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On the Isomorphic Group Algebras of Isotype Subgroups of Warfield Abelian Groups

Peter Danchev

(*Presented by B. V. Novikov*)

Abstract. A new class of global mixed Abelian groups, called W-groups, is defined. The following isomorphism theorem for commutative modular group algebras of such groups is proved: If G is a p-mixed μ-elementary W-group for some arbitrary ordinal μ, then the F-isomorphism between the group algebras FG and FH of prime characteristic p for any group H implies that G and H are isomorphic.

This strengthens our recent results in (Bol. Soc. Mat. Mexicana, 2004) and (Acta Math. Sinica, 2005) as well as results due to Ullery in (Proc. Amer. Math. Soc., 1988 and 1990) and (Comm. Algebra, 1989).

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Introduction

The isomorphism problem for modular group algebras over various classes of Abelian groups whose structures have been completely characterized possesses important, if not fascinating, role. In our present context, it is interesting to note that one instance of this is the well-read paper of W. Ullery [11]. He was the first author who study in his subsequent articles [9–11] the commutative group algebras of isotype subgroups of totally projective p-groups equipped with special properties.

Recently, in two independent works [1] and [2], we extend a classical result belonging to W. May [8] and established for p-local Warfield groups to the global, p-mixed, variant.

Combining both the ideas and the techniques from ([1–3]) and ([7,10]), we shall now generalize the main statements of ([9–11]) by proving an isomorphism claim for a larger class of Abelian groups than the mentioned

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ones. Before doing that, we need certain preliminary group-theoretical assertions.

1. Group Preliminaries

Throughout, p is an arbitrary, but fixed, prime number and all groups considered are Abelian, multiplicatively written, groups possibly mixed. Since we shall deal almost exclusively only with Abelian (= commutative) groups, there should be no confusion. For such a group G, G_p denotes the p-primary component of G, and q is a various prime different from p. Notions, notation and terminology in all aspects not explicitly stated herein are standard or are in agreement with ([1, 2] and [3]).

By analogy with the nomenclature of ([4, 5] and [7]), a group G is said to be a *μ-elementary W-group* for some arbitrary ordinal number μ if there exists a Warfield group A of p-length at most μ containing G such that the following conditions hold together:

(1) $G \cap A^{p^\lambda} = G^{p^\lambda}$ for all primes p and all ordinals λ;

(2) $(A/G)^{p^\lambda} = A^{p^\lambda}G/G$ for all primes p and all ordinals $\lambda < \mu$;

(3) $\cap_{n<\omega}(GA_qA^{q^n}) = G(\cap_{n<\omega}(A_qA^{q^n}))$ for all primes $q \neq p$;

(4) A/G is Warfield.

It is worthwhile noticing that the condition (3) is new and it considers an inner treatment of the global niceness; e.g. Lemma 1.6 listed below.

To facilitate the discussion, we introduce some further terminology.

If G is a subgroup of a group A, we call G *isotype* in A provided that (1) is valid.

If G is a subgroup of a group A of p-length $\leq \mu$, we call G *almost nice* in A provided that (2) is fulfilled. Thus, if G is both an isotype and an almost nice subgroup of A, it is called *almost balanced*. As is well-known, the equality (2) is equivalent to the identities $\cap_{\alpha<\lambda}(GA^{p^\alpha}) = G(\cap_{\alpha<\lambda}A^{p^\alpha}) = GA^{p^\lambda}$ for each prime integer p and each limit ordinal $\lambda < \mu$. If they are satisfied for a single prime p, G is termed *almost p-nice*; so G is almost nice if and only if it is almost p-nice for every prime natural p.

Similarly, the subgroup G of a group A is named *q-periodically nice* in A provided the following intersection equality $\cap_{\alpha<\lambda}(GA_qA^{q^\alpha}) = G(\cap_{\alpha<\lambda}(A_qA^{q^\alpha}))$ is true for any limit ordinal λ. Apparently, if A is p-mixed, that is $A_q = 1, \forall\, q \neq p$, then the q-periodically nice property implies an almost q-nice property and conversely, whereas if A is torsion, that is $A = \coprod_{\forall\, p} A_p$, then $A_qA^{q^\alpha} = A$ and so the q-periodically niceness over each prime number q is ever fulfilled. To simplify the terms, if the required ratio holds for $\lambda = \omega$, the first infinite ordinal, G is said to be *weakly q-periodically nice* in A.

After all, we are ready to reformulate the above definition as follows: For any ordinal number μ, a group G is called a μ-elementary W-group if there is a Warfield group A of p-length not exceeding μ so that G is almost balanced and weakly q-periodically nice subgroup of A for all primes $q \neq p$ with a Warfield quotient A/G.

By a W-group, we shall mean any group which can be expressed as the direct sum of a Warfield group and μ-elementary W-groups for various μ's. Since as far as structure and classification are concerned, we may restrict our attention to W-groups that are reduced; hereafter we will assume in subsequent explorations that any given W-group decomposes into a direct sum of μ-elementary W-groups for various limit ordinals μ.

With this in hand and this that a direct sum of Warfield groups is a Warfield group too (see, for example, [12]), we observe that the W-groups properly encompass both the torsion A-groups of P. Hill [4], the p-local B-modules of P. Hill–M. Lane–C. Megibben [5] and the global mixed Warfield groups of R. Warfield, Jr. [12] as well.

Before proving the main affirmation, we need a group-theoretic isomorphism assertion. A major consequence from the remarkable attainment of Hill–Megibben [7], which is necessary for the evidence of our theorem, is like this:

Prorosition 1.1 (Uniqueness). *Let G and H be almost balanced subgroups of a Warfield group A with identical Ulm-Kaplansky and Warfield invariants such that $A/G \cong A/H$. Then $G \cong H$.*

This isomorphism claim improves the corresponding one in [6] obtained for the p-local case.

We also need in the sequel a series of crucial technical constructions. The proofs of the following three lemmas are rather routine and are therefore omitted.

Lemma 1.1. *If $G \leq A$ is a q-isotype subgroup, then*

$$G \cap (A^{q^\delta} A_q) = G^{q^\delta} G_q \quad \textit{and} \quad (GA_q) \cap A^{q^\delta} = (GA_q)^{q^\delta},$$

for each ordinal number δ.

Lemma 1.2. *If $G \leq A$ is a q-pure subgroup, then*

$$(GA^{q^n})_q = G_q A_q^{q^n},$$

for every positive integer n.

Lemma 1.3. *If C is a p-mixed group, then*

$$C^{q^{\omega}} = C^{q^{\omega+1}},$$

for all primes $q \neq p$.

We are now prepared to prove an elementary but, however, useful lemma.

Lemma 1.4. *Given A is a group and G is its pure subgroup. Then the factor-group A/G is p-mixed provided A is p-mixed and, even more, $A/G(\coprod_{q\neq p} A_q)$ is always p-mixed.*

Proof. Choose $x = aG$ with $a \in A$ so that $x^q = G$. Hence $a^q \in G$ and $a^q = b^q$ for some $b \in G$. Thus $a = b \in G$, so we are done.

Next, turning to the general situation, for $y = cG(\coprod_{q\neq p} A_q)$ with $c \in A$ and $y^q = G(\coprod_{q\neq p} A_q)$ we deduce that $c^q \in G(\coprod_{q\neq p} A_q)$. Therefore, there is $m \in \mathbb{N}$ such that $(m,p) = 1$ and $c^{qm} \in G^m \cap A^{qm} = G^{qm}$. Hence, there exists $g \in G$ such that $(cg^{-1})^{qm} = 1$. But we observe that $(qm,p) = 1$ and thus $cg^{-1} \in \coprod_{q\neq p} A_q$, i.e. $c \in G(\coprod_{q\neq p} A_q)$. Thereby $y = G(\coprod_{q\neq p} A_q)$ and everything is done. The proof is ended. □

Remark 1.1. If tK denotes the torsion part (= the maximal torsion subgroup) of any group K, we come now to the more precise version of the last lemma. Specifically, under the given circumstances, its evidence leads us to the isomorphism relations:

$$t(A/G) = (tA)G/G \cong tA/tG = A_p/G_p \cong (A/G)_p$$

and

$$t\Big(A/G\Big(\coprod_{q\neq p} A_q\Big)\Big) = tA\Big(G\Big(\coprod_{q\neq p} A_q\Big)\Big)/G\Big(\coprod_{q\neq p} A_q\Big)$$
$$\cong tA/\Big[tG\Big(\coprod_{q\neq p} A_q\Big)\Big] = \Big(A_p \times \coprod_{q\neq p} A_q\Big)/\Big(G_p \times \coprod_{q\neq p} A_q\Big) \cong A_p/G_p.$$

The following lemma is of independent interest.

Lemma 1.5. *If G is q-isotype in A, then G_q being nice in A implies that G_q is nice in G.*

Proof. Owing to the definition, for each limit ordinal τ, we have $\cap_{\alpha<\tau}(G_q G^{q^{\alpha}}) \subseteq \cap_{\alpha<\tau}(G_q A^{q^{\alpha}}) = G_q A^{q^{\tau}}$, hence the modular law ensures that $\cap_{\alpha<\tau}(G_q G^{q^{\alpha}}) \subseteq (G_q A^{q^{\tau}}) \cap G = G_q(A^{q^{\tau}} \cap G) = G_q G^{q^{\tau}}$, which gives the desired implication. The proof is over. □

The following four technicalities play a central role.

Lemma 1.6. *If G is q-isotype and q-periodically nice in A and G_q is nice in A, then G is q-nice in A.*

Proof. Referring to the definition, for every limit ordinal τ, we have

$$\bigcap_{\alpha<\tau}(GA^{q^\alpha}) \subseteq \bigcap_{\alpha<\tau}(GA_qA^{q^\alpha}) = G\Big(\bigcap_{\alpha<\tau}(A_qA^{q^\alpha})\Big).$$

Hence Lemma 1.1 along with the modular law guarantee that

$$\bigcap_{\alpha<\tau}(GA^{q^\alpha}) \subseteq \Big[G\Big(\bigcap_{\alpha<\tau}(A_qA^{q^\alpha})\Big)\Big]\bigcap\Big[\bigcap_{\alpha<\tau}(GA^{q^\alpha})\Big]$$
$$= G\Big[\bigcap_{\alpha<\tau}(A_qA^{q^\alpha})\bigcap\Big(\bigcap_{\alpha<\tau}(GA^{q^\alpha})\Big)\Big] = G\Big[\bigcap_{\alpha<\tau}\Big((A_qA^{q^\alpha})\bigcap(GA^{q^\alpha})\Big)\Big]$$
$$= G\Big[\bigcap_{\alpha<\tau}\Big(A^{q^\alpha}\Big(G\bigcap(A_qA^{q^\alpha})\Big)\Big)\Big] = G\Big[\bigcap_{\alpha<\tau}(A^{q^\alpha}G_q)\Big]$$
$$= G\Big[\bigcap_{\alpha<\tau}A^{q^\alpha}\Big] = GA^{q^\tau}.$$

The proof is complete. □

Lemma 1.7. *If G is isotype in A, then $G(\coprod_{q\neq p}A_q)/\coprod_{q\neq p}A_q$ is so in $A/\coprod_{q\neq p}A_q$.*

Proof. Foremost, since it is clear that A_q is p-divisible for each $q\neq p$ and thereby $\coprod_{q\neq p}A_q$ is p-nice in A, appealing to the modular law it follows at once that

$$\Big[G\Big(\coprod_{q\neq p}A_q\Big)/\coprod_{q\neq p}A_q\Big]\bigcap\Big(A/\coprod_{q\neq p}A_q\Big)^{p^\alpha}$$
$$= \Big[G\Big(\coprod_{q\neq p}A_q\Big)/\coprod_{q\neq p}A_q\Big]\bigcap\Big[A^{p^\alpha}/\coprod_{q\neq p}A_q\Big]$$
$$= \Big[\Big(G\Big(\coprod_{q\neq p}A_q\Big)\Big)\bigcap A^{p^\alpha}\Big]/\coprod_{q\neq p}A_q = G^{p^\alpha}\Big(\coprod_{q\neq p}A_q\Big)/\coprod_{q\neq p}A_q$$
$$\subseteq \Big[G\Big(\coprod_{q\neq p}A_q\Big)\Big]^{p^\alpha}/\coprod_{q\neq p}A_q \subseteq \Big[G\Big(\coprod_{q\neq p}A_q\Big)/\coprod_{q\neq p}A_q\Big]^{p^\alpha},$$

for any ordinal number α.

Next, to establish the case for primes q distinct from p, applying Lemma 1.3, we can bound the attention to ordinals no more than the ordinal ω. Furthermore, by making use of Lemma 1.1 and the modular law, we subsequently compute

$$\Big[G\Big(\coprod_{q\neq p}A_q\Big)/\coprod_{q\neq p}A_q\Big]\bigcap\Big[A/\coprod_{q\neq p}A_q\Big]^{q^\omega}$$
$$=\Big[G\Big(\coprod_{q\neq p}A_q\Big)/\coprod_{q\neq p}A_q\Big]\bigcap\bigcap_{n<\omega}\Big[A^{q^n}\Big(\coprod_{q\neq p}A_q\Big)/\coprod_{q\neq p}A_q\Big]$$
$$=\bigcap_{n<\omega}\Big[\Big(G\Big(\coprod_{q\neq p}A_q\Big)\Big)\bigcap(A^{q^n}A_q)\Big]/\coprod_{q\neq p}A_q$$
$$=\bigcap_{n<\omega}\Big[\Big(\coprod_{q\neq p}A_q\Big)\Big(G\bigcap(A^{q^n}A_q)\Big)\Big]/\coprod_{q\neq p}A_q$$
$$=\bigcap_{n<\omega}\Big[G^{q^n}\Big(\coprod_{q\neq p}A_q\Big)\Big]/\coprod_{q\neq p}A_q=\bigcap_{n<\omega}\Big[G\Big(\coprod_{q\neq p}A_q\Big)/\coprod_{q\neq p}A_q\Big]^{q^n}$$
$$=\Big(G\Big(\coprod_{q\neq p}A_q\Big)/\coprod_{q\neq p}A_q\Big)^{q^\omega}.$$

The case for natural numbers is included in the previous one for the ordinal ω. The proof is completed. □

Lemma 1.8. *If G is weakly q-periodically nice and almost balanced in A, then $G(\coprod_{q\neq p}A_q)/\coprod_{q\neq p}A_q$ is so (or equivalently is almost balanced) in $A/\coprod_{q\neq p}A_q$.*

Proof. By exploiting the previous Lemma 1.7, it is enough to show only the almost niceness.

And so, it is not hard to verify that $G(\coprod_{q\neq p}A_q)/\coprod_{q\neq p}A_q$ is ever almost p-nice in $A/\coprod_{q\neq p}A_q$ since $G(\coprod_{q\neq p}A_q)$ is almost p-nice in A.

After this, to check the almost q-nice property for every prime integer $q\neq p$, invoking Lemma 1.3, whence we treat only the case for ordinal ω, we formally calculate that

$$\bigcap_{n<\omega}\Big[\Big(G\Big(\coprod_{q\neq p}A_q\Big)/\coprod_{q\neq p}A_q\Big)\Big(A/\coprod_{q\neq p}A_q\Big)^{q^n}\Big]$$
$$=\bigcap_{n<\omega}\Big[\Big(G\Big(\coprod_{q\neq p}A_q\Big)A^{q^n}\Big)/\coprod_{q\neq p}A_q\Big]=\Big[\bigcap_{n<\omega}\Big(G\Big(\coprod_{q\neq p}A_q\Big)A^{q^n}\Big)\Big]/\coprod_{q\neq p}A_q$$
$$=\Big[\bigcap_{n<\omega}(GA_qA^{q^n})\Big]/\coprod_{q\neq p}A_q=G\Big[\bigcap_{n<\omega}\Big(A^{q^n}\Big(\coprod_{q\neq p}A_q\Big)\Big)\Big]/\coprod_{q\neq p}A_q$$

$$= \Big[G\Big(\coprod_{q\neq p} A_q\Big)/\coprod_{q\neq p} A_q\Big]\Big[\Big(\bigcap_{n<\omega}\Big(A^{q^n}\Big(\coprod_{q\neq p} A_q\Big)\Big)\Big)/\coprod_{q\neq p} A_q\Big]$$
$$= \Big[G\Big(\coprod_{q\neq p} A_q\Big)/\coprod_{q\neq p} A_q\Big]\Big[\bigcap_{n<\omega}\Big(A^{q^n}\Big(\coprod_{q\neq p} A_q\Big)/\coprod_{q\neq p} A_q\Big)\Big]$$
$$= \Big[G\Big(\coprod_{q\neq p} A_q\Big)/\coprod_{q\neq p} A_q\Big]\Big[\bigcap_{n<\omega}\Big(A/\coprod_{q\neq p} A_q\Big)^{q^n}\Big]$$
$$= \Big[G\Big(\coprod_{q\neq p} A_q\Big)/\coprod_{q\neq p} A_q\Big]\Big(A/\coprod_{q\neq p} A_q\Big)^{q^\omega}.$$

The proof is concluded. □

Lemma 1.9. *Given that G is a q-pure subgroup of A. If A/G is Warfield, then so does $A/G(\coprod_{q\neq p} A_q)$.*

Proof. Since $A/G/G(\coprod_{q\neq p} A_q)/G \cong A/G(\coprod_{q\neq p} A_q)$ and, by virtue of Lemma 1.2, $\coprod_{q\neq p}(A/G)_q = \coprod_{q\neq p}(A_qG/G) = G(\coprod_{q\neq p} A_q)/G$, the claim follows by the same group arguments described in the proof of the main result from [1], namely that $A/G/\coprod_{q\neq p}(A/G)_q$ is Warfield because so is A/G. The proof is finished. □

Combining Lemmas 1.6 to 1.9, we establish the following key statement.

Prorosition 1.2. *If G is a μ-elementary W-group, then the same is $G/\coprod_{q\neq p} G_q$.*

Proof. Since $\coprod_{q\neq p} A_q$ is always p-nice in A and A possesses p-length at most μ, it simple follows that $A/\coprod_{q\neq p} A_q$ has also p-length less than or equal to μ. Next, we obviously observe that $G/\coprod_{q\neq p} G_q \cong G(\coprod_{q\neq p} A_q)/\coprod_{q\neq p} A_q$ since by the modular law $G \cap (\coprod_{q\neq p} A_q) = (\coprod_{q\neq p} G_q \times G_p) \cap (\coprod_{q\neq p} A_q) = (\coprod_{q\neq p} G_q)(G_p \cap \coprod_{q\neq p} A_q) = \coprod_{q\neq p} G_q$; notice that $z \in \coprod_{q\neq p} G_q \iff z \in \coprod_{\forall p} G_p = \coprod_{q\neq p} G_q \times G_p$ and $(order(z), p) = 1$. The proof is closed. □

The author feels that the following query is of some interest and importance.

Question: Does it follow that G is a (μ-elementary) W-group if and only if $G/\coprod_{q\neq p} G_q$ is a (μ-elementary) W-group for all primes p? In particular, whether G is a Warfield group if and only if so does $G/\coprod_{q\neq p} G_q$ for every prime number p?

2. Group Algebras of W-Groups

In all that follows, F is a field of characteristic p. As usual, the letter FG is saved for the group algebra of a group G over F with a group of normalized units, abbreviated by, $V(FG)$. We denote by $S(FG)$ the Sylow p-subgroup of $V(FG)$. For a subgroup M of G, we designate $I(FG; M)$ as the relative fundamental ideal of augmentation 0 in FG with respect to M. All other unexplained exclusively notions, notations and terminology follow essentially the cited in the bibliography research sources.

In this paper, we resolve the isomorphism question in the affirmative for the alluded to above W-groups. Actually, we extract a more strong version of the isomorphism theorem, which includes the p-mixed case, thus insuring the determination of the group basis from its group algebra.

Specifically, we are able to proceed by arguing the following.

Theorem 2.1 (Isomorphism Structure). *Suppose G is a μ-elementary W-group for some arbitrary ordinal μ such that $FG \cong FH$ as F-algebras for any group H. Then,*

$$G/\coprod_{q\neq p} G_q \cong H/\coprod_{q\neq p} H_q.$$

In particular, when G is p-mixed, $G \cong H$.

Proof. Since $FG \cong FH$ yields $F(G/\coprod_{q\neq p} G_q) \cong F(H/\coprod_{q\neq p} H_q)$, we may precisely assume via Proposition 1.2 that G is p-mixed. It is no harm in assuming that $FG = FH$ as well, so H is another linear basis. Consequently, it is well-known that H is p-mixed and jointly with G have equal Ulm-Kaplansky and Warfield invariants (see, for instance, [1] or [2]).

The next assertion, that we can presume the containing Warfield group A as p-mixed, is our principal tool. In order to obtain this, we see with the help of Lemmas 1.8, 1.9 and [1] that $G \cong G(\coprod_{q\neq p} A_q)/\coprod_{q\neq p} A_q$ is an almost balanced subgroup of the Warfield group $A/\coprod_{q\neq p} A_q$ with a Warfield factor isomorphic to $A/G(\coprod_{q\neq p} A_q)$, because by definition G is almost balanced in the Warfield group A with A/G Warfield.

Moreover, in virtue of [2] and [3], the p-mixed group A is balanced in $V(FA) = AS(FA)$, whence G is almost balanced in $V(FA)$ and thus $V(FG)$ is almost balanced in $V(FA)$ as well. But, again from [2] and [3], H is balanced in $V(FH) = V(FG)$. It easily follows now, via transitivity, that H is almost balanced in $V(FA)$. Finally, we summarize that there exists a p-mixed group $V(FA)$ containing both G and H as its almost balanced subgroups.

On the other hand, we have detected in [3] that $V(FA) \cong A \times V(FA)/A$ where $V(FA)/A$ is totally projective p-torsion. Therefore, $V(FA)/G \cong A/G \times V(FA)/A$. Henceforth, according to Lemma 1.4 and [12], $V(FA)/G$ must be p-mixed Warfield.

Besides, one can derive that $F(V(FA)/G) \cong F(V(FA)/H)$ because of the validity of the following isomorphism sequence

$$\begin{aligned} F(V(FA)/G) &\cong F(V(FA))/I(F(V(FA)); G) \\ &= F(V(FA))/F(V(FA)).I(FG; G) \\ &= F(V(FA))/F(V(FA)).I(FH; H) \\ &= F(V(FA))/I(F(V(FA)); H) \cong F(V(FA)/H). \end{aligned}$$

Thereby, with the aid of [1] or [2], we establish that $V(FA)/G \cong V(FA)/H$. As a final step, the above listed uniqueness corollary assures that $G \cong H$, as wanted.

The proof is deduced. $\square$

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Contact information

Peter Danchev
Mathematical Department,
Plovdiv University,
4000 Plovdiv,
Bulgaria
E-Mail: pvdanchev@yahoo.com