightarrow{\text{weakly}} X_0$ (here $\{X_n\}$ is a sequence or a net). Then, by the well-known property of the weak limit, we have $\|R'X_0^*x\| \leq \underline{\lim}_n \|R'X_n^*x\|$. Passing in (2.22) to the lower limit (which coincides with the limit on the right-hand side of this inequality), we hence see that inequality (2.22) also holds for the operator X_0.

Therefore, the set $\{X\}$ is closed in the weak operator topology, and the proof of the theorem is complete.

It is easy to show that, for an arbitrary self-adjoint operator D, the condition "$D = R + C$, where $R \geq 0$ and $C \in \gamma_\infty$", is equivalent to the following condition: "$D_{\in}\gamma_{\infty}$, where $D = D_+ + D_-$ is the spectral resolution of the operator D (here $D_- = \int_{-\infty}^{0} t\,dE_t$ and $D_+ = \int_0^{\infty} t\,dE_t$, where E_t is the partition of unity corresponding to D)". □

Corollary 2.2. *Suppose that $T \in L(\mathfrak{H})$ and $T_{12} \in \gamma_\infty$. Then $\operatorname{im} F_T$ is compact in the weak operator topology.*

Corollary 2.3. *Suppose that $T \in L(\mathfrak{H})$ and $T_{21} \in \gamma_\infty$. Then $\operatorname{im} F_T^*$ is compact in the weak operator topology.*

Proof. It follows from the relation $T^c = JT^*J$ that $(T^c)_{12}^c = -T_{21}^* \in \gamma_\infty$, and hence $\operatorname{im} F_T^*$ is compact in the weak operator topology. Thus, by Theorem 2.1, we readily see that the set $\operatorname{dom} H_T$ is compact in the weak operator topology. □

The following assertion is a generalization of Theorems 1.2 ([49]) and 2.3 ([36]).

Theorem 2.4. *Suppose that $T \in L(\mathfrak{H})$ and $D \geq 0$. Then the set $\operatorname{dom} H_T$ is compact in the weak operator topology on the space $\mathcal{L}(\mathcal{H}_2, \mathcal{H}_1)$ and convex.*

Proof. The compactness in the weak operator topology of $\operatorname{dom} H_T$ follows from Theorems 2.1 and 2.3 of the present paper, and the convexity of this set in the case $C = 0$ follows from Lemma 1.1 [49]. □

A closely related condition ensures that the set $\operatorname{dom} H_T$ is also nonempty. Namely, we have the following assertion.

Theorem 2.5. *Let $T \in L(\mathfrak{H})$. The set $\operatorname{dom} H_T$ is nonempty precisely if there exists an operator $K \in \mathcal{K}_1$ such that*

$$D + K^*S + S^*K + K^*EK \geq 0, \tag{2.23}$$

where $S = T_{22}T_{12}^* - T_{21}T_{11}^*$, $E = T_{21}T_{21}^* - T_{22}T_{22}^*$.

Proof. It follows from Theorem 2.1 that the nonemptiness of $\operatorname{dom} H_T$ is equivalent to the nonemptiness of $\operatorname{im} F_{T^c}$, i.e., to the existence of an operator $K \in \mathcal{K}_+$ such that $T^c(P_1 + K)\mathfrak{H}_1 \subset \mathcal{P}_+$. We rewrite the last condition as

$$\begin{aligned}\big(((T^c)_{11} + (T^c)_{12}K)x,\ ((T^c)_{11} + (T^c)_{12}K)x\big)& \\ \geq \big(((T^c)_{21} + (T^c)_{22}K)x,\ ((T^c)_{21} + (T^c)_{22}K)x\big)&\end{aligned}$$

for all $x \in \mathfrak{H}$. ($T^c = JT^*J,\ J = \left(\begin{smallmatrix} crI & 0 \\ 0 & -I \end{smallmatrix}\right)$.)

Transforming this inequality and taking into account the fact that $(T^c)_{11} = T_{11}^*$, $(T^c)_{12} = -T_{21}^*$, $(T^c)_{21} = -T_{12}^*$, and $(T^c)_{22} = T_{22}^*$, we obtain inequality (2.23).

Thus, the inequality $D \geq 0$ means that inequality (2.23) holds already for $K = 0$.

For the case in which $D \geq 0$ and A is a plus-operator, the nonemptiness of the set $\operatorname{dom} H_T$ was proved in [36] (see Theorem 2.3). □

Now let A be a plus-operator. In this case, the set $\operatorname{dom} H_T$ is compact in the weak operator topology and convex ([36, Theorem 2.1]). Conditions for compactness in the weak operator topology and convexity of the set $\operatorname{im} F_T$ were studied in several papers (e.g., see [36, 49, 53]). The following assertion is a generalization of Theorems 1.2 ([49]) and 2.3 ([36]), which is somewhat different than Theorem 2.4.

Theorem 2.6. *Suppose that T is a plus-operator, the operator $D = T_{11}T_{11}^* - T_{12}T_{12}^*$ is semidefinite (i.e., $D \geq 0$ or $D \leq 0$). Then the set $\operatorname{im} F_T$ is compact in the weak operator topology and convex. In this case, if $D \leq 0$, then the operator T annihilates a subspace $(P_+ + Q)\mathfrak{H}_+$, where $Q \in \mathcal{K}_+$, and $\operatorname{im} F_T = \mathcal{F}_T(Q) = \mathcal{K}_+$. Moreover, the block-matrix of the operator A with respect to the canonical decomposition of the space $\mathfrak{H}$ has the form*

$$\begin{pmatrix} -T_{12}Q & T_{12} \\ -ST_{12}Q & ST_{12} \end{pmatrix},$$

where $S \in \mathcal{K}_+$.

Proof. The case $D \geq 0$ was studied in Theorem 2.4. Let $D \leq 0$; this means that $T^c\mathfrak{H}_1 \subset \mathcal{P}_-$, that is $T^c\mathfrak{H}_1$ is the set of nonpositive vectors [6]. There exists a maximal nonnegative subspace $\mathfrak{L}_+$ which is J-orthogonal to $T^c\mathfrak{H}_1$ [6]; we have $[T\mathfrak{L}_+, \mathfrak{H}_1] = [\mathfrak{L}_+, T^c\mathfrak{H}_1] = 0$. This means that $T\mathfrak{L}_+ \subseteq \mathfrak{H}_2$, and hence T is a plus-operator, $T\mathfrak{L}_+ = \{0\}$. We have obtained $\operatorname{im} F_T = \mathcal{F}_T(Q) = \mathcal{K}_+$, where Q is an angular operator of the subspace $\mathfrak{L}_+$.

Now we describe the block-matrix of a plus-operator T such that $D \leq 0$. Since $T_{11} + T_{12}Q = 0$, we have $T_{11} = -T_{12}Q$. Next, a plus-operator annihilating a nonzero nonnegative vector is not strict [82]. Thus, we have $T_{21} = ST_{11}$ and $T_{22} = ST_{12}$, where $S \in \mathcal{K}_+$ [6], which completes the proof of the theorem. □

We note that the inequality $D \geq 0$ holds for any operator $T \in L(\mathfrak{H})$ such that T^* is a plus-operator. This remark, Theorem 2.1 ([36]) and Theorem 2.6 imply the following assertion.

Proposition 2.1. *Let T be a bilateral plus-operator, that is both T and T^* are plus-operators. Then the sets* $\operatorname{im} F_T$ *and* $\operatorname{dom} H_T$ *are nonempty, compact in the weak operator topology and convex.*

This proposition is a generalization of Corollary 2.4 ([36]).

In conclusion of Section 2.4 we would like to mention that the Riccati equations $P(X) = 0$ are used in the control theory (see, for example [64]). On the other hand, solutions of Problems 2.13–2.15 will provide a criterion for non-emptiness, convexity, and compactness in the weak operator topology of the image and the domain of LFR (see (2.15) and (2.16)). In a special case of LFT F_T convexity and compactness of $\operatorname{im} F_T$ were established in [49]. The same paper contains the result showing that this property characterize Hilbert spaces in the following sense: if E is a Banach space and in $\mathcal{L}(E)$ the image of each LFT is convex, then E is isometric to a Hilbert space.

The paper [35] contains an example of a unitary, with respect to the usual inner product on $\mathcal{H}$, plus-operator T for which H_T (hence, by Theorem 2.1, also F_{T^*}) is empty. Examples of plus-LFR with non-convex and non-compact images can be found in [44].

3. Applications to dynamical systems

In this section we consider the expected significance of results on Problems 2.13–2.15 for the theory of dynamical systems.

3.1. The case of continuos time

Consider a uniformly well-posed problem

$$\begin{aligned} x' &= A(t)x \\ x(t_0) &= x_0 \end{aligned} \tag{3.1}$$

in a Hilbert space $\mathcal{H}$, where the operators $A(t)$ are self-adjoint with the common dense domain D. Uniform well-posedness implies the existence of the evolution operator $U(t): \ x(t) = U(t)x_0$, where $x(t)$ is the solution of (3.1) (see [36, 49, 51–53]). Let operators $A(t)$ be indefinite (that is, with spectra having non-empty intersections both with the positive and the negative semi-axis). We introduce a family of indefinite metrics $[\cdot,\cdot]_t$ by:

$$[x, y]_t = (J(t)x, y),$$

where $(\cdot,\cdot)$ is the usual inner product on $\mathcal{H}$, $J(t) = P_1(t) - P_2(t)$, $P_1(t) = \int_{0^+}^{\infty} dE_\lambda(t)$, and $P_2(t) = \int_{-\infty}^{0} dE_\lambda(t)$.[1] Assume that $\dim P_1(t)$ and

[1] $E_\lambda(t)$ is the spectral function of $A(t)$

$\dim P_2(t)$ are stabilizing for large t. Under rather general conditions the operator $U(t) : (\mathcal{H}, [\cdot,\cdot]_0) \to (\mathcal{H}, [\cdot,\cdot]_t)$ is a plus-operator (as an example consider a strong sufficient condition of positivity of the derivative of the solution $x(t)$ along the trajectory, that is, $\frac{d}{dt}([x(t), x(t)]_t) \geq 0$). We can write the condition "$U(t)$ is a plus-operator" as an equivalent condition (2.15).

Let us study the dichotomy of the solutions: consider the set of all $y \in \mathcal{H}$ such that

$$[U(t)y, U(t)y]_t \leq 0. \tag{3.2}$$

Since $U(t)$ is a plus-operator, the vector y is non-positive: $\|P_2(0)y\| \geq \|P_1(0)y\|$. Defining the operator K_- by the equality $K_-P_2(0)y = P_1(0)y$, and extending it to the whole space $\mathcal{H}_2$ as a contraction, we get $K_- \in \mathcal{K}_-$. By (3.2) such K_- are in $\operatorname{dom} H_T$. By Theorem 2.1 $\operatorname{dom} H_T = -(\operatorname{im} F_{T^*})^*$. Consider the intersection of all such sets over all $t \geq 0$. Let us show that if all of these sets are non-empty, then their intersection is also non-empty. Since $U(t)$ is a plus-operator, the leading coefficient of the corresponding pencil is non-negative, therefore all these sets are compact and convex. It implies the non-emptiness of their intersection. Let us denote the intersection by Z. By (3.2) all trajectories $x(t)$ starting in this set remain non-positive for all t (with respect to $[\cdot,\cdot]_t$), all $x(t)$, starting at non-negative vectors x_0 remain non-negative for all t since $U(t)$ is a plus-operator. Finally, all trajectories $x(t)$ starting at non-positive vectors which are not in Z, at some moment t (see (3.2)) become non-negative (change their sign). This is a generalization of the classical method of Weyl circles.

Now let us consider a particular case of dynamical problems. In [36], problem (1) is studied under the following conditions. The operators $A(t)$ are self-adjoint in the Hilbert space $\mathfrak{H}$ with the inner product $(\cdot,\cdot)$ and have the same domain of definition $D \subseteq \mathfrak{H}$ for all $t \in \mathbb{R}^+ = [0, \infty)$. The Cauchy problem (3.1) is assumed to be uniformly well-posed, i.e., there exists a bounded linear operator (an evolution operator) $U(t)$ such that the relation $x(t) = U(t)x_0$ holds for any solution $x(t)$ of problem (ref(1)) with $x(0) = x_0 \in D$. If $y_0 \notin D$, then $y(t) = U(t)y_0$ is called a *generalized solution.*

Let $L_{2,\omega}(\mathbb{R}^+, \mathfrak{H})$ be the set of functions x: $\mathbb{R}^+ \to \mathfrak{H}$ square-integrable in the sense of Bochner with respect to a positive locally integrable weight $w = w(t)$. By $\mathfrak{N}$ we denote the set of all generalized solutions belonging to $L_{2,\omega}(\mathbb{R}^+, \mathfrak{H})$. We set $\mathfrak{N}_0 = \{h \in \mathfrak{H} : h = y(0),\, y \in \mathfrak{N}\}$. Let $[x, y]_t$ be an indefinite (depending on t) metric in $\mathfrak{H}$ determined as follows:

$$[x, y]_t = (J(t)x, y), x, y \in \mathfrak{H},$$

where $J(t) = P_1(t) - P_2(t)$, $P_1(t) = \int_0^{+\infty} dE_\lambda(t)$, $P_2(t) = \int_{-\infty}^{0} dE_\lambda(t)$, and $E_\lambda(t)$ is the spectral function of the operator $A(t)$. For each $t \in \mathbb{R}^+$, we denote

$$C_t^- = \{y_0 \in \mathfrak{H} : [U(t)y_0, U(t)y_0] \leq 0\}.$$

We shall say that the bicone C_t^- has rank d, $d \leq \infty$, if it contains a subspace $\mathfrak{L} \subset \mathfrak{H}$ such that $\dim \mathfrak{L} = d$ and does not contain subspaces of larger dimensions.

We assume that $J(t)$ is strongly differentiable and consider the derivative of the solution $x(t)$ along the trajectory:

$$[x(t), x(t)]_t' = 2\mathrm{Re}\,[A(t)x(t), x(t)]_t + (J'(t)x(t), x(t)).$$

We assume that this derivative is qualified to be positive, i.e.,

$$\inf_{\|z\|=1} \left\{ \mathrm{Re}\,[A(t)z, z]_t + \frac{1}{2}(J'(t)z, z) \right\} \geq \omega(t) > 0 \quad \text{for} \quad t \in \mathbb{R}^+.$$

Under these conditions, the dichotomy of solutions to the Cauchy problem 3.1) was established in [36]. The proof was essentially based on the nonemptiness and compactness in the weak operator topology of the sets $\mathrm{dom}\, H_{U(t)}$; but, in the proof of the nonemptiness of these sets, the key point was the condition

$$U_{11}(t)U_{11}^*(t) \geq U_{12}(t)U_{12}^*(t)$$

(condition (3.3) in Theorem 3.1 [36]) which is equivalent to $D = D(t) \geq 0$ in the corresponding pencil (2.12). Recall that $D(t) = U_{11}(t)U_{11}^*(t) - U_{12}(t)U_{12}^*(t)$. Following the argument used in the proof of Theorem 3.1 [36] and replacing its condition (3.3) by a more general inequality (2.23), we obtain the following result.

Theorem 3.1. *Suppose that the Cauchy problem* (3.1) *is uniformly well-posed; $J(t)$ is strongly differentiable; the limit*

$$\lim_{t\to\infty} \dim (P_2(t)\mathfrak{H}) = d_-$$

exists; the derivative of the solution along a trajectory is qualified to be positive; for any $t \in \mathbb{R}^+$ there exists an operator $K = K(t) \in \mathcal{K}_1$, such that the operator $U(t)$ satisfies inequality (2.23), *where $S = U_{22}(t)U_{12}^*(t) - U_{21}(t)U_{11}^*(t)$ and $E = U_{21}(t)U_{21}^*(t) - U_{22}(t)U_{22}^*(t)$.*

Then the generalized solution $y(t) = U(t)y_0$, $y_0 \in \mathfrak{H}$, has the following properties:

(1) $\mathfrak{N}_0 \supset C_\infty^- = \bigcap_{t\in\mathbb{R}^+} C_t^-$, *where C_∞^- is a bicone of rank d_-;*

(2) *the inequality*

$$\int_t^\infty \omega(s)\|y(s)\|^2\,ds \le I(y)\exp\left(-2\int_0^t \omega(s)\,ds\right), \tag{3.3}$$

where $I(y) = \int_0^\infty \omega(s)\|y(s)\|^2\,ds$, *holds for any* $y(t) \in \mathfrak{N}$;

(3) *for any* $y_0 \notin C_\infty^-$ *there exists* $C(y_0) > 0$, *such that the inequality*

$$\|y(t)\| \ge C(y_0)\exp\left(2\int_0^t \omega(s)\,ds\right), \qquad t \in \mathbb{R}^+ \tag{3.4}$$

holds.

As in [36], we establish the following assertion.

Corollary 3.1. *If in the conditions of Theorem* 3.1

$$\int_0^\infty \omega(t)\,dt = \infty, \tag{3.5}$$

then $\mathfrak{N}_0 = C_\infty^-$ *is a closed subspace of* $\mathfrak{H}$ *and* $\dim \mathfrak{N}_0 = d_-$.

In [53] we have proved the following theorem on a non-autonomous perturbation of an autonomous dynamical system:

Theorem 3.2. *Assume that the following conditions hold: in* (3.1) *the operator* $A(t)$ *has the form* $A_0 + B(t)$, *where* A_0 *is a self-adjoint operator defined on* D_0 *and bounded from the right; for any* $t \in \mathbb{R}_+$ $B(t) \in \mathcal{L}(\mathfrak{H})$ *and the operator functions* $B(t)$, $B^\star(t)$ *are strongly continuously differentiable on* D_0. *Furthermore, assume that the point 0 belongs to a gap of the spectrum* $\sigma(A_0)$ *and the following estimate holds:*

$$\forall\, t \in \mathbb{R}_+ : \|B(t)\| \le d - \omega(t), \tag{3.6}$$

where $d = dist(0, \sigma(A_0))$ *and* $\omega(t)$ *is a positive locally integrable function satisfying the condition* (3.5). *Let* P^+, P^- *be the orthogonal projections onto the invariant subspaces of the operator* A_0, *corresponding respectively to the positive and to the negative parts of* $\sigma(A_0)$. *Then the Cauchy problem* (3.1), *is uniformly well-posed and the following assertions are valid:*

(1) *the set* C_∞^- *coincides with a maximal non-positive subspace* L_∞^- *of the Krein space* $(\mathfrak{H}, V_0)$, *where* $V_0 = P^+ - P^-$*;*

(2) *for any* $y_0 \in L_\infty^-$ *the generalized solution* $y(t) = U(t,0)y_0$ *to* (3.1) *satisfies the condition:*

$$I(y) = \int_0^\infty \omega(s)\|y(s)\|^2 ds < \infty$$

and estimate (3.3) *holds;*

(3) *for any* $y_0 \notin L_\infty^-$ *there exists* $C(y_0) > 0$, *such that the generalized solution* $y(t) = U(t,0)y_0$ *to* (3.1) *satisfies the inequality* (3.4).

Let us describe an application of the latter theorem to the following diffusion equation (see [53])

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - Q(x,t)u \tag{3.7}$$

on the axis $-\infty < x < +\infty$ and $t \geq 0$. Denote $\mathfrak{H} = L_2(\mathbb{R})$. We will find a solution to (3.7) satisfying the initial condition:

$$u(x,t)|_{t=0} = \phi(x), \tag{3.8}$$

where $\phi \in \mathfrak{H}$, and the additional condition:

$$u(\cdot,t) \in \mathfrak{H} \quad \forall\, t > 0, \tag{3.9}$$

which plays the role of boundary conditions at $-\infty$ and $+\infty$. We can rewrite the boundary value problem (3.7), (3.8), (3.9) in the form of an abstract Cauchy problem:

$$\frac{dy}{dt} = A(t)y, \tag{3.10}$$

$$y(0) = \phi, \tag{3.11}$$

where $y(t)$ is the function $u(x,t)$, which is considered as a vector function with the range in the space $\mathfrak{H} : y(t) = u(\cdot,t)$. For each $t \geq 0$ $A(t)$ is an operator defined in $\mathfrak{H}$ by the differential operation $a(t) = \frac{d^2}{dx^2} - Q(x,t)\,\cdot$ $(t \geq 0)$. This means that $A(t)$ is the closure of the operator, defined by $a(t)$ on the linear set $C_0^\infty(\mathbb{R})$ of smooth functions with compact supports.

Assume that the function $Q(x,t)$ satisfies the following condition:

(I) $Q(x,t) = p(x) + q(x,t)$, *where* $p(x)$ *is a real-valued continuous* T*-periodic function with* $T > 0$, $q(x,t)$ *is a real-valued, continuous and* *bounded function on the half-plane* $\Pi_+ = \mathbb{R} \times \mathbb{R}_+$, *and there exists the partial derivative* $\frac{\partial}{\partial t} q(x,t)$ *on* Π_+, *continuously depending on* t *uniformly with respect to* $x \in \mathbb{R}$.

Represent $A(t)$ in the form $A(t) = A_0 + B(t)$, where A_0 is the operator, defined in $\mathfrak{H}$ by the operation $a_0 = \frac{d^2}{dx^2} - p(x)\cdot$, and for each $t \geq 0$ $B(t)$ is the operator of multiplication by the function $q(x,t)$ in the space $\mathfrak{H}$: $B(t)v(x) = q(x,t)v(x)$. It is easy to check that the operator A_0 is self-adjoint and bounded from the right, for any $t \in \mathbb{R}_+$ $B(t) \in \mathcal{L}(\mathfrak{H})$ and $B(t) = B^{\star}(t)$, and the operator function $B(t)$ is strongly continuously differentiable on D_0. Therefore by Theorem 3.2 the Cauchy problem (3.10), (3.11) is uniformly well-posed on $\mathbb{R}_+$. We will call any generalized solution to this problem *a generalized solution to the boundary value problem* (3.7), (3.8), (3.9).

Observe that since $p(x)$ is a periodic function, the spectrum $\sigma(A_0)$ consists of the closures of the zones of stability to the equation $\frac{d^2 v}{dx^2} - p(x)v = \lambda v$.

Assume that the following condition is fulfilled:

(J) *The point 0 belongs to a gap of the spectrum* $\sigma(A_0)$.

Denote by P^+, P^- the orthogonal projections on the invariant subspaces of A_0, corresponding to the positive and to the negative parts of $\sigma(A_0)$ respectively. Set $V_0 = P^+ - P^-$ and let $\mathcal{P}_0^-$ be the non-positive bicone of the Krein space $(\mathfrak{H}, V_0)$.

The following theorem is an immediate consequence of Theorem 3.2.

Theorem 3.3. *Assume that conditions* (I), (J) *are fulfilled and the function* $q(x,t)$ *satisfies the following inequality:*

$$|q(x,t)| \leq d - \omega(t) \qquad \forall\, t \geq 0,$$

where $d = \operatorname{dist}(0, \sigma(A_0)), \omega(t)$ *is a positive locally integrable function satisfying the condition* (3.5). *Then there exists a maximal non-negative subspace* L_∞^- *of the Krein space* $(\mathfrak{H}, V_0)$ *such that the following statements are true:*

(1) *for any initial state* $\phi \in L_\infty^-$ *the corresponding generalized solution* $u(x,t)$ *to the problem* (3.7), (3.8), (3.9) *satisfies the following condition:*

$$I(u) = \int_0^\infty \omega(\tau) d\tau \int_{-\infty}^{\infty} |u(x,\tau)|^2 \, dx < \infty,$$

and the following estimate holds:

$$\int_t^{\infty} \omega(\tau)d\tau \int_{-\infty}^{\infty} |u(x,\tau)|^2 dx \le I(u)exp\left(-2\int_0^t \omega(s)\,ds\right);$$

(2) *for any initial state* $\phi \notin L_{\infty}^{-}$ *there exists* $C(\phi) > 0$*, such that the corresponding generalized solution* $u(x,t)$ *to the problem* (3.7)*,* (3.8)*,* (3.9) *satisfies the inequality:*

$$\int_{-\infty}^{\infty} |u(x,t)|^2 dx \ge C(\phi)exp\left(2\int_0^t \omega(s)\,ds\right).$$

3.2. The case of discrete time

Now consider the case of discrete time.

In the paper [53] we considered a linear fractional transformation of the ball $\mathcal{K}^-$ of all angular operators corresponding to the set of all maximal non-positive subspaces of a Krein space, i.e, of a Hilbert space $\mathfrak{H}$ with an indefinite metric, for which both positive and negative components can be infinite-dimensional in general ([7, 31, 34, 59]). This transformation is generated by a continuous linear operator U in $\mathfrak{H}$ (so called bistrict plus-operator: see for example [7, 31, 34, 35, 59]). We did not suppose U to be continuously invertible. The weak compactness of the image and the domain of $\mathcal{K}^-$ by the generated linear fractional transformation F_U has been established ([53, Theorem 2.1]). We apply the above results to the study of dichotomous behavior of solutions to a non-autonomous linear difference equation in a Hilbert space $\mathfrak{H}$

$$y_{n+1} = A_n y_n \qquad (y_n \in \mathfrak{H},\ n = 0, 1, 2, \dots), \tag{3.12}$$

where A_n are linear bounded operators acting in the space $\mathfrak{H}$. This equation describes a non-autonomous dynamical system, in which the integer n plays a role of the discrete time. We do not assume that the operators A_n are continuously invertible. This means that the corresponding dynamical system is irreversible in general. The dichotomous behavior means that solutions with initial values belonging to some subspace of the phase space $\mathfrak{H}$ (the stable subspace) stabilize themselves to zero at infinity, but all the other solutions grow infinitely. For differential equations in Hilbert and Banach spaces this property was studied in [35, 49, 51, 53, 73, 74].

We turn now to the linear difference equation of the form (3.12) in a Hilbert space $\mathfrak{H}$. Assume that that all the operators A_n in this equation belong to the class $\mathcal{L}(\mathfrak{H})$. Consider the evolution operator $U(n,m)$ $(n, m \in \mathbb{Z}_+,\ n \geq m)$ of equation (3.12). Recall that this is the operator which associates with each $y_0 \in \mathfrak{H}$ the solution y_n of equation (3.12) satisfying the initial condition $y_n|_{n=m} = y_0$, that is

$$U(n,m)y_0 = A_{n-1} \cdot A_{n-2} \cdot \dots \cdot A_{m+1} \cdot A_m y_0, \tag{3.13}$$

if $n > m$ and $U(n,n) = I$, We shall denote briefly $U(n) = U(n,0)$. Along with equation (3.12) consider the following sequence of difference equations in $\mathfrak{H}$:

$$z_{n+1} = A^{\star}_{N-n} z_n, \tag{3.14}$$

where $N \in \mathbb{Z}_+,\ n = 0, 1, \dots N$. For each fixed $N \in \mathbb{Z}_+$ denote by $\tilde{U}_N(n,m)$ the evolution operator of this equation. In view of (3.13), these operators are connected with the evolution operator of equation (3.12) in the following manner:

$$\tilde{U}_{n+m-1}(n,m) = U^{\star}(n,m),$$

hence for $m = 0$

$$U^{\star}(n) = \tilde{U}_{n-1}(n). \tag{3.15}$$

In the space $\mathfrak{H}$ consider a sequence of indefinite metrics of the form

$$[x,y]_n = (V_n x, x) \quad (n \in \mathbb{Z}_+), \tag{3.16}$$

where each operator V_n belongs to $\mathcal{L}(\mathfrak{H})$, it is self-adjoint, it is continuously invertible and satisfies the following conditions:

(A) *The numbers* $d^+_{V_n}$, $d^-_{V_n}$ *do not depend on* n (*see subsection* 2.1^0);

(B) *The sequence of operators* $\{V_n\}_{n \in \mathbb{Z}_+}$ *is uniformly bounded with respect to the operator norm and moreover:*

$$\sup_{n \in \mathbb{Z}_+} \|V_n\| = 1.$$

Let us introduce the following

Definition 3.1. *We call the sequence of quadratic forms*

$$[U(n+1,n)y_0, U(n+1,n)y_0]_{n+1} - [y_0, y_0]_n \ \ (n \in \mathbb{Z}_+)$$

the increment of the sequence of the quadratic forms $[y_0, y_0]_n$ *with respect to the difference equation* (3.12) *and denote it by* $\Delta_{(3.12)}([y_0, y_0]_n)$.

From (3.16), (3.12) we obtain that

$$\Delta_{(3.12)}([y_0, y_0]_n) = [A_n y_0, A_n y_0]_{n+1} - [y_0, y_0]_n \\ = ((A_n^\star V_{n+1} A_n - V_n) y_0, y_0). \quad (3.17)$$

In the analogous manner we obtain:

$$\Delta_{(3.14)}([y_0, y_0]_n) = [A_{N-n}^\star y_0, A_{N-n}^\star y_0]_{n+1} - [y_0, y_0]_n \\ = ((A_{N-n} V_{n+1} A_{N-n}^\star - V_n) y_0, y_0). \quad (3.18)$$

In the sequel we shall assume the following condition to be satisfied for both the metrics (3.16) and equation (3.12):

(C) *There exists a non-increasing sequence of positive numbers* $\{\epsilon_n\}_{n\in\mathbb{Z}_+}$ *such that*

$$\Delta_{(3.12)}([y_0, y_0]_n) \geq \epsilon_n \|y_0\|^2 \qquad \forall\, n \in \mathbb{Z}_+,\ y_0 \in \mathfrak{H},$$

that is

$$A_n^\star V_{n+1} A_n - V_n \geq \epsilon_n I \qquad \forall\, n \in \mathbb{Z}_+.$$

As a consequence of condition (C) we obtain that

$$[U(n+1)y_0, U(n+1)y_0]_{n+1} - [U(n_0)y_0, U(n_0)y_0]_{n_0} \geq \sum_{k=n_0}^{n} \epsilon_k \|U(k)y_0\|^2 \\ \forall\, y_0 \in \mathfrak{H},\ n \geq n_0. \quad (3.19)$$

Consider the following bicones, connected both with the metrics (3.16) and the evolution operator of equation (3.12):

$$C_n^- = \{y_0 \in \mathfrak{H} : [U(n)y_0, U(n)y_0]_n \leq 0\} \quad (n \in \mathbb{Z}_+). \qquad (3.20)$$

Using the latter definition and property (3.19) it is easy to show that for the family of these bicones the property of monotonicity holds:

$$C_{n+1}^- \subseteq C_n^-. \qquad (3.21)$$

We set

$$C_\infty^- = \bigcap_{n\in\mathbb{Z}_+} C_n^-. \qquad (3.22)$$

Lemma 3.1. *Assume that, besides conditions (A), (B) and (C), the condition*

$$\Delta_{(3.14)}([y_0, y_0]_k) \geq 0 \qquad \forall\, k \in \{0, 1, \ldots, N\},\ y_0 \in \mathfrak{H} \tag{3.23}$$

is satisfied, that is

$$A_{N-k} V_{k+1} A^{\star}_{N-k} \geq V_k \qquad \forall\, k \in \{0, 1, \ldots, N\},\ N \in \mathbb{N}.$$

Then the set C^-_∞, defined by (3.22), contains a maximal non-positive subspace L^-_∞ of the Krein space $(\mathfrak{H}, V_0)$.

Proof. In view of condition (C), condition (3.23) with $N = n - 1$ and equality (3.15), the inequalities

$$[U(n)y, U(n)y]_n \geq [y, y]_0, \quad [U^\star(n)y, U^\star(n)y]_n \geq [y, y]_0 \quad \forall\, y \in \mathfrak{H},\ n \in \mathbb{N}.$$

are valid. This means that each operator $U(n)$ is a J-biexpansive operator. Let M^-_n be the set of maximal subspaces of the bicone C^-_n, defined by (3.20), and $\mathcal{M}^-_n$ be the set of angular operators K^- in the Krein space $(\mathfrak{H}, V_0)$, corresponding to the subspaces $L^- \in M^-_n$. By Theorem 2.1 of [53], each of the sets $\mathcal{M}^-_n$ is compact with respect to the weak operator topology defined in $\mathcal{L}(\mathfrak{H}^-_0, \mathfrak{H}^+_0)$, where $\mathfrak{H}^-_0, \mathfrak{H}^+_0$ are the negative and the positive components of the canonical decomposition of Krein space $(\mathfrak{H}, V_0)$. At the same time, property (3.21) implies that $\mathcal{M}^-_{n+1} \subseteq \mathcal{M}^-_n$. Then the intersection $\bigcap_{n \in \mathbb{N}} \mathcal{M}^-_n$ is non-empty. Hence a subspace $L^-_\infty \in \bigcap_{n \in \mathbb{N}} M^-_n$ is the desired maximal non-positive subspace of the Krein space $(\mathfrak{H}, V_0)$ contained in C^-_∞. $\square$

Let $\{\epsilon_n\}_{n \in \mathbb{Z}_+}$ be the sequence of numbers used in condition (C) and $L_{2,\epsilon}(\mathbb{Z}_+, \mathfrak{H})$ be the Hilbert space of sequences $\bar{y} = \{y_n\}_{n \in \mathbb{Z}_+}$ of vectors $y_n \in \mathfrak{H}$ satisfying the condition

$$\sum_{n=1}^{\infty} \epsilon_n \|y_n\|^2 < \infty$$

with the inner product

$$(\bar{y}, \bar{z})_\epsilon = \sum_{n=1}^{\infty} \epsilon_n (y_n, z_n).$$

Denote by $\mathcal{N}$ the set of solutions of the difference equation (3.12) which belong to $L_{2,\epsilon}(\mathbb{Z}_+, \mathfrak{H})$, i.e.,

$$\mathcal{N} = \{\bar{y} = \{y_n\}_{n \in \mathbb{Z}_+} \in L_{2,\epsilon}(\mathbb{Z}_+, \mathfrak{H}) \mid y_{n+1} = A_n y_n \quad \forall\, n \in \mathbb{Z}_+\}. \tag{3.24}$$

Let $\mathcal{N}_0$ be the "slice" of the set $\mathcal{N}$ at the moment $n = 0$, i.e.

$$\mathcal{N}_0 = \{y_0 \mid (y_0, y_1, \dots, y_n, \dots) \in \mathcal{N}\}. \tag{3.25}$$

We turn now to the main result of this subsection.

Theorem 3.4. *Assume that besides conditions (A), (B) and (C), the condition*

$$\sum_{n=0}^{\infty} \epsilon_n = +\infty \tag{3.26}$$

is satisfied, where $\{\epsilon_n\}_{n\in\mathbb{Z}_+}$ is the sequence used in condition (C). Then:

(i) *The set C_∞^- is a closed subspace of $\mathfrak{H}$ and, moreover,*

$$C_\infty^- = \mathcal{N}_0; \tag{3.27}$$

(ii) *For any $y_0 \in C_\infty^-$ for the solution $y_n = U(n)y_0$ of equation* (3.12) *the following estimate holds:*

$$\sum_{k=n+1}^{\infty} \epsilon_k \|y_k\|^2 \le I(y_0) \prod_{k=0}^{n} (1 + \epsilon_k)^{-1}, \tag{3.28}$$

where

$$I(y_0) = \sum_{k=1}^{\infty} \epsilon_k \|y_k\|^2; \tag{3.29}$$

(iii) *For any $y_0 \notin C_\infty^-$ there exists $C(y_0) > 0$ such that the solution $y_n = U(n)y_0$ of equation* (3.12) *satisfies the following inequality:*

$$\|y_n\|^2 \ge C(y_0) \prod_{k=0}^{n} (1 + \epsilon_k); \tag{3.30}$$

(iv) *If, in addition, condition* (3.23) *is satisfied, the set C_∞^- is a maximal non-positive subspace of the Krein space $(\mathfrak{H}, V_0)$.*

Proof. First of all, let us prove assertion (iii). Assume that $y_0 \notin C_\infty^-$. Then, in view of (3.20), (3.21), (3.22), there exists $n_0 \in \mathbb{Z}_+$ such that

$$[U(n_0)y_0, U(n_0)y_0]_{n_0} > 0. \tag{3.31}$$

Using condition (B), we obtain from property (3.19) the following inequality:

$$\|U(n+1)y_0\|^2 \ge [U(n_0)y_0, U(n_0)y_0]_{n_0} + \sum_{k=n_0}^{n} \epsilon_k \|U(k)y_0\|^2 \quad \forall\, n \ge n_0, \tag{3.32}$$

which can be rewritten in the form:

$$Y_{n+1} - Y_n \geq \epsilon_{n+1} Y_n \quad (n \geq n_0), \tag{3.33}$$

where

$$Y_n = [U(n_0)y_0, U(n_0)y_0]_{n_0} + \sum_{k=n_0}^{n} \epsilon_k \|U(k)y_0\|^2. \tag{3.34}$$

Inequalities (3.31) and (3.33) imply that

$$Y_n \geq [U(n_0)y_0, U(n_0)y_0]_{n_0} \prod_{k=n_0}^{n} (1 + \epsilon_{k+1}) \quad (n \geq n_0).$$

Taking into account (3.31), (3.32) and (3.34), we obtain from the latter inequality the desired estimate (3.30) with some positive constant $C(y_0)$. So, we have proved assertion (iii).

Let us prove assertion (i). Observe that estimate (3.30) and condition (3.26) imply that, if $y_0 \notin C_\infty^-$, the solution $y_n = U(n)y_0$ $(n \in \mathbb{Z}_+)$ of equation (3.12) does not belong to the space $L_{2,\epsilon}(\mathbb{Z}_+)$. This means that

$$\mathcal{N}_0 \subseteq C_\infty^-. \tag{3.35}$$

Let us prove the inverse inclusion. Assume that $y_0 \in C_\infty^-$. Then, by definitions (3.20), (3.21) of the sets C_n^- and C_∞^-,

$$[U(n)y_0, U(n)y_0]_n \leq 0 \qquad \forall\, n \in \mathbb{Z}_+.$$

This fact and property (3.19) imply that

$$\sum_{k=n_0}^{n} \epsilon_k \|U(k)y_0\|^2 \leq [U(n_0)y_0, U(n_0)y_0]_{n_0} \qquad \forall\, n \geq n_0, \tag{3.36}$$

hence the solution $y_n = U(n)y_0$ $(n \in \mathbb{Z}_+)$ of equation (3.12) belongs to the space $L_{2,\epsilon}(\mathbb{Z}_+)$, i.e., $y_0 \in \mathcal{N}_0$. So, we have proved the inclusion $C_\infty^- \subseteq \mathcal{N}_0$ which, together with inclusion (3.35), implies equality (3.27). Since the set C_∞^- is closed in $\mathfrak{H}$ as the intersection of the closed sets C_n^-, and the set $\mathcal{N}_0$ is linear (by (3.24), (3.25)), then C_∞^- is a closed subspace of the space $\mathfrak{H}$. We have proved assertion (i).

We turn now to the proof of assertion (ii). Assume, as above, that $y_0 \in C_\infty^-$. Let us turn $n \to \infty$ in inequality (3.36) and afterwards substitute there n for n_0. Then, in view of condition (B), we obtain the inequality:

$$\sum_{k=n+1}^{\infty} \epsilon_k \|U(k)y_0\|^2 \leq \|U(n)y_0\|^2 \qquad \forall\, n \in \mathbb{Z}_+.$$

Denote

$$Z_n = \sum_{k=n+1}^{\infty} \epsilon_k \|U(k)y_0\|^2. \tag{3.37}$$

Then the latter inequality can be rewritten in the form:

$$Z_{n+1} \leq (1+\epsilon_{n+1})^{-1} Z_n \qquad \forall\, n \in \mathbb{Z}_+.$$

This estimate and (3.37) imply the desired estimate (3.28), in which $I(y_0))$ is expressed by (3.29). We have proved assertion (ii).

Assertion (iv) follows from the obvious inclusion $C_\infty^- \subseteq C_0^-$ and Lemma 3.1. □

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